Research Article

Effects of Fiber Orientation and Material Isotropy on the Analytical Elastic Solution of a Stiffened Orthotropic Panel Subjected to a Combined Loading

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Using the displacement potential approach of orthotropic composite materials for the plane stress conditions, an orthotropic panel subjected to a combined loading at its right lateral edge is solved. Effects of fiber orientations and material isotropy on the elastic field of an orthotropic panel subjected to a combined loading are discussed. The analytical elastic solutions at different sections of the panel with fiber orientation $\theta = 90^\circ$ are compared with those of finite element predictions to ensure the reliability of our present solutions.

1. Introduction

Recently in lightweight structures, the use of orthotropic composite material is increasing due to its higher strength weight ratio compared to its corresponding isotropic material. Aircraft wings which are generally made of aluminium are stiffened. Nowadays, boron/epoxy, graphite/epoxy orthotropic composite materials are used for manufacturing of aircraft panels. In structures, the stiffened panel experiences tension and bending at the same time. Effect of fiber orientation and material isotropy on the exact elastic field is necessary for designing purpose because strength, toughness, and all other mechanical properties of any orthotropic composite depend on its fiber orientation. Under loading, the stress and displacement components of any orthotropic composite vary with the variation of fiber orientation. This type of analysis is carried out by the finite element method. Exact analytical solution of this type of problem is hardly possible to carry out till now due to lack of effective elasticity formulations of orthotropic composite materials and boundary management techniques.

The limited progress in solution of the two-dimensional problems of elasticity theory for anisotropic bodies was

achieved by means of Lekhnitskii formalism [1, 2] and Stroh's formalism [3, 4]. Some solutions of the two-dimensional problems for anisotropic bodies were obtained by these approaches. An exposition of Stroh's formalism and its application to a limited number of two-dimensional problems was given by Ting [5]. The proof of the equivalence between the Lekhnitskii and Stroh formalisms was obtained by Barnett and Kirchner [6]. The stress analysis problems are still suffering from a lot of shortcomings in spite of becoming the fundamental subject in the field of elasticity and thus are being constantly investigated [7–15].

Elasticity problems are usually formulated either in terms of deformation parameters or stress parameters. Among the existing mathematical models of plane boundary-value stress problems, the stress function approach [15] and the displacement formulation [16] are noticeable. The application of the stress function formulation in conjunction with finite-difference technique has been reported for the solution of plane elastic problems where all of the boundary conditions are prescribed in terms of stresses only [7–15, 17, 18]. Further, Conway and Ithaca [18] modified the stress function formulation in the form of Fourier integrals to the case where the material is orthotropic and obtained analytical solutions for

a number of ideal problems. The shortcoming of the stress function approach is that it accepts boundary conditions only in terms of loadings of the structure. Boundary restraints specified in terms of the displacement components of the structure cannot be satisfactorily imposed on the stress function approach.

Stress analysis of composite structures is carried out using different analytical, semianalytical approaches where loading is applied along the perpendicular plane of the lamina of laminated composite [19–29] with limited mixed boundary conditions but along the directions of the plane of the composite laminas of the composite should be analyzed for the perfect design of composite structures with mixed boundary conditions. Kulikov and Plotnikova [19] focused on the use of efficient approach to exact elastic solutions of cross-ply and angle-ply composite plates. Interlaminar stresses resulting from bending of thick rectangular laminated plates with arbitrary laminations and boundary conditions are analyzed analytically based on a three-dimensional multiterm extended Kantorovich method (3DMTEKM) [20]. Using the principle of minimum total potential energy, three systems of coupled ordinary differential equations with nonhomogeneous boundary conditions are obtained [20]. Zhang and Hoa [21] proposed a limit-based approach to provide solution to the composite tube having cylindrical orthotropy. The Taylor series expansion method was used to drive new equations to replace those that are identically satisfied, while the other equations are kept to be same [21]. Singh and Shukla [22] presented the nonlinear response of laminated composite plates. The mathematical formulation of the actual physical problem of the laminated composite plate subjected to mechanical loading is presented utilizing higher order shear deformation theory and von-Karman nonlinear kinematics [10]. These nonlinear governing differential equations of equilibrium are linearized using quadratic extrapolation technique [22]. A meshfree technique based on multiquadric RBF is used for analysis of the problems, and isotropic, orthotropic, and laminated composite plates with immovable simply supported and clamped edges are analyzed [22]. A semianalytical method for bending of corrugated-core, honeycomb-core, and X-core sandwich panel is presented [23]. In the global displacement field, the governing equations of three sandwich panels are derived using energy variation principle and solved by employing Fourier series and Galerkin approach [23]. Shahbazi et al. [24] introduced a meshfree approach for analysis of isotropic/orthotropic cross-ply laminated plates with symmetric/non-symmetric layers. A series of exponential basis functions, satisfying the partial differential equations were used to approximate the solution on the whole domain [24]. The boundary conditions are enforced through a collocation approach on a set of boundary points [24]. Aifantis strain gradient elasticity theories and Zhang's two-variable method are used to study elastic bending problems of bilayered microcantilever beams, containing a gradient layer, subjected to a transverse concentrated load [25]. Based on three-dimensional theory of elasticity, axisymmetric static analysis of functionally graded circular and annular plates imbedded in piezoelectric layers is investigated using differential quadrature method [26].

