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Research Article

Complex Dynamics of Some Hamiltonian Systems: Nonintegrability of Equations of Motion

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The main purpose of this paper is to study the complexity of some Hamiltonian systems from the view of nonintegrability, including the planar Hamiltonian with Nelson potential, double-well potential, and the perturbed elliptic oscillators Hamiltonian. Some numerical analyses show that the dynamic behavior of these systems is very complex and in fact chaotic in a large range of their parameter. I prove that these Hamiltonian systems are nonintegrable in the sense of Liouville. My proof is based on the analysis of normal variational equations along some particular solutions and the investigation of their differential Galois group.

1. Introduction

The topic of integrability and nonintegrability for systems of nonlinear differential equations is one of the central problems in the qualitative theory of ODEs. A system can be regarded as integrable if it admits such a number of first integrals or other tensor invariants that it is solvable by quadratures. The existence of k functionally independent first integrals can help us reduce the study of the considered n-dimensional system to $(n-k)$-dimensional one. Restricted on the level sets of first integrals, we can understand the complexity and topological structure of the considered dynamical system. For example, if a Hamiltonian system with n degrees of freedom is integrable in the sense of Liouville, i.e., it has n functionally independent first integrals in involution, then the invariant sets associated with first integrals are generically diffeomorphic to tori, cylinders, or planes inside the phase space. If the system is not integrable, we can expect it may exhibit a variety of chaotic phenomena or complex dynamical behavior.

In general, it is difficult to determine whether a system is integrable or not; see [1, 2] for instance. For all we know, the first results on the nonintegrability should go back to Poincaré [3], who found the influence of resonances of the eigenvalues of the monodromy matrix associated to a periodic orbit for small generic perturbations of integrable systems. Then, in the study of the rigid body with one fixed point [4], Kovalevskaya showed that a given system whose singularities are only poles turns out to be integrable in some sense. The idea of Kovalevskaya was the first trying to detect the integrability of dynamical systems by studying the singular structure of their general solutions.

Ziglin [5, 6] investigated the existence of meromorphic first integrals for analytic Hamiltonian systems. Based on properties of the monodromy group of a normal variational equation (NVE) along a complex integral curve, he presented necessary conditions for a $2n$-dimensional complex analytic Hamiltonian system to have n functionally independent rational first integrals. It should be pointed out that Ziglin's theorem does not deal with the integrability in the Liouville sense of the Hamiltonian system with arbitrary number of degrees of freedom in his original formulation since the considered first integrals may not be in involution. However, since the Poisson bracket of first integrals and the Hamiltonian always vanishes, Ziglin's results can be applied to study nonintegrability of the Hamiltonian system with

two degrees of freedom, such as Toda lattice Hamiltonian [7], Kepler problems [8], Lie-Poisson system [9], and Euler-Poisson system [6].

An important progress was made by Morales, Ramis, Simó [10, 11], and Baider, Churchill, Rod, and Singer [12] in the end of 20th century. They replaced the monodromy group of NVE with the differential Galois group of NVE and showed that the Liouville integrability of the nonlinear Hamiltonian system implies the integrability of the linear system NVE in the sense of expressibility of solutions in closed form.

Consider an analytic Hamiltonian system X_H on a complex analytic symplectic manifold M, and the equation of the motion for H reads

$$\frac{d}{dt}z = X_H, \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^{2n}, \ t \in \mathbb{C}. \tag{1}$$

The variational equation (VE) along a particular solution $z = \phi(t)$ of (1) has the form of

$$\frac{d}{dt}\xi = A(t)\xi, \quad A(t) = \frac{\partial X_H}{\partial z}(\phi(t)), \ \xi \in \mathbb{C}^{2n}. \tag{2}$$

By using the linear first integral $dH(\phi(t))$ of VE(2), (2) can be reduced into the NVE

$$\frac{d}{dt}\eta = B(t)\eta, \quad \eta \in \mathbb{C}^{2(n-1)}, \tag{3}$$

in suitable coordinates [10].

Theorem 1 (see [10]). *Let H be a Hamiltonian in $\mathbb{C}^{2n}$, and let Γ be a particular solution. Assume that there exist n rational first integrals of X_H (1) which are in involution and functionally independent in a neighborhood of Γ. Then, the identity component of the differential Galois group of NVE(3) is Abelian.*

Remark 2. As was pointed in Ref. [10], if the variational equation is Fuchsian, i.e., means that all singularities including infinity are regular, one can extend the class of first integrals and get the meromorphic version of Theorem 1.

Compared with Ziglin's theorem, the Morales-Ramis theory is more effective to study the nonintegrability of the Hamiltonian systems. On the one hand, the differential Galois group is bigger than the monodromy group and there are linear differential equations with trivial monodromy group but with nontrivial differential Galois group. On the other hand, the differential Galois group is an algebraic group, in particular, a Lie group, and can be calculated by infinitesimal methods. Since then, Morales-Ramis theory has been applied successfully for studying the nonintegrability of large numbers of physical models, such as the planar three-body problem [13–17], Hill's problem [18], generalized Yang-Mills Hamiltonian [19], Wilberforce spring-pendulum problem [20], and double pendula problem [21]. It should be pointed out that the differential Galois group can also be used to investigate the nonintegrability of general dynamical systems which may be non-Hamiltonian [22–24].

