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Research Article

Optimality Conditions and Duality of Three Kinds of Nonlinear Fractional Programming Problems

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Some assumptions for the objective functions and constraint functions are given under the conditions of convex and generalized convex, which are based on the F-convex, ρ-convex, and (F, ρ)-convex. The sufficiency of Kuhn-Tucker optimality conditions and appropriate duality results are proved involving (F, ρ)-convex, (F, α, ρ, d)-convex, and generalized (F, α, ρ, d)-convex functions.

1. Introduction

Multiobjective optimization theory is a development of numerical optimization and related to many subjects, such as nonsmooth analysis, convex analysis, nonlinear analysis, and the theory of set value. It has a wide range of applications in the fields of industrial design, economics, engineering, military, management sciences, financial investment, transport, and so forth, and now it is an interdisciplinary science branch between applied mathematics and decision sciences. Convexity plays an important role in optimization theory, and it becomes an important theoretical basis and useful tool for mathematical programming and optimization theory.

Convex function theory can be traced back to the works of Holder, Jensen, and Minkowski in the beginning of this century, but the real work that caught the attention of people is the research on game theory and mathematical programming by von Neumann and Morgenstern [1], Dantzing, and Kuhn and Tucker in the forties to fifties, and people have done a lot of intensive research about convex functions from the fifties to sixties. In the middle of the sixties convex analysis was produced, and the concept of convex function is promoted in a variety of ways, and the notion of generalized convex is given.

Fractional programming has an important significance in the optimization problems; for instance, in order to measure the production or the efficiency of a system, we should minimize a ratio of functions between a given period of time and a utilized resource in engineering and economics.

Preda [2] has established the concept of (F, ρ)-convex based on F-convex [3] and ρ-convex [4] and obtained some results, which are the expansion of F-convex and ρ-convex. Motivated by various concepts of convexity, Liang et al. [5] have put forward a generalized convexity, which was called (F, α, ρ, d)-convex, which extended (F, ρ)-convex, and Liang et al. [6], Weir and Mond [7], Weir [8], Jeyakumar and Mond [9], Egudo [10], Preda [2], and Gulati and Islam [3] obtained some corresponding optimality conditions and applied these optimality conditions to define dual problems and derived duality theorems for single objective fractional problems and multiobjective problems. Then the definition of generalized (F, α, ρ, d)-convex is given under the condition of (F, α, ρ, d)-convex. However, in general, fractional programming problems are nonconvex and the Kuhn-Tucker optimality conditions are only necessary. Under what conditions are the Kuhn-Tucker conditions sufficient for the optimality of problems? This question appeals to the interests of many researchers, and those are what we should probe. Based on the former conclusions, by adding conditions to objective functions and constraint functions and by changing K-T conditions [11], the optimality conditions and dual are given involving weaker convexity conditions. The main results in this paper are based on convex and generalized convex functions and the properties of sublinear functions.

In this paper, we will discuss sufficient optimality conditions and dual problems for three kinds of nonlinear fractional programming problems, and the paper is organized as follows.

In Sections 3.1 and 3.2, we present the Kuhn-Tucker sufficient optimality conditions and dual for nonlinear fractional programming problem and multiobjective fractional programming problem based on generalized (F, α, ρ, d)-convex. Section 3.3 contains optimality conditions and dual for multiobjective fractional programming problem under (F, ρ)-convex. In these sections, I present some assumptions for the objective functions and constraint functions such that the Kuhn-Tucker optimality conditions are sufficient and obtain the corresponding duality theorem.

2. Preliminaries

Let E^n be the n-dimensional real vector space, that is, n-dimensional Euclidean space, where $y = (y_1, y_2, \ldots, y_n)^T$, $z = (z_1, z_2, \ldots, z_n)^T \in E^n$, and provides as follows, (see [12]),

$$\begin{aligned} y = z &\Longleftrightarrow y_i = z_i, \quad i = 1, 2, \ldots, n; \\ y > z &\Longleftrightarrow y_i > z_i, \quad i = 1, 2, \ldots, n; \\ y \geqq z &\Longleftrightarrow y_i \geqq z_i, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

Definition 1 (see [12]). Suppose that $x_0 \in X_0$; that is, if $x \notin X_0$, such that $f(x) \leqq f(x_0)$, x_0 is an efficient solution of multiobjective programming problem.

Definition 2 (see [12]). Suppose that $x_0 \in X_0$; that is, if $x \notin X_0$, such that $f(x) < f(x_0)$, x_0 is a weakly efficient solution of multiobjective programming problem.

