Research Article

Plane Wave-Perturbative Method for Evaluating the Effective Speed of Sound in 1D Phononic Crystals

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A method for calculating the effective sound velocities for a 1D phononic crystal is presented; it is valid when the lattice constant is much smaller than the acoustic wave length; therefore, the periodic medium could be regarded as a homogeneous one. The method is based on the expansion of the displacements field into plane waves, satisfying the Bloch theorem. The expansion allows us to obtain a wave equation for the amplitude of the macroscopic displacements field. From the form of this equation we identify the effective parameters, namely, the effective sound velocities for the transverse and longitudinal macroscopic displacements in the homogenized 1D phononic crystal. As a result, the explicit expressions for the effective sound velocities in terms of the parameters of isotropic inclusions in the unit cell are obtained: mass density and elastic moduli. These expressions are used for studying the dependence of the effective, transverse and longitudinal, sound velocities for a binary 1D phononic crystal upon the inclusion filling fraction. A particular case is presented for 1D phononic crystals composed of W-Al and Polyethylene-Si, extending for a case solid-fluid.

1. Introduction

At present there is a great interest in fabricating artificial materials, having extraordinary properties, which considerably extend those of natural materials. Such a new class of materials is known as metamaterials. From the beginning, the photonic metamaterials, possessing negative index of refraction, have intensively been investigated. It was established that such an unusual optical property is found in periodic structures (photonic crystals) whose dielectric function is spatially modulated. A peculiarity of the photonic metamaterials is the high dielectric contrast between the components. The negative refraction has been observed in double negative metamaterials with simultaneously negative effective permittivity and permeability [1–6]. However, this

phenomenon is also manifested in anisotropic materials, as simple as a one-dimensional (1D) photonic crystal or superlattice, having elements in the effective permittivity tensor of different sign [7–10]. Analogously, metamaterials with uncommon acoustic properties have also searched among phononic crystals (PCs), that is, materials with periodic modulation of their elastic properties. As in the photonic case, the calculation of the effective parameters of a phononic crystal is an important task, for designing of resonant elastic metamaterials and acoustic lenses; there is a great interest in investigating the effective mass density and compliance tensors, as well as the effective sound velocity.

Several homogenization theories, which are valid when the acoustic wavelength is much longer than the lattice constant of the phononic crystal, have been proposed

[11–15]. Among homogenization theories, we can identify two commonly used approaches. One of them provides effective acoustic parameters within the framework of multiple scattering [11–13] in which the effective parameters are not obtained according to the direction of propagation; for example, for a sonic crystal the solid material is simply considered rigid; the study is done on the propagation of sound in the air of the structure's matrix, neglecting the modes that can propagate in the solid. The second, which is based on the Fourier formalism, makes use of the expansion of the microscopic acoustic field into plane waves and allows calculating the effective parameters as a function of the direction of propagation [14, 15]. The latter approach is of particular interest because it can be applied to different geometries of the inclusions inside the unit cell. Here, we will precisely use the Fourier formalism to calculate the effective velocities of elastic wave propagation in 1D solid-solid and fluid-solid phononic crystals. The very first known experimental observation of 1D phononic crystals was when Narayanamurti et al. investigated the propagation of high-frequency phonons through a GaAs/AlGaAs superlattice; other domains in which 1D phononic crystals have potential applications are crystal sensors, waveguide, and resonant transmission [16–19]. As in the case of 1D photonic crystals, such an inherently anisotropic elastic system is a potential metamaterial. However, the main goal of the present work is to establish the basis for the development of a general homogenization theory of (1D, 2D, or 3D) elastic phononic crystals based on the plane wave expansion.

The work is divided as follows: in Section 2, the Fourier formalism applied to a binary elastic superlattice is presented. The proposed method for calculating the effective acoustic parameters for a phononic superlattice, which is based upon the derivation of the wave equation for the macroscopic displacements field, is described in Section 3. Finally, we shall apply the derived explicit formulas for the effective sound velocities to binary superlattices composed of W-Al and Polyethylene-Si and thus study the behavior of this parameter effective in crystals with high and low contrast between the mechanical properties of its components; also a case solid-fluid is presented.

