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Research Article

Multiobjective Two-Stage Stochastic Programming Problems with Interval Discrete Random Variables

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Most of the real-life decision-making problems have more than one conflicting and incommensurable objective functions. In this paper, we present a multiobjective two-stage stochastic linear programming problem considering some parameters of the linear constraints as interval type discrete random variables with known probability distribution. Randomness of the discrete intervals are considered for the model parameters. Further, the concepts of best optimum and worst optimum solution are analyzed in two-stage stochastic programming. To solve the stated problem, first we remove the randomness of the problem and formulate an equivalent deterministic linear programming model with multiobjective interval coefficients. Then the deterministic multiobjective model is solved using weighting method, where we apply the solution procedure of interval linear programming technique. We obtain the upper and lower bound of the objective function as the best and the worst value, respectively. It highlights the possible risk involved in the decision-making tool. A numerical example is presented to demonstrate the proposed solution procedure.

1. Introduction

The input parameters of the mathematical programming model are not exactly known because relevant data are inexistent or scarce, difficult to obtain or estimate, the system is subject to changes, and so forth, that is, input parameters are uncertain in nature. This type of situations are mainly occurs in real-life decision-making problems. These uncertainties in the input parameters of the model can characterized by random variables with known probability distribution. The occurrence of randomness in the model parameters can be formulated as stochastic programming (SP) model. SP is widely used in many real-world decision-making problems of management science, engineering, and technology. Also, it has been applied to a wide variety of areas such as, manufacturing product and capacity

planning, electrical generation capacity planning, financial planning and control, supply chain management, airline planning (fleet assignment), water resource modeling, forestry planning, dairy farm expansion planning, macroeconomic modeling and planning, portfolio selection, traffic management, transportation, telecommunications, and banking.

An efficient method known as two-stage stochastic programming (TSP) in which policy scenarios is desired for studying problems with uncertainty. In TSP paradigm, the decision variables are partitioned into two sets. The decision variables which are decided before the actual realization of the uncertain parameters are known as first stage variables. Afterward, once the random events have exhibited themselves, further decision can be made by selecting the values of the second-stage, or recourse variables at a certain cost that is, a second-stage decision variables can be made to minimize "penalties" that may occurs due to any infeasibility [1]. The formulation of two-stage stochastic programming problems was first introduced by Dantzig [2]. Further it was developed by Beale [3] and Dantzig and Madansky [4]. However, TSP can barely deal with independent uncertainties of the left-hand side coefficients in each constraint or the objective function. It also requires probabilistic specifications for uncertain parameters while in many pragmatic problems, the quality of information that can be obtained is usually not satisfactory enough to be presented as probability distributions.

Interval Linear programming (ILP) is an alternative approach for handling uncertainties in the constraints as well as in the objective functions. It can deal with uncertainties that cannot be quantified as distribution functions, since interval numbers (a lower- and upper-bounded range of real numbers) are acceptable as uncertain inputs. The ILP can be transformed into two deterministic submodels, which correspond to worst lower bound and best upper bounds of desired objective function value. For this we develop methods that find the best optimum (highest maximum or lowest minimum as appropriate), and worst optimum (lowest maximum or highest minimum as appropriate), and the coefficient settings (within their intervals) which achieve these two extremes.

Interval analysis was introduced by Moore [5]. The growing efficiency of interval analysis for solving some deterministic real-life problems during the last decade enabled extension at the formalism to the probabilistic case. Thus, instead of using a single random variable, we adopt an interval random variable, which has the ability to represent not only the randomness via probability theory, but also imprecision and nonspecificity via intervals. In this context, interval random variables plays an important role in optimization theory.

2. Literature Review

Some of the important literatures related to TSP and ILP have been presented below.

Tong [6] focussed on two types of linear programming problems such as, interval number and fuzzy number linear programming, respectively, and described their solution procedures. Li and Huang [7] proposed an interval-parameter two-stage mixed integer linear programming model is developed for supporting long-term planning of waste management activities in the City of Regina. Molai and Khorram [8] introduced lower- and upper-satisfaction functions to estimates the degree to which arithmetic comparisons between two interval values are satisfied and apply these functions to present a new interpretation of inequality constraints with interval coefficients in an interval linear programming problem. Zhou et al. [9] presented an interval linear programming model and its solution procedures. A two-stage fuzzy random programming or fuzzy random programming with recourse problem along with its deterministic equivalent model is presented by [10]. Han et

al. [11] presented an interval-parameter multistage stochastic chance-constrained mixed integer programming method for interbasin water resources management systems under uncertainty. Xu et al. [12] presented an inexact two-stage stochastic robust programming model for water resources allocation problems under uncertainty. Li and Huang [13] developed an interval-parameter two-stage stochastic nonlinear programming method for supporting decisions of water-resources allocation within a multireservoir system. Suprajitno and Mohd [14] presented some interval linear programming problems, where the coefficients and variables are in the form of intervals. Su et al. [15] presented an inexact chance-constraint mixed integer linear programming model for supporting long-term planning of waste management in the City of Foshan, China. A new class of fuzzy stochastic optimization models called two-stage fuzzy stochastic programming with value-at-risk criteria is presented by [16].

3. Multiobjective Stochastic Programming

Stochastic or probabilistic programming deals with situations where some or all of the parameters of the optimization problem are described by stochastic (or random or probabilistic) variables rather than by deterministic quantities. In recent years, multiobjective stochastic programming problems have become increasingly important in scientifically based decision making involved in practical problem arising in economic, industry, health care, transportation, agriculture, military purposes, and technology. Mathematically, a multiobjective stochastic programming problem can be stated as follows:

$$\begin{aligned}
\max \quad & z^t = \sum_{j=1}^{n} c_j^t x_j, \quad t = 1,2,\ldots,T, \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1,2,\ldots,m_1, \\
& \sum_{j=1}^{n} d_{ij} x_j \le b_{m_1+i}, \quad i = 1,2,\ldots,m_2, \\
& x_j \ge 0, \quad j = 1,2,\ldots,n,
\end{aligned} \tag{3.1}$$

where some of the parameters a_{ij}, $(i = 1,2,\ldots,m_1;\ j = 1,2,\ldots,n)$ and b_i, $(i = 1,2,\ldots,m_1)$ are discrete random variables with known probability distribution. Rest of the parameters x_j, $(j = 1,2,\ldots,n)$, c_j^t, $(j = 1,2,\ldots,n;\ t = 1,2,\ldots,R)$, d_{ij}, $(i = 1,2,\ldots,m_2;\ j = 1,2,\ldots,n)$ and b_{m_1+i}, $i = 1,2,\ldots,m_2$ are considered as known intervals.

