Research Article

Generalized Differentiable E -Invex Functions and Their Applications in Optimization

S. Jaiswal and G. Panda

Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India

Correspondence should be addressed to G. Panda, geetanjali@maths.iitkgp.ernet.in

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The concept of E -convex function and its generalizations is studied with differentiability assumption. Generalized differentiable E -convexity and generalized differentiable E -invexity are used to derive the existence of optimal solution of a general optimization problem.

1. Introduction

E -convex function was introduced by Youness [1] and revised by Yang [2]. Chen [3] introduced Semi- E -convex function and studied some of its properties. Syau and Lee [4] defined E -quasi-convex function, strictly E -quasi-convex function and studied some basic properties. Fulga and Preda [5] introduced the class of E -preinvex and E -prequasi-invex functions. All the above E -convex and generalized E -convex functions are defined without differentiability assumptions. Since last few decades, generalized convex functions like quasiconvex, pseudoconvex, invex, B -vex, (p, r) -invex, and so forth, have been used in nonlinear programming to derive the sufficient optimality condition for the existence of local optimal point. Motivated by earlier works on convexity and E -convexity, we have introduced the concept of differentiable E -convex function and its generalizations to derive sufficient optimality condition for the existence of local optimal solution of a nonlinear programming problem. Some preliminary definitions and results regarding E -convex function are discussed below, which will be needed in the sequel. Throughout this paper, we consider functions $E : R^n \rightarrow R^n$, $f : M \rightarrow R$, and M are nonempty subset of R^n .

Definition 1.1 (see [1]). M is said to be E -convex set if $(1 - \lambda)E(x) + \lambda E(y) \in M$ for $x, y \in M$, $\lambda \in [0, 1]$.

Definition 1.2 (see [1]). $f : M \rightarrow R$ is said to be E -convex on M if M is an E -convex set and for all $x, y \in M$ and $\lambda \in [0, 1]$,

$$f((1 - \lambda)E(x) + \lambda E(y)) \leq (1 - \lambda)f(E(x)) + \lambda f(E(y)). \quad (1.1)$$

Definition 1.3 (see [3]). Let M be an E -convex set. f is said to be semi- E -convex on M if for $x, y \in M$ and $\lambda \in [0, 1]$,

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(x) + (1 - \lambda)f(y). \quad (1.2)$$

Definition 1.4 (see [5]). M is said to be E -invex with respect to $\eta : R^n \times R^n \rightarrow R^n$ if for $x, y \in M$ and $\lambda \in [0, 1]$, $E(y) + \lambda\eta(E(x), E(y)) \in M$.

Definition 1.5 (see [6]). Let M be an E -invex set with respect to $\eta : R^n \times R^n \rightarrow R^n$. Also $f : M \rightarrow R$ is said to be E -preinvex with respect to η on M if for $x, y \in M$ and $\lambda \in [0, 1]$,

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)). \quad (1.3)$$

Definition 1.6 (see [7]). Let M be an E -invex set with respect to $\eta : R^n \times R^n \rightarrow R^n$. Also $f : M \rightarrow R$ is said to be semi- E -invex with respect to η at $y \in M$ if

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (1.4)$$

for all $x \in M$ and $\lambda \in [0, 1]$.

Definition 1.7 (see [7]). Let M be a nonempty E -invex subset of R^n with respect to $\eta : R^n \times R^n \rightarrow R^n$, $E : R^n \rightarrow R^n$. Let $f : M \rightarrow R$ and $E(M)$ be an open set in R^n . Also f and E are differentiable on M . Then, f is said to be semi- E -quasiinvex at $y \in M$ if

$$f(x) \leq f(y) \quad \forall x \in M \implies (\nabla(f \circ E)(y))^T \eta(E(x), E(y)) \leq 0, \quad (1.5)$$

or

$$(\nabla(f \circ E)(y))^T \eta(E(x), E(y)) > 0 \quad \forall x \in M \implies f(x) > f(y). \quad (1.6)$$

Lemma 1.8 (see [1]). If a set $M \subseteq R^n$ is E -convex, then $E(M) \subseteq M$.

Lemma 1.9 (see [5]). If M is E -invex, then $E(M) \subseteq M$.

Lemma 1.10 (see [5]). If $\{M_i\}_{i \in I}$ is a collection of E -invex sets and $M_i \subseteq R^n$, for all $i \in I$, then $\bigcap_{i \in I} M_i$ is E -invex.