The aim of this paper is to investigate some Hamiltonian systems, including Nelson Hamiltonian, a double-well potential Hamiltonian, and perturbed elliptic oscillators Hamiltonian. By using Morales-Ramis theory and Kovacic's algorithm, we will contribute to the understanding of the complexity or the topological structure of these systems from the point of view of nonintegrability.

In order to apply the Morales-Ramis theory, it is necessary to know how to check if the identity component of the differential Galois group of a given linear system is Abelian. Generally speaking, this is a hard work. However, effective methods to investigate this question for lower dimensional systems with rational coefficients do exist. In particular, for second order equations over $\mathbb{C}(x)$ the algorithm of Kovacic [25] allows to decide whether the identity component of the differential Galois group is solvable. Based on the considerations above, we can solve the problems in this paper according to the following steps.

Firstly, finding a nonconstant particular solution Γ of the considered Hamiltonian system. Secondly, calculating the variational equation along Γ, furthermore, we can get the normal variational equation. Then, we transform the normal variational equation to an equation with rational coefficients by using a transformation which does not change the identity component of the differential Galois group. Finally, we investigate the differential Galois group of the equation with rational coefficients with the help of Kovacic's algorithm.

2. Main Results

2.1. Nonintegrability of Nelson Hamiltonian. Consider the Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(y - \frac{1}{2}x^2\right)^2 + \mu\frac{x^2}{2}, \tag{4}$$

where $x, y \in \mathbb{C}$, μ is an arbitrary real parameter. Its associated Hamiltonian system is

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= p_y, \\ \dot{p}_x &= x\left(2y - x^2 - \mu\right), \\ \dot{p}_y &= x^2 - 2y. \end{aligned} \tag{5}$$

This differential system is known as the Hamiltonian system with Nelson potential. Interest in it is coming from nuclear physics, where the deep valley could represent a collective degree of freedom which remains coupled to other types of excitation [26].

The Nelson Hamiltonian resembles the Hénon-Heiles potential and it also has a rich well-known periodic orbit structure [27]. Furthermore, the motion which it generates can be shown to be bounded and has been widely studied, in terms of periodic orbits [28], bifurcations [29], and quantum mechanically [30, 31].

The main goal of this section is to study the nonintegrability of system (5) by using Morales-Ramis theory and Kovacic's algorithm. Our result is formulated in the following theorem.

Theorem 3. *The system (5) is nonintegrable in the sense of Liouville.*

Proof of Theorem 3. It is easy to find that the manifold

$$\mathcal{N} = \left\{\left(x, y, p_x, p_y\right) \in \mathbb{C}^4 \mid x = p_x = 0\right\} \tag{6}$$

is invariant with respect to the flow of (5), and system (5) has a solution

$$\begin{aligned} \Gamma : x(t) &= 0, \\ y(t) &= \cos\sqrt{2}t, \\ p_x(t) &= 0, \\ p_y(t) &= -\sqrt{2}\sin\sqrt{2}t, \end{aligned} \tag{7}$$

which lies in $\mathcal{N}$. Let $(\xi_1, \xi_2, \eta_1, \eta_2)$ be variations in (x, y, p_x, p_y). Then, the variational equation of (4) along Γ reads

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2\cos\sqrt{2}t - \mu & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}. \tag{8}$$

Since the particular solution lies on $\mathcal{N}$, the equation for variables (ξ_1, η_1) forms a closed subsystem

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\eta}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\cos\sqrt{2}t - \mu & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}, \tag{9}$$

which is the so-called normal variational equation and is equivalent to

$$\ddot{\xi}_1 = \left(2\cos\sqrt{2}t - \mu\right)\xi_1. \tag{10}$$

Making a change of variable $z = \cos\sqrt{2}t$, we can transform (10) into

$$\xi_1'' = a(z)\,\xi_1' + b(z)\,\xi_1. \tag{11}$$

The prime denotes the derivative with respect to z, where

$$\begin{aligned} a(z) &= \frac{z}{1 - z^2}, \\ b(z) &= \frac{2z - \mu}{2\left(1 - z^2\right)}. \end{aligned} \tag{12}$$

Putting

$$\xi_1 = X \exp\left\{\frac{1}{2}\int a(\tau)\,d\tau\right\}, \tag{13}$$

then (11) can be changed to the reduced form

$$X'' = r(z)\,X, \tag{14}$$

where

$$\begin{aligned} r(z) = &-\frac{3}{16(z-1)^2} - \frac{3}{16(z+1)^2} - \frac{4\mu + 9}{16(z+1)} \\ &+ \frac{4\mu - 7}{16(z-1)}. \end{aligned} \tag{15}$$