3.1. Multiobjective Two-Stage Stochastic Programming

In two-stage stochastic programming (TSP), decision variables are divided into two subsets: (1) a group of variables determined before the realizations of random events are known as first stage decision variables, and (2) another group of variables known as recourse variables

which are determined after knowing the realized values of the random events. A general model of TSP with simple recourse can be formulated as follows [17–19]:

$$\begin{aligned}
\max \quad & \overline{z} = \sum_{j=1}^{n} c_j x_j - E\left(\sum_{i=1}^{m_1} q_i |y_i|\right), \\
\text{subject to} \quad & y_i = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m_1, \\
& \sum_{j=1}^{n} d_{ij} x_j \le b_{m_1+i}, \quad i = 1, 2, \ldots, m_2, \\
& x_j \ge 0, \quad j = 1, 2, \ldots, n; \qquad y_i \ge 0, \quad i = 1, 2, \ldots, m_1,
\end{aligned} \tag{3.2}$$

where x_j, $j = 1, 2, \ldots, n$ and y_i, $i = 1, 2, \ldots, m_1$ are the first-stage decision variables and second-stage decision variables, respectively. Further q_i, $i = 1, 2, \ldots, m_1$ are defined as the penalty cost associated with the discrepancy between $\sum_{j=1}^{n} a_{ij} x_j$ and b_i and E is used to represent the expected value of the discrete random variables.

Multiobjective optimization problems are appeared in most of the real-life decision-making problems. Thus, a general model of multiobjective two-stage stochastic programming of the multiobjective stochastic programming model (3.1) can be stated as follow [20]:

$$\begin{aligned}
\max \quad & z^t = \sum_{j=1}^{n} c_j^t x_j - E\left(\sum_{i=1}^{m} q_i^t |y_i|\right), \quad t = 1, 2, \ldots, T, \\
\text{subject to} \quad & y_i = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m_1, \\
& \sum_{j=1}^{n} d_{ij} x_j \le b_{m_1+i}, \quad i = 1, 2, \ldots, m_2, \\
& x_j \ge 0, \quad j = 1, 2, \ldots, n; \qquad y_i \ge 0, \quad i = 1, 2, \ldots, m_1.
\end{aligned} \tag{3.3}$$

In the next Section some useful definitions related to interval arithmetic are presented.

3.2. Real Interval Arithmetic and Basic Properties with Operations

Many operations (unary and binary) on sets or pairs of real numbers can be immediately applied to intervals. We can define the classical set operations [21] as follows.

suppose $[x] = [x^l, x^u]$ and $[y] = [y^l, y^u]$ are two real intervals such that

(i) equality: $[x] = [y]$ if and only if $x^l = y^l$ and $x^u = y^u$;

(ii) intersection: $[x] \bigcap [y] = [\max\{x^l, y^l\}, \min\{x^u, y^u\}]$;

(iii) union:

$$[x] \cup [y] = \begin{cases} [\min\{x^l, y^l\},\ \max\{x^u, y^u\}], & [x] \cap [y] \neq \emptyset; \\ \text{undefined} & \text{otherwise}; \end{cases} \tag{3.4}$$

(iv) inequality: $[x] < [y]$ if $x^u < y^l$ and $[x] > [y]$ if $x^l > y^u$;

(v) inclusion: $[x] \subseteq [y]$ if and only if $y^l \leq x^l$ and $x^u \leq y^u$;

(vi) maximum: $\max\{[x], [y]\} = [k]$ where $k^l = \max\{x^l, y^l\}$ and $k^u = \max\{x^u, y^u\}$;

(vii) minimum: $\min\{[x], [y]\} = [k]$ where $k^l = \min\{x^l, y^l\}$ and $k^u = \min\{x^u, y^u\}$.

An interval is said to be positive if $x^l > 0$ and negative if $x^u < 0$.

Using unary operations, we can define the following:

(i) width: $w([x]) = x^u - x^l$;

(ii) midpoint: $\text{mid}([x]) = (x^l + x^u)/2$;

(iii) radius: $\text{rad}([x]) = (x^u - x^l)/2$;

(iv) absolute value: $|[x]| = \{|y| : x^l \leq y \leq x^u\}$;

(v) interior: $\text{int}([x]) = (x^l, x^u)$.

For pairs of real numbers, the some classical binary operators are defined as:

(i) addition: $[x] + [y] = [x^l + y^l, x^u + y^u]$;

(ii) subtraction: $[x] - [y] = [x^l - y^u, x^u - y^l]$;

(iii) multiplication: $[x] \cdot [y] = [\min\{x^l y^l, x^l y^u, x^u y^l, x^u y^u\}, \max\{x^l y^l, x^l y^u, x^u y^l, x^u y^u\}]$;

(iv) division: If $0 \notin [y]$, then $[x]/[y] = [x] \cdot [1/y^u\ , 1/y^l]$;

(v) scalar Multiplication of $[x]$: Scalar multiplication of interval for $\alpha \in \mathbb{R}$ is given as:

$$\alpha[x] = \begin{cases} [\alpha x^l, \alpha x^u], & \text{if } \alpha \geq 0; \\ [\alpha x^u, \alpha x^l], & \text{if } \alpha < 0; \end{cases} \tag{3.5}$$

(vi) power of $[x]$: Power of interval for $n \in \mathbb{Z}$ is given as: when n positive and odd or $[x]$ is positive, then $[x]^n = [(x^l)^n, (x^u)^n]$ when n positive and even, then

$$[x]^n = \begin{cases} \left[(x^l)^n, (x^u)^n\right], & \text{if } x^l \geq 0; \\ \left[(x^u)^n, (x^l)^n\right], & \text{if } x^u < 0; \\ \left[0, \max\left\{(x^l)^n, (x^u)^n\right\}\right], & \text{otherwise}, \end{cases} \tag{3.6}$$

when n negative, $[x]^n = 1/[x]^{|n|}$;

(vii) square root of $[x]$: For an interval $[x]$ such that $x^l \geq 0$, define the square root of $[x]$ denoted by $\sqrt{[x]}$ as: $\sqrt{[x]} = \{\sqrt{y} : x^l \leq y \leq x^u\}$.

