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Research Article

Left- and Right-Shifted Fractional Legendre Functions with an Application for Fractional Differential Equations

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Two new orthogonal functions named the left- and the right-shifted fractional-order Legendre polynomials (SFLPs) are proposed. Several useful formulas for the SFLPs are directly generalized from the classic Legendre polynomials. The left and right fractional differential expressions in Caputo sense of the SFLPs are derived. As an application, it is effective for solving the fractional-order differential equations with the initial value problem by using the SFLP tau method.

1. Introduction

Legendre polynomials are a family of complete and orthogonal functions discovered by Adrien-Marie Legendre in 1782. As a very important application, Legendre spectral methods are successfully used to obtain numerical solutions of the various differential equations. Through Google Scholar search, there are almost 54,000 articles from 1980 to 2019 on the use of the Legendre spectral methods in the study of various problems, such as numerical solving for integrodifferential equations ([1–6]) and ordinary differential equations with fractional order ([7–9]) and integer order ([10]). Recently, the Legendre spectral method was proved to be an effective method to solve fractional differential equations, which has been studied by many scholars ([7, 8, 11–13]). More recently, many authors ([14–19]) applied Müntz orthogonal polynomials to solve the fractional-order differential equations (FDEs). Motivated by this literature, we define the left SFLPs by introducing the change of variable $z_L = 2((x-a)/(b-a))^{\alpha} - 1$. In particular, when $a = -1$, $b = 1$, and $\alpha = 1$, the left SFLPs degenerate the classic Legendre polynomials; while $a = 0$, $b = 1$, and $0 < \alpha < 1$, the left SFLPs are transformed into the fractional-order Legendre polynomials proposed in [7]. Similarly, the right SFLPs can be also obtained by introducing the change of variable $z_R = -2((b-x)/(b-a))^{\alpha} + 1$. Furthermore, to solve some FDEs, the SFLP tau method is better than the method based on the other orthogonal polynomials.

2. Shifted Fractional-Order Legendre Polynomials

In this section, we introduce some definitions, notations, and useful formulas about the shifted fractional-order Legendre polynomials. For the properties of classical Legendre polynomials, please refer to the literature [7, 11]. Now, we begin with the definition of Caputo fractional derivative.

Definition 1. (see [20]). For $m-1 < \alpha \le m$, $m \in Z_+$, $a, b \in R$, the left side and the right side Caputo fractional derivative is defined by

$$\begin{aligned} {}_a^C D_x^{\alpha} u(x) &= \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\xi)^{m-1-\alpha_u(m)} (\xi) d\xi, \\ {}_x^C D_b^{\alpha} u(x) &= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (x-\xi)^{m-1-\alpha_u(m)} (\xi) d\xi. \end{aligned} \tag{1}$$

Then, for $\alpha, \beta > 0$ and constant C, we have the following properties:

$$ {}_a^C D_x^{\alpha} C = {}_x^C D_b^{\alpha} C = 0, \tag{2}$$

$$ {}_a^C D_x^\alpha (x-a)^\beta = \begin{cases} 0, & \beta \in N_+, \quad \alpha > \beta, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-a)^{\beta-\alpha}, \end{cases} \tag{3} $$

$$ {}_x^C D_b^\alpha (b-x)^\beta = \begin{cases} 0, & \beta \in N_+, \quad \alpha > \beta, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-x)^{\beta-\alpha}. \end{cases} \tag{4} $$

Next, let us recall the definition of the classic Legendre polynomials. The classic Legendre polynomials, denoted by $L_n(z)$, $n = 0,1,\ldots$, are orthogonal on the interval $[-1, 1]$ with the orthogonality property

$$ \int_{-1}^{1} L_n(z) L_m(z) dz = \frac{2}{2n+1}\delta_{nm}, \tag{5} $$

where δ_{nm} is the Kronecker function. Now, in order to apply Legendre polynomials on the finite interval $[a, b]$, we define the left and right SFLPs by introducing the change of variable $z = z_L$ and $z = z_R$, respectively. Then, these two functions, denoted by $LL_n^\alpha(x)$ and $RL_n^\alpha(x)$, $n = 0,1,\ldots$, are orthogonal polynomials with the weight function $w_L(x) = (x-a)^{\alpha-1}$ and $w_R(x) = (b-x)^{\alpha-1}$, respectively, those are

