The Existence of the Sign-Changing Solutions for the Kirchhoff-Schrödinger-Poisson System in Bounded Domains

Cun-bin An, Jiangyan Yao, and Wei Han

1School of Mathematics and Statistics, Shanxi Datong University, Datong, Shanxi 037009, China
2Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China

Correspondence should be addressed to Cun-bin An; dt_ancunbin@126.com

Received 6 September 2019; Accepted 18 March 2020; Published 19 May 2020

1. Introduction and the Main Results

In this paper, the following Kirchhoff-Schrödinger-Poisson system is considered:

\[-a \Delta u + \frac{b}{2} \int_{\Omega} |\nabla u|^2 \, dx \Delta u + \lambda \phi x = g(u), \quad x \in \Omega,\]

\[-\Delta \phi = u^2, \quad x \in \Omega,\]

\[u = \phi = 0, \quad x \in \partial \Omega,\]

where \(\Omega \subset \mathbb{R}^3\) is a bounded domain with a smooth boundary \(\partial \Omega\), \(a, b, \lambda \in \mathbb{R}^+ = (0, +\infty)\), and \(g \in C(\mathbb{R}, \mathbb{R})\) satisfies some basic assumptions.

For \(b = 0\), problem (1) reduces to the following Schrödinger-Poisson system:

\[-a \Delta u + \lambda \phi(x) u = g(u), \quad x \in \Omega,\]

\[-\Delta \phi = u^2, \quad x \in \Omega,\]

\[\phi, u = 0, x \in \partial \Omega.\]

Alves and Souto [1] studied the above Schrödinger-Poisson system for \(a = \lambda = 1\). Under some suitable assumptions on the nonlinearity \(g(u)\), by using the deformation lemma and Brouwer’s topological degree theory, they proved that the above system possessed a least-energy sign-changing solution, which changed sign only once.

For \(\lambda = 0\), the problem (1) reduces to the following problem:

\[-a \int_{\Omega} |\nabla u|^2 \, dx \Delta u = g(u), \quad x \in \Omega,\]

\[u = 0, \quad x \in \partial \Omega.\]

The problem (3) has been studied in [2, 3]. Under different assumptions on \(g(u)\), the authors in [2, 3] obtained the existence and some qualitative properties of the sign-changing solution by using the Non-Nehari manifold method and deformation lemma. We can find that the results in [3] improve and generalize the results in [2]. In fact, the studies about the existence of the positive solutions, sign-changing solutions for a class of elliptic equations, have been studied extensively. For more details about such problems, we refer the reader to [4–19].

To our best knowledge, the results of the sign-changing solutions for the Kirchhoff-Schrödinger-Poisson system under a weak assumption that \(g \in C(\mathbb{R}, \mathbb{R})\) have not been studied yet. This paper attempts to fill this gap in the literature. Motivated by the above papers, we study the problem...
In this paper, we assume $g \in C(\mathbb{R}, \mathbb{R})$ satisfies the following four conditions:

1. $\lim_{s \to 0} (g(s)/s) = 0$
2. $\lim_{|s| \to \infty} g(s)/s^3 = \infty$
3. There exists a unique $s_0 \in \mathbb{R}$, where $c_0$ is a constant
4. There exists $\theta_0 \in (0, 1)$ such that for any $s > 0$ and $\tau \in \mathbb{R} \setminus \{0\}$,

\[
\frac{g(\tau) - g(s)}{\tau^3} \geq c_0 (1 + |s|^4)\]

where $\lambda_1$ is the first eigenvalue for the following problem:

\[
\begin{align*}
-\Delta u &= \lambda_1 u, & x &\in \Omega, \\
u &= 0, & x &\in \partial \Omega.
\end{align*}
\]  

Throughout this paper, we will use the following notations.

Let $H = H_0^1(\Omega)$ be the usual Sobolev space equipped with the following norm:

\[
||u|| = \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.
\]

The usual $L^p$ norm is denoted by $||u||_{L^p} = \left( \int_{\Omega} |u|^p dx \right)^{1/p}$. In this way, we know $||u||_2 = ||\nabla u||_2$.

For the Poisson system,

\[
\begin{align*}
-\Delta \phi &= u^2, & x &\in \Omega, \\
\phi &= 0, & x &\in \partial \Omega,
\end{align*}
\]

where there exists a unique $\phi_\theta = 1/4\pi \int_{\Omega} (u^2(y)/|x-y|) \, dy \in H$, such that $\phi_\theta$ satisfies the above system. It is known that $\phi_\theta$ satisfies the following conditions [17–19]:

