Research Article

New Generalized Soliton Solutions for a (3 + 1)-Dimensional Equation

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In this paper, we investigate the nonlinear wave solutions for a (3 + 1)-dimensional equation which can be reduced to the potential KdV equation. We present generalized \( N \)-soliton solutions in which some arbitrarily differentiable functions are involved by using a simplified Hirota’s method. Our work extends some previous results.

1. Introduction

The research of the exact solutions for nonlinear equations plays a significant role in the investigation of nonlinear science. For nonlinear science, owing to containing more information and describing many natural phenomena, such as plasmas, gas dynamics, fluid mechanics, and elastic media, many nonlinear equations have been investigated [1–10]. Compared with the lower-dimensional nonlinear equations, the higher-dimensional nonlinear equations are more difficult to study because of the complex construction and multiple variables. Clarkson and Mansfield [3] introduced the following equation:

\[
\frac{u_{xxxt}}{\beta} - \frac{3}{2} \left( u_{xx} + u_{xt} \right) = 0,
\]

(1)

which they called the SWWI equation, and another equation

\[
\frac{u_{xxxt}}{\beta} - \frac{3}{2} \left( 2u_{xx} + u_{xt} \right) = 0,
\]

(2)

which they called the SWWII equation.

For Equation (1), Clarkson and Mansfield [3] obtained an interesting solution

\[
u(x, t) = \frac{3}{\beta} \tanh \frac{x + f(t)}{2} + \frac{3}{\beta} \tanh \frac{x - f(t)}{2} + \frac{t}{\beta},
\]

(3)

which contains an arbitrarily differentiable function \( f(t) \).

For Equation (2), it seems that there is no solution which is similar to (3). Several other equations and their multiple soliton solutions were investigated by Tian and Zhang [11, 12], Yue and Chen [13], and Wazwaz [14, 15]. Furthermore, the other new and interesting solutions of some important nonlinear equations were investigated, such as the NLS\((m, n)\) equation [16], the GP\((m, n)\) equation [17], and new nonlocal models [18].

Based on Equations (1) and (2), a (3 + 1)-dimensional equation

\[
u_{ztt} + \nu_{xxzt} - 6u_xu_{xyz} - 6u_{xy}u_{xz} = 0
\]

(4)

was introduced by Wazwaz [19] as a higher-dimensional shallow water wave equation. It is easy to see that Equation (4) can be reduced to the potential KdV equation for \( z = y = x \).

In [19], Wazwaz investigated multiple soliton solutions and multiple singular soliton solutions of Equation (4). For Equation (4), Wazwaz gave the solutions combined by \( e^{k_i x + r_j y + z - t s_j} i (i = 1, 2, 3) \), where \( k_i, r_j, \) and \( s_j \) are arbitrary constants. In [20], we succeed in constructing the following generalized soliton solutions:
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In this section, we state the main results for Equation (4) by form:

\[
u = \frac{-2k_1 \psi_1 e^{\psi_1}}{p(y, z) + \psi_1 e^{\psi_1}} + \gamma(t, y, z) \tag{5}\]

(1) The generalized 1-soliton solution

\[
u = \frac{-2(k_1 \psi_1 e^{\psi_1} + k_2 \psi_2 e^{\psi_2} + a_0(k_1 + k_2) \phi_1 \phi_2 e^{\psi_1 + \psi_2}}{p(y, z) + \psi_1 e^{\psi_1} + \psi_2 e^{\psi_2} + a_0 \phi_1 \phi_2 e^{\psi_1 + \psi_2}} + \gamma(t, y, z) \tag{6}\]

(2) The generalized 2-soliton solution

where \(k_1\) and \(k_2\) are arbitrary constants; \(g(t, y), h(t, z), \phi_1(y, z), p(y, z), \phi_1 = \phi_1(y, z),\) and \(\phi_2 = \phi_2(y, z)\) are arbitrarily differentiable functions; \(\gamma(t, y, z) = g(t, y) + h(t, z) + e(y, z);\) and \(a_0 = (k_1 - k_2)^2/(k_1 - k_2)^2[p(y, z)]\)

However, we failed to gain the generalized \(N\)-soliton solutions \((N \geq 3)\) with the same form.