$$ \begin{aligned} \int_a^b LL_n^\alpha(x) LL_m^\alpha(x) w_L(x) dx &= \frac{(b-a)^\alpha}{\alpha(2n+1)}\delta_{nm}, \\ \int_a^b RL_n^\alpha(x) RL_m^\alpha(x) w_R(x) dx &= \frac{(b-a)^\alpha}{\alpha(2n+1)}\delta_{nm}. \end{aligned} \tag{6} $$

Let $a = -1$, $b = 1$, we plot the first six terms of the left and right SFLPs for $\alpha = 0.5$ in Figure 1, and in Figure 2(b), the sixth term of the left SFLPs is shown for different values of the order α. For the classic Legendre polynomials, see Figure 2(a). The following are some useful formulas about the left and right SFLPs on the interval $[a, b]$, which are directly generalized from the classic Legendre polynomials.

(1) The analytic forms of the left SFLPs

$$ \begin{aligned} LL_n^\alpha(x) &= \sum_{j=0}^{n} Lc_{j,n}(x-a)^{j\alpha}, \\ Lc_{j,n} &= (-1)^{n+j}\frac{(n+j)!}{(b-a)^{j\alpha}(n-j)!(j!)^2}, \end{aligned} \tag{7} $$

and the right SFLPs

$$ \begin{aligned} RL_n^\alpha(x) &= \sum_{j=0}^{n} Rc_{j,n}(b-x)^{j\alpha}, \\ Rc_{j,n} &= (-1)^{j}\frac{(n+j)!}{(b-a)^{j\alpha}(n-j)!(j!)^2} \end{aligned} \tag{8} $$

(2) Three-term recurrence relations for the left SFLPs

$$ \begin{aligned} (n+1)LL_{n+1}^\alpha(x) = (2n+1)\left(2\left(\frac{x-a}{b-a}\right)^\alpha - 1\right)LL_n^\alpha(x) \\ - nLL_{n-1}^\alpha(x), \end{aligned} \tag{9} $$

with $LL_0^\alpha(x) = 1$ and $LL_1^\alpha(x) = 2((x-a)/(b-a))^\alpha - 1$, and the right SFLPs

$$ \begin{aligned} (n+1)RL_{n+1}^\alpha(x) = (2n+1)\left(-2\left(\frac{b-x}{b-a}\right)^\alpha + 1\right)RL_n^\alpha(x) \\ - nRL_{n-1}^\alpha(x), \end{aligned} \tag{10} $$

with $RL_0^\alpha(x) = 1$ and $RL_1^\alpha(x) = -2((b-x)/(b-a))^\alpha + 1$

(3) Derivative recurrence relations for the left SFLPs

$$ (4n+2)LL_n^\alpha(x) = (b-a)\left(\frac{x-a}{b-a}\right)^{1-\alpha}\left(LL_{n+1}^\alpha(x) - LL_{n-1}^\alpha(x)\right)', \tag{11} $$

and the right SFLPs

$$ (4n+2)RL_n^\alpha(x) = (b-a)\left(\frac{b-x}{b-a}\right)^{1-\alpha}\left(RL_{n+1}^\alpha(x) - LL_{n-1}^\alpha(x)\right)' \tag{12} $$

(4) The boundary values of the left and right SFLPs

$$ \begin{aligned} LL_n^\alpha(a) &= RL_n^\alpha(a) = (-1)^n, \\ LL_n^\alpha(b) &= RL_n^\alpha(b) = 1, \\ n &= 0, 1, 2, \cdots \end{aligned} \tag{13} $$