1. $\int_{\Omega} \phi_\theta u^2 dx = \int_{\Omega} |\nabla \phi_\theta|^2 dx \leq C||\nabla u||_2^4$
2. $\phi_\theta \geq 0$ and $\phi_\theta > 0$ for $u \neq 0$
3. for $u = u^+ + u^-$, $\phi_\theta = \phi_\theta^+ + \phi_\theta^-$ and for $t \neq 0$, $\phi_\theta = t^2 \phi_\theta$
4. if $u_n \to u$ in $H_0^1(\Omega)$, then $\phi_\theta u_n \to \phi_\theta u$ in $H_0^1(\Omega)$ and $\lim_{n \to \infty} \int_{\Omega} \phi_\theta u_n^2 dx = \int_{\Omega} \phi_\theta u^2 dx$; if $u_n^+ \to u^+$ in $H_0^1(\Omega)$, then $\liminf_{n \to \infty} \int_{\Omega} \phi_\theta u_n^+ dx = \int_{\Omega} \phi_\theta (u^+)^2 dx$.

Consequently, $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of problem (1), that is, $\phi = \phi_\theta$ and $u \in H_0^1(\Omega)$ are a solution of the following problem:

\[
\begin{align*}
\begin{cases}
-\left(\frac{a + b}{2} \int_{\Omega} |\nabla u|^2 dx\right) \Delta u + \lambda \phi_\theta u = g(u), & x \in \Omega, \\
u = 0, & x \in \partial \Omega.
\end{cases}
\end{align*}
\]

In this paper, $u \in H_0^1(\Omega)$ is called a solution for problem (1), which implies $(u, \phi_\theta) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of problem (1).

Next, we can define the energy functional corresponding to problem (1): $F : H_0^1(\Omega) \to \mathbb{R}$ by

\[
F(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u|^2 dx \right)^2 + \frac{\lambda}{4} \int_{\Omega} \phi_\theta^2 u^2 dx
- \int_{\Omega} G(u) dx,
\]

where $G(s) = \int_0^s g(t) dt$. Obviously, the functional $F$ is well defined and belongs to $C^1(\mathbb{R}, \mathbb{R})$. By a simple computation, we have that for any $u, \phi \in H_0^1(\Omega)$,

\[
\begin{align*}
\langle F'(u), \phi \rangle &= \int_{\Omega} a \nabla u \nabla \phi dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u|^2 dx \right) \int_{\Omega} \nabla u \nabla \phi dx \\
&\quad + \int_{\Omega} \lambda \phi_\theta \phi dx - \int_{\Omega} g(u) \phi dx.
\end{align*}
\]

It is clear that the critical points of $F$ are the weak solutions for the problem (1). If $u \in H$ is a sign-changing solution of problem (1), then $u^+ \neq 0$ and for any $\phi \in H$, $\langle F'(u), \phi \rangle = 0$, where $u^+(x) = \max \{u(x), 0\}$, $u^-(x) = \min \{u(x), 0\}$.

For $u \in H$ and $u = u^+ + u^-$, by (9) and (10), we have
To get the main results, we restrict \(u\) in the following sets:

\[
\mathcal{M} = \left\{ u \in H : u^+ \neq 0, \langle F'(u), u^+ \rangle = 0, \langle F'(u), u^- \rangle = 0 \right\},
\]

\[
\mathcal{N} = \left\{ u \in H : u \neq 0, \langle F'(u), u \rangle = 0 \right\}.
\]

(14)

To get the energy doubling property, we define \(m = \inf_{u \in \mathcal{M}} F(u)\) and \(c = \inf_{u \in \mathcal{N}} F(u)\).

To prove the convergence property, we give the following definitions. Firstly, we define the energy functional corresponding to (2) \(F_{b_0} : H^1_0(\Omega) \to \mathbb{R}\) by

\[
F_{b_0}(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{\lambda}{4} \int_{\Omega} \phi_u u^2 dx - \int_{\Omega} G(u) dx.
\]

(15)

Similarly, we have

\[
\langle F_{b_0}'(u), \phi \rangle = \int_{\Omega} a \nabla u \nabla \phi dx + \int_{\Omega} \lambda \phi_u u \phi dx - \int_{\Omega} g(u) \phi dx.
\]

(16)

The set \(\mathcal{M}_{b_0}\) is defined by \(\mathcal{M}_{b_0} = \{ u \in H : u^+ \neq 0, \langle F_{b_0}'(u), u^+ \rangle + \langle F_{b_0}'(u), u^- \rangle = 0 \} \)

The energy functional \(F_{\lambda_n} : H^1_0(\Omega) \to \mathbb{R}\) corresponding to (3) can be defined by

\[
F_{\lambda_n}(u) = \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\Omega} |\nabla u|^2 dx \right)^2 - \int_{\Omega} G(u) dx.
\]

(17)

Also, we can compute that

\[
\langle F_{\lambda_n}'(u), \phi \rangle = \int_{\Omega} a \nabla u \nabla \phi dx + b \int_{\Omega} |\nabla u|^2 dx \int_{\Omega} \nabla u \nabla \phi dx - \int_{\Omega} g(u) \phi dx.
\]

(18)

To seek the sign-changing solution of (3), we define the set

\[
\mathcal{M}_{\lambda_n} = \left\{ u \in H : u^+ \neq 0, \langle F_{\lambda_n}'(u), u^+ \rangle + \langle F_{\lambda_n}'(u), u^- \rangle = 0 \right\}.
\]

(19)

The main results of this paper are described as follows.