In this paper, using a simplified Hirota’s method [21] which is different from classical Hirota’s method [22–25], we obtain generalized \(N\)-soliton solutions and their dispersion relations. For Equation (4), these solutions are combined by \(p(y, z)\) and \(\phi_1(y, z) e^{k_i x - c_i t}\), where \(\phi_1(y, z) (i = 1, 2, \ldots, n)\) are arbitrarily differentiable functions and \(k_i (i = 1, 2, \ldots, n)\) are arbitrary constants. This construction of \(N\)-soliton solution is novel and has not been seen before.

The structure of this paper is organized as follows: In Section 2, our main results are presented. A short conclusion is given in Section 3.

2. The New Generalized \(N\)-Soliton Solutions

In this section, we state the main results for Equation (4) by means of simplified Hirota’s method [21]. Depending on the different constructions of Equation (4), we use more complex test functions.

Firstly, we consider the linear part of Equation (4)

\[
u_{yzt} + \nu_{xxxx} = 0. \tag{7}\]

Substituting \(u = \phi_1 e^{k_i x - c_i t} (i = 1, 2, \ldots, n)\) and \(c_i\) should be determined) into Equation (7), we obtain \(c_i = k_i^2\).

For \(N = 1,\) based on the different constructions of Equation (4) and \(c_i = k_i^2,\) we can let

\[
F_1 = p(y, z) + \phi_1 e^{\psi_1} \tag{8}
\]

and assume that Equation (4) has solution of the following form:

\[
u = \frac{(\partial \phi_2/\partial x)}{\partial \phi_2} + \gamma(t, y, z), \tag{9}\]

where \(R\) is to be determined. Substituting (9) into Equation (4), we acquire \(R = -2\) and the following generalized 1-soliton solution:

\[
u = \frac{-2k_1 \psi_1 e^{\psi_1}}{p(y, z) + \psi_1 e^{\psi_1}} + \gamma(t, y, z), \tag{10}\]

which is equal to Equation (5).

For \(N = 2,\) let

\[
F_2 = p(y, z) + \phi_1 e^{\psi_1} + \phi_2 e^{\psi_2} + \phi_3 e^{\psi_3} + \frac{a_0}{p(y, z)} \phi_1 \phi_2 \phi_3 e^{\psi_1 + \psi_2 + \psi_3}, \tag{11}\]

where \(a_0\) is to be determined. Because of \(R = -2,\) we can assume that Equation (4) has solution of the following form:

\[
u = \frac{2(\partial F_2/\partial x)}{F_2} + \gamma(t, y, z). \tag{12}\]

Substituting (12) into Equation (4), then we obtain \(a = (k_1 - k_2)^2/(k_1 + k_2)\), \(a_{ij} = a_{ij}(p(y, z)).\) and the following generalized 2-soliton solution:

\[
u = \frac{-2(k_1 \psi_1 e^{\psi_1} + k_2 \psi_2 e^{\psi_2} + a_0(k_1 + k_2) \phi_1 \phi_2 e^{\psi_1 + \psi_2}}{p(y, z) + \psi_1 e^{\psi_1} + \psi_2 e^{\psi_2} + a_0 \phi_1 \phi_2 e^{\psi_1 + \psi_2}} + \gamma(t, y, z), \tag{13}\]

which is equal to Equation (6).

For \(N = 3,\) based on the result of the generalized 1-soliton and 2-soliton solutions above and \(a = ((k_1 - k_2)/(k_1 + k_2))^2,\) we should let

\[
\theta_3 = k_3 x - k_3 t, \tag{14}\]

\[
a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \quad (1 \leq i < j \leq 3),
\]

\[
F_3 = p(y, z) + \phi_1 e^{\psi_1} + \phi_2 e^{\psi_2} + \phi_3 e^{\psi_3} + \frac{a_{12}}{p(y, z)} \phi_1 \phi_2 e^{\psi_1 + \psi_2} + \frac{a_{13}}{p(y, z)} \phi_1 \phi_3 e^{\psi_1 \psi_3} + \frac{a_{23}}{p(y, z)} \phi_2 \phi_3 e^{\psi_2 \psi_3} + d_{123} \phi_1 \phi_2 \phi_3 e^{\psi_1 + \psi_2 + \psi_3}, \tag{14}\]

and suppose that Equation (4) has solution of the following form:

\[
u = \frac{-2(\partial F_3/\partial x)}{F_3} + \gamma(t, y, z), \tag{15}\]

where \(d_{123}\) is to be determined. Substituting (15) into Equation (4), we have

\[
d_{123} = \frac{a_{12} a_{13} a_{23}}{p(y, z)^3}. \tag{16}\]
This implies that the function in (15) \((N = 3)\) is a generalized 3-soliton solution of Equation (4). For \(N > 3\), we can obtain the \(N\)-soliton solution with the parallel manner.