In what follows, we will show that the differential Galois group of (14) is $SL(2, \mathbb{C})$. Clearly, the set of singularities of equation (14) is $\mathcal{P} = \{-1, 1, \infty\}$, the order at $z = -1$ and $z = 1$ is two, and the order at $z = \infty$ is one; necessary conditions for Cases 1 and 3 are not satisfied and only Case 2 should be analysed; see Theorem A.2 in Appendix. Indeed, by simple computation, we get

$$\begin{aligned} E_\infty &= \{1\}, \\ E_{-1} &= \{2, 1, 3\}, \\ E_1 &= \{2, 1, 3\}. \end{aligned} \tag{16}$$

However, for any $e_\infty \in E_\infty$, $e_{-1} \in E_{-1}$, $e_1 \in E_1$, $d = (1/2)(e_\infty - e_{-1} - e_1) \le (1/2)(1 - 1 - 1) < 0$, so Case 2 in Theorem A.2 in Appendix cannot hold, too. By Theorem A.2 in Appendix, the differential Galois group G of (14) is in Case 4, so the corresponding identity component of (14) is also $SL(2, \mathbb{C})$.

Assume system (5) is integrable in the sense of Liouville, then by Theorem 1 we conclude that the identity component of the differential Galois group of (10) is Abelian. Note that the transformation $z = \cos\sqrt{2}t$ does not change the identity component of this group (page 31, Theorem 2.5 [10]). Thus the identity component of (11) is Abelian. Since an Abelian group is always a solvable group and the solvability of (11) coincides with that of (14), we show that the identity component of (14) is solvable. However, $SL(2, \mathbb{C})$ is not solvable. We can complete the proof of this theorem. □

We remark that (10) is a particular case of Mathieu Equation and its solvability has been studied by P. Acosta-Humánez *et al.* [32]. In their works, they also investigated a large class of Hamiltonians with rational potentials which is close to Nelson potential.

By using Symplectic algorithm and Matlab one can observe a rich variety of behavior of the Nelson system in Poincaré cross section as the energy increases. For example, we consider the Poincaré cross section with $y = 0.01$; Figure 1 shows that the section changes from regular pattern to irregular pattern as the energy level increases from 0.01 to 0.15; this indeed implies the phenomena of chaos. Theorem 3 shows that (5) is nonintegrable. We all know that lack of first integrals can imply a complex behavior of phase curves of the system, and the numerical result just illustrates this point.

2.2. Nonintegrability of a Double-Well Potential Hamiltonian. Next, we consider the planar Hamiltonian with a double-well potential

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) - \alpha y^2 \exp\left(-\beta\left(x^2 + y^2\right)\right), \tag{17}$$