TABLE I: Properties of composites used to obtain numerical results.

Material	Property	Boron/Epoxy
Fiber	E_f (10^3 MPa)	414
	ν_f	0.20
Resin	E_r (10^3 MPa)	3.45
	ν_r	0.35
Composite	E_1 (10^3 MPa)	282.9
	E_2 (10^3 MPa)	24.2
	G_{12} (10^3 MPa)	10.4
	ν_{12}	0.27
	ν_{21}	0.023

Liew et al. [27] discussed mainly on the developments of element free or meshless methods and their applications in the analysis of composite structures. Rodrigues et al. [28] proposed to use the Murakami's zig-zag theory for the static and vibration analysis of laminated plates, by local collocation with radial basis functions in a finite differences framework. Stürzenbecher and Hofstetter [29] presented an axiomatic equivalent single layer plate theory for cross-ply laminated composites, which is based on the work of Lekhnitskii and Ren and delivers accurate deformation and stress prognoses at the cost six solution variables.

Deb Nath et al. [30, 31] analyzed the elastic field of different stiffened composite structures analytically, with the verifications of the solution of FE (finite element) and FD (finite difference) methods. Deb Nath et al. [30, 31] studied the effect of fiber orientation on the elastic field of a stiffened orthotropic panel under tension. For the case of combined effects of bending and tension at the ends of the stiffened panel, effects of fiber orientation and material isotropy are not studied analytically till now which are necessary for the optimum design of orthotropic members in any loaded structure because during loading members of any structure experience tension and bending together. The present study presents the effects of fiber orientation and material isotropy on the elastic field of an orthotropic composite panel subjected to a combined loading at its tip. This analysis is carried out analytically using Fourier integral and the displacement potential formulation of orthotropic composite materials. Two types of fiber orientation are considered in the present analysis such as $\theta = 0^\circ$ and $\theta = 90^\circ$. Appendix A illustrates the boundary and governing equations for isotropic panel and Appendix B illustrates the equations to derive the isotropic mechanical properties from orthotropic composite materials. The soundness and reliability of the present solution are shown comparing the present analytical solution at some particular sections with those of finite element predictions.

2. Analytical Model of the Present Problem

The geometry of a stiffened boron/epoxy panel subjected to a combined loading whose fiber orientation $\theta = 90^\circ$ is shown

FIGURE 1: Model of an orthotropic composite panel subjected to a combined loading (uniform tension and shear).

in Figure 1. The supporting edge of the panel is rigidly fixed, two opposite edges are stiffened, and at the tip of the panel, uniform tension and shear are applied. Mechanical properties of the boron/epoxy composite used are given by Table 1. The applied tensile and shear stress on the tip of the panel is 41.4 MPa.

3. Displacement Potential Formulation of Orthotropic Materials for Plane Stress Conditions

With reference to a rectangular coordinate system (x, y) , the differential equations of equilibrium for the plane stress problems of orthotropic composite materials having fiber orientation $\theta = 90^\circ$ [32] are as follows:

$$\begin{aligned} \frac{E_1 E_2}{E_1 - \nu_{12}^2 E_2} \frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\nu_{12} E_1 E_2}{E_1 - \nu_{12}^2 E_2} + G_{12} \right) \frac{\partial^2 u_y}{\partial x \partial y} \\ + G_{12} \frac{\partial^2 u_x}{\partial y^2} = 0, \\ \frac{E_1^2}{E_1 - \nu_{12}^2 E_2} \frac{\partial^2 u_y}{\partial y^2} + \left(\frac{\nu_{12} E_1 E_2}{E_1 - \nu_{12}^2 E_2} + G_{12} \right) \frac{\partial^2 u_x}{\partial x \partial y} \\ + G_{12} \frac{\partial^2 u_y}{\partial x^2} = 0. \end{aligned} \quad (1)$$

