Existence of Solution of Space–Time Fractional Diffusion-Wave Equation in Weighted Sobolev Space

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In this paper, we consider Cauchy problem of space-time fractional diffusion-wave equation. Applying Laplace transform and Fourier transform, we establish the existence of solution in terms of Mittag-Leffler function and prove its uniqueness in weighted Sobolev space by use of Mikhlin multiplier theorem. The estimate of solution also shows the connections between the loss of regularity and the order of fractional derivatives in space or in time.

1. Introduction

In this paper, we focus space-time fractional diffusion-wave equation

\[ C_0 \left( \frac{\partial}{\partial t} \right)^{\alpha_1} u + (-\Delta)^{\alpha_2/2} u = f(t, x), \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^n, \quad (1) \]

where \( C_0 \left( \frac{\partial}{\partial t} \right)^{\alpha_1} \) stands for the Caputo fractional partial derivative operator of order \( \alpha_1, \alpha_1 \in (0, 1) \cup (1, 2) \), \((-\Delta)^{\alpha_2/2}\) is the fractional Laplace differential operator of order \( \alpha_2, \alpha_2 \in (1, 2) \).

Fractional derivatives describe the property of memory and heredity of many materials, which is the major advantage compared with integer order derivatives. Fractional diffusion-wave equations are obtained from the classic diffusion equation and wave equation by replacing the integral order derivative terms by fractional derivatives of order \( \alpha \in (0, 1) \cup (1, 2) \). It has attracted considerable attention recently for various reasons, which include modeling of anomalous diffusive and subdiffusive systems, description of fractional random walk, wave propagation phenomenon, multiphase fluid flow problems, and electromagnetic theory. Nigmatullin [1, 2] pointed out that many of the universal electromagnetic, acoustic, and mechanical responses can be modeled accurately using the fractional diffusion-wave equations. Schneider and Wyss [3] presented the diffusion and wave equations in terms of integro-differential equations, and obtained the associated Greens functions in closed form in terms of the Foxs functions. Mbodje and Montseny [4] investigated the existence, uniqueness, and asymptotic decay of the wave equation with fractional derivative feedback, and showed that the method developed can easily be adapted to a wide class of problems involving fractional derivative or integral operators of the time variable. In fact, more numerical algorithms present an efficient method in solving the related problem [5–8]. The development of analytical methods is delayed since there are no analytic solutions in many cases [9–12]. Additional background, survey, and more applications of this field in science, engineering, and mathematics can be found in [13–20] and the references therein.

The fractional wave equation has been researched in all probability for the first time in [21] with the same order in space and in time, i.e., \( \alpha_1 = \alpha_2 \), where an explicit formula for the fundamental solution of this equation was established. Then this feature was shown to be a decisive factor for inheriting some crucial characteristics of the wave equation like a constant propagation velocity of both the maximum of its fundamental solution and its gravity and mass centers in [22]. Moreover, the first, the second, and the Smith centrovelocities of the damped waves described by the fractional wave equation are constant and depend just on the equation order.

While the fractional wave equation contains fractional derivatives of the same order in space and in time, we establish existence of solution of Cauchy problem to fractional wave equation (1) with different order in space and in time in weighted Sobolev spaces. The powers of the weighted show the connections between the loss of the regularity and the
orders of the fractional derivatives in space or in time. The main tools are Laplace transform, Fourier transform, Mikhlin multiplier theorem, and Mittag-Leffler functions. Applying the same technique, we also obtain the existence of solution of fractional diffusion equation.

This paper is organized as follows: In Section 2, the related fractional calculus definition and Laplace transform are introduced, the explicit solution of fractional differential equation is given by use of Mittag-Leffler functions. In Section 3, based on the main result given in Section 2, we show the existence and uniqueness of solution of space-time fractional diffusion-wave equation.

2. Laplace Transform and Fractional Calculus

In this section, we recall some necessary definitions and properties of fractional calculus, then use Laplace transform to consider initial value problem of the related fractional differential equation.