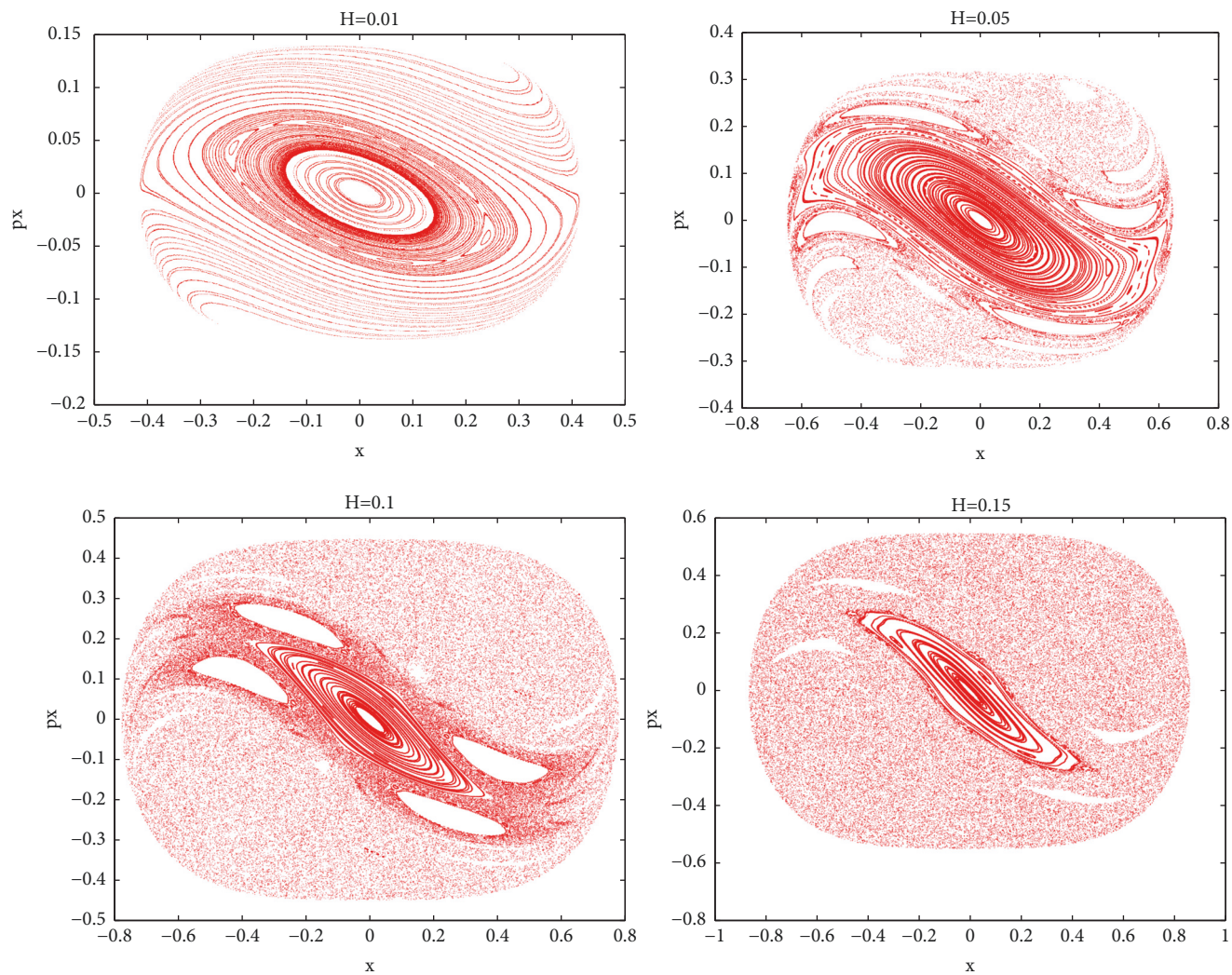

Figure 1: The Poincaré cross sections of the Nelson Hamiltonian system on the surface $y = 0.01$ when $\mu = 0.05$.

where α, β are real parameters. Its associated Hamiltonian equation is

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{y} &= p_y, \\ \dot{p}_x &= -2\alpha\beta x y^2 \exp\left(-\beta\left(x^2 + y^2\right)\right), \\ \dot{p}_y &= 2\alpha y\left(1 - \beta y^2\right)\exp\left(-\beta\left(x^2 + y^2\right)\right). \end{aligned} \tag{18}$$

The motivation for the choice of the potential $V = -\alpha y^2 \exp(-\beta(x^2 + y^2))$ comes from the interest of this potential in chaotic scattering [33, 34]. This system can be considered as a model to describe the electron scattering on the H_2^+ ion or the prototype for the trapping of incoming stellar objects by a collection of massive objects such as a solar system [35].

Numerical tests in [35] show that the motion of system (18) has sensitive dependence on initial conditions and chaotic scattering occurs when $\alpha = \beta = 1$. However, such tests do not exclude the possibility that the system (18) admits complex or chaotic behavior for other values of (α, β). In this work, from a view of the integrability, we study the nonintegrability of system (18) and contribute to understanding the complexity of system (18).

In the case of $\alpha\beta = 0$, obviously, the Hamiltonian system (18) is integrable. Suppose $\alpha\beta \neq 0$; we proceed to prove the theorem below.

Theorem 4. *For $\alpha\beta \neq 0$, the Hamiltonian system (18) is nonintegrable in the sense of Liouville.*

Proof of Theorem 4. It is easy to see that the manifold

$$\mathscr{M} = \left\{\left(x, y, p_x, p_y\right) \in \mathbb{C}^4 \mid x = p_x = 0\right\} \tag{19}$$

is invariant with respect to the flow generated by system (18). Indeed, system (18) restricted to $\mathscr{M}$ is given by

$$\begin{aligned} \dot{y} &= p_y, \\ \dot{p}_y &= -2\alpha y\left(\beta y^2 - 1\right)\exp\left(-\beta y^2\right), \end{aligned} \tag{20}$$

which is completely integrable with the first integral $H|_{\mathcal{M}} = (1/2)p_y^2 - \alpha y^2 \exp(-\beta y^2)$. Fixing the level set $H|_{\mathcal{M}} = 0$, we obtain a particular solution $(x, y, p_x, p_y) = (0, y(t), 0, \dot{y}(t))$ with $\dot{y}^2 = 2\alpha y^2 \exp(-\beta y^2)$ and denote by Γ the phase curve corresponding to this solution. The variational equation of (18) along the solution $(0, y(t), 0, \dot{y}(t))$ is given by

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\alpha\beta y^2 \exp\left(-\beta y^2\right) & 0 & 0 & 0 \\ 0 & M & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \tag{21}$$

where $M = (4\alpha\beta^2 y^4 - 10\alpha\beta y^2 + 2\alpha)\exp(-\beta y^2)$. The corresponding normal variational equation reads

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\eta}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\alpha\beta y^2 \exp\left(-\beta y^2\right) & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}, \tag{22}$$

or equivalently,

$$\ddot{\xi}_1 = \left(-2\alpha\beta y^2 \exp\left(-\beta y^2\right)\right)\xi_1. \tag{23}$$