3.3. Interval Linear Programming

A linear programming model having the coefficients as real intervals is known as interval linear programming (ILP). Mathematically, an ILP model can be stated as follows [22, 23]:

$$\begin{aligned} \max \quad & z' = \sum_{j=1}^{n} [c_j] x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} [a_{ij}] x_j \le [b_i], \quad i = 1, 2, \ldots, m_1 \\ & \sum_{j=1}^{n} [d_{ij}] x_j \ge [b_{m_1+i}], \quad i = 1, 2, \ldots, m_2 \\ & x_j \ge 0, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{3.7}$$

where x_j, $j = 1, 2, \ldots, n$ are the decision variables. However, $[c_j]$, $[a_{ij}]$, $[d_{ij}]$, $[b_i]$, and $[b_{m_1+i}]$, $i = 1, 2, \ldots, m_2$, $j = 1, 2, \ldots, n$ are real intervals. These interval parameters are defined as follows:

$[c_j] = [c_j^l, c_j^u]$, where $c_j^l, c_j^u \in R$, c_j^l and c_j^u are called lower and upper bounds of $[c_j]$, respectively.

$[a_{ij}] = [a_{ij}^l, a_{ij}^u]$, where $a_{ij}^l, a_{ij}^u \in R$, a_{ij}^l and a_{ij}^u are called lower and upper bounds of $[a_{ij}]$, respectively.

$[d_{ij}] = [d_{ij}^l, d_{ij}^u]$, where $d_{ij}^l, d_{ij}^u \in R$, d_{ij}^l and d_{ij}^u are called lower and upper bounds of $[d_{ij}]$, respectively.

$[b_{m_1+i}] = [b_{m_1+i}^l, b_{m_1+i}^u]$, where $b_{m_1+i}^l, b_{m_1+i}^u \in R$, $b_{m_1+i}^l$ and $b_{m_1+i}^u$ are called lower and upper bounds of $[b_{m_1+i}]$, respectively.

$[b_i] = [b_i^l, b_i^u]$, where $b_i^l, b_i^u \in R$, b_i^l and b_i^u are called lower and upper bounds of $[b_i]$, respectively.

4. Some Definitions on Interval Random Variable

In this Section, we have given some important definitions related to interval random variable.

Definition 4.1 (interval random variable [21]). Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. Interval random variable $[X]$ is a function $[X]: \Omega \to \mathbb{R}$ defined by $[X](\omega) = [X^l(\omega), X^u(\omega)]$, $\forall \omega \in \Omega$, specified by a pair of $\mathcal{F}$-measurable functions X^l, $X^u: \Omega \to \mathbb{R}$ such as $X^l \le X^u$ almost surely.

Definition 4.2 (interval discrete random variable [21]). interval random variable is said to be discrete if it takes values in a finite subset of $\mathbb{R}$ with probability mass function $f: \mathbb{R} \to [0, 1]$ defined by $f([x]) = \Pr([X] = [x])$.

Definition 4.3 (expected value of an interval discrete random variable [21]). Let $[X]$ be an interval discrete random variable which assumes interval values $[x_1]$, $[x_2], \ldots, [x_K]$ with

probabilities $p_1, p_2, \ldots, p_K$. Then the expected value of an interval discrete random variable is defined as follows:

$$E([X]) = \sum_{[x]:f([x])>0} [x] f([x]) = \sum_{k=1}^{K} [x_k] p_k. \tag{4.1}$$

Definition 4.4 (variance of an interval discrete random variable [21]). the variance of an interval discrete random variable is defined by the following:

$$V([X]) = \sum_{[x]:f([x])>0} E\left([X]^2\right) - (E([X]))^2, \tag{4.2}$$

where $E([X]^2) = \sum_{k=1}^{K} [x_k]^2 p_k$.

5. Random Interval Multiobjective Two-Stage Stochastic Programming

Optimization model incorporating some of the input parameters as interval random variables is modeled as random interval multiobjective two-stage stochastic programming (RIMTSP) to handle the uncertainties within TSP optimization platform with simple recourse. Mathematically, it can be presented as follows:

$$\begin{aligned}
\max \quad & \hat{z}^t = \sum_{j=1}^{n} \left[c_j^t\right] x_j - E\left(\sum_{i=1}^{m_1} q_i^t |y_i|\right), \quad t = 1, 2, \ldots, T, \\
\text{subject to} \quad & y_i = [b_i] - \sum_{j=1}^{n} [a_{ij}] x_j, \quad i = 1, 2, \ldots, m_1, \\
& \sum_{j=1}^{n} [d_{ij}] x_j \le [b_{m_1+i}], \quad i = 1, 2, \ldots, m_2, \\
& y_i \ge 0, \quad i = 1, 2, \ldots, m_1, \qquad x_j \ge 0, \quad j = 1, 2, \ldots, n,
\end{aligned} \tag{5.1}$$

where x_j, $j = 1, 2, \ldots, n$, y_i, $i = 1, 2, \ldots, m_1$ are the first stage decision variables and second stage decision variables, respectively. Further, $[c_j^t]$, $j = 1, 2, \ldots, n$, $t = 1, 2, \ldots, T$ are the cost associated with the first stage decision variables and q_i^t, $i = 1, 2, \ldots, m_1$, $t = 1, 2, \ldots, T$ are the penalty cost associated with the discrepancy between $\sum_{j=1}^{n} [a_{ij}] x_j$ and $[b_i]$ of the kth objective function. The left hand side parameter $[a_{ij}]$ and the right hand side parameter $[b_i]$ are interval discrete random variables with known probability distribution. E is used to represent the expected value associated with the interval random variables.

5.1. Multiobjective Two-Stage Stochastic Programming Problem Where Only $[b_i]$, $i = 1, 2, \ldots, m_1$ Are Interval Discrete Random Variables

It is assumed that $[b_i]$, $i = 1, 2, \ldots, m_1$ are interval discrete random variables which takes interval values v_i^k, $k = 1, 2, \ldots, K$ with known probabilities p_i^k, $k = 1, 2, \ldots, K$.