The governing equation of the orthotropic material with fiber orientation 90° in terms of the function ψ is as follows [31]:

$$\begin{aligned} E_2 G_{12} \frac{\partial^4 \psi}{\partial x^4} + E_2 (E_1 - 2\nu_{12} G_{12}) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} \\ + E_1 G_{12} \frac{\partial^4 \psi}{\partial y^4} = 0. \end{aligned} \quad (2)$$

The corresponding governing equation for isotropic materials can be obtained when the respective conditions $[E_1 = E_2 = E, \nu_{12} = \nu_{21} = \nu, G_{12} = G = E/2(1 + \nu)]$ are substituted in (2) which is as follows:

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0. \quad (3)$$

The components of different displacement components with fiber orientation 90° [31] are as follows:

$$\begin{aligned} u_x(x, y) &= \frac{\partial^2 \psi}{\partial x \partial y}, \\ u_y(x, y) &= -\frac{1}{Z_{11}} \left[E_1 E_2 \frac{\partial^2 \psi}{\partial x^2} + G_{12} (E_1 - \nu_{12}^2 E_2) \frac{\partial^2 \psi}{\partial y^2} \right]. \end{aligned} \quad (4)$$

The stress-strain relations of orthotropic composite lamina with fiber orientation $\theta = 90^\circ$ in terms of the potential function ψ are obtained as follows [31]:

$$\begin{aligned} \sigma_{xx}(x, y) &= \frac{E_1 E_2 G_{12}}{Z_{11}} \left[\frac{\partial^3 \psi}{\partial x^2 \partial y} - \nu_{12} \frac{\partial^3 \psi}{\partial y^3} \right], \\ \sigma_{yy}(x, y) &= \frac{E_1}{Z_{11}} \left[E_2 (\nu_{12} G_{12} - E_1) \frac{\partial^3 \psi}{\partial x^2 \partial y} - E_1 G_{12} \frac{\partial^3 \psi}{\partial y^3} \right], \\ \sigma_{xy}(x, y) &= \frac{E_1 E_2 G_{12}}{Z_{11}} \left[\nu_{12} \frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial^3 \psi}{\partial x^3} \right]. \end{aligned} \quad (5)$$

4. Solution of the Present Problem

4.1. Solution of the Panel ($\theta = 90^\circ$). A stiffened rectangular boron/epoxy orthotropic composite panel is considered. Its fiber orientation is $\theta = 90^\circ$. Its supporting edge is rigidly fixed and the two opposing edges are stiffened. The panel is considered to be of unit thickness, and its configuration with respect to coordinate axes is illustrated in Figure 1. In this case, if the potential function ψ is assumed to be

$$\psi = \sum_{m=1}^{\infty} Y_m \sin \alpha x + Bx^3 + Cxy^2, \quad (6)$$

where Y_m is a function of y only, and $\alpha = m\pi/a$, then Y_m has to satisfy the ordinary differential equation

$$Y_m'''' - \left(\frac{E_2}{G_{12}} - \frac{2\nu_{12}E_2}{E_1} \right) \alpha^2 Y_m'' + \frac{E_2}{E_1} \alpha^4 Y_m = 0. \quad (7)$$

The general solution of the differential equation (7) can be given by

$$Y_m = A_m e^{m_1 y} + B_m e^{m_2 y} + C_m e^{m_3 y} + D_m e^{m_4 y}, \quad (8)$$

where

$$\begin{aligned} m_1, m_2 &= \frac{\alpha}{\sqrt{2}} \left[\left\{ K_1 \pm \sqrt{K_1^2 - \frac{4E_2}{E_1}} \right\} \right]^{1/2}, \\ m_3, m_4 &= -\frac{\alpha}{\sqrt{2}} \left[\left\{ K_1 \pm \sqrt{K_1^2 - \frac{4E_2}{E_1}} \right\} \right]^{1/2}, \\ K_1 &= \frac{E_2}{G_{12}} - \frac{2\nu_{12}E_2}{E_1}, \end{aligned} \quad (9)$$

where A_m, B_m, C_m , and D_m are arbitrary constants.