2. One-Dimensional Phononic Crystals

Let us consider a one-dimensional phononic crystal (or elastic superlattice) composed of alternating layers of isotropic elastic materials, A and B (Figure 1). Their thicknesses are, respectively, d and $a - d$, where a is the lattice constant. Assuming that the z -axis is parallel to the superlattice growth direction, the mass density ρ , as well as the longitudinal and transverse sound velocities, C_l and C_t , turn out to be functions of the z -coordinate only.

In this case, the second Newton's law for the displacement vector $\vec{u}(z, t)$, in the absence of external forces, acquires the form [20, 21]:

$$\frac{\partial \sigma_{ik}}{\partial x_k} = \rho(z) \frac{\partial^2 u_i}{\partial t^2}. \quad (1)$$

FIGURE 1: Scheme of the binary superlattice.

Here, the Cauchy stress tensor σ_{ij} is expressed in terms of the strain tensor

$$u_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \quad (2)$$

according to the relation

$$\sigma_{ik} = 2\rho(z) C_t^2(z) u_{ik} + \rho(z) (C_l^2(z) - 2C_t^2(z)) u_{ll} \delta_{ik}, \quad (3)$$

where δ_{ik} is Kronecker's delta. Substituting (2) and (3) into (1), the wave equation for an elastic inhomogeneous medium is obtained [22, 23]:

$$\begin{aligned} \rho \frac{\partial^2 u_i}{\partial t^2} = & \nabla \cdot (\rho C_t^2 \nabla u_i) + \nabla \cdot \left(\rho C_t^2 \frac{\partial \vec{u}}{\partial x_i} \right) \\ & + \frac{\partial}{\partial x_i} [(\rho C_l^2 - 2\rho C_t^2) \nabla \cdot \vec{u}]. \end{aligned} \quad (4)$$

Due to the periodicity of the functions $\rho(z)$, $C_l(z)$, and $C_t(z)$, we can expand them into Fourier series as

$$\rho(z) = \sum_{G_z} \rho(G_z) e^{iG_z z}, \quad (5)$$

$$C_{11}(z) = \rho(z) C_l^2(z) = \sum_{G_z} \Lambda(G_z) e^{iG_z z}, \quad (6)$$

$$C_{44}(z) = \rho(z) C_t^2(z) = \sum_{G_z} \tau(G_z) e^{iG_z z}, \quad (7)$$

where C_{11} and C_{44} are the longitudinal and transversal elastic constants for an isotropic elastic medium, and the components for the vectors of the reciprocal lattice are expressed as

$$G_z = \frac{2\pi}{a} n, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots \quad (8)$$

FIGURE 2: Unit cell of the binary superlattice.

The component C_{12} for an isotropic medium is given by the relation:

$$C_{12}(z) = C_{11}(z) - 2C_{44}(z). \quad (9)$$

According to our geometry (see Figures 1 and 2), the Fourier coefficients $\rho(G_z)$, appearing in (5), can be calculated with the formula

$$\rho(G_z) = \frac{1}{a} \int_{-a/2}^{a/2} \rho(z) e^{-iG_z z} dz \quad (10)$$

$$= \rho_B \delta_{0G_z} + (\rho_A - \rho_B) F(G_z),$$

where $F(G_z)$ is the form factor of the inclusion A,

$$F(G_z) = f \frac{\sin(G_z d/2)}{G_z d/2}, \quad (11)$$

and the ratio $f = d/a$ is its filling fraction. Similarly, the Fourier coefficients $\Lambda(G_z)$ and $\tau(G_z)$ of the expansions (6) and (7) can, respectively, be written as

$$\begin{aligned} \Lambda(G_z) &= \Lambda_B \delta_{0G_z} + (\Lambda_A - \Lambda_B) F(G_z), \\ \tau(G_z) &= \tau_B \delta_{0G_z} + (\tau_A - \tau_B) F(G_z). \end{aligned} \quad (12)$$