Thus, the probability mass function (pmf) of the interval discrete random variable $[b_i]$ is given by:

$$f\left(\left[v_i^k\right]\right) = \Pr\left([b_i] = \left[v_i^k\right]\right) = p_i^k, \quad k = 1, 2, \ldots, K. \tag{5.2}$$

Let

$$g_i(x) = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m_1, \tag{5.3}$$

where $x = x_1, x_2, \ldots, x_{m_1}$ and $g_i(x) \geq 0$.

We compute $E(q_i^t|y_i|) = q_i^t E(|[b_i] - g_i(x)|)$, $i = 1, 2, \ldots, m_1$, $t = 1, 2, \ldots, T$ in two different cases as follows.

Case 1. When $([b_i] - g_i(x)) \geq 0$, we compute

$$\begin{aligned} q_i^t E(|[b_i] - g_i(x)|) &= q_i^t E([b_i]) - q_i^t E(g_i(x)) \\ &= q_i^t \sum_{k=1}^{K} \left[v_i^k\right] p_i^k - q_i^t g_i(x), \quad i = 1, 2, \ldots, m_1, \ t = 1, 2, \ldots, T. \end{aligned} \tag{5.4}$$

On simplification, we have

$$q_i^t E(|[b_i] - g_i|) = q_i^t \sum_{k=1}^{K} \left[v_i^k\right] p_i^k - q_i^t \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m_1, \ t = 1, 2, \ldots, T. \tag{5.5}$$

Using (5.5) in the RIMTSP model (5.1), we establish the deterministic model as follows:

$$\begin{aligned} \max \quad & \hat{z}^t = \sum_{j=1}^{n} \left[c_j^t\right] x_j - \sum_{i=1}^{m_1} \left[q_i^t \left(\sum_{k=1}^{K} \left[v_i^k\right] p_i^k \right) - q_i^t \left(\sum_{j=1}^{n} a_{ij} x_j \right) \right], \quad t = 1, 2, \ldots, T, \\ \text{subject to} \quad & \sum_{j=1}^{n} [d_{ij}] x_j \leq [b_{m_1+i}], \quad i = 1, 2, \ldots, m_2, \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{5.6}$$

Case 2. When $([b_i] - g_i(x)) < 0$, we compute

$$\begin{aligned} q_i^t E(|[b_i] - g_i(x)|) &= q_i^t E(g_i(x)) - q_i^t E([b_i]) \\ &= q_i^t g_i(x) - q_i^t \sum_{k=1}^{K} \left(\left[v_i^k\right] p_i^k \right), \quad i = 1, 2, \ldots, m_1, \ t = 1, 2, \ldots, T. \end{aligned} \tag{5.7}$$

On simplification, we have

$$q_i^t E(|[b_i] - g_i(x)|) = q_i^t \sum_{j=1}^{n} a_{ij} x_j - q_i^t \sum_{k=1}^{K} \left(\left[v_i^k\right] p_i^k \right), \quad i = 1, 2, \ldots, m_1, \ t = 1, 2, \ldots, T. \tag{5.8}$$

Using (5.8) in the RIMTSP model (5.1), we establish the deterministic model as follows:

$$\begin{aligned}
\max \quad & \hat{z}^t = \sum_{j=1}^{n}\left[c_j^t\right]x_j - \sum_{i=1}^{m}\left[q_i^t\left(\sum_{j=1}^{n}a_{ij}x_j\right) - q_i^t\left(\sum_{k=1}^{K}\left[v_i^k\right]p_i^k\right)\right], \quad t = 1,2,\ldots,T, \\
\text{subject to} \quad & \sum_{j=1}^{n}[d_{ij}]x_j \le [b_{m_1+i}], \quad i = 1,2,\ldots,m_2, \\
& x_j \ge 0, \quad j = 1,2,\ldots,n.
\end{aligned} \tag{5.9}$$

5.2. Multiobjective Two-Stage Stochastic Programming Problem Where Only $[a_{ij}]$, $i = 1,2,\ldots,m_1$, $j = 1,2,\ldots,n$ Are Interval Discrete Random Variables

It is assumed that $[a_{ij}]$, $i = 1,2,\ldots,m_1$, $j = 1,2,\ldots,n$ are interval discrete random variables which takes interval values w_{ij}^r, $r = 1,2,\ldots,R$ with known probabilities p_{ij}^r, $r = 1,2,\ldots,R$.

Let

$$f_i(x) = \sum_{j=1}^{n}[a_{ij}]x_j, \quad i = 1,2,\ldots,m_1, \tag{5.10}$$

where $f_i(x) \ge 0$.

Thus, the probability mass function (pmf) of the interval discrete random variable $[a_{ij}]$ is given by the following:

$$f\left(\left[w_{ij}^r\right]\right) = \Pr\left([a_{ij}] = \left[w_{ij}^r\right]\right) = p_{ij}^r, \quad r = 1,2,\ldots,R,\ i = 1,2,\ldots,m_1,\ j = 1,2,\ldots,n. \tag{5.11}$$

We compute $E(q_i^t|y_i|) = q_i^t E(|b_i - f_i(x)|)$, $i = 1,2,\ldots,m_1$, $t = 1,2,\ldots,T$ in two different cases as follows.

Case 1. When $(b_i - f_i(x)) \ge 0$, we compute

$$\begin{aligned}
q_i^t E(|b_i - f_i(x)|) &= q_i^t E(b_i) - q_i^t E(f_i(x)) = q_i^t b_i - q_i^t \sum_{j=1}^{n} E([a_{ij}])x_j \\
&= q_i^t b_i - q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j, \quad i = 1,2,\ldots,m_1, \quad t = 1,2,\ldots,T.
\end{aligned} \tag{5.12}$$

Hence,

$$q_i^t E(|b_i - f_i(x)|) = q_i^t b_i - q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j, \quad i = 1,2,\ldots,m_1,\ t = 1,2,\ldots,T. \tag{5.13}$$

Using (5.13) in the RIMTSP model (5.1), we establish the deterministic model as:

$$\begin{aligned}
\max \quad & \hat{z}^t = \sum_{j=1}^{n}\left[c_j^t\right]x_j - \sum_{i=1}^{m}\left[q_i^t b_i - q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j\right], \quad t = 1,2,\ldots,T, \\
\text{subject to} \quad & \sum_{j=1}^{n}[d_{ij}]x_j \le [b_{m_1+i}], \quad i = 1,2,\ldots,m_2, \\
& x_j \ge 0, \quad j = 1,2,\ldots,n.
\end{aligned} \tag{5.14}$$