Now, combining (4) to (5) and (6), the expressions of displacement and stress components are obtained as follows:

$$\begin{aligned} u_x(x, y) &= \sum_{m=1}^{\infty} \alpha Y_m' \cos \alpha x + 2Cy, \\ u_y(x, y) &= \frac{1}{Z_{11}} \sum_{m=1}^{\infty} [-Z_{22} Y_m'' + E_1 E_2 \alpha^2 Y_m] \sin \alpha x \\ &\quad + (6BE_1 E_2 + 2CZ_{33}) x, \\ \sigma_{xx}(x, y) &= -\frac{E_1 E_2 G_{12}}{Z_{11}} \sum_{m=1}^{\infty} [\nu_{12} Y_m''' + \alpha^2 E_1 Y_m'] \sin \alpha x, \\ \sigma_{yy}(x, y) &= \frac{E_1 E_2}{Z_{11}} \sum_{m=1}^{\infty} [Y_m' \alpha^2 (E_1 - \nu_{12} G_{12}) \\ &\quad - G_{12} Y_m'''] \sin \alpha x, \\ \sigma_{xy}(x, y) &= \frac{E_1 E_2 G_{12}}{Z_{11}} \sum_{m=1}^{\infty} [\alpha \nu_{12} Y_m'' + \alpha^3 Y_m] \cos \alpha x \\ &\quad - \frac{6BE_1 E_2 G_{12}}{Z_{11}} + \frac{2C\nu_{12} E_1 E_2 G_{12}}{Z_{11}}, \end{aligned} \quad (10)$$

where $Z_{22} = G_{12}(E_1 - \nu_{12}^2 E_2)$, $Z_{33} = (\nu_{12} E_2 Z_{11} - E_1^2 E_2)/(E_1 - \nu_{12}^2 E_2)$.

Substituting the different derivatives of Y_m in the expressions of the displacement and stress components equations (10), we get

$$\begin{aligned} u_x(x, y) &= \sum_{m=1}^{\infty} [(m_1 A_m e^{m_1 y} + m_2 B_m e^{m_2 y} + m_3 C_m e^{m_3 y} \\ &\quad + m_4 D_m e^{m_4 y})] \alpha \cos \alpha x + 2Cy, \\ u_y(x, y) &= -\frac{1}{Z_{11}} \sum_{m=1}^{\infty} \left[\begin{aligned} &(m_1^2 Z_{22} - \alpha^2 E_1 E_2) A_m e^{m_1 y} \\ &+ (m_2^2 Z_{22} - \alpha^2 E_1 E_2) B_m e^{m_2 y} \\ &+ (m_3^2 Z_{22} - \alpha^2 E_1 E_2) C_m e^{m_3 y} \\ &+ (m_4^2 Z_{22} - \alpha^2 E_1 E_2) D_m e^{m_4 y} \end{aligned} \right] \\ &\quad \times \sin \alpha x + (6BE_1 E_2 + 2CZ_{33}) x, \\ \sigma_{xx}(x, y) &= -\frac{E_1 E_2 G_{12}}{Z_{11}} \\ &\quad \times \sum_{m=1}^{\infty} \left[\begin{aligned} &(m_1 \alpha^2 + m_1^3 \nu_{12}) A_m e^{m_1 y} \\ &+ (m_2 \alpha^2 + m_2^3 \nu_{12}) B_m e^{m_2 y} \\ &+ (m_3 \alpha^2 + m_3^3 \nu_{12}) C_m e^{m_3 y} \\ &+ (m_4 \alpha^2 + m_4^3 \nu_{12}) D_m e^{m_4 y} \end{aligned} \right] \sin \alpha x, \\ \sigma_{yy}(x, y) &= -\frac{E_1 E_2}{Z_{11}} \\ &\quad \times \sum_{m=1}^{\infty} \left[\begin{aligned} &\{m_1 \alpha^2 (\nu_{12} G_{12} - E_1) + m_1^3 G_{12}\} A_m e^{m_1 y} \\ &+ \{m_2 \alpha^2 (\nu_{12} G_{12} - E_1) + m_2^3 G_{12}\} B_m e^{m_2 y} \\ &+ \{m_3 \alpha^2 (\nu_{12} G_{12} - E_1) + m_3^3 G_{12}\} C_m e^{m_3 y} \\ &+ \{m_4 \alpha^2 (\nu_{12} G_{12} - E_1) + m_4^3 G_{12}\} D_m e^{m_4 y} \end{aligned} \right] \sin \alpha x, \\ \sigma_{xy}(x, y) &= \frac{E_1 E_2 G_{12}}{Z_{11}} \sum_{m=1}^{\infty} \left[\begin{aligned} &(\alpha^3 + \alpha m_1^2 \nu_{12}) A_m e^{m_1 y} \\ &+ (\alpha^3 + \alpha m_2^2 \nu_{12}) B_m e^{m_2 y} \\ &+ (\alpha^3 + \alpha m_3^2 \nu_{12}) C_m e^{m_3 y} \\ &+ (\alpha^3 + \alpha m_4^2 \nu_{12}) D_m e^{m_4 y} \end{aligned} \right] \cos \alpha x \\ &\quad - \frac{6BE_1 E_2 G_{12}}{Z_{11}} + \frac{2C\nu_{12} E_1 E_2 G_{12}}{Z_{11}}. \end{aligned} \quad (11)$$