**Definition 1 ([19]).** The Riemann-Liouville fractional integral of order \( \alpha \) of a function \( f : (0, \infty) \to \mathbb{R} \) is defined by

\[
\mathcal{I}_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \Re(\alpha) > 0. \tag{2}
\]

**Definition 2 ([19]).** The Riemann-Liouville fractional derivative of order \( \alpha \in (0, 1) \cup (1, 2) \) of a function \( f : (0, \infty) \to \mathbb{R} \) is defined by

\[
\mathcal{D}_0^\alpha f(t) = \frac{d}{dt} \mathcal{I}_0^{\alpha-1} f(t). \tag{3}
\]

**Definition 3 ([19]).** The Caputo fractional derivative of order \( \alpha \) of a function \( f : (0, \infty) \to \mathbb{R} \) is defined by

\[
\mathcal{D}_0^\alpha f(t) = \begin{cases} \frac{d}{dt} \mathcal{I}_0^{\alpha-1} f(t), & \alpha \in (0, 1) \\ \mathcal{I}_0^{\alpha-2} f(t) - tf''(0^+), & \alpha \in (1, 2). \end{cases} \tag{4}
\]

The Mittag-Leffler function \( E_{\alpha,\beta}(z) \) [23] is represented by

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \tag{5}
\]

where \( \Re(\alpha) \) and \( \Re(\beta) \) denotes the real part of the complex numbers \( \alpha \) and \( \beta \), respectively.

**Lemma 1 ([23]).**

\[
\frac{d}{dy} E_{\alpha,\beta}(y) = \frac{E_{\alpha,\beta-1}(y) - (\beta - 1)E_{\alpha,\beta}(y)}{\alpha y}, \tag{6}
\]

\[
\frac{d^m}{dy^m} \left( y^{\beta-1} E_{\alpha,\beta}(y) \right) = y^\beta - m E_{\alpha,\beta-m}(y^\alpha), \quad \Re(\beta - m) > 0, \quad m \in \mathbb{N}. \tag{7}
\]

**Lemma 2 ([23]).** Let \( \alpha < 2, \beta \in \mathbb{R} \) and \( (\pi\alpha/2) < \mu < \min(\pi, \pi\alpha) \). Then we have the following estimate

\[
|E_{\alpha,\beta}(y)| \leq \frac{M}{1 + |y|}, \quad \mu \leq |\arg y| \leq \pi. \tag{8}
\]

where \( M \) denotes a positive constant.

**Lemma 3 ([24]).** For any \( \alpha > 0, \beta > 0 \) and \( \lambda \in \mathbb{C} \), there is

\[
\mathcal{L} \left[ t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) \right] = s^{\alpha-\beta} (s^\alpha - \lambda)^{-1}, \tag{9}
\]

with \( \Re(s) > ||\lambda||^{1/\alpha} \), where \( \Re(s) \) denotes the real part of the complex number \( s \), the Laplace transform of a function \( f(t) \) is defined by

\[
\mathcal{L} \left[ f(t) \right](s) = \int_0^\infty e^{-st} f(t) dt. \tag{10}
\]

The initial value problem of fractional differential equation for \( \alpha \in (0, 1) \),

\[
\begin{cases}
\mathcal{C}_0^\alpha D_t^\alpha u(t) = \lambda u(t) + f(t), & t > 0, \\
u(0) = u_0,
\end{cases} \tag{11}
\]

where \( \mathcal{C}_0^\alpha D_t^\alpha \) stands for a Caputo fractional derivative operator, \( u_0 \) is a constant number.

**Theorem 1 ([24]).** Consider the problem (11), then there is a explicit solution which is given in the integral form

\[
u(t) = \nu_0 E_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) f(s) ds. \tag{12}
\]

The initial value problem of fractional differential equation for \( \alpha \in (1, 2) \),

\[
\begin{cases}
\mathcal{C}_0^\alpha D_t^\alpha u(t) = \lambda u(t) + f(t), & t > 0, \\
u(0) = u_0, \\
u'(0) = u_1,
\end{cases} \tag{13}
\]

where \( \mathcal{C}_0^\alpha D_t^\alpha \) denotes a Caputo fractional derivative operator, \( u_i \) (\( i = 0, 1 \)) is a constant number.