2. E -Convexity and Its Generalizations with Differentiability Assumption

E -convexity and convexity are different from each other in several contests. From the previous results on E -convex functions, as discussed by our predecessors, one can observe the following relations between E -convexity and convexity.

- (1) All convex functions are E -convex but all E -convex functions are not necessarily convex. (In particular, E -convex function reduces to convex function in case $E(x) = x$ for all x in the domain of E .)
- (2) A real-valued function on R^n may not be convex on a subset of R^n , but E -convex on that set.
- (3) An E -convex function may not be convex on a set M but E -convex on $E(M)$.
- (4) It is not necessarily true that if M is an E -convex set then $E(M)$ is a convex set.

In this section we study E -convex and generalized E -convex functions with differentiability assumption.

2.1. Some New Results on E -Convexity with Differentiability

E -convexity at a point may be interpreted as follows.

Let M be a nonempty subset of R^n , $E : R^n \rightarrow R^n$. A function $f : M \rightarrow R$ is said to be E -convex at $\bar{x} \in M$ if M is an E -convex set and

$$f(\lambda E(x) + (1 - \lambda)E(\bar{x})) \leq \lambda(f \circ E)(x) + (1 - \lambda)(f \circ E)(\bar{x}) \quad (2.1)$$

for all $x \in N_\delta(\bar{x})$ and $\lambda \in [0, 1]$, where $N_\delta(\bar{x})$ is δ -neighborhood of $\bar{x}$, for small $\delta > 0$.

It may be observed that a function may not be convex at a point but E -convex at that point with a suitable mapping E .

Example 2.1. Consider $M = \{(x, y) \in R^2 \mid y \geq 0\}$. $E : R^2 \rightarrow R^2$ is $E(x, y) = (0, y)$ and $f(x, y) = x^3 + y^2$. Also f is not convex at $(-1, 1)$. For all $(x, y) \in N_\delta(-1, 1)$, $\delta > 0$, and $\lambda \in [0, 1]$, $f(\lambda E(x, y) + (1 - \lambda)E(-1, 1)) - \lambda(f \circ E)(x, y) - (1 - \lambda)(f \circ E)(-1, 1) = -\lambda(1 - \lambda)(y - 1)^2 \leq 0$. Hence, f is E -convex at $(-1, 1)$.

Proposition 2.2. Let $M \subseteq R^n$, $E : R^n \rightarrow R^n$ be a homeomorphism. If $f : M \rightarrow R$ attains a local minimum point in the neighborhood of $E(\bar{x})$, then it is E -convex at $\bar{x}$.

Proof. Suppose f has a local minimum point in a neighborhood $N_\epsilon(E(\bar{x}))$ of $E(\bar{x})$ for some $\bar{x} \in M$, $\epsilon > 0$. This implies f is convex on $N_\epsilon(E(\bar{x}))$. That is,

$$f(\lambda z + (1 - \lambda)E(\bar{x})) \leq \lambda f(z) + (1 - \lambda)f(E(\bar{x})) \quad \forall z \in N_\epsilon(E(\bar{x})). \quad (2.2)$$

Since $E : R^n \rightarrow R^n$ is a homeomorphism, so inverse of the neighborhood $N_\epsilon(E(\bar{x}))$ is a neighborhood of $\bar{x}$ say $N_\delta(\bar{x})$ for some $\delta > 0$. Hence, there exists $x \in N_\delta(\bar{x})$ such that $E(x) = z$, $E(x) \in N_\epsilon(E(\bar{x}))$. Replacing z by $E(x)$ in the above inequality, we conclude that f is E -convex at $\bar{x}$. $\square$

In the above discussion, it is clear that if a local minimum exists in a neighborhood of $E(\bar{x})$, then f is E -convex at $\bar{x}$. But it is not necessarily true that if f is E -convex at $\bar{x}$ then $E(\bar{x})$ is local minimum point. Consider the above example where f is E -convex at $(-1, 1)$ but $E(-1, 1)$ is not local minimum point of f .

Theorem 2.3. *Let M be an open E -convex subset of R^n , f and E are differentiable functions, and let E be a homeomorphism. Then, f is E -convex at $\bar{x} \in M$ if and only if*

$$(f \circ E)(x) \geq (f \circ E)(\bar{x}) + (\nabla(f \circ E)(\bar{x}))^T (E(x) - E(\bar{x})) \quad (2.3)$$

for all $E(x) \in N_\epsilon(E(\bar{x}))$ where $N_\epsilon(E(\bar{x}))$ is ϵ -neighborhood of $E(\bar{x})$, $\epsilon > 0$.