Theorem 1. Assume that \((g_1) - (g_4)\) hold, then problem (1) possesses a least-energy sign-changing solution \(u_0 \in \mathcal{M}\) such that \(F(u_0) = \inf_{u \in \mathcal{M}} F > 0\), which changes sign only once.

Theorem 2. Assume that \((g_1) - (g_4)\) hold. Then problem (1) possesses a solution \(u_t \in \mathcal{N}\) such that \(F(u_t) = \inf_{u \in \mathcal{N}} F\). Moreover, \(m > 2c\).

Theorem 3. Assume that \((g_1) - (g_4)\) hold. Then problem (2) possesses a sign-changing solution \(v_0 \in \mathcal{M}_{b_0}\) such that \(F_{b_0}(v_0) = \inf_{u \in \mathcal{M}_{b_0}} F_{b_0} > 0\), which changes sign only once. Moreover, for any sequence \(\{b_n\}\) with \(b_n \to 0\) as \(n \to \infty\), there exists a subsequence of \(\{u_{b_n}\}\), still denoted by \(\{u_{b_n}\}\), such that \(u_{b_n} \to u_{b_0} \in H^1_0(\Omega)\), where \(u_{b_0} \in \mathcal{M}_{b_0}\) is a sign-changing solution of problem (2) with \(F_{b_0}(u_{b_0}) = \inf_{u \in \mathcal{M}_{b_0}} F_{b_0} > 0\).

Theorem 4. Assume that \((g_1) - (g_4)\) hold. Then problem (3) has a sign-changing solution \(w_0 \in \mathcal{M}_{\lambda_n}\) such that \(F_{\lambda_n}(w_0) = \inf_{u \in \mathcal{M}_{\lambda_n}} F_{\lambda_n} > 0\), which changes sign only once. Moreover, for any sequence \(\{\lambda_n\}\) with \(\lambda_n \to 0\) as \(n \to \infty\), there exists a subsequence of \(\{u_{\lambda_n}\}\), still denoted by \(\{u_{\lambda_n}\}\), such that \(u_{\lambda_n} \to u_{\lambda_0} \in H^1_0(\Omega)\), where \(u_{\lambda_0} \in \mathcal{M}_{\lambda_0}\) is a sign-changing solution of (3) with \(F_{\lambda_0}(u_{\lambda_0}) = \inf_{u \in \mathcal{M}_{\lambda_0}} F_{\lambda_0} > 0\).

The rest of the paper is organized as follows. In Section 2, we will give several estimates. In Section 3, some critical lemmas are proved. In Section 4, we will give the proof of the existence of the least-energy sign-changing solution. In section 5, the energy doubling property is proved. Section 6 is devoted to proving the convergence property.

2. Several Estimates

Lemma 5. If the assumptions \((g_1) - (g_4)\) hold, then

\[
F(u) \geq F(su^+ + tu^-) + \frac{1 - s^4}{4} \left\langle F'(u), u^+ \right\rangle + \frac{1 - t^4}{4} \left\langle F'(u), u^- \right\rangle + \frac{a(1 - s^2)(1 - \theta_0)}{4} \|u^+\|^2_2
+ \frac{a(1 - t^2)(1 - \theta_0)}{4} \|u^-\|^2_2 + \frac{b(s^2 - t^2)^2}{4} \|u^+\|^2_2 \|u^-\|^2_2
+ \frac{\lambda}{4} \int_{\Omega} \left( (s^2 - t^2)^2 (s^2 - t^2) \phi_u (u^+) \right)^2 + (s^2 - t^2)^2 (s^2 - t^2) \phi_u (u^-) \right\rangle dx, \forall u, u^+ + u^- \in H, s, t \geq 0.
\]

(20)

Proof. According to \((g_4)\), we can deduce that

\[
\frac{1 - t^4}{4} g(r \tau) + G(r \tau) - G(r) + \frac{a \theta_0 C_1 (1 - t^2)^2}{4} \int_0^\tau \left[ \frac{g(r \tau - g(\tau s^2)) + \frac{a \theta_0 C_1 (1 - s^2)^2}{(\tau s^2)^2} \int_0^\tau s^4 ds \geq 0, \forall \tau \geq 0, r \in \mathbb{R} \setminus \{0\}.
\]

(21)
From (9), (12), (13), and (21), we have

\[
F(u) - F(u^+ + tu^-) \geq \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right) - \frac{b}{4} \|Vu^+\|^2 + \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right) + \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right) + \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right) + \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right) + \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right) + \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right) + \frac{a}{4} \left( \|Vu^+\|^2 - \|Vu^+ + Vu^-\|^2 \right)
\]

The above inequality implies that (20) holds.