**Theorem 2.** Consider the problem (13), then there is a explicit solution which is given in the integral form

\[
u(t) = \nu_0 E_{\alpha,1}(\lambda t^\alpha) + u_1 t E_{\alpha,2}(\lambda t^\alpha)
+ \int_0^t \left( (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-s)^\alpha) f(s) ds. \right. \tag{14}
\]

**Proof.** According to Definition 1–3, taking Laplace transform with respect to \( t \) in both sides of Eq. (13), we obtain

\[
\mathcal{L}[\nu(s)](s) = s^{\alpha-1} (s^\alpha - \lambda)^{-1} \nu_0 + s^{\alpha-2} (s^\alpha - \lambda)^{-1} u_1 + (s^\alpha - \lambda)^{-1} \mathcal{L}[f(t)]. \tag{15}
\]

The inverse Laplace transform using Lemma 3 yields

\[
\mathcal{L}^{-1} \left[ s^{\alpha-1} (s^\alpha - \lambda)^{-1} \right] = E_{\alpha,1}(\lambda t^\alpha) \tag{16}
\]

\[
\mathcal{L}^{-1} \left[ s^{\alpha-2} (s^\alpha - \lambda)^{-1} \right] = t E_{\alpha,2}(\lambda t^\alpha), \tag{17}
\]
\[ \mathcal{L}^{-1}[(s^\alpha - \lambda)^{-1}\mathcal{L}[f(t)]] \\
= \mathcal{L}^{-1}[(s^\alpha - \lambda)^{-1}] \ast f(t) \\
= t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) \ast f(t) \\
= \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda (t-s)^\alpha) f(s)ds. \]

Then substitute (15–18) into (13) which yields Theorem 2. 

\[ \text{3. Fourier Transform and the Main Result} \]

In this section, based on the results of Theorem 2, Mikhlin multiplier theorem, Mattag-Leffler function and Fourier transform, we establish the existence and uniqueness of solution of Cauchy problem of space-time fractional diffusion-wave equation in weighted Sobolev space.

Definition 4 ([25]). The Fourier transform of fractional laplace operator \((-\Delta)^\alpha\) is defined by

\[ \hat{\mathcal{F}}[(-\Delta)^\alpha f](\xi) = |\xi|^{2\alpha} \hat{f}(\xi), \]

where \( f \) satisfies \((-\Delta)^\alpha f \in L^2(\mathbb{R}^n), \rho \in [1, +\infty). \)

For more details of Fourier transformation, one can refer to [26, 27].

First, we consider the fractional wave equation, i.e., the Laplace operator

\[ \frac{\partial^\alpha u}{\partial t^\alpha} \in \mathcal{D}^a \mathcal{D}_t u(0, x) = \phi_1(x), \]

(\( \phi_2(\cdot) \) yields

\[ \left\{ \begin{array}{ll}
\frac{\partial^\alpha \hat{u}}{\partial t^\alpha}(0, \xi) = (-|\xi|^{2\alpha} \hat{u}(t, \xi) + \hat{f}(t, \xi), & \text{in } (-\infty, \infty) \times \mathbb{R}^n, \\
\hat{u}(0, \xi) = \hat{\phi}(\xi), & \end{array} \right. \]

(20)

where \( \hat{u}(t, \xi) = \mathcal{F}[u(t, \cdot)]. \)

Set \( \lambda = -|\xi|^{2\alpha} \) in (13), according to Theorem 2, the solution of (20) is given by

\[ \hat{u}(t, \xi) = \hat{\phi}_1(\xi)E_{\alpha,\alpha}(-|\xi|^{2\alpha} t^\alpha) + \hat{\phi}_2(\xi)tE_{\alpha,\alpha}(-|\xi|^{2\alpha} t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-|\xi|^{2\alpha} (t-s)^\alpha) \hat{f}(s, \xi)ds. \]

In terms of (2) in Lemma 1 and Lemma 2, by mathematical induction, we conclude.

Lemma 4. For each \( k \in \mathbb{Z}^+ \) and any \( \Re(\alpha) > 0, \beta \in \mathbb{R}, 0 < \delta < 1 \), there exists a positive constant \( C_k \) such that

\[ |\xi|^{\alpha k} \left| \frac{\partial^k}{\partial t^k} (y^\delta E_{\alpha,\beta}(y)) \right| \leq C_k. \tag{22} \]

Proof. For \( k = 1 \), Lemmas 1 (6) and 2.5 imply (22) holding.

For \( k = 2 \), \( y^2 (\frac{d}{dy} y^\delta) = (y(d/dy)y^\delta)^2 - y(d/dy) \) Then it is enough to show \( (y(d/dy)y^\delta E_{\alpha,\beta}(y)) \) is bounded since \( k = 1 \) holds. By a direct computation in terms of (6) that

\[ \left( y \frac{d}{dy} y^\delta \right) (y^\delta E_{\alpha,\beta}(y)) = \frac{1}{\alpha} y \frac{d}{dy} (y^\delta E_{\alpha,\beta}(1,y) - (\beta-1)E_{\alpha,\beta}(y)) \]

+ \delta y \frac{d}{dy} (y^\delta E_{\alpha,\beta}(y)). \tag{23} \]

This also reduces to the case for \( k = 1 \). Hence, (22) holds for \( k = 2 \).