For a periodic system, the solution of (4) should satisfy the Bloch theorem [24]. Therefore, we can write the displacement vector in the form

$$\vec{u}(\vec{r}, t) = e^{-i\omega t} e^{i\vec{K} \cdot \vec{r}} \vec{u}_{\vec{K}}(z), \quad (13)$$

where $\vec{u}_{\vec{K}}(z)$ is a periodic function with the same period as for the superlattice and ω is the frequency. We should note that the component K_z is here the Bloch wave number and $\vec{K}_{||}$ is an independent vector parallel to the x - y plane. After expanding the function $\vec{u}_{\vec{K}}(z)$ into Fourier series, an expression can be obtained

$$\vec{u}(\vec{r}, t) = e^{-i\omega t} e^{i\vec{K} \cdot \vec{r}} \sum_{G_z} \vec{u}_{\vec{K}}(G_z) e^{iG_z z}. \quad (14)$$

It is worth emphasizing that the Fourier formalism for solving the wave equation (4) allows us to avoid the application

of boundary conditions for the displacement vector $\vec{u}(\vec{r})$, since the structural information of the system is explicitly contained in the Fourier coefficients $\vec{u}_{\vec{K}}(G_z)$. This is owing to the fact that the boundary conditions for $\vec{u}(\vec{r})$ are consistent with the wave equation (4).

Substituting (5)–(7) and (14) into (4), we obtain

$$\begin{aligned} \sum_{G'_z} \{ & \tau(G_z - G'_z) \vec{u}(G'_z) (\vec{K}_{||} + (K_z + G'_z) \hat{z}) \\ & \cdot (\vec{K}_{||} + (K_z + G_z) \hat{z}) + \tau(G_z - G'_z) \vec{u}(G'_z) \\ & \cdot (\vec{K}_{||} + (K_z + G_z) \hat{z}) (\vec{K}_{||} + (K_z + G'_z) \hat{z}) \\ & + \Lambda(G_z - G'_z) \vec{u}(G'_z) \\ & \cdot (\vec{K}_{||} + (K_z + G_z) \hat{z}) (\vec{K}_{||} + (K_z + G_z) \hat{z}) \\ & - 2\tau(G_z - G'_z) \vec{u}(G'_z) \\ & \cdot (\vec{K}_{||} + (K_z + G'_z) \hat{z}) (\vec{K}_{||} + (K_z + G_z) \hat{z}) \\ & - \omega^2 \rho(G_z - G'_z) \vec{u}(G'_z) \} = 0. \end{aligned} \quad (15)$$

We can rewrite (15) in dyadic form as

$$\sum_{G'_z} \left[\overleftrightarrow{B}(G_z, G'_z) - \omega^2 \rho(G_z - G'_z) \overleftrightarrow{I} \right] \cdot \vec{u}(G'_z) = 0, \quad (16)$$

where

$$\begin{aligned} \overleftrightarrow{B}(G_z, G'_z) &= \tau(G_z - G'_z) (\vec{K}_{||} + (K_z + G'_z) \hat{z}) \\ & \cdot (\vec{K}_{||} + (K_z + G_z) \hat{z}) \overleftrightarrow{I} \\ & + \tau(G_z - G'_z) (\vec{K}_{||} + (K_z + G'_z) \hat{z}) \\ & \cdot (\vec{K}_{||} + (K_z + G_z) \hat{z}) + \Lambda(G_z - G'_z) \\ & \cdot (\vec{K}_{||} + (K_z + G_z) \hat{z}) \\ & \cdot (\vec{K}_{||} + (K_z + G'_z) \hat{z}) - 2\tau(G_z - G'_z) \\ & \cdot (\vec{K}_{||} + (K_z + G_z) \hat{z}) \\ & \cdot (\vec{K}_{||} + (K_z + G'_z) \hat{z}), \end{aligned} \quad (17)$$

and $\overleftrightarrow{I}$ is the unit dyadic $[(\overleftrightarrow{I})_{ij} = \delta_{ij}]$.