Case 2. When $(b_i - f_i(x)) < 0$, we compute

$$\begin{aligned}
q_i^t E(|\, b_i - f_i(x)) &= q_i^t E(f_i(x)) - q_i^t E(b_i) = q_i^t \sum_{j=1}^{n} E([a_{ij}])x_j - q_i^t b_i \\
&= q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j - q_i^t b_i, \quad i = 1,2,\ldots,m_1,\ t = 1,2,\ldots,T.
\end{aligned} \tag{5.15}$$

On simplification, we have

$$q_i^t E(|b_i - f_i(x)|) = q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j - q_i^t b_i, \quad i = 1,2,\ldots,m_1,\ t = 1,2,\ldots,T. \tag{5.16}$$

Using (5.16) in the RIMTSP model (5.1), we establish the deterministic model as follows:

$$\begin{aligned}
\max \quad & \hat{z}^t = \sum_{j=1}^{n}\left[c_j^t\right]x_j - \sum_{i=1}^{m}\left[q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j - q_i^t b_i\right], \quad t = 1,2,\ldots,T, \\
\text{subject to} \quad & \sum_{j=1}^{n}[d_{ij}]x_j \le [b_{m_1+i}], \quad i = 1,2,\ldots,m_2 \\
& x_j \ge 0, \quad j = 1,2,\ldots,n.
\end{aligned} \tag{5.17}$$

5.3. *Multiobjective Two-Stage Stochastic Programming Problem Where Both* $[b_i]$ *and* $[a_{ij}]$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ *Are Interval Discrete Random Variables*

It is assumed that both $[b_i]$ and $[a_{ij}]$, $i = 1,2,\ldots,m_1$, $j = 1,2,\ldots,n$ are independent interval discrete random variables which takes interval values v_i^k, $i = 1,2,\ldots,m_1$, $k = 1,2,\ldots,K$ with known probabilities p_i^k, $i = 1,2,\ldots,m_1$, $k = 1,2,\ldots,K$ and w_{ij}^r, $i = 1,2,\ldots,m_1$, $r = 1,2,\ldots,R$ with known probabilities p_{ij}^r, $i = 1,2,\ldots,m_1$, $r = 1,2,\ldots,R$.

Let

$$h_i(x) = [b_i] - \sum_{j=1}^{n} [a_{ij}] x_j, \tag{5.18}$$

where $h_i(x) \geq 0$.

Thus, the probability mass function (pmf) of the interval discrete random variables $[b_i]$ and $[a_{ij}]$ are given by the following:

$$\begin{aligned} f\left(\left[v_i^k\right]\right) &= \Pr\left([b_i] = \left[v_i^k\right]\right) = p_i^k, \quad k = 1, 2, \ldots, K, \\ f\left(\left[w_{ij}^r\right]\right) &= \Pr\left([a_{ij}] = \left[w_{ij}^r\right]\right) = p_{ij}^r, \quad r = 1, 2, \ldots, R. \end{aligned} \tag{5.19}$$

We compute $E(q_i^t|y_i|) = q_i^t E(|h_i(x)|)$, $i = 1, 2, \ldots, m_1$, $t = 1, 2, \ldots, T$ in two different cases as follows.

Case 1. When $h_i(x) \geq 0$, we compute

$$\begin{aligned} q_i^t E(|h_i|) &= q_i^t E\left([b_i] - \sum_{j=1}^{n} [a_{ij}] x_j\right) \\ &= q_i^t \sum_{k=1}^{K} \left[v_i^k\right] p_i^k - q_i^t \sum_{j=1}^{n} \left(\sum_{r=1}^{R} \left[w_{ij}^r\right] p_{ij}^r\right) x_j, \quad i = 1, 2, \ldots, m_1, \ t = 1, 2, \ldots, T. \end{aligned} \tag{5.20}$$

Hence,

$$q_i^t E(|h_i(x)|) = q_i^t \sum_{k=1}^{K} \left[v_i^k\right] p_i^k - q_i^t \sum_{j=1}^{n} \left(\sum_{r=1}^{R} \left[w_{ij}^r\right] p_{ij}^r\right) x_j, \quad i = 1, 2, \ldots, m_1, \ t = 1, 2, \ldots, T. \tag{5.21}$$

Using (5.21) in the RIMTSP model (5.1), we establish the deterministic model as follows:

$$\begin{aligned} \max \quad & \hat{z}^t = \sum_{j=1}^{n} \left[c_j^t\right] x_j - \sum_{i=1}^{m_1} \left[q_i^t \sum_{k=1}^{K} \left[v_i^k\right] p_i^k - q_i^t \sum_{j=1}^{n} \left(\sum_{r=1}^{R} \left[w_{ij}^r\right] p_{ij}^r\right) x_j\right], \quad t = 1, 2, \ldots, T, \\ \text{subject to} \quad & \sum_{j=1}^{n} [d_{ij}] x_j \leq [b_{m_1+i}], \quad i = 1, 2, \ldots, m_2, \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{5.22}$$

Case 2. When $h_i(x) < 0$, we compute

$$q_i^t E(|h_i(x)|) = q_i^t E\left(\sum_{j=1}^{n}[a_{ij}]x_i - [b_i]\right) = q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j - q_i^t \sum_{k=1}^{K}\left[v_i^k\right]p_i^k, \quad i = 1,2,\ldots,m_1,\ t = 1,2,\ldots,T. \tag{a}$$

Hence,

$$q_i^t E(|h_i(x)|) = q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j - q_i^t \sum_{k=1}^{K}\left[v_i^k\right]p_i^k, \quad i = 1,2,\ldots,m_1,\ t = 1,2,\ldots,T. \tag{5.23}$$

Using (5.23) in the RIMTSP model (5.1), we establish the deterministic model as follows:

$$\begin{aligned}
\max \quad & \hat{z}^t = \sum_{j=1}^{n}\left[c_j^t\right]x_j - \sum_{i=1}^{m_1}\left[q_i^t \sum_{j=1}^{n}\left(\sum_{r=1}^{R}\left[w_{ij}^r\right]p_{ij}^r\right)x_j - q_i^t \sum_{k=1}^{K}\left[v_i^k\right]p_i^k\right], \quad t = 1,2,\ldots,T, \\
\text{subject to} \quad & \sum_{j=1}^{n}[d_{ij}]x_j \le [b_{m_1+i}]\ , \quad i = 1,2,\ldots,m_2, \\
& x_j \ge 0, \quad j = 1,2,\ldots,n.
\end{aligned} \tag{5.24}$$

6. Solution Procedures

The proposed RIMTSP model is very difficult to solved directly due to presence of discrete random intervals in the model input parameters. In order to solve the model, first we remove the randomness from the model input parameters and then formulate an equivalent deterministic multiobjective linear programming problem involving discrete real interval parameters. After that we transform the multiobjective linear programming problem into a single linear programming problem involving discrete real interval parameters using weighting method [24]. Further, ILP solution procedure is used to solve the deterministic model. The steps of the solution procedure of ILP is presented as follows [22].