**Corollary 6.** If the assumptions \((g_1) - (g_4)\) hold and \(u = u^+ + u^- \in \mathcal{M}\), then

\[
F(u) \geq F(su^+ + tu^-) + \frac{a(1-t^2)^2(1-\theta_0)}{4} \|Vu^+\|^2 + \frac{b(1-t^2)^2}{4} \|Vu^+\|^2 + \frac{\lambda}{4} \int_\Omega \left( (s^2 - t^2)\phi_{u^+}(u^-)^2 + (t^2 - s^2)\phi_{u^-}(u^+)^2 \right) dx \
\]

From \(u \in \mathcal{M}\), we have \(<F'(u), u^-> = <F'(u), u^-> = 0\). Therefore, we can immediately get the above conclusion by (20).

\[
F(u^+ + u^-) = \max_{u \in \mathcal{M}} F(su^+ + tu^-) \quad \text{(24)}
\]

**Lemma 7.** Assume that \((g_4)\) holds, then

\[
\frac{1}{4} \phi(t) - \frac{a(1-t^2)^2(1-\theta_0)}{4} \|Vu^+\|^2 \quad \text{(25)}
\]

We can get (25) by taking \(t = 0\) in (21).

**Lemma 8.** If the assumptions \((g_1) - (g_4)\) hold, then for any \(u \in H\), we have

\[
F(u) \geq F(tu) + \frac{1-t^4}{4} <F'(u), u> + \frac{a(1-t^2)^2(1-\theta_0)}{4} \|Vu^+\|^2 \quad \text{(26)}
\]

We can get the conclusion by a similar deduction as Lemma 5.

**Corollary 9.** If the assumptions \((g_1) - (g_4)\) hold and \(u \in \mathcal{N}\), then

\[
F(u) \geq F(tu) + \frac{a(1-t^2)^2(1-\theta_0)}{4} \|Vu^+\|^2 \quad \text{(27)}
\]

**3. Some Critical Preliminaries**

**Lemma 10.** If the assumptions \((g_1) - (g_4)\) hold and \(u \in H\) with \(u^+ \neq 0\), then there exists a unique pair \((s_u, t_u)\) of positive numbers such that \(s_u u^+ + t_u u^- \in \mathcal{M}\).

**Proof.** From the definition of the set \(\mathcal{M}\), \(s_u u^+ + t_u u^- \in \mathcal{M}\) implies that \(<F'(s_u u^+ + t_u u^-), s_u u^-> = <F'(s_u u^+ + t_u u^-), t_u u^-> = 0\). Thus, we assume
Lemma 12. If the assumptions \((g_1) - (g_4)\) hold, then

\[
\inf_{u \in \mathcal{M}} F(u) = m = \inf_{u \in H^1(\Omega) \setminus \{0\}} \max_{t \geq 0} F(su^+ + tu^-). \tag{34}
\]

Proof. Firstly, by Corollary 6, one has

\[
\inf_{u \in H^1(\Omega) \setminus \{0\}} \max_{t \geq 0} F(su^+ + tu^-) \leq \inf_{u \in \mathcal{M}} \max_{t \geq 0} F(su^+ + tu^-) = \inf_{u \in \mathcal{M}} F(u) = m. \tag{35}
\]

Secondly, for any \(u \in H\) with \(u^+ \neq 0\), it follows from Lemma 10 that

\[
\max_{t \geq 0} F(su^+ + tu^-) \geq F(s_uu^+ + t_uu^-) \geq \inf_{v \in \mathcal{M}} F(v) = m. \tag{36}
\]

Combining (35) and (36), we can get Lemma 12.

Lemma 13. If the assumptions \((g_1) - (g_4)\) hold, then \(m > 0\) can be achieved.

Proof. For all \(u \in \mathcal{M}\), we have \(-F'(u), u > 0\). According to \((g_1), (g_3)\), and the Sobolev embedding theorem, we can get

\[
a\|Vu\|_2^2 \leq a\|Vu\|_2^2 + b\|Vu\|_2^2 + \lambda \int_{\Omega} \phi_u u^2 dx
\]

\[
= \int_{\Omega} g(u)udx - \frac{a\lambda_1}{2} \|Vu\|_2^2 + c\|Vu\|_2^2
\]

\[
\leq \frac{a}{2} \|Vu\|_2^2 + c\|Vu\|_2^2,
\]

where \(c\) and \(c_2\) are positive constants. Thus we have \(\|Vu\|_2^2 \geq (a/2c_2)^{2/3} > 0\). Therefore, by (9), (10), (25), and (37), we have

\[
F(u) = F(u) - \frac{1}{4} < F'(u), u >
\]

\[
= \frac{a}{4} \|Vu\|_2^2 + \int_{\Omega} \frac{1}{4} g(u)udx - G(u)dx
\]

\[
\geq \frac{a}{4} \|Vu\|_2^2 - \frac{a\theta_0 \lambda_1}{4} \|Vu\|_2^2
\]

\[
\geq \frac{a(1 - \theta_0)}{4} \|Vu\|_2^2
\]

\[
\geq \frac{a(1 - \theta_0)}{4} \left( \frac{a}{2c_2} \right)^{2/3} > 0.
\]