The inverse $v(G_z, G'_z)$ of matrix $\rho(G_z, G'_z) \equiv \rho(G_z - G'_z)$ in Eq. (17) satisfies the relation

$$\sum_{G'_z} v(G_z, G'_z) \rho(G'_z, G''_z) = \delta_{G_z, G''_z}, \quad (18)$$

and its elements are directly calculated by using the expression

$$\begin{aligned} v(G_z, G'_z) &= v(G_z - G'_z) \\ &= \frac{1}{a} \int_{-a/2}^{a/2} \frac{1}{\rho(z)} e^{-i(G_z - G'_z)z} dz \\ &= \frac{1}{\rho_B} \delta_{G_z, G'_z} + \left(\frac{1}{\rho_A} - \frac{1}{\rho_B} \right) F(G_z - G'_z). \end{aligned} \quad (19)$$

Multiplying (17) by $v(G_z, G'_z)$ (19), we get

$$\sum_{G'_z} \left[\vec{M}(G_z, G'_z) - \omega^2 \delta_{G_z, G'_z} \vec{T} \right] \cdot \vec{u}(G'_z) = 0, \quad (20)$$

where we have introduced the dyadic $\vec{M}(G_z, G'_z)$:

$$\vec{M}(G_z, G'_z) = \sum_{G''_z} v(G_z, G''_z) \vec{B}(G''_z, G'_z). \quad (21)$$

The system (20) represents a homogeneous system of algebraic equations, which has a nontrivial solution if the determinant of the associated matrix is identical to zero; that is,

$$\det \left\{ \vec{M}(G_z, G'_z) - \omega^2 \delta_{G_z, G'_z} \vec{T} \right\} = 0. \quad (22)$$

The condition (22) provides the dispersion relation $\omega(K_z)$ for vibrational eigenmodes in the PC. It should be noted that such a phononic dispersion parametrically depends upon the wave vector component $\vec{K}_{||}$. Thus, the phononic dispersion can be straightforwardly calculated by numerically diagonalizing the matrix $\vec{M}(G_z, G'_z)$. In doing that, we should use a finite matrix of sufficiently large size to guarantee accurate results.

3. Effective Sound Velocity

In the case when the wave length of sound is much larger than the lattice constant of the phononic crystal, this artificial periodic heterostructure can be modeled as a homogeneous medium with effective acoustic parameters. In this section, we shall calculate the effective sound velocities for both transverse and longitudinal vibrational modes, propagating along the growth direction of a superlattice like that considered in the previous sections.

From the Bloch theorem, given by (14), we can write the displacements field as

$$\begin{aligned} \vec{u}(\vec{r}, t) &= e^{-i\omega t} \left\{ e^{i\vec{K} \cdot \vec{r}} \vec{u}_{\vec{K}}(0) + e^{i\vec{K} \cdot \vec{r}} \sum_{G_z \neq 0} \vec{u}_{\vec{K}}(G_z) e^{iG_z z} \right\}. \end{aligned} \quad (23)$$

Assuming $|K_z a| \ll 1$ and $\vec{K}_{||} = 0$, the first term on the right-hand side of (23) describes the smooth variations of the displacement vector $\vec{u}(\vec{r}, t)$ as a function of the coordinates, whereas the second term therein is associated with the rapid

oscillations (fluctuations) over the unit cell. So, we can define the average (macroscopic) displacements field as

$$\vec{U}(\vec{r}, t) \equiv e^{-i\omega t} e^{i\vec{K} \cdot \vec{r}} \vec{u}_{\vec{K}}(0) = e^{-i\omega t} e^{i\vec{K} \cdot \vec{r}} \vec{U}_0. \quad (24)$$