The right-hand side of this equation is nontrivial provided $\alpha \neq 0$. Making a change of variable $z = y^2(t)$, then (23) becomes

$$\xi_1'' = a(z)\xi_1' + b(z)\xi_1, \tag{24}$$

and

$$a(z) = \frac{\beta z - 2}{2z}, \qquad b(z) = -\frac{\beta}{4z}. \tag{25}$$

Furthermore, let

$$\xi_1 = X \exp\left\{\frac{1}{2}\int a(\tau)\,d\tau\right\}. \tag{26}$$

Then (24) becomes

$$X'' = r(z)X, \tag{27}$$

where

$$r(z) = \frac{\beta^2}{16} - \frac{\beta}{2z} - \frac{1}{4z^2}. \tag{28}$$

Then we show that the differential Galois group of (27) is $SL(2,\mathbb{C})$. It follows from Theorem A.1 in Appendix that we need only to check that Cases 1, 2, and 3 cannot hold. Due to $\beta \neq 0$, we see that the function $r(z)$ admits two singular points $z_1 = 0$ with order two, and $z_2 = \infty$ with order zero. We can easily know that Case 3 in Theorem A.2 cannot hold. Moreover, the difference of exponents at $z = 0$ is zero. Then in a neighbourhood of $z = 0$, two independent local solutions have the form

$$\omega_1(z) = z^{1/2} f(z), \qquad \omega_2(z) = \omega_1(z)\ln(z) + z^{1/2} g(z), \tag{29}$$

where $f(z)$ and $g(z)$ are analytic at $z = 0$ and $f(0) \neq 0$. It follows that the differential Galois group can be only full triangular of $SL(2,C)$ (for details see [36, 37]). Hence, only Case 1 or 4 of the Kovacic's algorithm is possible.

For Case 1, a simple computation leads to

$$[\sqrt{r}]_0 = 0, \qquad \alpha_0^+ = \alpha_0^- = \frac{1}{2} \tag{30}$$

and

$$[\sqrt{r}]_\infty = \frac{|\beta|}{4}, \quad \nu = 0, \quad a = \frac{|\beta|}{4}, \quad b = -\frac{\beta}{2}, \quad \alpha_\infty^\pm \in \{1, -1\}. \tag{31}$$

It is not difficult to check that, for any $s(0), s(\infty) \in \{+,-\}$, $d = \alpha_\infty^{s(\infty)} - \alpha_0^{s(\infty)}$ is not a nonnegative integer. Therefore, Case 1 cannot hold.

Based on above discussion, we can make the conclusion that Case 4 in Theorem A.1 in Appendix holds; namely, the differential Galois group of (27) is $SL(2,\mathbb{C})$. Using the same argument as in the proof of Theorem 3, we can easily complete the proof of this theorem. □

2.3. Nonintegrability of Perturbed Elliptic Oscillators Hamiltonian. We study the following Hamiltonian which appeared in [38]:

$$H = \frac{1}{2}\left(x^2 + y^2 + z^2 + p_x^2 + p_y^2 + p_z^2\right) + \varepsilon\left(x^2y^2 + x^2z^2 + y^2z^2 - x^2y^2z^2\right), \tag{32}$$

where ε is an arbitrary real parameter. The Hamiltonian we consider above consists of three coupled harmonic oscillators

known as perturbed elliptic oscillators. Its associated Hamiltonian system is

$$\begin{aligned}\dot{x} &= p_x,\\ \dot{y} &= p_y,\\ \dot{z} &= p_z,\\ \dot{p}_x &= -x-\varepsilon\left(2xy^2+2xz^2-2xy^2z^2\right),\\ \dot{p}_y &= -y-\varepsilon\left(2x^2y+2yz^2-2x^2yz^2\right),\\ \dot{p}_z &= -z-\varepsilon\left(2x^2z+2y^2z-2x^2y^2z\right).\end{aligned} \tag{33}$$

The reason for the choice of this Hamiltonian is that perturbed elliptic oscillators appear very often in galactic dynamics and atomic physics [39–41]. Perturbed elliptic oscillators display exact periodic orbits, interesting sticky orbits together with large chaotic regions [42]. We want to show that the system there may cause complex behavior from the view of nonintegrability.

If $\varepsilon = 0$, it is easy to get the conclusion that the Hamiltonian system (33) is integrable. If $\varepsilon \neq 0$, we obtain the following theorem.