4.2. Boundary Conditions and Its Application. For the present problem, it is observed that the boundary conditions on the two stiffened edges as observed in Figure 1 are

$$\begin{aligned} u_y(0, y) &= 0, & \sigma_{xx}(0, y) &= 0, \\ u_y(a, y) &= 0, & \sigma_{xx}(a, y) &= 0, \end{aligned} \quad (12)$$

satisfied automatically.

The associated boundary conditions of the supporting edge are

$$u_x(x, 0) = 0, \quad u_y(x, 0) = 0. \quad (13)$$

Now, the axial loading on the right lateral edge of the panel $y = b$ can be expressed mathematically as follows:

$$\begin{aligned}\sigma_{yy}(x, b) &= P = \sum_{m=1}^{\infty} E_m \sin \alpha x, \\ \sigma_{xy}(x, b) &= P = E_0 + \sum_{m=1}^{\infty} \overline{E}_m \cos \alpha x,\end{aligned}\quad (14)$$

where $E_m = (2/a) \int_0^a P \sin \alpha x dx = -4P/m\pi$, $E_0 = (1/a) \int_0^a P dx = P$, $\overline{E}_m = (2/a) \int_0^a P \cos \alpha x dx = 0$.

Substituting the boundary conditions of (13) to (14) into the general expressions of (11), we get the following six equations in terms of the six unknown coefficients:

$$m_1 A_m + m_2 B_m + m_3 C_m + m_4 D_m = 0, \quad (15)$$

$$\begin{aligned}& (m_1^2 Z_{22} - \alpha^2 E_1 E_2) A_m + (m_2^2 Z_{22} - \alpha^2 E_1 E_2) B_m \\ & + (m_3^2 Z_{22} - \alpha^2 E_1 E_2) C_m + (m_4^2 Z_{22} - \alpha^2 E_1 E_2) D_m = 0,\end{aligned}\quad (16)$$

$$\begin{aligned}& (m_1 \alpha^2 Z_{33} + m_1^3 G_{12} E_1) A_m e^{m_1 b} \\ & + (m_2 \alpha^2 Z_{33} + m_2^3 G_{12} E_1) B_m e^{m_2 b} \\ & + (m_3 \alpha^2 Z_{33} + m_3^3 G_{12} E_1) C_m e^{m_3 b}\end{aligned}\quad (17)$$

$$+ (m_4 \alpha^2 Z_{33} + m_4^3 G_{12} E_1) D_m e^{m_4 b} = \frac{-E_m Z_{11}}{E_1},$$

$$\begin{aligned}& (\alpha^3 + \alpha m_1^2 \nu_{12}) A_m e^{m_1 b} + (\alpha^3 + \alpha m_2^2 \nu_{12}) B_m e^{m_2 b} \\ & + (\alpha^3 + \alpha m_3^2 \nu_{12}) C_m e^{m_3 b} + (\alpha^3 + \alpha m_4^2 \nu_{12}) D_m e^{m_4 b} = 0,\end{aligned}\quad (18)$$

$$-3B + C \nu_{12} = \frac{P Z_{11}}{2E_1 E_2 G_{12}}, \quad (19)$$

$$3BE_1 E_2 + CZ_{33} = 0. \quad (20)$$

Solution of the previous algebraic equations (15) to (18) yields the unknown constants A_m , B_m , C_m , and D_m .

From (19) and (20), we get the values of B and C as

$$\begin{aligned}B &= -\frac{P Z_{11} Z_{33}}{6G_{12} E_1 E_2 (\nu_{12} E_1 E_2 + Z_{33})}, \\ C &= \frac{P Z_{11}}{2G_{12} (\nu_{12} E_1 E_2 + Z_{33})}.\end{aligned}\quad (21)$$

Once the values of the unknowns are known, they are directly substituted into (11) to obtain the explicit expressions for the different parameters of interest, namely, the two displacement and the three stress components. Following the previous procedure, the panel having the orientation of fibers $\theta = 0^\circ$ and isotropic material are solved.