According to (20), the Fourier coefficients $\vec{u}(G_z)$ satisfy the system of equations

$$\sum_{G'_z} \overleftrightarrow{D}(G_z, G'_z) \cdot \vec{u}(G'_z) = 0, \quad (25)$$

where we have introduced the dyadic

$$\overleftrightarrow{D}(G_z, G'_z) = \vec{M}(G_z, G'_z) - \omega^2 \delta_{G_z, G'_z} \vec{T}. \quad (26)$$

One can express the coefficients $\vec{u}(G_z \neq 0)$ in terms of $\vec{U}_0$ by using (25) for $G_z \neq 0$. Thus,

$$\vec{u}(G_z) = - \sum_{G'_z \neq 0} \overleftrightarrow{D}_s^{-1}(G_z, G'_z) \overleftrightarrow{D}(G'_z, 0) \cdot \vec{U}_0. \quad (27)$$

Here, $\overleftrightarrow{D}_s(G_z, G'_z)$ is a submatrix, obtained from $\overleftrightarrow{D}(G_z, G'_z)$ (26) after eliminating its block rows (columns) with $G_z = 0$ ($G'_z \neq 0$). As it was demonstrated in [25], the inverse of the submatrix $\overleftrightarrow{D}_s(G_z, G'_z)$ is related with the inverse of the matrix $\overleftrightarrow{D}(G_z, G'_z)$ according to the formula:

$$\begin{aligned} \overleftrightarrow{D}_s^{-1}(G_z, G'_z) &= \overleftrightarrow{D}^{-1}(G_z, G'_z) \\ &\quad - \overleftrightarrow{D}^{-1}(G_z, 0) \left\{ \overleftrightarrow{D}^{-1}(0, 0) \right\}^{-1} \overleftrightarrow{D}^{-1}(0, G'_z), \end{aligned} \quad (28)$$

where $\overleftrightarrow{D}^{-1}(0, 0)$ is a 3×3 block, which is obtained from the original matrix $\overleftrightarrow{D}^{-1}(G_z, G'_z)$ (26) of infinite size, and $\{\dots\}^{-1}$ symbolizes the inverse of the 3×3 matrix block.

Let us rewrite (25) for $G_z = 0$ as

$$\omega^2 \vec{U}_0 = \vec{M}(0, 0) \cdot \vec{U}_0 + \sum_{G'_z \neq 0} \overleftrightarrow{D}(0, G'_z) \cdot \vec{u}(G'_z). \quad (29)$$

Substituting (27) and (28) into (29), we obtain the macroscopic wave equation:

$$\omega^2 \vec{U}_0 = \vec{M}_{\text{eff}}(K_z) \cdot \vec{U}_0, \quad (30)$$

where the effective matrix $\vec{M}_{\text{eff}}$ is given by

$$\vec{M}_{\text{eff}}(K_z) = \left\{ \overleftrightarrow{D}^{-1}(K_z; 0, 0) \right\}^{-1} + \omega^2 \vec{T}. \quad (31)$$

From the macroscopic wave equation (30), we can define an effective dyadic representing the Christoffel tensor, divided by the mass density, as

$$\left(\frac{\vec{T}}{\rho} \right)_{\text{eff}} = \frac{\vec{M}_{\text{eff}}(K_z)}{K_z^2}. \quad (32)$$

FIGURE 3: Effective sound velocity for transverse vibrational modes propagating along the growth direction of a Polyethylene-Si superlattice. Here, solid line was obtained by using formula (33) and squares were obtained by using formula (35).

Thus,

$$\left(\frac{\vec{\Gamma}}{\rho} \right)_{\text{eff}} = \frac{1}{K_z^2} \left\{ \vec{D}^{-1}(K_z; 0, 0) \right\}^{-1} + \left(\frac{\omega}{K_z} \right)^2 \vec{T}. \quad (33)$$

In the situation, considered here, of acoustic waves propagating along the growth direction, the effective tensor (33) turns out to be diagonal; that is, $(\vec{\Gamma}/\rho)_{\text{eff},ij} = C_{\text{eff},i} \delta_{ij}$ with $C_{\text{eff},x} = C_{\text{eff},y}$. The principal values $C_{\text{eff},x} (= C_{\text{eff},y})$ and $C_{\text{eff},z}$ are, in fact, the effective sound velocities for transverse and longitudinal modes, respectively. It should be emphasized that formula (33) is valid in the long-wavelength limit, that is, when both $K_z \rightarrow 0$ and $\omega \rightarrow 0$.