(5) The left and right Legendre's differential equations of fractional order

$$ \begin{aligned} &\left(\left(\frac{x-a}{b-a} - \left(\frac{x-a}{b-a}\right)^{\alpha+1}\right)\left(LL_n^\alpha(x)\right)'\right)' \\ &\quad + n(n+1)\left(\frac{\alpha}{b-a}\right)^2\left(\frac{x-a}{b-a}\right)^{\alpha-1}LL_n^\alpha(x) = 0, \\ &\left(\left(\frac{b-x}{b-a} - \left(\frac{b-x}{b-a}\right)^{\alpha+1}\right)\left(RL_n^\alpha(x)\right)'\right)' \\ &\quad + n(n+1)\left(\frac{\alpha}{b-a}\right)^2\left(\frac{b-x}{b-a}\right)^{\alpha-1}RL_n^\alpha(x) = 0 \end{aligned} \tag{14} $$

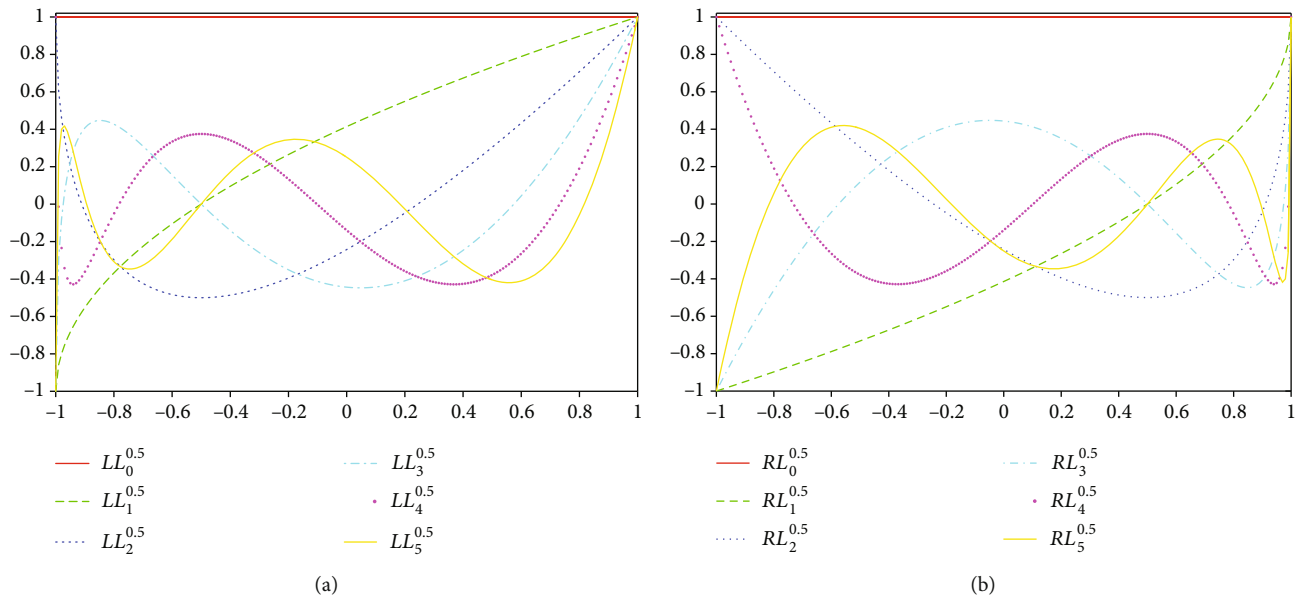

Figure 1: (a) The first six terms of the left SFLPs for $\alpha = 0.5$. (b) The first six terms of the right SFLPs for $\alpha = 0.5$.