Definition 3 (see [5]). Given an open set $X_0 \subset R^n$, a functional $F : X_0 \times X_0 \times R^n \to R$ is called sublinear if, for any $x, x_0 \in X_0$,

$$\begin{aligned} F(x, x_0; a_1 + a_2) &\leqq F(x, x_0; a_1) + F(x, x_0; a_2), \\ &\qquad \forall a_1, a_2 \in R^n, \\ F(x, x_0; \alpha a) &= \alpha F(x, x_0; a), \quad \forall \alpha \in R,\ \alpha \geqq 0,\ \forall a \in R^n. \end{aligned} \tag{2}$$

It follows from the second equality that

$$\begin{aligned} F(x, x_0; 0) = F(x, x_0; 0 \times a) = 0 \times F(x, x_0; a) = 0, \\ \text{for any } a \in R^n. \end{aligned} \tag{3}$$

Let $F : X_0 \times X_0 \times R^n \to R$ be a sublinear function, and let $\alpha : X_0 \times X_0 \to R_+ \setminus \{0\}$, $\rho = (\rho_1, \rho_2, \ldots, \rho_m)^T$, $\rho_i \in R$, $d : X_0 \times X_0 \to R$, and the function $f = (f_1, f_2, \ldots, f_m) : X_0 \to R^m$ is differentiable at $x_0 \in X_0$.

Definition 4 (see [3]). Let $\phi(x)$ be a differentiable function defined on $X_0 \subset E^n$. The function $\phi(x)$ is said to be F-convex on X_0 with respect to F, if $\phi(x) - \phi(y) \geqq F_{x,y}[\nabla\phi(y)]$.

Definition 5 (see [4, 12]). Let $f(x)$ be a real-valued function defined on the convex set $X_0 \subset E^n$, if there exists a real number $\rho \in R$, such that

$$\begin{aligned} f(\lambda x^1 + (1-\lambda) x^2) \leqq \lambda f(x^1) + (1-\lambda) f(x^2) \\ - \rho\lambda(1-\lambda) \left\| x^1 - x^2 \right\|^2 \end{aligned} \tag{4}$$

for any $x^1, x^2 \in X_0$ and any $\lambda \in [0, 1]$, then the function $f(x)$ is said to be ρ-convex on X_0.

Especially, if $\rho = 0$, then we obtain the definition of convex.

If $\rho > 0$ (or $\rho < 0$) in the above definition, then we have strong convex (or weak convex).

Definition 6 (see [2]). The function $f_i : X_0 \to R$ is said to be (F, ρ)-convex at $x_0 \in X_0$, if for any $x_0 \in X_0$, $f_i(x)$ satisfies the following condition:

$$f_i(x) - f_i(x_0) \geqq F(x, x_0; \nabla f_i(x_0)) + \rho_i d^2(x, x_0). \tag{5}$$

Definition 7 (see [5]). The function f_i is said to be (F, α, ρ_i, d)-convex at $x_0 \in X_0$, if

$$\begin{aligned} f_i(x) - f_i(x_0) \geqq F(x, x_0; \alpha(x, x_0) \nabla f_i(x_0)) + \rho_i d^2(x, x_0), \\ \forall x \in X_0. \end{aligned} \tag{6}$$

The function f is said to be (F, α, ρ, d)-convex at x_0, if each component f_i of f is (F, α, ρ_i, d)-convex at x_0.

The function f is said to be (F, α, ρ, d)-convex on X_0, if it is (F, α, ρ, d)-convex at every point in X_0.

Definition 8. The function f_i is said to be (F, α, ρ_i, d)-quasiconvex at x_0, if $f_i(x) \leqq f_i(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0)\nabla f_i(x_0)) \leqq -\rho_i d^2(x, x_0)$.

The function f is said to be (F, α, ρ, d)-quasiconvex at x_0, if each component f_i of f is (F, α, ρ_i, d)-quasiconvex at x_0.

Definition 9. The function f_i is said to be (F, α, ρ_i, d)-pseudoconvex at x_0, if for all $x \in X_0$, $f_i(x) < f_i(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0)\nabla f_i(x_0)) < -\rho_i d^2(x, x_0)$.

The function f is said to be (F, α, ρ, d)-pseudoconvex at x_0, if each component f_i of f is (F, α, ρ_i, d)-pseudoconvex at x_0.

Definition 10. The function f is said to be strictly (F, α, ρ_i, d)-pseudoconvex at $x_0 \in X_0$, if $f(x) \leqq f(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0)\nabla f(x_0)) < -\rho d^2(x, x_0)$, where $F(x, x_0; \alpha(x, x_0)\nabla f(x_0)) = (F(x, x_0; \alpha(x, x_0)\nabla f_1(x_0)), \ldots, F(x, x_0; \alpha(x, x_0)\nabla f_m(x_0)))$.