4. Numerical Results and Validation

Let us apply the derived formula (33) for calculating the effective sound velocities of modes propagating along the growth direction of specific binary 1D phononic crystals. First, we shall consider a Polyethylene-silicon superlattice having a period $a = 0.2$ cm. The parameters used in the calculations are mass densities $\rho_A = 2.32$ gr/cm³ and $\rho_B = 0.89$ gr/cm³, transverse sound velocities $C_{t,A} = 5.86 \times 10^5$ cm/s and $C_{t,B} = 0.5 \times 10^5$ cm/s, and longitudinal sound velocities $C_{l,A} = 8.45 \times 10^5$ cm/s and $C_{l,B} = 1.9 \times 10^5$ cm/s for silicon and Polyethylene, respectively (the material parameters were taken from [24]). In the numerical calculations, 300 plane waves were needed to achieve good convergence of the results.

Figures 3 and 4 (continuous line) show the dependencies of the transverse ($C_{\text{eff},t}$) and longitudinal ($C_{\text{eff},l}$) effective sound velocities upon the silicon filling fraction f ($f = d/a$). As it is seen, both transverse and longitudinal effective sound velocities slowly vary with increasing the silicon filling fraction f up to $f \approx 0.9$. In contrast, at $f > 0.9$, the slopes of the curves $C_{\text{eff},t}(f)$ and $C_{\text{eff},l}(f)$ are relatively large, because

FIGURE 4: Effective sound velocity for longitudinal vibrational modes propagating along the growth direction of a Polyethylene-Si superlattice. Here, solid line was obtained by using formula (33) and squares were obtained by using formula (36).

FIGURE 5: Effective sound velocity for transverse vibrational modes propagating along the growth direction of a W-Al superlattice. Here, solid line was obtained by using formula (33) and squares were obtained by using formula (35).

of the large contrast between the acoustic impedance $\rho_A C_{i,A}$ and $\rho_B C_{i,B}$ ($i = t, l$) of the system.

Figures 5 and 6 (continuous line) exhibit numerically calculated effective parameters, $C_{\text{eff},t}$ and $C_{\text{eff},l}$, for a W-Al 1D phononic crystal. The parameters used here are [24] $\rho_A = 2.69$ gr/cm³ and $\rho_B = 19.3$ gr/cm³, transverse sound velocities $C_{t,A} = 3.22 \times 10^5$ cm/s and $C_{t,B} = 2.88 \times 10^5$ cm/s, and longitudinal sound velocities $C_{l,A} = 6.45 \times 10^5$ cm/s and $C_{l,B} = 5.21 \times 10^5$ cm/s for aluminium and tungsten, respectively. In this case, the effective sound velocity for transverse [longitudinal] modes decreases with the Al filling fraction f up to its minimum value at $f \approx 0.45$. For larger values of f (i.e., in the interval $0.5 < f < 1$) the effective parameter $C_{\text{eff},t}$ [$C_{\text{eff},l}$] increases with f .