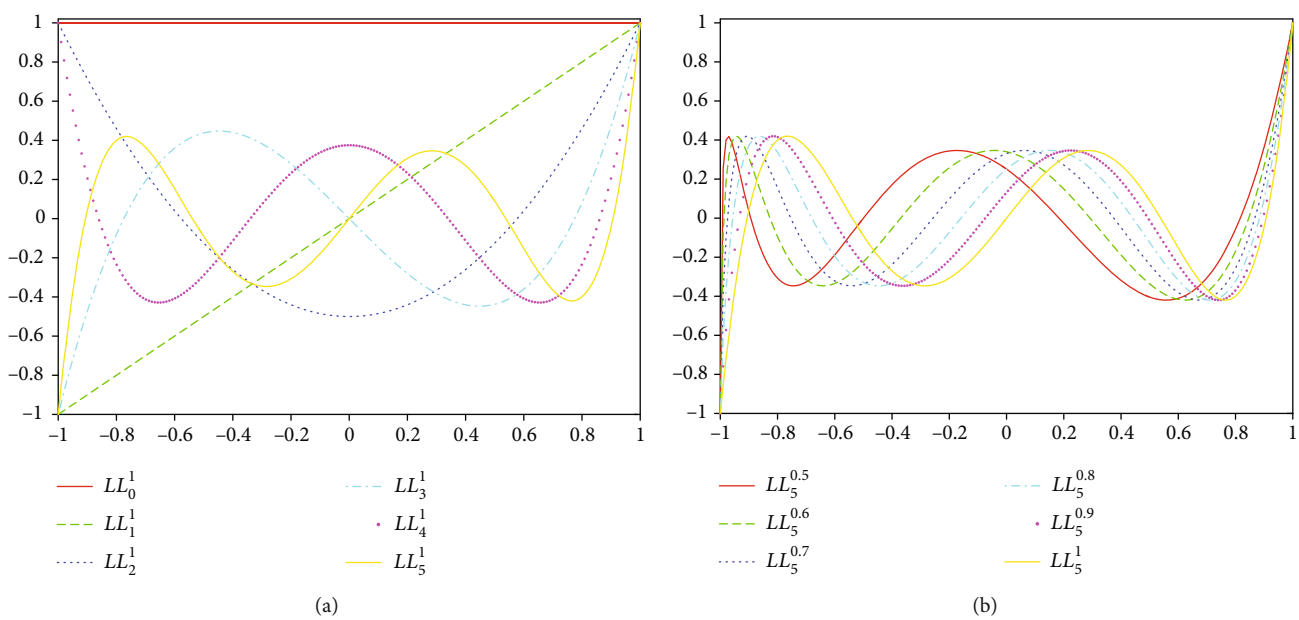

Figure 2: (a) The first six terms of the left SFLPs for $\alpha = 1$. (b) The sixth term of the left SFLPs for $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$, and 1.

In the following lemmas, we derive the fractional differential expressions of the left and right SFLPs in Caputo sense.

Lemma 2. *Let $\boldsymbol{\alpha} > 0$ and*

$$Ld_{i,j}^{\alpha} = \frac{\alpha(2j+1)}{(b-a)^{\alpha}} \int_a^b {}_a^C D_x^{\alpha} LL_i^{\alpha} \cdot (x) LL_j^{\alpha}(x) w_L(x) dx, \quad i, j = 0, 1, 2, \cdots. \tag{15}$$

Then, we have

$$Ld_{0,j} = 0,$$

$$Ld_{i,j}^{\alpha} = \sum_{n=1}^{i} \sum_{m=0}^{j} Lc_{n,i} Lc_{m,j} \cdot \frac{\alpha(2j+1)}{(b-a)^{\alpha}} \frac{\Gamma(n\alpha+1)(b-a)^{\alpha(n+m)}}{\alpha\Gamma((n-1)\alpha+1)(n+m)}, \quad i \geq 1, \tag{16}$$

where $Lc_{n,i}$ and $Lc_{m,j}$ are given by (7).

Proof. By (2), we have $Ld_{0,j} = 0$. Then, for $i \geq 1$, formulas (7), (2), and (3) lead to

$$\begin{aligned} Ld_{i,j}^{\alpha} &= \int_a^b \sum_{n=0}^{i} Lc_{n,i} {}_a^C D_x^{\alpha} (x-a)^{n\alpha} \sum_{m=0}^{j} Lc_{m,j} (x-a)^{m\alpha} w_L(x) dx \\ &= \int_a^b \sum_{n=1}^{i} Lc_{n,i} \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} \\ &\quad \cdot (x-a)^{(n-1)_\alpha} \sum_{m=0}^{j} Lc_{m,j} (x-a)^{m\alpha} w_L(x) dx \\ &= \sum_{n=1}^{i} \sum_{m=0}^{j} Lc_{n,i} Lc_{m,j} \left(\frac{\alpha(2j+1)}{(b-a)^{\alpha}} \right) \\ &\quad \cdot \frac{\Gamma(n\alpha+1)(b-a)^{\alpha(n+m)}}{\alpha\Gamma((n-1)\alpha+1)(n+m)}. \end{aligned} \tag{17}$$