Theorem 5. *Assume $\varepsilon \neq 0$; then the system (33) is nonintegrable in the sense of Liouville.*

Proof of Theorem 5. Obviously, system (33) has a periodic solution

$$\begin{aligned}\Gamma : x(t) &= 0,\\ y(t) &= 0,\\ z(t) &= \sin t,\\ p_x(t) &= 0,\\ p_y(t) &= 0,\\ p_z(t) &= \cos t.\end{aligned} \tag{34}$$

It is easy to check that the variational equation of (32) along Γ is

$$\begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\\ \dot{\eta}_1\\ \dot{\eta}_2\\ \dot{\eta}_3\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ -1-2\varepsilon\sin^2 t & 0 & 0 & 0 & 0 & 0\\ 0 & -1-2\varepsilon\sin^2 t & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \eta_1\\ \eta_2\\ \eta_3\end{pmatrix}, \tag{35}$$

and corresponding normal variational equation is

$$\begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\eta}_1\\ \dot{\eta}_2\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1-2\varepsilon\sin^2 t & 0 & 0 & 0\\ 0 & -1-2\varepsilon\sin^2 t & 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \eta_1\\ \eta_2\end{pmatrix}. \tag{36}$$

Making a change of the location of the variable, we get the following equation:

$$\begin{pmatrix}\dot{\xi}_1\\ \dot{\eta}_1\\ \dot{\xi}_2\\ \dot{\eta}_2\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ -1-2\varepsilon\sin^2 t & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1-2\varepsilon\sin^2 t & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \eta_1\\ \xi_2\\ \eta_2\end{pmatrix}. \tag{37}$$

Clearly, the normal variational equation (37) along Γ is a system of two uncoupled second order linear differential equations

$$\ddot{\xi}_1 = -\left(1+2\varepsilon\sin^2 t\right)\xi_1, \tag{38}$$

$$\ddot{\xi}_2 = -\left(1+2\varepsilon\sin^2 t\right)\xi_2. \tag{39}$$

We denote by G_1, G_2, and G the differential Galois group of the equations (38), (39), and (37), respectively, over the field of rational functions on Γ. As a representation of an element of G is of the form

$$\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}, \tag{40}$$

where 0 is the 2×2 null matrix, $A \in G_1$, and $B \in G_2$, the identity component of G is not Abelian if the identity component of G_1 or G_2 is not Abelian. Then, we turn to consider the normal variational equation (38) and analyze the differential Galois group G_1.

In order to do that, we make the usual change of variable $z = \sin^2 t$; then (38) becomes

$$\xi_1'' = a(z)\xi_1' + b(z)\xi_1, \tag{41}$$

where

$$\begin{aligned}a(z) &= \frac{2z-1}{2z-2z^2},\\ b(z) &= -\frac{1+2\varepsilon z}{4z-4z^2}.\end{aligned} \tag{42}$$

Introducing the change of variable

$$\xi_1 = X \exp\left\{\frac{1}{2}\int a(\tau)\,d\tau\right\}, \tag{43}$$

we have that the equation (41) becomes

$$X'' = r(z)\,X, \tag{44}$$

where

$$r(z) = -\frac{3}{8z} - \frac{3}{16z^2} + \frac{3+4\varepsilon}{8(z-1)} - \frac{3}{16(z-1)^2}. \tag{45}$$

This theorem will follow from Theorem 1, if we can show that the differential Galois group of (44) is $SL(2,\mathbb{C})$.

Note that the set of singularities of equation (44) is $\mathscr{P} = \{0, 1, \infty\}$. The order at $z = 0$ and $z = 1$ is 2. It follows from $\varepsilon \neq 0$ that the order at $z = \infty$ is 1. We now proceed as in the proof of Theorem 3. By Theorem A.2 in Appendix Cases 1 and 3 do not occur.

For Case 2, by simple computation, we get

$$\begin{aligned} E_\infty &= \{1\}, \\ E_0 &= \{2, 1, 3\}, \\ E_1 &= \{2, 1, 3\}. \end{aligned} \tag{46}$$

However $d = (1/2)(e_\infty - e_{-1} - e_1) \leq (1/2)(1-1-1) < 0$; it is not a nonnegative integer, so Case 2 cannot happen. By Theorem A.2 in Appendix, we know that Cases 1, 2, and 3 in Theorem A.1 in Appendix cannot hold; we have thus proved the theorem. □

Appendix

A. Kovacic's Algorithm

Consider the differential equation

$$\xi'' = r\xi, \quad r \in \mathbb{C}(x), \tag{A.1}$$

where $\mathbb{C}(x)$ is the field of rational functions defined on the complex plane $\mathbb{C}$.

We recall that any general second order linear differential equation can be transformed to the form (A.1) by change of variable. Indeed, for equation

$$y'' = ay' + by, \quad a, b \in \mathbb{C}(x), \tag{A.2}$$

if we make the change of variable $y = e^{(1/2)\int a}\xi$, then (A.2) becomes

$$\xi'' = \left(\frac{a^2}{4} - \frac{a'}{2} + b\right)\xi. \tag{A.3}$$

To avoid triviality, we assume that r is not a constant. Then, one has the following.

Theorem A.1 (see [25]). *There are four cases that can occur.*

Case 1. G is conjugate to a triangular group. Then (A.1) has a solution of the form $e^{\int\omega}$ with $\omega \in \mathbb{C}(x)$.

