Research Article

Flexural Vibration Test of a Beam Elastically Restrained at One End: A New Approach for Young's Modulus Determination

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A new vibration beam technique for the fast determination of the dynamic Young modulus is developed. The method is based on measuring the resonant frequency of flexural vibrations of a partially restrained rectangular beam. The strip-shaped specimen fixed at one end to a force sensor and free at the other forms the Euler Bernoulli cantilever beam with linear and torsion spring on the fixed end. The beam is subjected to free bending vibrations by simply releasing it from a flexural position and its dynamic response detected by the force sensor is processed by FFT analysis. Identified natural frequencies are initially used in the frequency equation to find the corresponding modal numbers and then to calculate the Young modulus. The validity of the procedure was tested on a number of industrial materials by comparing the measured modulus with known values from the literature and good agreement was found.

1. Introduction

The Young modulus is a fundamental material property and its determination is common in science and engineering [1, 2]. It is a key parameter in mechanical engineering design to predict the behavior of the material under deformation forces or more to get an idea of the quality of the material. Young's moduli are determined from static and dynamic tests. In static measurements [3, 4] such as the classical tensile or compressive test, a uniaxial stress is exerted on the material, and the elastic modulus is calculated from the transverse and axial deformations as the slope of the stress-strain curve at the origin. Dynamic methods [5–12] are more precise and versatile since they use very small strains, far below the elastic limit and therefore are virtually nondestructive allowing repeated testing of the same sample. These include the ultrasonic pulse-echo [6, 7] or bar resonance methods [4, 8–14]. In the sonic pulse technique, the dynamic Young modulus is determined by measuring the sound velocity in the sample. In the resonance method, the linear elastic, uniform, and isotropic material of density ρ usually in the form of a bar of known dimensions is subjected to transverse

or flexural vibrations, the natural frequency of n th mode of which f_n related to Young's modulus E by the relation [15, 16]

$$f_n = \frac{\lambda_n^2}{2\pi l^2} \sqrt{\frac{EI}{\rho A}}, \quad n = 1, 2, 3, \dots \quad (1)$$

can be accurately measured. In (1) λ_n is the modal eigenvalue that depends on boundary conditions, l is the vibrating length of the bar, A is its cross-sectional area, and I is the second moment of the cross-section, equal to $\pi r^4/4$ for a rod of radius r and $dh^3/12$ for a rectangular beam with width d and depth h . Knowing the modal numbers, by simply measuring the resonance frequencies, geometry, and density of the specimen, the Young modulus can be determined from (1) as

$$E = \frac{4\pi^2 f_n^2 l^4 \rho A}{\lambda_n^4 I}. \quad (2)$$

The test sample is usually arranged in a manner to simulate free-free or clamped-free end conditions [10–12], when λ_n , associated with the n th flexural mode is a constant.

FIGURE 1: Model of a cantilever beam elastically restrained at $x = 0$.

In the present paper, we develop a new approach, in which a rectangular strip-shaped sample attached to a force sensor forms the Euler Bernoulli beam with partial translational and rotational restraints at the fixed end. This feature expands the capabilities of the resonant beam method making it suitable for materials with high stiffness and low density in which case, it is difficult to ascertain the flexural resonance frequencies with high certainty.

2. Theoretical Background

Consider a rectangular bar of uniform density ρ , cross-section dimensions of which width d and depth h are much less than length l . The bar is fixed at one end ($x = 0$) to the force sensor with a linear (K_T) and torsion (K_R) springs constants (see Figure 1) and is otherwise free to move in the transverse z -direction. For small deflections, that is, $\partial z(x, t)/\partial x \ll 1$, the effects of rotary inertia and shear deformation can be ignored. In this case, neglecting the deflection due to the weight, the flexural displacement of a bar, $z(x, t)$ at point x is governed by the Euler-Bernoulli equation [10]

$$\rho A \frac{\partial^2 z}{\partial t^2} + EI \frac{\partial^4 z}{\partial x^4} = 0 \quad (3)$$

with boundary conditions at $x = 0$:

$$-K_T z(0, t) = EI \left. \frac{\partial^3 z}{\partial x^3} \right|_{x=0}, \quad K_R \left. \frac{\partial z}{\partial x} \right|_{x=0} = EI \left. \frac{\partial^2 z}{\partial x^2} \right|_{x=0} \quad (4)$$

which correspond to the force and moment balance, respectively. At the free end $x = l$ there are no moment and shear force acting, that is,

$$\left. \frac{\partial^2 z}{\partial x^2} \right|_{x=l} = 0, \quad \left. \frac{\partial^3 z}{\partial x^3} \right|_{x=l} = 0. \quad (5)$$