FIGURE 2: Deformed shape of the panel (deformation is magnified by multiplying factor of 1000).

5. Results and Discussions

5.1. Elastic Analysis of the Stiffened Panel ($\theta = 90^\circ$). Here, the deformed shape of the orthotropic beam panel subjected to a combined loading having fiber orientation $\theta = 90^\circ$ under uniform tension and shear at its tip is shown in Figure 2. The effect of fiber orientation and material isotropy on the elastic field of the stiffened orthotropic panel is analyzed. Finally, the present elastic solutions of the supporting edge and the loaded tip are compared with those of finite element predictions.

5.2. Effect of Fiber Orientation and Material Isotropy. Effects of fiber orientation and material isotropy on the displacement and stress components are analyzed in this paper. Two different fiber orientations, such as $\theta = 0^\circ$ and 90° , are taken into account for the purpose of the present analysis. Here, a serious attention is taken on the effect of fiber orientation and material isotropy of a boron/epoxy-stiffened panel subjected to a combined loading (tension and shear) at the tip. The aspect ratio of the panel b/a , as observed in Figure 1, is 3. The effective materials properties are given in Table 1. Effects of fiber orientation $\theta = 90^\circ$ and $\theta = 0^\circ$ are considered as case 1 and case 2, respectively, and material isotropy is considered in case 3.

The distribution of the lateral displacement component at the section $y/b = 1.0$ for three different cases is shown in Figure 3(a). In this figure, there is observed slight effect on fiber orientations. Over the range $0 < x/a < 1$, for case 1, the lateral displacement with x/a at the section $y/b = 1$ remains same with x/a . For case 2, the lateral displacement with x/a at the section $y/b = 1$ increases. At the neutral axis, $x/a = 0.5$, the lateral displacement for case 1 and case 2 is equal. The lateral displacement at the section $y/b = 1$ of the isotropic panel increases with x/a . The magnitude of lateral displacement component u_x of the isotropic panel at the section $y/b = 1$ is the lowest. Figure 3(b) illustrates the effect of fiber orientation and material isotropy on the normalized axial displacement component u_y at the section $y/b = 1.0$. At two stiffened edges, the axial displacement component u_y is zero, which satisfies the physical boundary conditions of the problem. The axial displacement component u_y of the

FIGURE 3: Effect of fiber orientation and material isotropy on the normalized displacement components at the section $y/b = 1.0$ of the panel: (a) lateral displacement; (b) axial displacement.

FIGURE 4: Effect of fiber orientation and material isotropy on the normalized lateral stress component at the section $y/b = 1.0$ of the panel.

orthotropic panel is the highest for case 2 because in this case, the stiffness of the panel in y -direction is the lowest, and for case 1, the opposite phenomenon occurs due to having the highest stiffness in the same directions of the panels among three cases, and the axial displacement component u_y of the isotropic panel remains in between them due to having intermediate stiffness. From Figure 4, it is observed that at the section $y/b = 1$, for case 2, case 1, and case 3, the panel shows the highest, lowest, and intermediate lateral stress σ_{xx} , and the variation of the lateral stress occurs due to having different stiffness in x -direction of the panel; for example, for case 2, case 1, and case 3, in x -direction, the panel shows the highest, lowest, and intermediate stiffness. From Figure 5(a), it is observed that the lateral displacement at the section $y/b = 0.5$ with x/a for case 2 remains constant with x/a but it decreases with x/a for case 1. At the neutral axis, $x/a = 0.5$, the lateral displacement at the section $y/b = 0.5$ for case 1 and 2 is same. For case 3, the lateral displacement at the section $y/b = 0.5$ is the lowest in compared to other two cases, 1 and 2. Figure 5(b) illustrates the comparison of the axial displacement components u_y at the section $y/b = 0.5$ at different fiber orientation and material isotropy of the panel. The axial displacement component u_y at the section $y/b = 0.5$ is the highest for the 90° fiber orientation (case 1) and the lowest for the isotropic material (case 3), and the displacement component of 90° fiber orientation (case 2) remains in between them. From Figure 6(a), it is shown that at the section $y/b = 0.5$, the isotropic panel shows the

FIGURE 5: Effect of fiber orientation and material isotropy on the normalized displacement components at the section $y/b = 0.5$ of the panel: (a) lateral displacement; (b) axial displacement.

highest lateral stress, σ_{xx} but effect of fiber orientation on the stress distribution is little. The panel whose fiber orientation is 90° shows highest axial stress σ_{yy} at the section $y/b = 0.5$ as shown in Figure 6(b). From Figure 6(c), it is observed that at the section $y/b = 0.5$, the shear stress is the highest for case 1 and the lowest for case 2. The shear stress for case 3 at the section $y/b = 0.5$ remains in between that of other two cases. Figure 7 shows the effect of fiber orientation and material isotropy on the shear stress distribution at the section $x/a = 0.0$. There is also observed effect of fiber orientation and material isotropy on the shear stress distribution.