Proof. Since M is an E -convex set, by Lemma 1.8, $E(M) \subseteq M$. Also, $E(M)$ is an open set as E is a homeomorphism. Hence, there exists $\epsilon > 0$ such that $E(x) \in N_\epsilon(E(\bar{x}))$ for all $x \in N_\delta(\bar{x})$, $\delta > 0$, very small. So, f is differentiable on $E(M)$. Using expansion of f at $E(\bar{x})$ in the neighborhood $N_\epsilon(E(\bar{x}))$,

$$\begin{aligned} f(E(\bar{x}) + \lambda(z - E(\bar{x}))) &= f(E(\bar{x})) + \lambda \nabla f(E(\bar{x}))^T (z - E(\bar{x})) \\ &\quad + \alpha[E(\bar{x}), \lambda(z - E(\bar{x}))] \lambda \|z - E(\bar{x})\|, \end{aligned} \quad (2.4)$$

where $z \in N_\epsilon(E(\bar{x}))$ and $\lim_{\lambda \rightarrow 0} \alpha[E(\bar{x}), \lambda(z - E(\bar{x}))] = 0$. Since f is E -convex at $\bar{x} \in M$, so for all $x \in N_\delta(\bar{x})$, $\lambda \in (0, 1]$, $x \neq \bar{x}$,

$$\lambda((f \circ E)(x) - (f \circ E)(\bar{x})) \geq f(E(\bar{x}) + \lambda(E(x) - E(\bar{x}))) - (f \circ E)(\bar{x}). \quad (2.5)$$

Since E is a homeomorphism, there exists $x \in N_\delta(\bar{x})$ such that $E(x) = z$. Replacing z by $E(x)$ in (2.4) and using above inequality, we get

$$\begin{aligned} (f \circ E)(x) - (f \circ E)(\bar{x}) &\geq (\nabla(f \circ E)(\bar{x}))^T (E(x) - E(\bar{x})) \\ &\quad + \alpha[E(\bar{x}), \lambda(E(x) - E(\bar{x}))] \|E(x) - E(\bar{x})\|, \end{aligned} \quad (2.6)$$

where $\lim_{\lambda \rightarrow 0} \alpha[E(\bar{x}), \lambda(E(x) - E(\bar{x}))] = 0$. Hence, (2.3) follows.

The converse part follows directly from (2.4). □

It is obvious that if $E(\bar{x})$ is a local minimum point of f , then $\nabla(f \circ E)(\bar{x}) = 0$. The following result proves the sufficient part for the existence of local optimal solution, proof of which is easy and straightforward. We leave this to the reader.

Corollary 2.4. *Let $M \subseteq R^n$ be an open E -convex set, and let f be a differentiable E -convex function at $\bar{x}$. If $E : R^n \rightarrow R^n$ is a homeomorphism and $\nabla(f \circ E)(\bar{x}) = 0$, then $E(\bar{x})$ is the local minimum of f .*

2.2. Some New Results on Generalized E -Convexity with Differentiability

Here, we introduce some generalizations of E -convex function like semi- E -convex, E -invex, semi- E -invex, E -pseudoinvex, E -quasi-invex and so forth, with differentiability assumption and discuss their properties.

2.2.1. Semi- E -Convex Function

Chen [3] introduced a new class of semi- E -convex functions without differentiability assumption. Semi- E -convexity at a point may be understood as follows:

$f : M \rightarrow R$ is said to be semi- E -convex at $\bar{x} \in M$ if M is an E -convex set and

$$f(\lambda E(x) + (1 - \lambda)E(\bar{x})) \leq \lambda f(x) + (1 - \lambda)f(\bar{x}) \quad (2.7)$$

for all $x \in N_\delta(\bar{x})$ and $\lambda \in [0, 1]$, where $N_\delta(\bar{x})$ is δ -neighborhood of $\bar{x}$.

The following result proves the necessary and sufficient condition for the existence of a semi- E -convex function at a point.

Theorem 2.5. Suppose $f : M \rightarrow R$ and $E : R^n \rightarrow R^n$ are differentiable functions. Let E be a homeomorphism and let $\bar{x}$ be a fixed point of E . Then, f is semi- E -convex at $\bar{x} \in M$ if and only if

$$f(x) \geq f(\bar{x}) + (\nabla(f \circ E)(\bar{x}))^T (E(x) - E(\bar{x})) \quad (2.8)$$

for all $E(x) \in N_\epsilon(E(\bar{x}))$, very small $\epsilon > 0$.