Further, f is said to be weakly strictly (F, α, ρ, d)-pseudoconvex at $x_0 \in X_0$, if $f(x) \leqq f(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0)\nabla f(x_0)) < -\rho d^2(x, x_0)$.

In order to prove our main result, we need a lemma which we present in this section.

Lemma 11 (see [13]). *Suppose that differentiable real-valued functions $h_j(x)$ $(j = 1, 2, \ldots, m)$ are (F, α, ρ_j, d)-quasiconvex at $\overline{x} \in S$; then $V^T h(x)$ is $(F, \alpha, \sum_{j=1}^{m} v_j \rho_j, d)$-quasiconvex at $\overline{x} \in S$, where $\rho_j \in R$, $V \geqq 0$ and V^T denote the transpose of the m-dimensional column vector V; that is, $V^T = (v_1, v_2, \ldots, v_m)$.*

3. Optimality Conditions and Duality

3.1. Nonlinear Fractional Programming Problem Involved Inequality and Equality Constraints Based on Generalized (F,α,ρ,d)-Convex. Consider the nonlinear fractional programming problem (FP)

$$\begin{aligned} \min \quad & \frac{f(x)}{g(x)} \\ \text{s.t.} \quad & h(x) \leqq 0,\ l(x) = 0,\ x \in X_0, \end{aligned} \tag{7}$$

where X_0 is an open set of R^n, $f(x)$ and $g(x)$ are real-valued functions defined on X_0, $h(x)$ is an m-dimensional vector-valued functions defined also on X_0, and $l(x)$ a q-dimensional vector-valued function.

Let

$$S = \{x \in X_0 \mid h(x) \leqq 0, l(x) = 0\} \tag{8}$$

denotes the set of all feasible solutions for (FP) and assume that $f(x)$, $g(x)$, $h_j(x)$ $(j = 1, 2, \ldots, m)$, and $l_i(x)$ $(i = 1, 2, \ldots, q)$ are continuously differentiable over X_0 and that $f(x) \geqq 0$, $g(x) > 0$, for all $x \in X_0$.

If $\overline{x} \in X_0$ is a solution for problem (FP) and if a constraint qualification [14] holds, then the Kuhn-Tucker necessary conditions are given below: there exists $V_0 \in R^m$ and $W_0 \in R^q$ such that

$$\begin{gathered} \nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right) + \nabla h(\overline{x}) V_0 + \nabla l(\overline{x}) W_0 = 0, \\ V_0^T h(\overline{x}) = 0, \\ V_0 \geqq 0, \qquad h(\overline{x}) \leqq 0, \qquad l(\overline{x}) = 0. \end{gathered} \tag{9}$$

Theorem 12. *Suppose that $\overline{x}$ is a feasible solution of (FP), that the Kuhn-Tucker conditions hold at $\overline{x}$, that $f(x)/g(x)$ in problem (FP) is (F, α, ρ, d)-pseudoconvex on S, $h_j(x)$ $(j = 1, 2, \ldots, m)$ are (F, α, ρ_j, d)-quasiconvex on S, and that $l_i(x)$ $(i = 1, 2, \ldots, q)$ are (F, α, ρ_i, d)-quasiconvex over S, ρ, ρ_i, $\rho_j \in R$, $\rho + V_0^T \rho' + W_0^T \rho'' \geqq 0$, where $V_0^T \rho'$ is the inner product about V_0 and ρ' and $W_0^T \rho''$ the inner product about W_0 and ρ''. Then, $\overline{x}$ is an optimality solution for problem (FP).*

Proof. Suppose that $\overline{x}$ is not an optimality solution of (FP). Then, there exists a feasible solution $x \in S$ such that $f(x)/g(x) < f(\overline{x})/g(\overline{x})$.

By the (F, α, ρ, d)-pseudoconvexity assumption of $f(x)/g(x)$, we have

$$F\left(x, \overline{x}; \alpha(x, \overline{x}) \nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right)\right) < -\rho d^2(x, \overline{x}). \tag{10}$$

For each j $(j = 1, 2, \ldots, m)$, by the (F, α, ρ_j, d)-quasiconvexity assumption of $h_j(x)$ and Lemma 11, we have that $V_0^T h(x)$ is $(F, \alpha, \sum_{j=1}^{m} v_j \rho_j, d)$-quasiconvex on S. Therefore,

$$F\left(x, \overline{x}; \alpha(x, \overline{x}) \nabla h(\overline{x}) V_0\right) \leqq -\sum_{j=1}^{m} \rho_j v_j d^2(x, \overline{x}), \tag{11}$$

that is, $F(x, \overline{x}; \alpha(x, \overline{x}) \nabla h(\overline{x}) V_0) \leqq -V_0^T \rho' d^2(x, \overline{x})$.