Step 1. Find the best optimal solution by solving the following LPP:

$$\begin{aligned}
\min \quad & z_1' = \sum_{j=1}^{n} c_j^l x_j, \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij}^l x_j \le b_i^u, \quad i = 1,2,\ldots,m_1, \\
& \sum_{j=1}^{n} d_{ij}^u x_j \ge b_{m_1+i}^l, \quad i = 1,2,\ldots,m_2, \\
& x_j \ge 0, \quad j = 1,2,\ldots,n.
\end{aligned} \tag{6.1}$$

If the objective function is maximization type, then solve the following LPP to find the best optimal solution:

$$\begin{aligned}
\max \quad & z_1'' = \sum_{j=1}^{n} c_j^u x_j, \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij}^l x_j \le b_i^u, \quad i = 1, 2, \ldots, m_2, \\
& \sum_{j=1}^{n} d_{ij}^u x_j \ge b_{m_1+i}^l, \quad i = 1, 2, \ldots, m_2, \\
& x_j \ge 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{6.2}$$

Step 2. Find the worst optimal solution by solving the following LPP:

$$\begin{aligned}
\min \quad & z_2' = \sum_{j=1}^{n} c_j^u x_j, \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij}^u x_j \le b_i^l, \quad i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} d_{ij}^l x_j \ge b_{m_1+i}^u, \quad i = 1, 2, \ldots, m_2, \\
& x_j \ge 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{6.3}$$

If the objective function is maximization type, then solve the following LPP to find the worst optimal solution:

$$\begin{aligned}
\max \quad & z_2'' = \sum_{j=1}^{n} c_j^l x_j, \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij}^u x_j \le b_i^l, \quad i = 1, 2, \ldots, m_1, \\
& \sum_{j=1}^{n} d_{ij}^l x_j \ge b_{m_1+i}^u, \quad i = 1, 2, \ldots, m_2, \\
& x_j \ge 0, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{6.4}$$

7. Numerical Example

In this section, a numerical example with two objective functions along with four constraints among which two of them are deterministic constraints and another two contains the discrete random interval parameters with known probability distributions is presented as follows:

$$\begin{aligned}
\max \quad & \check{z}_1 = [18,20]x_1 + [15,16]x_2 + [14,15]x_3 + [14,16]x_4, \\
\max \quad & \check{z}_2 = [13,14]x_1 + [9,10]x_2 + [16,17]x_3 + [12,14]x_4. \\
\text{subject to} \quad & [4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4 \le [b_1], \\
& [4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4 \le [b_2], \\
& [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34], \\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42], \\
& x_1, x_2, x_3, x_4 \ge 0,
\end{aligned} \tag{7.1}$$

where $[b_1]$ and $[b_2]$ are the interval discrete random variables with known probability distributions.

Further, using the above model (7.1), a RIMTSP model with simple unit recourse cost (i.e., $q_i = 1, \forall i$) is formulated as follows:

$$\begin{aligned}
\max \quad & \check{z}_1 = [18,20]x_1 + [15,16]x_2 + [14,15]x_3 + [14,16]x_4 - E(|y_1|) - E(|y_2|), \\
\max \quad & \check{z}_2 = [13,14]x_1 + [9,10]x_2 + [16,17]x_3 + [12,14]x_4 - E(|y_1|) - E(|y_2|), \\
\text{subject to} \quad & y_1 = [b_1] - ([4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4), \\
& y_2 = [b_2] - ([4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4)\ , \\
& [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34], \\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42], \\
& x_1, x_2, x_3, x_4 \ge 0,
\end{aligned} \tag{7.2}$$

where $[b_1]$ and $[b_2]$ are the interval discrete random variables takes the interval values associated with the specified probabilities as given in the following:

$$\begin{gathered}
\Pr([b_1] = [20,22]) = \frac{2}{5}, \qquad \Pr([b_1] = [18,20]) = \frac{3}{5}, \\
\Pr([b_2] = [22,24]) = \frac{2}{9}, \qquad \Pr([b_2] = [19,21]) = \frac{1}{3}, \qquad \Pr([b_2] = [23,25]) = \frac{4}{9}.
\end{gathered} \tag{7.3}$$

On simplification, the above model (7.2) can be written as follows:

$$\begin{aligned}
\max \quad & \breve{z}_1 = [18,20]x_1 + [15,16]x_2 + [14,15]x_3 + [14,16]x_4 \\
& \quad - E(|[b_1] - ([4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4)|) \\
& \quad - E(|[b_2] - ([4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4)|), \\
\max \quad & \breve{z}_2 = [13,14]x_1 + [9,10]x_2 + [16,17]x_3 + [12,14]x_4 \\
& \quad - E(|[b_1] - ([4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4)|) \\
& \quad - E(|[b_2] - ([4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4)|), \\
\text{subject to} \quad & [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34], \\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42] \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{7.4}$$

Further, using the interval values associated with the probability of occurrence, the above model (7.4) can be transformed into two equivalent deterministic multiobjective linear programming models with interval coefficients as follows.

Case 1. When $([b_1] - ([4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4)) \ge 0$, $([b_2] - ([4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4)) \ge 0$.