Case 2. G is not of Case 1, but it is conjugate to a subgroup of

$$\begin{aligned} D = &\left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix} \mid c \in \mathbb{C},\ c \neq 0\right\} \\ &\cup \left\{\begin{pmatrix} 0 & c \\ -c^{-1} & 0\end{pmatrix} \mid c \in \mathbb{C},\ c \neq 0\right\}. \end{aligned} \tag{A.4}$$

Then (A.1) has a solution of the form $e^{\int\omega}$ with ω algebraic over $\mathbb{C}(x)$ of degree 2.

Case 3. G is not of Cases 1 and 2, but it is a finite group. Then all solutions of (A.1) are algebraic over $\mathbb{C}(x)$.

Case 4. $G = SL(2,\mathbb{C})$. Then (A.1) is not integrable in Liouville sense.

Furthermore, corresponding to the four cases listed in Theorem A.1, there are some easy necessary conditions for Cases 1, 2, and 3 which consequently form a sufficient condition for Case 4. Since $r \in \mathbb{C}(x)$, it can be represented as $r = s/t$ with $s, t \in \mathbb{C}[x]$ relatively prime. Then the poles of r are the zeros of t and we mean the order of the pole by the multiplicity of the zero of t. By the order of r at ∞ we shall refer the order of ∞ as a zero of r, i.e., $\deg t - \deg s$. Then we have the following.

Theorem A.2 (see [25]). *The following conditions are necessary for the respective cases of Theorem A.1 to hold.*

Case 1. Every pole of r must have even order or else have order 1. The order of r at ∞ must be even or else be greater than 2.

Case 2. r must have at least one pole that either has odd order greater than 2 or else has order 2.

Case 3. The order of a pole of r cannot exceed 2 and the order of r at ∞ must be at least 2. If the partial fraction expansion of r is

$$r = \sum_i \frac{\alpha_i}{(x-c_i)^2} + \sum_j \frac{\beta_j}{x-d_j}, \tag{A.5}$$

then $\sqrt{1+4\alpha_i} \in \mathbb{Q}$, for each i, and $\sum_j \beta_j = 0$. Furthermore, if we let

$$\gamma = \sum_i \alpha_i + \sum_j \beta_j d_j, \tag{A.6}$$

then $\sqrt{1+4\gamma} \in \mathbb{Q}$.

In what follows, we describe the Kovacic's algorithm [25]. Let $\mathscr{P}$ be the set of singularities of linear differential equation (A.1) containing the poles of $r(z)$ and ∞.

A.1. Case 1 of Kovacic's Algorithm.

Step 1. For each $c \in \mathscr{P}$ we define $[\sqrt{r}]_c$, $\alpha_c^{\pm}$ as follows.

(a) If c is a pole of order 1, then $[\sqrt{r}]_c = 0$, $\alpha_c^+ = \alpha_c^- = 1$.

(b) If c is a pole of order 2, then $[\sqrt{r}]_c = 0$, $\alpha_c^{\pm} = 1/2 \pm (1/2)\sqrt{1+4b}$, where b is coefficients of $1/(x-c)^2$ in the partial fraction expansion for r.

(c) If c is a pole of order $2\nu \geq 4$, then $[\sqrt{r}]_c = a/(x-c)^{\nu} + \cdots + d/(x-c)^2$ of negative order part of the Laurent series expansion of $\sqrt{r}$ at c, $\alpha_c^{\pm} = (1/2)(\pm(b/a)+\nu)$, where b is the coefficient of $1/(x-c)^{\nu+1}$ in r minus the coefficient of $1/(x-c)^{\nu+1}$ in $[\sqrt{r}]_c^2$.

(d) If the order of r at ∞ is > 2, then $[\sqrt{r}]_\infty = 0$, $\alpha_\infty^+ = 0$, $\alpha_\infty^- = 1$.

(e) If the order of r at ∞ is 2, then $[\sqrt{r}]_\infty = 0$, $\alpha_\infty^{\pm} = 1/2 \pm (1/2)\sqrt{1+4b}$, where b is coefficients of $1/x^2$ in the Laurent series expansion of r at ∞.

(f) If ∞ is a pole of order $-2\nu \leq 0$, then $[\sqrt{r}]_\infty = ax^{\nu} + \cdots + d$ of positive order part of the Laurent series expansion of $\sqrt{r}$ at ∞, $\alpha_\infty^{\pm} = (1/2)(\pm(b/a)+\nu)$, where b is the coefficient of $x^{\nu-1}$ in r minus the coefficient of $x^{\nu-1}$ in $[\sqrt{r}]_\infty^2$.

Step 2. Let $d = \sum_{c\in\mathscr{P}} t(c)\alpha_c^{s(c)}$, where $s(c) \in \{+,-\}$ for any $c \in \mathscr{P}$, $t(\infty) = +$ and $t(c) = -$ for any $c \in \mathscr{P} \setminus \{\infty\}$. If d is a nonnegative integer, then let $w = \sum_{c\in\mathscr{P}}(s(c)[\sqrt{r}]_c + \alpha_c^{s(c)}/(x-c))$; otherwise, the family is discarded. If no families remain under consideration, Case 1 of Theorem A.1 cannot happen.