Since \(\theta_0 \in (0, 1)\), thus, for any \(u \in \mathcal{M}\), \(F(u) > 0\) and \(m > 0\). Let \(\{u_n\} \subseteq \mathcal{M}\) be such that \(F(u_n) \to m\). For large \(n \in \mathbb{N}\), one has

\[
m + 1 \geq F(u_n) - \frac{1}{4} < F'(u_n), u_n > \geq \frac{a(1 - \theta_0)}{4} \|Vu_n\|_2^2. \tag{39}
\]
Thus, \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \) for \( \theta_0 \in (0, 1) \), then there exists \( u_0 \in H \) such that \( u_n^+ \to u_0^+ \) in \( H \). From \( u_n \in M \), we have \( <F'(u_n), u_n^+> = 0 \), that is
\[
\tag{49}
<F'(u_0), u_0^+> \leq 0.
\]

Since \( \|u_n\|_2^2 - \theta_0 \lambda_1 \|u_n\|_1^2 \geq (1 - \theta_0) \|u\|_2^2 \), for all \( u \in H \), by (9), (10), (20), and (25), the weak semicontinuity of norm, Fatou’s lemma, and Lemma 12, we can get
\[
\tag{50}
m = \lim_{n \to \infty} \left\{ F(u_n) - \frac{1}{4} <F'(u_n), u_n> \right\} = \lim_{n \to \infty} \left\{ \frac{a}{4} \|u_n\|_2^2 + \frac{1}{4} \int g(u_n) u_n - G(u_n) dx \right\} \geq \frac{a}{4} \liminf_{n \to \infty} \left( \|u_n\|_2^2 - \theta_0 \lambda_1 \|u_n\|_1^2 \right) + \liminf_{n \to \infty} \int \left( \frac{1}{4} g(u_n) u_n - G(u_n) + \frac{a \theta_0 \lambda_1}{4} |u_n|^2 \right) dx \geq \frac{a}{4} \left( \|u_0\|_2^2 - \theta_0 \lambda_1 \|u_0\|_1^2 \right) + \int \left( \frac{1}{4} g(u_0) u_0 - G(u_0) + \frac{a \theta_0 \lambda_1}{4} |u_0|^2 \right) dx = F(u_0) - \frac{1}{4} <F'(u_0), u_0> \geq \sup_{s, t \geq 0} \left[ F(su_0^+ + tu_0) + \frac{1 - s^2}{4} <F'(u_0), u_0> \right] \geq \max_{s, t \geq 0} F(su_0^+ + tu_0) \geq m.
\]

Thus, \( \lim_{n \to \infty} (\|u_n\|_2^2 - \theta_0 \lambda_1 \|u_n\|_1^2) = \|u_0\|_2^2 - \theta_0 \lambda_1 \|u_0\|_1^2 \). Consequently, \( u_n \to u_0 \) in \( H^1_0(\Omega) \) and \( F(u_0) = m, u_0 \in M \).

**Corollary 14.** Assume that \((g_1) - (g_4)\) hold. Then
\[
\inf_{u \in E} F(u) = c = \inf_{u \in H, x \in (0, 1]} \max F(tu)
\]
and \( c > 0 \).

**Lemma 15** (See for example [3]). Assume that \((g_1) - (g_4)\) hold. Then there exists a constant \( c_* \in (0, c] \) and a sequence \( \{u_n\} \subset E \) satisfying
\[
F(u_n) \longrightarrow c_* \|F'(u_n)\|_2(1 + \|u_n\|) \longrightarrow 0.
\]
Lemma 16. If the assumptions \((g_1) - (g_d)\) hold and \(u_0 \in \mathcal{M}\) with \(F(u_0) = m\), then \(u_0\) is a critical point of \(F\).