The model (7.4) can be transformed into an equivalent deterministic multiobjective linear programming model with interval coefficients as follows:

$$\begin{aligned}
\max \quad & \breve{z}_{11} = [18,20]x_1 + [15,16]x_2 + [14,15]x_3 + [14,16]x_4 \\
& \quad - \left([20,22] \times \frac{2}{5} + [18,20] \times \frac{3}{5} - [4,5]x_1 - [2,4]x_2 - [5,6]x_3 - [3,4]x_4\right) \\
& \quad - \left([22,24] \times \frac{1}{2} + [19,21] \times \frac{1}{3} + [23,25] \times \frac{1}{6} - [4,6]x_1 - [3,4]x_2 \right. \\
& \qquad \left. -[5,7]x_3 - [2,4]x_4\right) \\
\max \quad & \breve{z}_{12} = [13,14]x_1 + [9,10]x_2 + [16,17]x_3 + [12,14]x_4 \\
& \quad - \left([20,22] \times \frac{2}{5} + [18,20] \times \frac{3}{5} - [4,5]\mathrm{x}_1 - [2,4]x_2 - [5,6]x_3 - [3,4]x_4\right) \\
& \quad - \left([22,24] \times \frac{1}{2} + [19,21] \times \frac{1}{3} + [23,25] \times \frac{1}{6} - [4,6]x_1 - [3,4]x_2 \right. \\
& \qquad \left. -[5,7]x_3 - [2,4]x_4\right)
\end{aligned}$$

$$\begin{aligned}
\text{subject to}\quad & [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34],\\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42],\\
& x_1, x_2, x_3, x_4 \ge 0\ .
\end{aligned} \tag{7.5}$$

On simplification, we get

$$\begin{aligned}
\max\quad & \check{z}_{11} = [26,31]x_1 + [20,24]x_2 + [24,28]x_3 + [19,24]x_4 - [39.96,43.96],\\
\max\quad & \check{z}_{12} = [21,25]x_1 + [14,18]x_2 + [26,30]x_3 + [17,22]x_4 - [39.96,43.96],\\
\text{subject to}\quad & [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34],\\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42],\\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{7.6}$$

Case 2. When $([b_1] - ([4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4)) < 0$, $([b_2] - ([4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4)) < 0$.

The model (7.4) can be transformed into an equivalent deterministic multiobjective linear programming model with real interval parameters as follows:

$$\begin{aligned}
\max\quad & \check{z}_{21} = [18,20]x_1 + [15,16]x_2 + [14,15]x_3 + [14,16]x_4\\
& - \left([4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4 - [20,22] \times \frac{2}{5} - [18,20] \times \frac{3}{5}\right)\\
& - \left([4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4 - [22,24] \times \frac{1}{2}\right.\\
& \quad \left. -[19,21] \times \frac{1}{3} - [23,25] \times \frac{1}{6}\right)\\
\max\quad & \check{z}_{22} = [13,14]x_1 + [9,10]x_2 + [16,17]x_3 + [12,14]x_4\\
& - \left([4,5]x_1 + [2,4]x_2 + [5,6]x_3 + [3,4]x_4 - [20,22] \times \frac{2}{5} - [18,20] \times \frac{3}{5}\right)\\
& - \left([4,6]x_1 + [3,4]x_2 + [5,7]x_3 + [2,4]x_4 - [22,24] \times \frac{1}{2}\right.\\
& \quad \left. -[19,21] \times \frac{1}{3} - [23,25] \times \frac{1}{6}\right)\\
\text{subject to}\quad & [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34],\\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42],\\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{7.7}$$

Table 1: Optimal solution for Case 1.

Types of problem	Weights w_1	Weights w_2	Optimal decision variables	Value of the objective function
Case 1	0.1	0.9	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 202.6246$
			$x_1 = 4.889, x_2 = 0, x_3 = 1.778, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 111.0178$
	0.2	0.8	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 205.3631$
			$x_1 = 4.889, x_2 = 0, x_3 = 1.778, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 113.1067$
	0.3	0.7	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 208.1015$
			$x_1 = 4.889, x_2 = 0, x_3 = 1.778, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 115.1956$
	0.4	0.6	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 210.84$
			$x_1 = 4.889, x_2 = 0, x_3 = 1.778, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 117.2844$
	0.5	0.5	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 213.5785$
			$x_1 = 4.889, x_2 = 0, x_3 = 1.778, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 119.3733$
	0.6	0.4	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 216.3169$
			$x_1 = 4.889, x_2 = 0, x_3 = 1.778, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 121.4622$
	0.7	0.3	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 219.0554$
			$x_1 = 4.889, x_2 = 0, x_3 = 1.778, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 123.5511$
	0.8	0.2	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 221.7938$
			$x_1 = 6.667, x_2 = 0, x_3 = 0, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 126.7067$
	0.9	0.1	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_1^{\text{Best}} = 224.5323$
			$x_1 = 6.667, x_2 = 0, x_3 = 0, x_4 = 0$	$\check{F}_1^{\text{Worst}} = 130.04$

On simplification, we get

$$\begin{aligned}
\max \quad & \check{z}_{21} = [7,12]x_1 + [7,11]x_2 + [1,5]x_3 + [6,11]x_4 + [39.96, 43.96], \\
\max \quad & \check{z}_{22} = [2,6]x_1 + [1,5]x_2 + [3,7]x_3 + [4,9]x_4 + [39.96, 43.96], \\
\text{subject to} \quad & [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34], \\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42], \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{7.8}$$

The above deterministic multiobjective linear programming models (7.6) and (7.8) with interval coefficient can be transformed into a single objective linear programming problem containing interval coefficient by using weighting method as given in the following:

$$\begin{aligned}
\max \quad & \check{F}_1 = w_1 \check{z}_{11} + w_2 \check{z}_{12}, \\
\text{subject to} \quad & [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34], \\
& [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42], \\
& x_1, x_2, x_3, x_4 \ge 0, \quad w_1 \ge 0, \quad w_2 \ge 0,
\end{aligned}$$

Table 2: Optimal solution for Case 2.