Equations (3)–(5) define completely the linear flexural vibration problem, in which the natural frequencies of the beam depend on spring constants K_T and K_R . Applying the separation of variables method, the solution of (3) can be cast in the following form:

$$z(x, t) = w(x) T(t) = \sum_{n=1}^{\infty} w_n(x) e^{i\omega_n t}, \quad (6)$$

where $w_n(x)$ describes the n th normal mode and ω_n ($=2\pi f_n$, f : resonant frequency) is the angular frequency of

the n th mode. Substituting (6) into (3) gives an eigenvalue problem in the form of a fourth-order ordinary differential equation

$$\frac{\partial^4 w_n(x)}{\partial x^4} - \kappa_n^4 w_n(x) = 0, \quad (7)$$

where κ_n is related to the angular frequency ω_n and the modal number λ_n by the dispersion relationship

$$\kappa_n^4 = \left(\frac{\lambda_n}{l} \right)^4 = \omega_n^2 \frac{\rho A}{EI}. \quad (8)$$

With the boundary condition, (4), the solution of (7) admits the form of n th flexural eigenmode:

$$w_n(x) = A_n (\cosh \kappa_n x + a_n \sinh \kappa_n x + b_n \sin \kappa_n x) + C_n (\cos \kappa_n x - b_n \sinh \kappa_n x - a_n \sin \kappa_n x), \quad (9)$$

where related coefficients b_n and a_n are defined as

$$a_n = \frac{1 - \lambda_n^4 R_n T_n}{2\lambda_n^3 R_n}, \quad b_n = \frac{1 + \lambda_n^4 R_n T_n}{2\lambda_n^3 R_n} \quad (10)$$

with R_n and T_n expressed through experimentally accessible quantities:

$$R_n = \frac{K_R}{(2\pi f_n)^2 \rho A l^3}, \quad T_n = \frac{K_T}{(2\pi f_n)^2 \rho A l}, \quad (11)$$

and integration constants A_n and C_n are related by the modal restriction equation (5) as

$$\frac{A_n}{C_n} = \frac{\cos \lambda_n + b_n \sinh \lambda_n - a_n \sin \lambda_n}{\cosh \lambda_n + a_n \sinh \lambda_n - b_n \sin \lambda_n} = -\frac{\sin \lambda_n - b_n \cosh \lambda_n + a_n \cos \lambda_n}{\sinh \lambda_n + a_n \cosh \lambda_n - b_n \cos \lambda_n}, \quad (12)$$

where the second equality implies the constraints on possible values of λ_n , known as the frequency equation:

$$\begin{aligned} & (1 + \lambda_n^4 R_n T_n) - (1 - \lambda_n^4 R_n T_n) \cos \lambda_n \cosh \lambda_n \\ & - \lambda_n [(\lambda_n^2 R_n + T_n) \sin \lambda_n \cosh \lambda_n \\ & + (\lambda_n^2 R_n - T_n) \sinh \lambda_n \cos \lambda_n] \\ & = 0. \end{aligned} \quad (13)$$

For given values of R_n and T_n , transcendental equation (13) has infinite (iterative or graphical) solutions. As $T_n \rightarrow 0$ and $R_n \rightarrow 0$ (13) becomes

$$1 - \cos \lambda_n \cosh \lambda_n = 0 \quad (14)$$

that is the frequency equation for the free-free beam. For $T_n \gg R_n$ (13) reduces to the frequency equation derived by Chun [17], which in our case is written in the form

$$\begin{aligned} & R_n \lambda_n^3 (1 + \cos \lambda_n \cosh \lambda_n) \\ & - \sin \lambda_n \cosh \lambda_n - \sinh \lambda_n \cos \lambda_n = 0 \end{aligned} \quad (15)$$

which, in turn, at $R_n \rightarrow \infty$ becomes the frequency equation for the clamped-free beam

$$1 + \cos \lambda_n \cosh \lambda_n = 0. \quad (16)$$

FIGURE 2: Equipment used for measuring Young's modulus with the force sensor.

3. Principles of Operation

The experimental setup consists of a commercial, Fourier Force Sensor DT 272 with rotation K_R ($=17.837 \text{ N}\cdot\text{m}$) and translation K_T ($=6400 \text{ N}\cdot\text{m}^{-1}$) spring constants, accuracy $\pm 2\%$, and resolution (12 bit) 0.005 N for a scale range $\pm 10 \text{ N}$. The force sensor mounted on a support is connected through data acquisition system and data studio software to a personal computer (PC) as is shown in Figure 2.