From Lemma 2, it is elementary to get

$${}_a^C D_x^{\alpha} LL_i^{\alpha}(x) = \sum_{j=0}^{i-1} Ld_{i,j}^{\alpha} LL_j^{\alpha}(x), \tag{18}$$

with $Ld_{i,j}^{\alpha}$ given by (17). The following lemma on fractional differential expressions for the right SFLPs can be obtained similarly.

Lemma 3. *Let $\alpha > 0$ and*

$$\begin{aligned} Rd_{i,j}^{\alpha} &= \left(\frac{\alpha(2j+1)}{(b-a)^{\alpha}} \right) \int_a^b {}_x^C D_b^{\alpha} RL_i^{\alpha} \\ &\quad \cdot (x) RL_j^{\alpha}(x) w_R(x) dx, \quad i, j = 0, 1, 2, \cdots. \end{aligned} \tag{19}$$

Then, we have

$$\begin{aligned} &Rd_{0,j} = 0, \\ &Rd_{i,j}^{\alpha} = \sum_{n=1}^{i} \sum_{m=0}^{j} Rc_{n,i} Rc_{m,j} \left(\frac{\alpha(2j+1)}{(b-a)^{\alpha}} \right) \\ &\qquad \cdot \frac{\Gamma(n\alpha+1)(b-a)^{\alpha(n+m)}}{\alpha\Gamma((n-1)\alpha+1)(n+m)}, \quad i \geq 1, \end{aligned} \tag{20}$$

where $Rc_{n,i}$ and $Rc_{m,j}$ are given by (8).

3. Application

In this section, we give two examples to illustrate that our methods are effective. First, we apply the left SFLP tau method to solve the fractional-order differential equation of the following form:

$$\begin{cases} {}_{-1}^C D_x^{\alpha} u(x) + u(x) = f(x), & -1 < x < b, \\ u(-1) = 0. \end{cases} \tag{21}$$

Suppose $f(x) = (\Gamma(2\alpha+1)/\Gamma(\alpha+1)) \cdot (x+1)^{\alpha} + (x+1)^{2\alpha}$. Then, the exact solution of (21) is $u(x) = (x+1)^{2\alpha}$. Now, we use the left SFLP tau method to obtain it. Let

$$u(x) \simeq \sum_{i=0}^{n-1} c_i LL_i^{\alpha}(x) = C^T \phi(x), \tag{22}$$

with $C^T = [c_0, c_1, \cdots, c_n - 1]$ and $\phi(x)^T = [LL_0^{\alpha}(x), LL_1^{\alpha}(x), \cdots, LL_{n-1}^{\alpha}(x)]$. From Lemma 2, we have

$${}_{-1}^C D_x^{\alpha} u(x) \simeq \sum_{i=0}^{n-1} c_i {}_{-1}^C D_x^{\alpha} LL_i^{\alpha}(x) = C^T D_n^{\alpha} \phi(x), \tag{23}$$

with the matrix $D_n^{\alpha} = \{Ld_{i,j}\}_{n \times n}$, where $Ld_{i,j}$ is given by (17). Assume

$$f(x) \simeq \sum_{i=0}^{n-1} f_i LL_i^{\alpha}(x) = F^T \phi(x), \tag{24}$$

with $F^T = [f_0, f_1, \cdots, f_n - 1]$, where

$$f_i = \frac{\alpha(2i+1)}{(b+1)^{\alpha}} \int_0^1 f(x) LL_i^{\alpha}(x) w(x) dx, \quad i = 0, 1, \cdots, n-1. \tag{25}$$