Types of problem	Weights w_1	Weights w_2	Optimal decision variables	Value of the objective function
Case 2	0.1	0.9	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_2^{\text{Best}} = 104.5446$
			$x_1 = 4, x_2 = 0, x_3 = 0, x_4 = 2$	$\check{F}_2^{\text{Worst}} = 58.36$
	0.2	0.8	$x_1 = 5.692308, x_2 = 0, x_3 = 3.384615, x_4 = 0$	$\check{F}_2^{\text{Best}} = 107.2831$
			$x_1 = 4, x_2 = 0, x_3 = 0, x_4 = 2$	$\check{F}_2^{\text{Worst}} = 60.76$
	0.3	0.7	$x_1 = 5.692308, x_2 = 3.384615, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Best}} = 111.3754$
			$x_1 = 6.667, x_2 = 0, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Worst}} = 63.2933$
	0.4	0.6	$x_1 = 5.692308, x_2 = 3.384615, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Best}} = 116.8215$
			$x_1 = 5, x_2 = 2, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Worst}} = 66.76$
	0.5	0.5	$x_1 = 5.692308, x_2 = 3.384615, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Best}} = 122.2677$
			$x_1 = 5, x_2 = 2, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Worst}} = 70.46$
	0.6	0.4	$x_1 = 5.692308, x_2 = 3.384615, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Best}} = 127.7138$
			$x_1 = 5, x_2 = 2, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Worst}} = 74.16$
	0.7	0.3	$x_1 = 5.692308, x_2 = 3.384615, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Best}} = 133.16$
			$x_1 = 5, x_2 = 2, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Worst}} = 77.86$
	0.8	0.2	$x_1 = 5.692308, x_2 = 3.384615, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Best}} = 138.6062$
			$x_1 = 5, x_2 = 2, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Worst}} = 81.56$
	0.9	0.1	$x_1 = 5.692308, x_2 = 3.384615, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Best}} = 144.0523$
			$x_1 = 5, x_2 = 2, x_3 = 0, x_4 = 0$	$\check{F}_2^{\text{Worst}} = 85.26$

$$\begin{aligned} \max \quad & \check{F}_2 = w_1 \check{z}_{21} + w_2 \check{z}_{22}, \\ \text{subject to} \quad & [3,4]x_1 + [5,6]x_2 + [5,7]x_3 + [6,8]x_4 \le [32,34], \\ & [5,6]x_1 + [4,5]x_2 + [4,6]x_3 + [7,8]x_4 \le [40,42], \\ & x_1, x_2, x_3, x_4 \ge 0, \quad w_1 \ge 0, \quad w_2 \ge 0, \end{aligned} \tag{7.9}$$

where w_1 and w_2 are the relative weights associated with the respective objective functions and $w_1 + w_2 = 1$. These weights are calculated by using AHP (analytic hierarchy process) [25].

Using the solution procedure described in Section 6, the above linear programming models with interval coefficient models (7.9) are solved by using GAMS [26] and LINGO (language for interactive general optimization) Version 11.0 [27]. The obtained best and worst optimal solutions are given in the Tables 1 and 2 and are shown in the Figures 1 and 2.

8. Conclusions

A multiobjective two-stage stochastic programming problem involving some interval discrete random variable has been presented in this paper. Before solving the problem the deterministic models are established. Then weighting method as well as interval programming method is applied to make the model solvable. Deterministic models are solved using GAMS and LINGO softwares and obtained the optimal solutions. From the results Table 1 and Table 2, we observed the following:

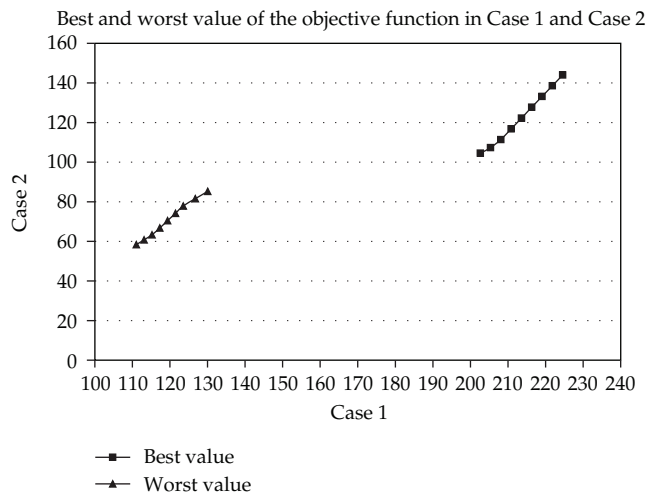

Figure 1: Best and worst value of the objective function for Cases 1 and 2.

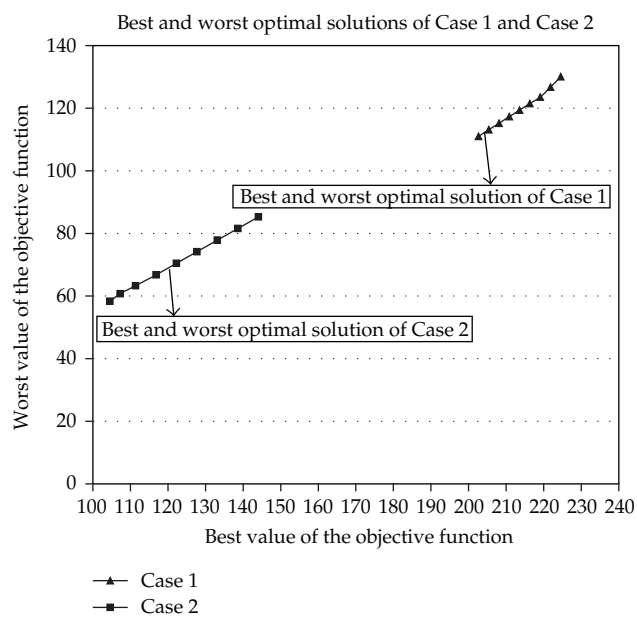

Figure 2: Optimal solutions of Cases 1 and 2.

(i) the solution obtained by assigning the first pair of weights (w_1, w_2) that is, 0.1 and 0.9 gives the best lower bound (i.e., 111.02) and worst upper bound (i.e., 202.62) for the objective function $\breve{F}_1$. However, the solution obtained by using the last pair of weights (w_1, w_2) that is, 0.9 and 0.1 gives the worst lower bound (i.e., 130.04) and best upper bound (i.e., 224.53) for the objective function in Case 1;

(ii) similarly, the solution obtained by assigning the first pair of weights (w_1, w_2) that is, 0.1 and 0.9 gives the best lower bound (i.e., 58.36) and worst upper bound (i.e., 104.54) of the objective function $\breve{F}_2$. However, the solution obtained by using

the last pair of weights (w_1, w_2) that is, 0.9 and 0.1 gives the worst lower bound (i.e., 85.26) and best upper bound (i.e., 144.05) for the objective function for Case 2.

Further both the tools (i.e., GAMS and LINGO) are giving the same optimal solutions with respect to comparison.

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