In order to verify our numerical results in Figures 3–6 (continuous line), we use the effective nonlocal-response

FIGURE 6: Effective sound velocity for longitudinal vibrational modes propagating along the growth direction of a W-Al superlattice. Here, solid line was obtained by using formula (33) and squares were obtained by using formula (36).

matrix $\bar{\bar{A}}_{\text{eff}}(\vec{K})$ in the quasistatic limit ($\omega \rightarrow 0, \vec{K} \rightarrow 0$). The block structure of the matrix $\bar{\bar{A}}_{\text{eff}}(\vec{K})$ for such systems has the form [26]

$$\bar{\bar{A}}_{\text{eff}}(\vec{K} \rightarrow 0) = \begin{pmatrix} \bar{\rho}_{\text{eff}}(\vec{K} \rightarrow 0) & 0_{3 \times 6}(\vec{K} \rightarrow 0) \\ 0_{6 \times 3}(\vec{K} \rightarrow 0) & \bar{\mathcal{S}}_{\text{eff}}(\vec{K} \rightarrow 0) \end{pmatrix}, \quad (34)$$

where $\bar{\rho}_{\text{eff}}(\vec{K} \rightarrow 0)$ and $\bar{\mathcal{S}}_{\text{eff}}(\vec{K} \rightarrow 0)$ are the effective mass density and compliance tensors within the long-wavelength limit.

Figure 7 exhibits graphs of the nonzero elements in the matrices $\bar{\rho}_{\text{eff}}$ (only principal value $\bar{\rho}_{\text{eff},zz}$) and $\bar{\mathcal{C}}_{\text{eff}} = \bar{\mathcal{S}}_{\text{eff}}^{-1}$ (only the effective stiffness constants $\bar{\mathcal{C}}_{\text{eff},44}$ and $\bar{\mathcal{C}}_{\text{eff},33}$) in (34) for a W/Al and Polyethylene/Si 1D phononic crystals versus the inclusion filling fraction f . The principal z -axis of the 1D phononic crystal has been oriented parallel to periodicity direction. Hence, the homogenized phononic crystal acquires tetragonal symmetry in the interval $0 < f < 1$, for which the effective sound velocity for transverse and longitudinal vibrational modes in the low frequency limit propagating along the growth direction is given according to the formulas (see [27], where Christoffel's equation for the tetragonal system can be factored to obtain the solutions for the transverse and longitudinal acoustic velocity):

$$C_{\text{eff},t} = \sqrt{\frac{C_{44,\text{eff}}}{\rho_{\text{eff},zz}}}, \quad (35)$$

$$C_{\text{eff},l} = \sqrt{\frac{C_{33,\text{eff}}}{\rho_{\text{eff},zz}}}. \quad (36)$$

Applying formulas (35) and (36), the effective sound velocity for transverse and longitudinal vibrational modes is shown in Figures 3–6 (squares). We have found that our numerical

results given by formula (33) coincide with those predicted by equations (35) and (36). The effective nonlocal-response matrix method is based on matrix inversion of $n \times n$ size (n : number of plane waves); the number of plane waves increases as the contrast between the mechanical properties of the components of phononic crystal is large and this represents a great disadvantage for the numerical convergence of the method. Formula (33) shows that the sums over $\vec{G}$ and $\vec{G}'$ converge more rapidly than the matrix inversion with finite number of plane waves; therefore numerical calculation of the sum is not a time consuming procedure at all.

Otherwise, formula (33) can be applied for determining the effective sound velocity of 1D-phononic crystals with a type fluid component, which has zero shear modulus; this is possible by using a very small value of C_{44} ($C_{44} \rightarrow 0$) in order to avoid numerical errors. With this mathematical artifice, we have calculated the effective sound velocity for the longitudinal vibrational mode for a water-aluminium superlattice having a period $a = 1$ cm (see Figure 8). The parameters used for water are $\rho = 1 \text{ gr/cm}^3$ and longitudinal sound velocity $C_l = 6.45 \times 10^5 \text{ cm/s}$. The graph shows the dependence of the effective longitudinal sound velocity upon the aluminium filling fraction f ; note that until filling fractions near 0.8 the values of the effective velocity remain close to the water, for later ascending quickly until reaching the values of the solid, thanks to the large mismatches at the interfaces of the water-aluminium building-block (due to high contrast of the acoustic impedance of the materials in the unit cell).