6. Comparison with the Available Results

To compare the present solutions with those of FEM (finite element modeling), the panel made of boron/epoxy having fiber orientation $\theta = 90^\circ$ is solved using the commercial software ANSYS. Rectangular orthotropic plane element is used to mesh the present model. To get higher accuracy of solutions, the numbers of elements are decreased or increased.

Figure 8 shows the comparison of solutions of lateral and axial displacement components at the section $y/b = 1$ by the analytical and FE predictions. The lateral displacement component of the section $y/b = 1$ obtained by FE and analytical method coincide each other as shown in Figure 8(a). Away from the stiffened edge to the neutral axis of the panel, the deviation between the solutions of axial displacement

obtained by FEM and analytical method increases, and axial displacement obtained by analytical method is higher than those of FEM as shown in Figure 8(b). From Figure 9, it is observed that the lateral stress of the section $y/b = 1$ obtained by analytical and FEM is almost same in most of the regions of this section. Figure 10 illustrates the comparison of different stress components of the section $y/b = 0$ by analytical and FE methods. From Figures 10(a) and 10(b), it is observed that the deviation between the solution of lateral stress obtained by the analytical and FE methods is significant; for the lateral stress, finite element method gives higher values than that of analytical method, but for the axial stress, the opposite phenomenon occurs. From Figure 10(c), it is clearly observed that the shear of the section $y/b = 0$ obtained by analytical and FE methods is almost same. From the comparative analysis of the solutions between those of analytical and FEM methods, it is observed that in most of the cases, the solutions obtained by the analytical and FE methods quantitatively agree well, but in some cases, the solutions do not agree quantitatively but agree well qualitatively. So, the present analytical solutions are highly reliable for designing such type of structures.

7. Conclusions

The effects of fiber orientation and material isotropy on the stress and displacement components of the panels under

FIGURE 6: Effect of fiber orientation and material isotropy on the normalized stress components at the section $y/b = 0.5$ of the panel: (a) lateral stress component; (b) axial stress component; (c) shear stress component.

FIGURE 7: Effect of fiber orientation and material isotropy on the normalized shear stress component at the section $x/a = 0$ of the panel.

uniform tension and shear at the tip of the panel are discussed. Effect of stiffeners is strongly observed in the solution. Due to stiffeners, maximum shear stress is observed at the stiffened edges. Significant effect of fiber orientation and material isotropy are observed on the displacement and stress components at different important sections of the panel. To check the reliability and soundness of the present solution, the present analytical solutions of the panel are compared with those of Finite element predictions.

Appendices

A. Solution of the Isotropic Panel

The mathematical background for the solution of isotropic stiffened panel using the present ψ -formulation is summarized in this appendix for ready reference of interested readers. Combining (3) and (6), the governing ordinary differential equation for the isotropic panel problem is obtained, which is

$$Y_m'''' - 2\alpha^2 Y_m'' + \alpha^4 Y_m = 0. \quad (\text{A.1})$$

(a)

(b)

FIGURE 8: Comparison of solutions of different displacement components at the section $y/b = 1$ of the beam by analytical and FE methods: (a) lateral displacement; (b) axial displacement.

The general solution of (A.1) is as follows:

$$Y_m = A_m \cosh \alpha y + B_m \alpha y \sinh \alpha y + C_m \sinh \alpha y + D_m \alpha y \cosh \alpha y. \quad (\text{A.2})$$

FIGURE 9: Comparison of solutions of the lateral stress component at the section, $y/b = 1$ by analytical and FE methods.