Set $n = 3$. Then

$$\begin{aligned} &LL_0^{\alpha}(x) = 1, \\ &LL_1^{\alpha}(x) = 2\left(\frac{x+1}{b+1}\right)^{\alpha} - 1, \\ &LL_2^{\alpha}(x) = 6\left(\frac{x+1}{b+1}\right)^{2\alpha} - 6\left(\frac{x+1}{b+1}\right)^{\alpha} + 1, \\ &\quad f_0 = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} (b+1)^{\alpha} + \frac{1}{3}(b+1)^{2\alpha}, \\ &\quad f_1 = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} (b+1)^{\alpha} + \frac{1}{2}(b+1)^{2\alpha}, \\ &\quad f_2 = \frac{1}{6}(b+1)^{2\alpha}. \end{aligned} \tag{26}$$

By (22), (23), and (24), we have

$$C^T D_3^{\alpha} + C^T = F^T, \tag{27}$$

with three-order matrix

$$D_3^{\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ d_{1,0} & d_{1,1} & d_{1,2} \\ d_{2,0} & d_{2,1} & d_{2,2} \end{pmatrix}, \tag{28}$$

which, in accordance with $u(-1)=\sum_{i=0}^{2}c_iLL_i^{\alpha}(-1)=c_0-c_1+c_2=0$, yields

$$C^T=(b+1)^{2\alpha}\left[\frac{1}{3},\frac{1}{2},\frac{1}{6}\right]. \tag{29}$$

Using (22), we obtain the exact solution of (21). Obviously, we cannot get this exact solution by the classic Legendre tau method.

Next, we apply the right SFLP tau method to solve the fractional-order differential equation of the following form:

$$\begin{cases} {}_x^CD_1^{2\alpha}u(x)+{}_x^CD_1^{\alpha}u(x)+u(x)=f(x), & a<x<1,\\ {}_x^CD_1^{\alpha}u(1)=u(1)=0, \end{cases} \tag{30}$$

where ${}_x^CD_1^{2\alpha}u(x):={}_x^CD_1^{\alpha}({}_x^CD_1^{\alpha}u(x))$ and $f(x)=(\Gamma(3\alpha+1)/\Gamma(\alpha+1))\cdot(1-x)^{\alpha}+(\Gamma(3\alpha+1)/\Gamma(2\alpha+1))\cdot(1-x)^{2\alpha}+(1-x)^{3\alpha}$. Then, the exact solution of (30) is $u(x)=(1-x)^{3\alpha}$. Now, we use the right SFLP tau method to obtain it. Let

$$u(x)\simeq\sum_{i=0}^{n-1}c_iRL_i^{\alpha}(x)=C^T\varphi(x), \tag{31}$$

with $C^T=[c_0,c_1,\cdots,c_{n-1}]$ and $\varphi(x)^T=[RL_0^{\alpha}(x),RL_1^{\alpha}(x),\cdots,RL_{n-1}^{\alpha}(x)]$. From Lemma 3, we have

$${}_{-1}^CD_x^{\alpha}u(x)\simeq\sum_{i=0}^{n-1}c_{i-1}{}^CD_x^{\alpha}RL_i^{\alpha}(x)=C^TD_n^{\alpha}\varphi(x), \tag{32}$$

with the matrix $D_n^{\alpha}=\{Rd_{i,j}\}_{n\times n}$, where $Rd_{i,j}$ is given by Lemma 3. Similar to the calculation of the matrix D_n^{α} above, we get the fractional derivative matrix $D_n^{2\alpha}$, that is

$${}_{-1}^CD_x^{2\alpha}u(x)\simeq\sum_{i=0}^{n-1}c_{i-1}{}^CD_x^{2\alpha}RL_i^{\alpha}(x)=C^TD_n^{2\alpha}\varphi(x). \tag{33}$$

Assume

$$f(x)\simeq\sum_{i=0}^{n-1}f_iRL_i^{\alpha}(x)=F^T\varphi(x), \tag{34}$$

with $F^T=[f_0,f_1,\cdots,f_n-1]$, where

$$f_i=\frac{\alpha(2i+1)}{(1-a)^{\alpha}}\int_0^1 f(x)RL_i^{\alpha}(x)w(x)dx,\quad i=0,1,\cdots,n-1. \tag{35}$$