5. Discussion

In the developed method in this work, the homogenization is achieved in the limit $\omega, \vec{K} \rightarrow 0$ and the effective Christoffel tensor $(\Gamma_{ij})_{\text{eff}}$ is calculated from the wave equation for an unidimensional elastic inhomogeneous medium; this tensor relates the elastic constants with the propagation velocities of the elastic waves according to different directions of the principal axes that are entirely dependent on the normal wave $\vec{n} = (n_i)$ on the direction of $\vec{K}$ but not its magnitude, so that the phase velocity is defined by the square roots of the eigenvalues of the effective matrix $(\vec{\Gamma}/\rho)_{\text{eff}}$. We solve the determinant of the effective matrix taking into account the solution method proposed by [27]. It should be noted that [27] does not present a methodology of homogenization for different layers of materials but only develops the general theory of wave propagation. In this way, we obtain the relationship between the propagation velocity in the direction $\vec{n}$ and the elastic and mechanical homogenized properties of the PC. Therefore, our formula simultaneously calculates the effective velocities to which the transverse and longitudinal elastic waves propagate in the PC. On the other hand, when the 1D-PC has fluid type components, close to zero values are ingressed for C_{44} (transversal elastic constant) to calculate the effective sound velocity for the longitudinal vibrational mode. This allows establishing a more general homogenization methodology for calculating the effective

FIGURE 7: Graphs of the effective mass density $\bar{\rho}_{\text{eff},zz}$ (c) and the effective stiffness constants $\bar{C}_{\text{eff},44}$ (a) and $\bar{C}_{\text{eff},33}$ (b) of Polyethylene-Si and W-Al superlattices.

velocity of propagation of sound waves in phononic crystals with elastic constituents of solid and fluid type, which is useful to define a methodology towards the development of a theory for two- and three-dimensional systems. In contrast to the methodology presented in [14, 15] which presents a separate analysis to study the propagation of acoustic waves in fluid-fluid and solid-solid phononic crystals, in such studies, the first case is based on solving the equation sound wave propagating in a gas for the longitudinal modes; in the second case the equation of motion for an elastic medium is solved considering only transverse modes. The effective sound velocity in the homogenized system is obtained by

taking the limit $\omega, \vec{K} \rightarrow 0$; in this limit the group velocity is equal to the phase velocity, which is calculated as $c_{\text{eff}} = \omega/K$. With the above considerations, the authors obtain for each case the formula for the effective sound velocity; also they do not show results for a solid-liquid PC.

6. Conclusions

We have derived explicit formulas for the calculation of the effective sound velocities in a 1D phononic crystal in the long-wavelength limit. The formulas were applied for analyzing

FIGURE 8: Dependence of the effective longitudinal sound velocity for a homogenized 1D phononic crystal of aluminium embedded in water upon the aluminium filling fraction f .

the dependence of the effective, transverse and longitudinal, sound velocities upon the inclusion filling fraction for binary superlattices composed of Polyethylene-Si and W-Al. In the latter case, the contrast of material parameters is relatively larger and, as a result, at Al filling fractions $f \approx 0.45$, the effective sound velocity for both transverse ($C_{\text{eff},t}$) and longitudinal ($C_{\text{eff},l}$) modes, propagating along the superlattice growth direction, takes values smaller than the sound velocity for each component (W or Al). For this reason, we can say that the homogenized 1D W-Al phononic crystal behaves as a metamaterial. We have verified that our results in the quasistatic limit for Polyethylene-Si and W-Al 1D phononic crystals coincide with the results predicted by the effective mass density and compliance tensors method. Finally, formula (33) can also be applied for determining the effective sound velocity of 1D-phononic crystals with a type fluid component having zero shear modulus.

Although the homogenization theory developed here is valid only for phononic crystals with one-dimensional periodicity and isotropic inclusions, it shows the usefulness of the plane wave expansion method to obtain explicit expressions for theoretical results of the effective sound velocity. The generalization of this approach to 2D and 3D periodic elastic structures with anisotropic inclusions and arbitrary contrast of the materials parameters will be presented elsewhere.

Competing Interests

The authors declare that they have no competing interests.

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