Proof. Proceeding as in Theorem 2.3, we get the following relation from the expansion of f at $E(\bar{x})$ in the neighborhood $N_\epsilon(E(\bar{x}))$, where $\bar{x}$ is the fixed point of E . (Since E is a homeomorphism, there exists $\epsilon > 0$ such that $E(x) \in N_\epsilon(E(\bar{x}))$ for all $x \in N_\delta(\bar{x})$, very small $\delta > 0$):

$$\begin{aligned} f(E(\bar{x}) + \lambda(E(x) - E(\bar{x}))) &= (f \circ E)(\bar{x}) + \lambda(\nabla(f \circ E)(\bar{x}))^T (E(x) - E(\bar{x})) \\ &\quad + \alpha[E(\bar{x}), \lambda(E(x) - E(\bar{x}))]\lambda\|E(x) - E(\bar{x})\|, \end{aligned} \quad (2.9)$$

where $E(x) \in N_\epsilon(E(\bar{x}))$, $\lim_{\lambda \rightarrow 0} \alpha[E(\bar{x}), \lambda(E(x) - E(\bar{x}))] = 0$. Since f is semi- E -convex at $\bar{x} \in M$, and $\bar{x}$ is a fixed point of E , so, for all $x \in N_\delta(\bar{x})$, $\lambda \in (0, 1]$, $x \neq \bar{x}$,

$$\lambda(f(x) - f(\bar{x})) \geq f(E(\bar{x}) + \lambda(E(x) - E(\bar{x}))) - (f \circ E)(\bar{x}). \quad (2.10)$$

Using (2.9), the above inequality reduces to

$$\begin{aligned} f(x) - f(\bar{x}) &\geq (\nabla(f \circ E)(\bar{x}))^T (E(x) - E(\bar{x})) \\ &\quad + \alpha[E(\bar{x}), \lambda(E(x) - E(\bar{x}))]\lambda\|E(x) - E(\bar{x})\|, \end{aligned} \quad (2.11)$$

where $\lim_{\lambda \rightarrow 0} \alpha[E(\bar{x}), \lambda(E(x) - E(\bar{x}))] = 0$. Hence Inequality (2.8) follows for all $E(x) \in N_\epsilon(E(\bar{x}))$.

Conversely, suppose Inequality (2.8) holds at the fixed point $\bar{x}$ of E for all $E(x) \in N_\epsilon(E(\bar{x}))$. Using (2.9) and $E(\bar{x}) = \bar{x}$ in (2.8), we can conclude that f is semi- E -convex at $\bar{x} \in M$. $\square$

2.2.2. Generalized E -Invex Function

The class of preinvex functions defined by Ben-Israel and Mond is not necessarily differentiable. Preinvexity, for the differential case, is a sufficient condition for invexity. Indeed, the converse is not generally true. Fulga and Preda [5] defined E -invex set, E -preinvex function, and E -prequasiinvex function where differentiability is not required (Section 1). Chen [3] introduced semi- E -convex, semi- E -quasiconvex, and semi- E -pseudoconvex functions without differentiability assumption. Jaiswal and Panda [7] studied some generalized E -invex functions and applied these concepts to study primal dual relations. Here, we define some more generalized E -invex functions with and without differentiability assumption, which will be needed in next section. First, we see the following lemma.

Lemma 2.6. *Let M be a nonempty E -invex subset of R^n with respect to $\eta : R^n \times R^n \rightarrow R^n$. Also $f : M \rightarrow R$ are differentiable on M . $E(M)$ is an open set in R^n . If f is E -preinvex on M then $(f \circ E)(x) \geq (f \circ E)(y) + (\nabla(f \circ E)(y))^T \eta(E(x), E(y))$ for all $x, y \in M$.*

Proof. If $E(M)$ is an open set, f and E are differentiable on M , then $f \circ E$ is differentiable on M . From Taylor's expansion of f at $E(y)$ for some $y \in M$ and $\lambda > 0$,

$$\begin{aligned} f(E(y) + \lambda \eta(E(x), E(y))) &= (f \circ E)(y) + \lambda (\nabla(f \circ E)(y))^T (\eta(E(x), E(y))) \\ &\quad + \lambda \|\eta(E(x), E(y))\| \alpha(E(y), \lambda \eta(E(x), E(y))), \end{aligned} \quad (2.12)$$

where $E(x) \neq E(y)$, $\lim_{\lambda \rightarrow 0} \alpha(E(y), \lambda \eta(E(x), E(y))) = 0$.