The strip-shaped specimen with a roughly mass of $m \approx 10 \text{ g}$ is attached horizontally to a force sensor at one end and is free at the other end. By simply displacing the free end in the transverse direction and abruptly releasing it, the sample is subjected to free flexural vibrations, so that the condition $\partial z/\partial x \ll 1$ was fulfilled. At a given sample rate, $f_s = 1000 \text{ Hz}$ the force sensor detects the dynamic response and through the data acquisition system displays the damped oscillations of the restoring force versus time (Figure 3(a)). This vibration signal is analyzed and processed by an FFT implemented in acquisition software like MultiLogPRO [18] providing the direct information about natural frequencies up to $f_s/2$, which appears as single peaks in the frequency spectrum (Figure 3(b)). At this point, using data on the geometrical dimensions of the sample, its density, and spring constants of the force sensor K_R and K_T , identified resonance frequencies f_n are used in (11) to calculate dimensionless parameters R_n and T_n . Knowing R_n and T_n , modal number λ_n was found from the graphical solution of (13) using the mathematical packages MATHEMATICA [19]. The Young modulus is then determined from

$$E = 48\pi^2 \frac{\rho f_n^2 l^4}{\lambda_n^4 h^2}. \quad (17)$$

The relative error of the method arises from the uncertainties in the measurement of the quantities in (17). The relative error in the density ρ determined by the hydrostatic method is $|d(\ln \rho)| \leq 0.7\%$. The uncertainty in the thickness and length measurement are, respectively, $|d(\ln h)| \leq 0.4\%$ and $|d(\ln l)| \leq 0.25\%$. The dispersion in the natural frequency

TABLE 1: Identified resonant frequencies f_n in Hz, modal numbers determined from (13), and Young's moduli in GPa evaluated by (17) for the brass specimens of different lengths, $d = 16 \text{ mm}$, $h = 1.5 \text{ mm}$, and $\rho = 8400 \text{ kg}\cdot\text{m}^{-3}$.

l/d	T_n/R_n	f_1	λ_1	E_1	f_2	λ_2	E_2
8.13	60.6	37.82	1.595	111.6	232.79	4.068	100
9.06	75.4	31.14	1.619	110.3	193.83	4.14	100
9.813	88.4	26.84	1.638	107.4	170.23	4.179	102
10.69	104.9	23.07	1.653	107.9	147.82	4.215	104.7
12.25	137.8	17.9	1.679	105.2	115.24	4.276	103.7
14.69	198.2	13.0	1.701	108.9	84.10	4.326	109
15.5	220.7	11.75	1.709	108.2	76.25	4.372	106.5

determination is $|d(\ln f_1)| \leq 0.25\%$. By applying the error propagation technique, given by

$$|d(\ln E)| \leq 2|d(\ln f_1)| + |d(\ln \rho)| + 4|d(\ln l)| + 2|d(\ln h)| \quad (18)$$

we find that the relative error in Young's modulus does not exceed $\pm 3\%$. The greatest inaccuracies occurred in the measurement of the specimen dimensions.

4. Results and Discussion

To test the accuracy and validity of the present method, the effect of the sample length on the resonance frequency and Young's modulus was studied. A commercial brass strip of width $d = 16 \text{ mm}$, thickness $h = 1.5 \text{ mm}$, and density $\rho = 8400 \text{ kg}\cdot\text{m}^{-3}$ was cut into samples of various length, so that one of the conditions of the Euler-Bernoulli beam theory $l/h > 10$ remained unchanged, while the second l/d ranged from 8 to 16. Since the ratio $T_n/R_n \gg 1$ (see Table 1) we used the approximated equation (15) to find the λ_1 . The validity of this approach is illustrated in Figure 4, which shows that the solutions of both (15) and (13) practically coincide.

Based on the results of measurements of the natural frequencies and modal numbers presented in Table 1, a double logarithm plot of f_i/λ_i^2 against l predicted by (1) to be linear with the gradient of -2 shows that the slope of the line that best fits these data in a least-squares sense is -2.02 for the first mode and -1.94 for the second one. Based on the same set of data we show in Figure 5 the plot of the ratio $f_i l^2/\lambda_i^2$ versus l/d . Most of the uncertainty in the ordinates of this plot arises from uncertainty of l rather than f_i . In the range of $l/d > 10$, the value of $f_i l^2/\lambda_i^2$ is constant to within the experimental uncertainties, showing that $f_i/\lambda_i^2 \propto l^{-2}$ is in agreement with (1). For $l/d < 10$, that is smaller for validity of the Euler-Bernoulli beam theory, the ratio $f_i l^2/\lambda_i^2$ decreases (increases) with l for the first (second) mode. For $l/d \geq 10$, the mean value of Young's modulus $E = 106 \pm 2 \text{ GPa}$ lies within the range $95 \div 110 \text{ GPa}$, listed in an extensive table of ASTM testing [20].