In conclusion, for \(N \geq 1\), arbitrarily given constants \(k_i\) and arbitrarily given differentiable functions \(g(t, y), h(t, z), e(y, z), p(y, z),\) and \(\varphi_i = \varphi_i(y, z)(i = 1, 2, \cdots, N,)\), let

\[
y = (t, y, z) = g(t, y) + h(t, z) + e(y, z), \\
\theta_i = k_i x - k_i^3 t \quad (i = 1, 2, \cdots, N), \\
a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2 \quad (1 \leq i < j \leq N), \\
f = p(y, z) + \sum_{i=1}^{N} f_i, \\
f_1 = \sum_{i=1}^{N} \varphi_i \exp \theta_i, \\
f_k = p(y, z)^{1-k} \sum_{i_1<i_2<\cdots<i_k}^{(N)} \prod_{j=1}^{k} \varphi_{i_j} \prod_{i_k<l_k} \exp \left[\left(\frac{\kappa_{i_k} - \kappa_{l_k}}{\kappa_{i_k} + \kappa_{l_k}}\right)^2 \theta_{i_k}\right], \quad (1 < k < N), \\
\cdots, \\
f_N = p(y, z)^{1-N} \prod_{i=1}^{N} \varphi_{i} \prod_{i<j}^{(N)} a_{ij} \exp \left[\sum_{i=1}^{N} \theta_i\right],
\]

where the notation \(\sum_{i_1<i_2<\cdots<i_k}^{(N)}\) means the summation over all possible combinations of \(0 < i_1 < i_2 \cdots < i_k \leq N, \ (1 < k < N)\). Then Equation (4) has the following generalized \(N\)-soliton solutions:

\[
u = -\frac{2(\partial f / \partial x)}{f} + y(t, y, z).
\]

**Remark 1.** The \(N\)-soliton solutions \((N > 2)\) contain arbitrarily differentiable functions \(p(y, z)\), and this phenomenon has not been found in previous works. Because the \(N\)-soliton solutions we gained include a lot of arbitrarily differentiable functions, they have some interesting dynamic properties. We take 1- and 2-solitons to display some in Figures 1 and 2.

**Remark 2.** Since these solutions contain arbitrary functions, we add the word "generalized" to each type of solutions.

### 3. Conclusion

In this paper, we succeed in presenting generalized \(N\)-soliton solutions by virtue of the special construction of Equation (4). We have obtained many new expressions of the solutions which are listed in Section 2. It is interesting that these expressions contain some arbitrary functions. Moreover, we can see that these generalized \(N\)-soliton solutions

**Figure 1:** The graph of generalized 1-soliton solution for Equation (4), where \(x = t = 0, k_1 = 1, p(0, y, z) = \cos^2(\gamma y) \cos^2(\varphi_1), \varphi_1(y, z) = \exp (y) \sin^2(\gamma y), \varphi_2(0, y, z) = \exp (z) \cos^2(\varphi_1), \text{ and } u(0, y, z) = \sin (y) \sin (z). \)** (a) Perspective view of the \(u\) in the \((y, z, u)\) space and (b) overhead view of the \(u\).

**Figure 2:** The graph of generalized 2-soliton solution for Eq.(4), where \(x = t = 0, k_1 = 2, k_2 = 1, p(0, y, z) = \cos^2(\gamma) \cos^2(\varphi_1), \varphi_1(y, z) = \exp (y) \sin^2(\gamma), \varphi_2(0, y, z) = \exp (z) \sin^2(\gamma), \text{ and } u(0, y, z) = \sin (y) \sin (z). \)** (a) Perspective view of the \(u\) in the \((y, z, u)\) space and (b) overhead view of the \(u\).
Data Availability

The data used to support this work are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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