If f is E -preinvex on M with respect to η (Definition 1.5), then as $\lambda \rightarrow 0^+$, the above inequality reduces to $(f \circ E)(x) \geq (f \circ E)(y) + (\nabla(f \circ E)(y))^T \eta(E(x), E(y))$ for all $x, y \in M$. $\square$

As a consequence of the above lemma, we may define E -invexity with differentiability assumption as follows.

Definition 2.7. Let M be a nonempty E -invex subset of R^n with respect to $\eta : R^n \times R^n \rightarrow R^n$. Also $f : M \rightarrow R$ are differentiable on M . $E(M)$ is an open set in R^n . Then, f is E -invex on M if $(f \circ E)(x) \geq (f \circ E)(y) + (\nabla(f \circ E)(y))^T \eta(E(x), E(y))$ for all $x, y \in M$.

From the above discussions on E -invexity and E -preinvexity, it is true that E -preinvexity with differentiability is a sufficient condition for E -invexity. Also a function which is not E -convex may be E -invex with respect to some η . This may be verified in the following example.

Example 2.8. $M = \{(x, y) \in R^2 \mid x, y > 0\}$, $E : R^2 \rightarrow R^2$ is $E(x, y) = (0, y)$ and $f : M \rightarrow R$ is defined by $f(x, y) = -x^2 - y^2$, and

$$\eta((x_1, y_1), (x_2, y_2)) = \begin{cases} \left(\frac{x_1^2}{2x_2}, \frac{y_1^2}{2y_2}\right), & x_2 \neq 0, y_2 \neq 0, \\ \left(0, \frac{y_1^2}{2y_2}\right), & x_2 = 0, y_2 \neq 0, \\ \left(\frac{x_1^2}{2x_2}, 0\right), & x_2 \neq 0, y_2 = 0, \\ (0, 0), & \text{otherwise.} \end{cases} \quad (2.13)$$

Definition 2.9. Let M be a nonempty E -invex subset of R^n with respect to $\eta : R^n \times R^n \rightarrow R^n$, let $E(M)$ be an open set in R^n . Suppose f and E are differentiable on M . Then, f is said to be E -quasiinvex on M if

$$(f \circ E)(x) \leq (f \circ E)(y) \quad \forall x, y \in M \implies (\nabla(f \circ E)(y))^T \eta(E(x), E(y)) \leq 0 \quad (2.14)$$

or

$$(\nabla(f \circ E)(y))^T \eta(E(x), E(y)) > 0 \implies (f \circ E)(x) > (f \circ E)(y). \quad (2.15)$$

A function may not be E -invex with respect to some η but E -quasiinvex with respect to same η . This may be justified in the following example.

Example 2.10. Consider $M = \{(x, y) \in R^2 \mid x, y < 0\}$, $E : R^2 \rightarrow R^2$ is $E(x, y) = (0, y)$, and $f : M \rightarrow R$ is $f(x, y) = x^3 + y^3$, $\eta : R^2 \times R^2 \rightarrow R^2$ is $\eta((x_1, y_1), (x_2, y_2)) = (x_1 - x_2, y_1 - y_2)$. Now for all $(x_1, y_1), (x_2, y_2) \in M$, $(f \circ E)(x_1, y_1) - (f \circ E)(x_2, y_2) - \nabla(f \circ E)(x_2, y_2)^T \eta(E(x_1, y_1), E(x_2, y_2)) = y_1^3 + y_2^3 - 3y_2^2(y_1 - y_2)$, which is not always positive. Hence, f is not E -invex with respect to η on M .

If we assume that $(f \circ E)(x_1, y_1) \leq (f \circ E)(x_2, y_2)$ for all $(x_1, y_1), (x_2, y_2) \in M$, then $(\nabla(f \circ E)(x_2, y_2))^T \eta(E(x_1, y_1), E(x_2, y_2)) = 3y_2^2(y_1 - y_2) \leq 0$. Hence, f is E -quasiinvex with respect to same η on M .

Definition 2.11. Let M be a nonempty E -invex subset of R^n with respect to $\eta : R^n \times R^n \rightarrow R^n$, let $E(M)$ be an open set in R^n . Suppose f and E are differentiable on M . Then, f is said to be E -pseudoinvex on M if

$$(\nabla(f \circ E)(y))^T \eta(E(x), E(y)) \geq 0 \quad \forall x, y \in M \implies (f \circ E)(x) \geq (f \circ E)(y) \quad (2.16)$$

or

$$(f \circ E)(x) < (f \circ E)(y) \quad \forall x, y \in M \implies (\nabla(f \circ E)(y))^T \eta(E(x), E(y)) < 0. \quad (2.17)$$

A function may not be E -invex with respect to some η but E -pseudoinvex with respect to same η . This can be verified in the following example.