Set $n=4$. Then

$$\begin{aligned}
RL_0^{\alpha}(x)&=1,\\
RL_1^{\alpha}(x)&=-2\left(\frac{1-x}{1-a}\right)^{\alpha}-1,\\
RL_2^{\alpha}(x)&=6\left(\frac{1-x}{1-a}\right)^{2\alpha}-6\left(\frac{1-x}{1-a}\right)^{\alpha}+1,\\
RL_3^{\alpha}(x)&=-20\left(\frac{1-x}{1-a}\right)^{3\alpha}+30\left(\frac{1-x}{1-a}\right)^{2\alpha}-12\left(\frac{1-x}{1-a}\right)^{\alpha}+1,\\
f_0&=\frac{5}{20}(1-a)^{3\alpha}+\frac{\Gamma(3\alpha+1)}{3\Gamma(2\alpha+1)}(1-a)^{2\alpha}\\
&\quad+\frac{\Gamma(3\alpha+1)}{2\Gamma(\alpha+1)}(1-a)^{\alpha},\\
f_1&=\frac{-9}{20}(1-a)^{3\alpha}-\frac{\Gamma(3\alpha+1)}{2\Gamma(2\alpha+1)}(1-a)^{2\alpha}\\
&\quad-\frac{\Gamma(3\alpha+1)}{2\Gamma(\alpha+1)}(1-a)^{\alpha},\\
f_2&=\frac{5}{20}(1-a)^{3\alpha}+\frac{\Gamma(3\alpha+1)}{6\Gamma(2\alpha+1)}(1-a)^{2\alpha},\\
f_3&=\frac{-1}{20}(1-a)^{3\alpha}.
\end{aligned} \tag{36}$$

By (31), (32), (33), and (34), we have

$$C^TD_4^{2\alpha}+C^TD_4^{\alpha}+C^T=F^T, \tag{37}$$

with the fourth-order matrix

$$\begin{aligned}
D_4^{\alpha}&=\begin{pmatrix} 0 & 0 & 0 & 0\\ Rd_{1,0} & Rd_{1,1} & Rd_{1,2} & Rd_{1,3}\\ Rd_{2,0} & Rd_{2,1} & Rd_{2,2} & Rd_{2,3}\\ Rd_{3,0} & Rd_{3,1} & Rd_{3,2} & Rd_{3,3}\end{pmatrix},\\
D_4^{2\alpha}&=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \tilde{R}d_{2,0} & \tilde{R}d_{2,1} & \tilde{R}d_{2,2} & \tilde{R}d_{2,3}\\ \tilde{R}d_{3,0} & \tilde{R}d_{3,1} & \tilde{R}d_{3,2} & \tilde{\tilde{R}}d_{3,3}\end{pmatrix},
\end{aligned} \tag{38}$$

which, in accordance with boundary value conditions

$$\begin{aligned}
u(1)&=\sum_{i=0}^{3}c_iRL_i^{\alpha}(1)=c_0+c_1+c_2+c_3=0,\\
{}_x^CD_1^{\alpha}u(1)&=\sum_{i=0}^{3}c_{ix}{}^CD_1^{\alpha}RL_i^{\alpha}(1)=c_1+3c_2+6c_3=0,
\end{aligned} \tag{39}$$

yields

$$C^T = (1-a)^{3\alpha}\left[\frac{5}{20}, \frac{-9}{20}, \frac{5}{20}, \frac{-1}{20}\right]. \tag{40}$$

Finally, using (31), we obtain the exact solution of (30).

4. Conclusions

In this paper, the left- and right-shifted fractional-order Legendre polynomials are proposed by substituting the variables of the classic Legendre polynomials. Correspondingly, the differential expressions of these new polynomials for the left and right fractional derivatives in Caputo sense are derived, based on which the tau method can be used to solve the FDEs on the arbitrary finite interval $[a, b]$. Moreover, the results in this article are easy to generalize to the case of the other orthogonal functions, e.g., Chebyshev polynomials, which will be studied later.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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