Proof. For \(F' (u_0) \neq 0\), there exist \(\sigma > 0\) and \(\rho > 0\) such that
\[
u H_0^1(\Omega), \|u - u_0\| \leq 3\sigma \implies \|F'(u)\| \geq \rho. \tag{53}\]

From (23), we have that
\[
F(su_0^* + tu_0^*) \leq F(u_0) - \frac{a(1-\theta_0)(1-s^2)^2}{4} \|\nabla u_0^*\|^2_2
- \frac{a(1-\theta_0)(1-t^2)^2}{4} \|\nabla u_0^*\|^2_2
= m - \frac{a(1-\theta_0)(1-s^2)^2}{4} \|\nabla u_0^*\|^2_2 \tag{54}\]
\[
\leq - \frac{a(1-\theta_0)(1-s^2)^2}{4} \|\nabla u_0^*\|^2_2.
\]

Let \(D = (1/2, 3/2) \times (1/2, 3/2)\), \(g(s, t) = su_0^* + tu_0^*\). It follows from (54) that
\[
\chi = \max_{(s,t)\in D} I(su_0^* + tu_0^*) < m. \tag{55}\]

For \(\varepsilon = \min \{(m - \chi)/3, 1, \rho\sigma/8\}, S = B(u_0, \sigma), \) [22].

Lemma 7 yields a deformation \(\eta \in C([0, 1] \times H, H)\) such that
\[
(i) \eta(1, u) = u, \text{ if } u \notin F^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_\sigma \]
\[
(ii) \eta(1, F^{m}\cap B(u_0, \sigma)) < F^{m\varepsilon}
\]
\[
(iii) F(\eta(1, u)) \leq F(u), \forall u \in H_0^1(\Omega).
\]

From Corollary 6, we have that \(F(su_0^* + tu_0^*) \leq F(u_0) = m\) for \(s, t \geq 0\). For \(s, t \geq 0, |s - 1|^2 + |t - 1|^2 \leq \sigma^2/\|u_0^*\|^2_2\), we know \(s u_0^* + tu_0^* \in F^{m\varepsilon}\cap B(u_0, \sigma)\), then it follows from (ii) that \(F(s u_0^* + tu_0^*) \leq m - \varepsilon\).

According to (iii) and (54), we have that
\[
F(\eta(1, su_0^* + tu_0^*)) \leq F(su_0^* + tu_0^*) \leq m - \frac{a(1-\theta_0)(1-s^2)^2}{4} \|\nabla u_0^*\|^2_2
- \frac{a(1-\theta_0)(1-t^2)^2}{4} \|\nabla u_0^*\|^2_2 \tag{56}\]
\[
\leq m - \frac{a(1-\theta_0)(1-s^2)^2}{4} \|\nabla u_0^*\|^2_2 \leq m - \frac{a(1-\theta_0)(1-s^2)^2}{4} \|\nabla u_0^*\|^2_2.
\]

Thus, \(m \geq \max_{(s,t)\in D} F(\eta(1, g(s, t))) < m\).

Next, we prove that \(\eta(1, g(D)) \cap \mathcal{M} \neq \emptyset\), which contradicts to the definition of \(m\). Let us define \(h(s, t) = \eta(1, g(s, t))\) and
\[
\Psi_0(s, t) = \left( F'(g(s, t))u_0^*, F'(g(s, t))u_0^* \right)
= \left( F'(su_0^* + tu_0^*)u_0^*, F'(su_0^* + tu_0^*)u_0^* \right),
\]
\[
\Psi_1(s, t) = \left( \frac{1}{2} F'(h(s, t))h^+(s, t), \frac{1}{2} F'(h(s, t))h^-(s, t) \right). \tag{57}\]

Lemma 10 and the degree theory yields \(\deg (\Psi_0(s, t), D, 0) = 1\). By (55), we can deduce that \(g = h\) on \(\partial \Omega\). Consequently, \(\deg (\Psi_1(s, t), D, 0) \neq 0\). Therefore, we have that \(\Psi_1(s_0, t_0) = 0\) for some \((s_0, t_0) \in D\), so that \(\eta(1, g(s_0, t_0)) = h(s_0, t_0) \in \mathcal{M}\), which is a contradiction. Thus, (53) does not hold. In other words, \(u_0\) is a critical point of \(F\), that is, \(u_0\) is a sign-changing solution for problem (1).

4. The Existence Result of the Sign-Changing Solutions

In this section, we mainly give the proof of Theorem 17.

Proof of Theorem 17. By Lemma 13 and Lemma 16, there is a \(u_0 \in \mathcal{M}\) such that \(F(u_0) = m\) and \(F'(u_0) = 0\). Therefore, \(u_0\) is exactly a sign-changing solution of problem (1). Now, we prove that \(u_0\) changes sign only once.

We assume by contradiction that \(u_0 = u_1 + u_2 + u_3\), where \(u_i \neq u_0, u_i \geq 0, u_i \leq 0\) and \(\sup p(u_i) \cap \text{supp}(u_i) = \emptyset, i \neq j, i, j = 1, 2, 3\).