Example 2.12. Consider $M = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\}$. $E : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is $E(x, y) = (0, y)$ and $f : M \rightarrow \mathbb{R}$ is $f(x, y) = -x^2 - y^2$. For $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by $\eta((x_1, y_1), (x_2, y_2)) = (x_1 - x_2, y_1 - y_2)$, and for all $(x_1, y_1), (x_2, y_2) \in M$, $y_1 \neq y_2$, $(f \circ E)(x_1, y_1) - (f \circ E)(x_2, y_2) - \nabla(f \circ E)(x_2, y_2)^T \eta(E(x_1, y_1), E(x_2, y_2)) = -(y_2 - y_1)^2 < 0$. Hence, f is not E -invex with respect to η on M . If $\nabla(f \circ E)(x_2, y_2)^T \eta(E(x_1, y_1), E(x_2, y_2)) \geq 0$, then $(f \circ E)(x_1, y_1) - (f \circ E)(x_2, y_2) - f(0, y_1) + f(0, y_2) = (y_2 + y_1)(y_2 - y_1) \geq 0$. Hence, f is E -pseudoinvex with respect to η on M .

If a function $f : M \rightarrow \mathbb{R}$ is semi- E -invex with respect to η at each point of an E -invex set M , then f is said to be semi- E -invex with respect to η on M . Semi- E -invex functions and some of its generalizations are studied in [7]. Here, we discuss some more results on generalized semi- E -invex functions.

Proposition 2.13. *If $f : M \rightarrow \mathbb{R}$ is semi- E -invex on an E -invex set M , then $f(E(y)) \leq f(y)$ for each $y \in M$.*

Proof. Since f is semi- E -invex on $M \subseteq \mathbb{R}^n$ and M is an E -invex set so for $x, y \in M$ and $\lambda \in [0, 1]$, we have $E(y) + \lambda \eta(E(x), E(y)) \in M$ and $f(E(y) + \lambda \eta(E(x), E(y))) \leq \lambda f(x) + (1 - \lambda)f(y)$. In particular, for $\lambda = 0$, $f(E(y)) \leq f(y)$ for each $y \in M$. $\square$

An E -invex function with respect to some η may not be semi- E -invex with respect to same η may be verified in the following example.

Example 2.14. Consider the previous example where $M = \{(x, y) \in \mathbb{R}^2 \mid x, y < 0\}$, $E : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is $E(x, y) = (0, y)$ and $f : M \rightarrow \mathbb{R}$ is $f(x, y) = x^3 + y^3$, $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is $\eta((x_1, y_1), (x_2, y_2)) = (x_1 - x_2, y_1 - y_2)$. It is verified that f is E -invex with respect to η on M . But $f(E(2, 0)) > f(2, 0)$. From Proposition 2.13 it can be concluded that f is not semi- E -invex with respect to same η . Also, using Definition 1.6, for all $(x_1, y_1), (x_2, y_2) \in M$, $\lambda \in [0, 1]$, $f(E(x_2, y_2) + \lambda \eta(E(x_1, y_1), E(x_2, y_2))) - \lambda f(x_1, y_1) - (1 - \lambda)f(x_2, y_2) = -(y_2 + \lambda y_1^2/2y_2)^2 + \lambda(x_1^3 + y_1^3) + (1 - \lambda)(x_2^3 + y_2^3)$, which is not always negative. Hence, f is not semi- E -invex with respect to η on M .

3. Application in Optimization Problem

In this section, the results of previous section are used to derive the sufficient optimality condition for the existence of solution of a general nonlinear programming problem. Consider a nonlinear programming problem

$$\begin{aligned} (P) \quad & \min \quad f(x) \\ & \text{subject to} \quad g(x) \leq 0, \end{aligned} \tag{3.1}$$

where $f : M \rightarrow \mathbb{R}$, $g_i : M \rightarrow \mathbb{R}^m$, $M \subseteq \mathbb{R}^n$, $g = (g_1, g_2, \dots, g_m)^T$. $M' = \{x \in M : g_i(x) \leq 0, i = 1, \dots, m\}$ is the set of feasible solutions.