Below, we compare the results of measurements of elastic moduli at ambient temperature for a wide class of industrial materials with that accepted in the literature. A set of test specimens used were cut from the commercial sheet materials

FIGURE 3: The flexural vibration test of partially restrained aluminum beam with $l = 0.282$ m, $d = 1.35$ cm, and $h = 1.55$ mm. (a) Dynamic response detected by the force sensor at sample rate 1 kHz. (b) Identified resonance frequencies: $f_1 = 14.7$ Hz; $f_2 = 92.8$ Hz; $f_3 = 249.5$ Hz.

FIGURE 4: Graphical solutions of (13) and (15) for the first mode of flexural vibrations of partially restrained aluminum beam are, respectively, $\lambda_1 = 1.7752$ and $\lambda_1 = 1.7745$ against $\lambda_1 = 1.8751$ for ideal clamped-free case.

into strips of the thickness (h) from 0.5 to 3.3 mm, width (d) from 5 to 16.5 mm, and length (l) from 15 to 30 cm. For each specimen, the length, width, and thickness were altered and the value of the Young modulus calculated by (17) for each set varied within the experimental error. Table 2 summarizes data of the specimen dimension, material

density, natural frequencies, modal numbers, and Young's modulus calculated from the first resonant frequency. It can be seen that test results are in good agreement with the accepted those in the literature data.

In all cases, identified natural frequencies of the partially restrained cantilever beam are lower than those for the clamped-free case. However, this does not necessary mean that the elastic modulus determined from these frequencies should be smaller than in ideal clamped-free case as the modal number, being in the denominator equation (17) in fourth power, decreases as well. Interestingly, the plot of λ_1^4 versus $1/R_1$ in Figure 5 shows a linear dependence

$$\lambda_1^4 = \lambda_\infty^4 \left(1 - \frac{1}{3R_1}\right), \quad (19)$$

where $\lambda_\infty = 1.8751$ is the first modal number for the clamped-free flexural vibrations of the beam. Taking into account (11), after substituting (19) in (17), we obtain the working equation for the Young modulus determination through the first resonance frequency of flexural vibrations of the partially restrained cantilever beam (Figure 6)

$$E = \frac{48\pi^2 f_1^2 \rho l^4}{h^2 \lambda_\infty^4 \left(1 - \left(4\pi^2 f_1^2 \rho A l^3 / 3K_R\right)\right)}. \quad (20)$$

5. Summary

A new technique for the fast determination of the dynamic Young modulus was developed, yielding a substantial modification of the classical cantilever beam method. The procedure

TABLE 2: Young's moduli of materials E in GPa, determined at room temperature by (17) and specimen parameters: length l in mm, thickness h in mm, density ρ in $\text{kg}\cdot\text{m}^{-3}$, fundamental frequency f_1 in Hz, and modal number λ_1 calculated by (13).

Material	l	h	ρ	f_1	λ_1	E	$E_{\text{Literature}}$
Al 6061 sheet	282	1.55	2715	14.42	1.778	70.0	70–72
Zn coated steel	193	0.55	7820	12.0	1.868	201	206
Sheet steel 304	193	0.5	7970	11.0	1.864	210	190–213
Cooper alloy	214	1.0	8790	11.89	1.848	106	110–120
Brass stripe	248	1.5	8400	11.75	1.709	108	96–110
Perspex	190	3.0	1190	23.2	1.798	4.2	2.4–4.6
Wood, oak	252	3.5	674	30.3	1.664	12.6	11–12
Composite ^a	232	2.5	1600	33.0	1.429	92.0	36–150

^a Graphite carbon epoxy.

FIGURE 5: Plot of $f_i(l/\lambda_i)^2$ versus l/d based on the flexural vibration test of a rectangular brass strip ($d = 16.0$ mm and $h = 1.5$ mm). For $l/d \geq 10$, the mean value 0.245 (broken line) is constant to within experimental uncertainty.

uses a rectangular beam, partially restrained at one end, flexural vibrations of which are detected with the aid of the force sensor. The relative experimental uncertainty is found to be less than 3%, which is mainly due to the uncertainty in the samples dimensions. The feasibility and accuracy of a new experimental procedure has been demonstrated by measuring the Young modulus for a number of test materials with different material properties. Comparison of obtained results with those accepted in the literature data is good. The relative deviation of measured values from the cited data is less than 5%. The method has potential advantages over other dynamic methods of being very simple and fast and requiring no additional equipment to excite resonance frequencies. It is particularly suitable for composite materials having a high

FIGURE 6: The modal number of the first mode of flexural vibrations of partially restrained cantilever beams in the fourth degree as a function of the inverse of the dimensionless frequency parameter, $1/R_1$. Symbols are experimental data taken from Tables 1 and 2. Solid line is by (19).

stiffness and low density, such as carbon fiber reinforced plastic. The accuracy can be significantly improved by more precise determination of specimen dimensions.

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