In the following we conclude that \((y(d/dy)y^\delta E_{\alpha,\beta}(y))\) is bounded for any \( k \in \mathbb{Z}^+ \) by mathematical induction. Assume for \( k - 1 \), there exist

\[ |\xi|^{\alpha k-1} \left| \frac{d^{k-1}}{dy^{k-1}} (y^\delta E_{\alpha,\beta}(y)) \right| \leq C_{k-1}, \]

\[ y^{k-1} \frac{d^{k-1}}{dy^{k-1}} = \sum_{i=1}^{k-1} b_i \left( y \frac{d}{dy} \right)^i, \]

where \( b_i \) are constants. Then by use of (25), we have

\[ y^{k} \left( \frac{d}{dy} \right)^k (y^\delta E_{\alpha,\beta}(y)) = y \frac{d}{dy} \left( \sum_{i=1}^{k-1} b_i \left( y \frac{d}{dy} \right)^i (y^\delta E_{\alpha,\beta}(y)) \right) \]

\[ = \sum_{i=1}^{k} d_i \left( y \frac{d}{dy} \right)^i (y^\delta E_{\alpha,\beta}(y)). \]

It follows (24) and (26) that (22) is holding. 

Corollary 1. For each \( c \in \mathbb{R}^n \) and any \( \gamma > 0, \beta \in \mathbb{R}, 0 < \delta < 1 \), there exists a positive constant \( C_\gamma \) such that

\[ \left| \xi \right|^{\alpha |\gamma|} \left| \frac{\partial^{|\gamma|}}{\partial^{|\gamma|}} (y^\delta E_{\alpha,\beta}(y)) \right| \leq C_\gamma, \]

\[ \left| \xi \right|^{\alpha |\gamma|} \left| \frac{\partial^{|\gamma|}}{\partial^{|\gamma|}} (y^\delta E_{\alpha,\beta}(y)) \right| \leq C_\gamma, \]

where \( y = -|\xi|^{\alpha} \).

Next, we choose the version of Mikhlin’s multiplier theorem given in [28] as our Lemma.

Lemma 5. Let \( a(\xi) \) be the symbol of a singular integral operator \( A \) in \( \mathbb{R}^n \). Suppose that \( a(\xi) \in C^\infty((0, +\infty), L^p(\mathbb{R}^n)), \) and there is some positive constant \( M \) for all \( \xi \neq 0 \) such that

\[ |\xi|^{1/p} \frac{\partial^{|\gamma|}}{\partial^{|\gamma|}} a(\xi) \mid \leq M, \quad 0 \leq |\gamma| \leq 1 + \left| \frac{n}{2} \right|. \tag{29} \]

Then, \( A \) is a bounded linear operator from \( L^p(\mathbb{R}^n) \) into itself for \( 1 < p < +\infty \), and its operator norm depends only on \( M, n, \) and \( p. \)

Theorem 3. Set \( 1 < p < +\infty, \alpha_i \in (1, 2), i = 1, 2 \). Suppose \( \phi_i \in L^p(\mathbb{R}^n), i = 1, 2 \), \( f \in C^\infty((0, +\infty), L^p(\mathbb{R}^n)) \), then there is a unique solution \( u \) of Cauchy problem of space-time fractional diffusion-wave equation which is represented by

\[ u(t, x) = \phi_1(x) \ast \mathcal{F}^{-1}[E_{\alpha,\alpha}(-|\xi|^{2\alpha} t^\alpha)] + \phi_2(x) \]

\[ \ast \mathcal{F}^{-1}(\mathcal{F}_{\alpha,\alpha}(-|\xi|^{2\alpha} t^\alpha)) \]

\[ + \int_0^t (t-s)^{-\alpha} \mathcal{F}^{-1}[E_{\alpha,\alpha}(-|\xi|^{2\alpha} (t-s)^\alpha)] \]

\[ \ast f(s, x) ds, \]

and satisfies
Let \( u(t, \cdot) \in L^p(R^n) \) for any positive number \( \varepsilon \ll 1 \), where \( \mathcal{H}^{k,p}(R^n) \) denotes the classical homogeneous Sobolev space.