Theorem 3.1 (sufficient optimality condition). *Let M be a nonempty open E -convex subset of $\mathbb{R}^n$, $f : M \rightarrow \mathbb{R}$, $g : M \rightarrow \mathbb{R}^m$, and $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are differentiable functions. Let E be*

a homeomorphism and let $\bar{x}$ be a fixed point of E . If f and g are semi- E -convex at $\bar{x} \in M'$ and $(\bar{x}, \bar{y}) \in M' \times R^m$ satisfies

$$\begin{aligned} \nabla[(f \circ E)(x) + \langle y, (g \circ E)(x) \rangle] &= 0, \\ \langle y, (g \circ E)(x) \rangle &= 0, \quad y \geq 0, \end{aligned} \quad (3.2)$$

then $\bar{x}$ is local optimal solution of (P) .

Proof. Since f and g are semi- E -convex at $\bar{x} \in M$ by Theorem 2.5,

$$\begin{aligned} f(x) - f(\bar{x}) &\geq (\nabla(f \circ E)(\bar{x}))^T (E(x) - E(\bar{x})) \quad \forall E(x) \in N_e(E(\bar{x})), \\ g(x) - g(\bar{x}) &\geq (\nabla(g \circ E)(\bar{x}))^T (E(x) - E(\bar{x})) \quad \forall E(x) \in N_e(E(\bar{x})). \end{aligned} \quad (3.3)$$

Adding the above two inequalities, we have

$$[f(x) - f(\bar{x})] + \bar{y}^T [g(x) - g(\bar{x})] \geq \nabla[(f \circ E)(\bar{x}) + \bar{y}^T ((g \circ E)(\bar{x}))]^T (E(x) - E(\bar{x})). \quad (3.4)$$

If (3.2) hold, then $f(x) - f(\bar{x}) + \bar{y}^T g(x) \geq 0$ for all $E(x) \in N_e(E(\bar{x}))$. Since $g(x) \leq 0$ for all $x \in M'$ and $\bar{y} \geq 0$, so $\bar{y}^T g(x) \leq 0$. Hence, $f(x) - f(\bar{x}) \geq 0$ for all $E(x) \in N_e(E(\bar{x}))$. Since E is a homeomorphism, there exists $\delta > 0$ such that $x \in N_\delta(\bar{x})$ for all $E(x) \in N_e(E(\bar{x}))$, which means $f(x) \geq f(\bar{x})$ for all $x \in N_\delta(\bar{x})$. Hence, $\bar{x}$ is a local optimal solution of (P) . $\square$

Also we see that a fixed point of E is a local optimal solution of (P) under generalized E -invexity assumptions.

Lemma 3.2. Let M be a nonempty E -invex subset of R^n with respect to some $\eta : R^n \times R^n \rightarrow R^n$. Let $g_i : M \rightarrow R, i = 1, \dots, m$ be semi- E -quasiinvex functions with respect to η on M . Then, M' is an E -invex set.

Proof. Let $M_i = \{x \in M : g_i(x) \leq 0\}, i = 1, \dots, m$. $M' = \cap_{i=1}^m M_i$ and $M' \subseteq M$. Since $g_i, i = 1, \dots, m$ are semi- E -quasiinvex function on M , so for all $x, y \in M_i$ and $\lambda \in [0, 1], g_i(E(y) + \lambda\eta(E(x), E(y))) \leq \max\{g_i(x), g_i(y)\} \leq 0$. Hence, $E(y) + \lambda\eta(E(x), E(y)) \in M_i$ for all $x, y \in M_i$. So M_i is E -invex with respect to same η . From Lemma 1.10, $M' = \cap_{i=1}^m M_i$ is E -invex with respect to same η . $\square$

Corollary 3.3. Let M be a nonempty E -invex subset of R^n with respect to some $\eta : R^n \times R^n \rightarrow R^n$. Let $g_i : M \rightarrow R, i = 1, \dots, m$, be semi- E -quasiinvex functions with respect to η on M . If x is a feasible solution of (P) , then $E(x)$ is also a feasible solution of (P) .

Proof. Since x is a feasible solution of (P) , so $x \in M' \Rightarrow E(x) \in E(M')$. Since each $g_i, i = 1, \dots, m$ is semi- E -quasiinvex function on M , from Lemma 3.2, M' is an E -invex set. Also $E(M') \subseteq M'$. Hence, $E(x) \in M'$. That is, $E(x)$ is a feasible solution of (P) . $\square$

Theorem 3.4 (sufficient optimality condition). Let M be a nonempty E -invex subset of R^n with respect to $\eta : R^n \times R^n \rightarrow R^n$. Let $E(M)$ be an open set in R^n . Suppose $f : R^n \rightarrow R, g : R^n \rightarrow R^m$

and E are differentiable functions on M . If f is E -pseudoinvex function with respect to η and for $u \geq 0$, $u^T g$ is semi- E -quasiinvex function with respect to the same η at $x \in M'$, where x is a fixed point of the map E and $(x, u) \in M' \times R^m$, $u \geq 0$ satisfies the following system:

$$\nabla [(f \circ E)(x) + \langle u, (g \circ E)(x) \rangle] = 0, \quad (3.5)$$

$$\langle u, (g \circ E)(x) \rangle = 0, \quad (3.6)$$

then x is a local optimal solution of (P) .