Let \(v = u_1 + u_2\), then \(v^+ = u_1, v^- = u_2\), and \(v^0 \neq 0\). Note that \(F'(u_0), v^+ > 0\) and \(F'(u_0), v^- > 0\), we have
\[
<F'(v), v^+ > = -b\|\nabla u_2\|_2^2 \|\nabla v^+\|_2^2 - \lambda \int_\Omega \phi(u_2)(v^+)^2 dx, \tag{58}\]
\[
<F'(v), v^- > = -b\|\nabla u_2\|_2^2 \|\nabla v^-\|_2^2 - \lambda \int_\Omega \phi(u_2)(v^-)^2 dx. \tag{59}\]

From (9)–(13), (23), (25), (58), and (59), we have
\[
m = F(u_0) - \frac{1}{4} < F'(u_0), u_0 >
= F(v) + F(u_3) + \frac{b}{2} \|\nabla v\|_2^2 \|\nabla u_3\|_2^2
+ \frac{1}{4} \left( \phi(u_3)^2 + \phi(u_3)(v^0)^2 \right) dx
- \frac{1}{4} \left( < F'(v), v^+ > + < F'(u_2), u_2 > + 2b\|\nabla v\|_2^2 \|\nabla u_3\|_2^2
+ \lambda \int_\Omega \phi(u_2)^2 + \phi(u_2)(v^0)^2 dx \right)
= F(v) + F(u_3) - \frac{1}{4} < F'(v), v^+ > - < F'(u_2), u_2 >
\geq \sup_{(s,t)\in D} \left\{ < F'(s^+ v^+ + t^+ v^-), v^- > + \frac{1 - s^2}{4} < F'(v), v^+ > + \frac{1 - t^2}{4} < F'(v), v^- > \right\} + F(u_3) - \frac{1}{4} < F'(v), v^+ >
\[
-\frac{1}{4} < F'(u_s), u_s >
\]
\[
\geq \sup_{s,t \geq 0} \left\{ F(sv^* + tv^*) + \frac{t^2}{4} \left( b\|\nabla u_s\|^2_2 \|\nabla v^*\|^2_2 + \lambda \right) \phi_{\theta_s}(v^*) dx \right\}
\]
Then, for any $b \in [0, 1]$, it follows from Lemma 12 and (65)-(66), we have

$$F_b(u_b) = m_b \leq \max_{s,t \geq 0} F_b(s u_b + t \phi_0)$$

$$= \max_{s,t \geq 0} \left( \frac{as^2}{2} \|\nabla u_b\|^2_2 + \frac{bs^4}{4} \|\nabla u_b\|^4_2 - \int_\Omega G(s u_b) dx + \frac{at^2}{2} \|\nabla u_b\|^2_2 + \frac{bt^4}{4} \|\nabla u_b\|^4_2 - \int_\Omega G(t \phi_0) dx + \frac{\lambda}{4} \int_\Omega \phi_{\nabla u_b, t \phi_0} (s u_b + t \phi_0)^2 dx \right)$$

$$\leq \max_{s,t \geq 0} \left( \frac{as^2}{2} \|\nabla u_b\|^2_2 + \frac{bs^4}{4} \|\nabla u_b\|^4_2 - \int_\Omega G(s u_b) dx + \frac{at^2}{2} \|\nabla u_b\|^2_2 + \frac{bt^4}{4} \|\nabla u_b\|^4_2 - \int_\Omega G(t \phi_0) dx + \frac{\lambda}{4} \int_\Omega \phi_{\nabla u_b, t \phi_0} (s u_b + t \phi_0)^2 dx \right)$$

$$\leq \max_{s,t \geq 0} \left( \frac{as^2}{2} \|\nabla u_b\|^2_2 + \frac{bs^4}{4} \|\nabla u_b\|^4_2 - \int_\Omega G(s u_b) dx + \frac{at^2}{2} \|\nabla u_b\|^2_2 + \frac{bt^4}{4} \|\nabla u_b\|^4_2 - \int_\Omega G(t \phi_0) dx + \frac{\lambda}{4} \int_\Omega \phi_{\nabla u_b, t \phi_0} (s u_b + t \phi_0)^2 dx \right)$$

$$= A_0 \in (0, \infty).$$

(67)

For any sequence $\{b_n\}$ with $b_n \rightarrow 0$ as $n \rightarrow \infty$, by (9), (10), (25), and (67), we have for large $n \in \mathbb{N}$

$$A_0 + 1 \geq F_{b_n}(u_{b_n}) - \frac{1}{4} < F'_{b_n}(u_{b_n}), u_{b_n} > = \frac{a(1 - \theta_n)}{4} \|\nabla u_{b_n}\|^2_2.$$

(68)