The corresponding expressions for the displacement and stress components in terms of the four arbitrary constants are as follows:

$$\begin{aligned}
 u_x(x, y) &= \sum_{m=1}^{\infty} \alpha^2 [(A_m + B_m) \sinh \alpha y \\
 &\quad + (C_m + D_m) \cosh \alpha y + B_m \alpha y \cosh \alpha y \\
 &\quad + D_m \alpha y \sinh \alpha y] \cos \alpha x + 2Cy,
 \end{aligned} \tag{A.3}$$

$$\begin{aligned}
 u_y(x, y) &= \sum_{m=1}^{\infty} \alpha^2 [(A_m - 2tB_m) \cosh \alpha y \\
 &\quad + (C_m - 2tD_m) \sinh \alpha y \\
 &\quad + B_m \alpha y \sinh \alpha y \\
 &\quad + D_m \alpha y \cosh \alpha y] \sin \alpha x \\
 &\quad + \{6B + C(1 - \nu)\} E^2 x,
 \end{aligned} \tag{A.4}$$

$$\begin{aligned}
 \sigma_{xx}(x, y) &= - \sum_{m=1}^{\infty} \frac{E\alpha^3}{(1 + \nu)} [(A_m + t'B_m) \sinh \alpha y \\
 &\quad + (C_m + t'D_m) \cosh \alpha y \\
 &\quad + B_m \alpha y \cosh \alpha y + D_m \alpha y \sinh \alpha y] \sin \alpha x,
 \end{aligned} \tag{A.5}$$

$$\begin{aligned}
 \sigma_{yy}(x, y) &= - \sum_{m=1}^{\infty} \frac{E\alpha^3}{(1 + \nu)} [(A_m - tB_m) \sinh \alpha y \\
 &\quad + (C_m - tD_m) \cosh \alpha y \\
 &\quad + B_m \alpha y \cosh \alpha y \\
 &\quad + D_m \alpha y \sinh \alpha y] \sin \alpha x,
 \end{aligned} \tag{A.6}$$

$$\begin{aligned}
 \sigma_{xy}(x, y) &= \sum_{m=1}^{\infty} \frac{E\alpha^3}{(1 + \nu)^2} [\{A_m(1 + \nu) + 2\nu B_m\} \cosh \alpha y \\
 &\quad + B_m(1 + \nu) \alpha y \sinh \alpha y \\
 &\quad + \{C_m(1 + \nu) + 2\nu D_m\} \sinh \alpha y \\
 &\quad + D_m(1 + \nu) \alpha y \cosh \alpha y] \\
 &\quad \times \cos \alpha x \\
 &\quad + \frac{E}{(1 + \nu)^2} (-6B + 2C\nu),
 \end{aligned} \tag{A.7}$$

where $t = (1 - \nu)/(1 + \nu)$ and $t' = (1 + 3\nu)/(1 + \nu)$.

Now, as before, solving (A.3), (A.4), (A.6), and (A.7) with the appropriate boundary conditions given by (12)–(14), the values of the four constants are evaluated.

B. Effect of Material Isotropy on the Solution

It can be noted here that the corresponding isotropic material properties of the boron/epoxy orthotropic composite material are obtained by solving the following set of equations [33]:

$$\begin{aligned}
 V_m \frac{K_m - K}{3K_m + 4G} + V_f \frac{K_f - K}{3K_f + 4G} &= 0, \\
 V_m \frac{G_m - G}{G_m + \beta} + V_f \frac{G_f - G}{G_f + \beta} &= 0,
 \end{aligned} \tag{B.1}$$

$$E = \frac{9KG}{3K + G},$$

$$G = \frac{E}{2(1 + \nu)},$$

where $\beta = G(9K + 8G)/(6K + 12G)$.

FIGURE 10: Comparison of solutions of different stress components at the section $y/b = 0$ by analytical and FE methods: (a) lateral stress; (b) axial stress and (c) shear stress.

The solutions of the previous equations are carried out for a composite composition of 60% boron fiber and 40% epoxy resin. The supporting mathematical treatments required to solve the isotropic stiffened panel problem are given in Appendix A.

Nomenclature

E_1 : Elastic modulus of the material in x -direction
 E_2 : Elastic modulus of the material in y -direction
 ν_{12} : Major Poisson's ratio

ν_{21} : Minor Poisson's ratio
 θ : Fiber orientation
 u_x : Displacement components in the x direction
 u_y : Displacement components in the y direction
 σ_{xx} : Normal stress components in the x direction
 σ_{yy} : Normal stress components in the y direction
 σ_{xy} : Shearing stress component in the xy plane
 ψ : Displacement potential function
 a : Width of the panel
 b : Length of the panel
 σ_{yy}^0 : Uniformly distributed axial loading on the panel
 σ_{xy}^0 : Uniform shear stress
 E : Elastic modulus of isotropic material
 ν : Poisson's ratio of isotropic material.

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