Proof. Suppose $(x, u) \in M' \times R^m$ satisfies (3.5) and (3.6). For all $y \in M'$, $g(y) \leq 0$. Also, $u \geq 0$. Hence, $u^T g(y) \leq 0$ for all $y \in M'$. From (3.6), $\langle u, (g \circ E)(x) \rangle = 0$, that is, $u^T g(E(x)) = 0$. x is a fixed point of E that is $E(x) = x$. So $u^T g(x) = 0$. Hence,

$$u^T g(y) \leq u^T g(x) \quad \forall y \in M'. \quad (3.7)$$

Since $u \geq 0$ and $u^T g$ is semi- E -quasiinvex function with respect to η at x , so the above inequality implies

$$\nabla u^T g(E(x)) \eta(E(y), E(x)) \leq 0. \quad (3.8)$$

From (3.5), $\nabla f(E(x)) = -\nabla u^T g(E(x))$. Putting this value in the above inequality, we have $\nabla f(E(x)) \eta(E(x), E(y)) \geq 0$.

f is E -pseudoinvex at x with respect to η . Hence, $\nabla f(E(x)) \eta(E(x), E(y)) \geq 0$ implies

$$f(E(y)) \geq f(E(x)) = f(x) \quad \forall y \in M'. \quad (3.9)$$

Hence, x is the optimal solution (P) on $E(M')$. $\square$

The following example justifies the above theorem.

Example 3.5. Consider the optimization problem,

$$\begin{array}{ll} (P) \min & -x^2 - y^2 \\ \text{subject to} & x^2 + y^2 - 4 \leq 0, \end{array} \quad (3.10)$$

where $M = \{(x, y) \in R^2 \mid x, y > 0\}$. $E : R^2 \rightarrow R^2$ is $E(x, y) = (0, y)$. This is not a convex programming problem. Consider $\eta : R^2 \times R^2 \rightarrow R^2$ defined by $\eta((x_1, y_1), (x_2, y_2)) = (x_1 - x_2, y_1 - y_2)$. Here, $M' = \{(x, y) \in M : x^2 + y^2 - 4 \leq 0\}$, and $E(M') = \{(0, y) : y \geq 0, y^2 - 4 \leq 0\}$. The sufficient conditions (7-8) reduce to

$$-2y + u2y = 0, \quad u(y^2 - 4) = 0, \quad u \geq 0, \quad (3.11)$$

whose solution is $y = 2, u = 1$ and $E(0, 2) = (0, 2)$.

In Example 2.12, we have already proved that $f(x, y) = -x^2 - y^2$ is E -pseudoinvex function with respect to η . Using Definition 1.7, one can verify that $ug(x, y)$ is semi- E -quasiinvex with respect to same η at $(0, 2) \in M'$, where $ug(x, y) = x^2 + y^2 - 4$. So $(0, 2)$ is the optimal solution of (P) on $E(M')$.

4. Conclusion

E -convexity and its generalizations are studied by many authors earlier without differentiability assumption. Here, we have studied the the properties of E -convexity, E -invexity, and their generalizations with differentiable assumption. From the developments of this paper, we conclude the following interesting properties.

- (1) A function may not be convex at a point but E -convex at that point with a suitable mapping E , and if a local minimum of f exists in a neighborhood of $E(x)$, then f is E -convex at x . But it is not necessarily true that if f is E -convex at x then $E(x)$ is local minimum point.
- (2) From the relation between E -invexity and its generalizations, one may observe that a function which is not E -convex may be E -invex with respect to some η and E -preinvexity with differentiability is a sufficient condition for E -invexity. Moreover, a function may not be E -invex with respect to some η but E -quasiinvex with respect to same η , a function may not E -invex with respect to some η but E -pseudoinvex with respect to the same η and an E -invex function with respect to some η may not be semi- E -invex with respect to same η .

Here, we have studied E -convexity for first-order differentiable functions. Higher-order differentiable E -convex functions may be studied in a similar manner to derive the necessary and sufficient optimality conditions for a general nonlinear programming problems.

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