Since $\theta_n \in (0, 1)$, $\{u_{b_n}\}$ is bounded in $H_0^1(\Omega)$. Therefore, there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$ and $u_{b_n} \in H$, such that $u_{b_n} \rightharpoonup u_{b_n}^0$ in $H_0^1(\Omega)$. By a standard argument, we can prove $u_{b_n}^0 \rightharpoonup u_{b_n}^0 \neq 0$ in $H_0^1(\Omega)$. Since

$$<F'_{b_n}(u_{b_n}), \phi> = a \int_\Omega \nabla u_{b_n} \nabla \phi dx + \int_\Omega \lambda \phi_{\nabla u_b} u_{b_n} \phi dx$$

$$- \int_\Omega g(u_{b_n}) \phi dx$$

$$= \lim_{n \rightarrow \infty} \left( a + b_n \|\nabla u_{b_n}\|^2_2 \right) \int_\Omega \nabla u_{b_n} \nabla \phi dx$$

$$+ \int_\Omega \lambda \phi_{\nabla u_b} u_{b_n} \phi dx - \int_\Omega g(u_{b_n}) \phi dx$$

$$= \lim_{n \rightarrow \infty} <F_{b_n}(u_{b_n}), \phi> = 0. \forall \phi \in C_0^\infty(\Omega).$$

(69)

Thus, $F'_{b_n}(u_{b_n}) = 0$, $u_{b_n} \in \mathcal{M}_{b_n}$, and $F_{b_n}(u_{b_n}) \geq m_{b_n}$. Next, we give the proof of $F_{b_n}(u_{b_n}) = m_{b_n}$. Choose $b_n \in [0, 1]$, from (66), there exists a $K_0$ such that

$$F_{b_n}(sv_0 + t v_0) = \frac{as^2}{2} \|v_0\|^2_2 + \frac{b_n s^4}{4} \|v_0\|^4_2 - \int_\Omega G(s v_0) dx$$

$$+ \frac{at^2}{2} \|v_0\|^2_2 + \frac{b_n t^4}{4} \|v_0\|^4_2 - \int_\Omega G(t v_0) dx + \frac{\lambda}{4} \int_\Omega \phi_{\nabla v_0, t v_0} (s v_0 + t v_0)^2 dx$$

$$\leq \frac{as^2}{2} \|v_0\|^2_2 + \frac{1 + \lambda c'}{2} \|v_0\|^2_2$$

$$- \int_\Omega G(s v_0) dx + \frac{at^2}{2} \|v_0\|^2_2$$

$$+ \frac{1}{2} \lambda c' \|v_0\|^2_2 - \int_\Omega \beta_1 (t v_0)^4 dx + 2 \beta_1 |\Omega|$$

$$\leq \frac{as^2}{2} \|v_0\|^2_2 - \frac{1 + \lambda c'}{2} \|v_0\|^4_2$$

$$+ \frac{at^2}{2} \|v_0\|^2_2 - \frac{1 + \lambda c'}{2} \|v_0\|^2_2$$

$$+ 2 \beta_1 |\Omega|$$

$$= \lambda_0 \in (0, \infty).$$

(70)

From Lemma 10, there exists $(s_n, t_n)$ such that $s_n v_0 + t_n v_0 \in \mathcal{M}_{b_n}$. (54) implies $0 < s_n, t_n < K_n$. Since $F'_{b_n}(v_0) = 0$, then from (9), (10), (15), (16), and (20), we have

$$m_{b_n} = F_{b_n}(v_0) - \frac{b_n}{4} \|v_0\|^2_2$$

$$\geq F_{b_n}(s_n v_0 + t_n v_0) - \frac{1 - s_n^4}{4} \|v_0\|^2_2$$

$$+ \frac{a t_n^2}{4} < F'_{b_n}(v_0), v_0 >$$

$$+ \frac{1 + \lambda c'}{4} \|v_0\|^2_2$$

$$\geq m_{b_n} - \frac{1 + K_0^4}{4} |< F'_{b_n}(v_0), v_0 >|$$

$$- \frac{1}{4} \|v_0\|^2_2 - \frac{b_n}{4} \|v_0\|^2_2$$

$$= m_{b_n} - \frac{1 + K_0^4}{4} b_n \|v_0\|^2_2 \|v_0\|^2_2$$

$$- \frac{1 - s_n^4}{4} \|v_0\|^2_2 - \frac{b_n}{4} \|v_0\|^2_2.$$
which implies
\[ \limsup_{n \to \infty} m_{b_n} \leq m_{b_h}. \] (72)

According to (9), (15), and (72), we have
\[ m_{b_h} \leq F_{b_h}(u_{b_h}) = \limsup_{n \to \infty} F_{b_n}(u_{b_n}) = \limsup_{n \to \infty} m_{b_n} \leq m_{b_h}. \] (73)

This shows \( F_{b_h}(u_{b_h}) = m_{b_h} \), and the convergence property of \( b_n \) is proved.

**Proof of Theorem 20.** Since the proof is similar as the proof of Theorem 19, we omit the details.

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declares that they have no conflicts of interest.

**Acknowledgments**

The authors would like to express their sincere gratitude to the anonymous referees for their invaluable comments and suggestions which helped improve the paper greatly. This research was supported by the Program for the Innovative Talents of Higher Education Institutions of Shanxi, the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (201802085), and the innovative research team of North University of China(TD201901), the Fund for Shanxi “1331KIRT.”

**References**