**Proof.** Taking inverse Fourier transform on (21), it is easy to obtain (30). Then, It follows (30) that
\[
\|u(t, \cdot)\|_{L^p(R^n)} \\
\leq \delta^{-1} \left( \int_{R^n} |\phi_\xi(t)\xi|^{\alpha} E_{\alpha,\beta}(y) \right) \|u(t, \cdot)\|_{L^p(R^n)} \\
+ \delta^{-1} \left( \int_{R^n} |\phi_\xi(t)\xi|^{\alpha} \sup_{s \in (0,t]} \|f(s, \cdot)\|_{L^p(R^n)} \right) \\
+ \delta^{-1} \left( \int_{R^n} |\phi_\xi(t)\xi|^{\alpha} \sup_{s \in (0,t]} \|f(s, \cdot)\|_{L^p(R^n)} \right) \\
+ \delta^{-1} \left( \int_{R^n} |\phi_\xi(t)\xi|^{\alpha} \sup_{s \in (0,t]} \|f(s, \cdot)\|_{L^p(R^n)} \right).
\]

Let \( y = -|\xi|^{\alpha} (t - s)^{\alpha} \), then (27) yields that
\[
\left| \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right| \leq C_{\gamma}.
\]

According to Corollary 1 and Lemma 5 for \( 0 \leq \delta \leq \alpha_\xi \) we have
\[
\left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \leq \Gamma(\alpha_\xi,\alpha_\beta) \|u(t, \cdot)\|_{L^p(R^n)}.
\]

Substitute (34–36) into (32), we get
\[
\left\|f(t, \cdot)\right\|_{L^p(R^n)} \\
\leq t^{-\alpha(\alpha/\beta)} \left( \left\|\phi_\xi(\xi)\xi\right\|_{L^p(R^n)} + \|\phi_\xi(\xi)\xi\|_{L^p(R^n)} + t^{\gamma}, \right.
\]
and satisfies
\[
\int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) = \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) = \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right).
\]

It is easy to verify that \( \alpha_\xi - 1 - (\alpha_\xi/\beta) > -1 \) holds for \( 0 \leq \delta \leq \alpha_\xi \) and \( 0 < \varepsilon \ll 1 \), then we have
\[
\left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \leq \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right).
\]

In terms of (37) and (39), sum up with \( \delta = 0, \alpha_\xi - \varepsilon \), we arrive at the following estimate
\[
\left\|f(t, \cdot)\right\|_{L^p(R^n)} \leq \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \leq \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right).
\]

Combining (39) and (40), we arrive at (31).

Then, we complete Theorem 3.

Next, we consider fractional diffusion equation (1), i.e., the case \( \alpha_\xi \in (0,1) \). Taking Fourier transform of space variables on Eq. (1) with the initial datum \( u(0, x) = \varphi(x) \) yields
\[
\left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \leq \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right).
\]

In terms of Theorem 1, we solve initial value problem of time fractional diffusion equation by taking a similar procedure in proving Theorem 3, then we directly give the conclusion without proof.

**Theorem 4.** Set \( 1 < p < +\infty, \alpha_\xi \in (0,1), \alpha_\beta \in (1,2) \). Suppose \( \varphi \in L^p(R^n), f \in C_{\alpha_\xi}(0, +\infty, L^p(R^n)) \), then there is a unique solution \( u \) of problem (41) which is represented by
\[
u(t, x) = \varphi_\xi(x) \ast \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right).
\]

and satisfies
\[
\int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \left( \int_{R^n} \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right) \right) \leq \left( \frac{\partial^\alpha}{\partial x^\alpha} y(x) E_{\alpha,\beta}(y) \right).
\]
\begin{align}
\|u(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \|u(\alpha_{1} / \alpha_{2})u(t, \cdot)\|_{H^{\alpha_{1} - \alpha_{2}}(\mathbb{R}^n)}
\leq \|f(t, \cdot)\|_{L^2(\mathbb{R}^n)} + \sup_{s \in (0, t)} \left\| f(s, \cdot) \right\|_{H^{\alpha_{1} - \alpha_{2}}(\mathbb{R}^n)}
\end{align}

for any positive number \( \varepsilon \ll 1 \), where \( H^{b, p}(\mathbb{R}^n) \) denotes the classical homogeneous Sobolev space.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interests.

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