Step 3. For each family retained from Step 2, we search for a monic polynomial P of degree d such that the equation $P'' + 2wP' + (w' + w^2 - r)P = 0$ holds. If such a polynomial exists, then $\xi = Pe^{\int w}$ is a solution of $\xi'' = r\xi$. If no such polynomial is found for any family retained from Step 2, Case 1 of Theorem A.1 cannot happen.

A.2. Case 2 of Kovacic's Algorithm.

Step 1. For each $c \in \mathscr{P}$ we define E_c as follows.

(a) If c is a pole of order 1, then $E_c = \{4\}$.

(b) If c is a pole of order 2, then $E_c = \{2 + k\sqrt{1+4b}, k = 0, \pm 2\} \cap \mathbb{Z}$, where b is coefficients of $1/(x-c)^2$ in the partial fraction expansion for r.

(c) If c is a pole of order $\nu > 2$, then $E_c = \{\nu\}$.

(d) If the order of r at ∞ is > 2, then $E_\infty = \{0, 2, 4\}$.

(e) If the order of r at ∞ is 2, then $E_\infty = \{2 + k\sqrt{1+4b}, k = 0, \pm 2\} \cap \mathbb{Z}$, where b is coefficients of $1/x^2$ in the Laurent series expansion of r at ∞.

(f) If ∞ is a pole of order $\nu < 2$, then $E_\infty = \{\nu\}$.

Step 2. Let $d = (1/2)\sum_{c\in\mathscr{P}} t(c)e_c$, where $e_c \in E_c$ for any $c \in \mathscr{P}$, $t(\infty) = +$ and $t(c) = -$ for any $c \in \mathscr{P} \setminus \{\infty\}$. If d is a nonnegative integer, then let $\theta = (1/2)\sum_{c\in\mathscr{P}} e_c/(x-c)$; otherwise, the family is discarded. If no families remain under consideration, Case 2 of Theorem A.1 cannot happen.

Step 3. For each family retained from Step 2, we search for a monic polynomial P of degree d such that the equation $P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0$ holds. If such a polynomial exists, let $\phi = \theta + P'/P$ and let w be a solution of the equation $w^2 + \phi w + ((1/2)\phi' + (1/2)\phi^2 - r) = 0$; then $\xi = e^{\int w}$ is a solution of $\xi'' = r\xi$. If no such polynomial is found for any family retained from Step 2, Case 2 of Theorem A.1 cannot happen.

A.3. Case 3 of Kovacic's Algorithm.

Step 1. For $n \in \{4, 6, 12\}$ fixed, we define E_c with $c \in \mathscr{P}$ as follows.

(a) If c is a pole of order 1, then $E_c = \{12\}$.

(b) If c is a pole of order 2, then $E_c = \{6 + (12k/n)\sqrt{1+4b}, k = 0, \pm 1, \pm 2, \ldots, \pm(n/2)\} \cap \mathbb{Z}$, where b is coefficients of $1/(x-c)^2$ in the partial fraction expansion for r.

(c) $E_\infty = \{6 + (12k/n)\sqrt{1+4b}, k = 0, \pm 1, \pm 2, \ldots, \pm(n/2)\} \cap \mathbb{Z}$, where b is coefficients of $1/x^2$ in the Laurent series expansion of r at ∞.

Step 2. Let $d = (n/12)\sum_{c\in\mathscr{P}} t(c)e_c$, where $e_c \in E_c$ for any $c \in \mathscr{P}$, $t(\infty) = +$ and $t(c) = -$ for any $c \in \mathscr{P} \setminus \{\infty\}$. If d is a nonnegative integer, then let $\theta = (n/12)\sum_{c\in\mathscr{P}} e_c/(x-c)$, $S = \prod_{c\in\mathscr{P}\setminus\{\infty\}}(x-c)$; otherwise, the family is discarded. If no families remain under consideration, Case 3 of Theorem A.1 cannot happen.

Step 3. For each family retained from Step 2, we search for a monic polynomial P of degree d such that the recursive equations

$$\begin{aligned} P_n &= P; \\ P_{i-1} &= -SP_i + \left((n-i)S' - S\theta\right)P_i \\ &\quad - (n-i)(i+1)S^2 r P_{i+1}; \\ &(i = n, n-1, \ldots, 0), \end{aligned} \tag{A.7}$$

with $P_{-1} \equiv 0$ hold. If such a polynomial exists, let w be a solution of the equation $\sum_{i=0}^{n}(S^i P_i/(n-i)!)w^i = 0$; then $\xi = e^{\int w}$ is a solution of $\xi'' = r\xi$. If no such polynomial is found for any family retained from Step 2, Case 3 of Theorem A.1 cannot happen.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

The author declares having no conflicts of interest.

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