By the (F, α, ρ_i, d)-quasiconvexity of $l_i(x)$ $(i = 1, 2, \ldots, q)$ over S, we have that $W_0^T l(x)$ is $(F, \alpha, \sum_{i=1}^{q} w_i \rho_i, d)$-quasiconvexity on S.

Then we obtain $F(x, \overline{x}; \alpha(x, \overline{x}) \nabla l(\overline{x}) W_0) \leqq -\sum_{i=1}^{q} \rho_i w_i d^2(x, \overline{x})$, that is,

$$F\left(x, \overline{x}; \alpha(x, \overline{x}) \nabla l(\overline{x}) W_0\right) \leqq -W_0^T \rho'' d^2(x, \overline{x}). \tag{12}$$

By (10), (11), and (12), and based on the sublinearity of F, we have

$$\begin{aligned} & F\left(x, \overline{x}; \alpha(x, \overline{x}) \left(\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right) + \nabla h(\overline{x}) V_0 + \nabla l(\overline{x}) W_0\right)\right) \\ & \quad < -\left(\rho + V_0^T \rho' + W_0^T \rho''\right) d^2(x, \overline{x}). \end{aligned} \tag{13}$$

Considering that $\rho + V_0^T \rho' + W_0^T \rho'' \geqq 0$, we get

$$F\left(x, \overline{x}; \alpha(x, \overline{x}) \left(\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right) + \nabla h(\overline{x}) V_0 + \nabla l(\overline{x}) W_0\right)\right) < 0. \tag{14}$$

By the K-T conditions, we have $\nabla(f(\overline{x})/g(\overline{x})) + \nabla h(\overline{x}) V_0 + \nabla l(\overline{x}) W_0 = 0$.

Hence, based on the sublinearity of F, we obtain

$$F\left(x, \overline{x}; \alpha(x, \overline{x}) \left(\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right) + \nabla h(\overline{x}) V_0 + \nabla l(\overline{x}) W_0\right)\right) = 0, \tag{15}$$

which contradicts (14). The proof is complete. □

Consider the dual problem of (FP):

$$\begin{aligned} \max \quad & \frac{f(y)}{g(y)} + u^T h(y) + v^T l(y) \\ \text{s.t.} \quad & \lambda^T \nabla\left(\frac{f(y)}{g(y)}\right) + u^T \nabla h(y) + v^T \nabla l(y) = 0, \\ & u^T h(y) \geqq 0, \\ & v^T l(y) = 0, \\ & u \geqq 0, \quad v \geqq 0. \end{aligned} \tag{FD}$$

Theorem 13. *Suppose that $\lambda^T(f(y)/g(y))$ is (F, α, ρ_1, d)-pseudoconvex at y, $u^T h(y) + v^T l(y)$ is (F, α, ρ_2, d)-quasiconvex at y in problem (FP) and (FD), and that $\rho_1 + \rho_2 \geqq 0$; then $\lambda^T(f(x)/g(x)) \geqq \lambda^T(f(y)/g(y))$, for any feasible solution x of (FP) and (y, λ, u, v) of (FD).*

Proof. Assume that the conclusion is not true; that is, $\lambda^T(f(x)/g(x)) < \lambda^T(f(y)/g(y))$.

By the (F,α,ρ_1,d)-pseudoconvex of $\lambda^T(f(x)/g(x))$ at y, we get

$$F\left(x,y;\alpha(x,y)\lambda^T\nabla\left(\frac{f(y)}{g(y)}\right)\right) < -\rho_1 d^2(x,y). \quad (16)$$

Using $u^T h(x) \leqq 0$, $v^T l(x) = 0$, $u^T h(y) \geqq 0$, $v^T l(y) = 0$, we have

$$u^T h(x) + v^T l(x) \leqq u^T h(y) + v^T l(y). \quad (17)$$

By the (F,α,ρ_2,d)-quasiconvex of $u^T h(y) + v^T l(y)$, we get

$$F\left(x,y;\alpha(x,y)\left(u^T\nabla h(y) + v^T\nabla l(y)\right)\right) \leqq -\rho_2 d^2(x,y). \quad (18)$$

By (16) and (18) and based on the sublinearity of F, we have

$$\begin{aligned} F\Bigg(x,y;\alpha(x,y)\Bigg(&\lambda^T\nabla\left(\frac{f(y)}{g(y)}\right) \\ &+u^T\nabla h(y) + v^T\nabla l(y)\Bigg)\Bigg) \\ &< -(\rho_1+\rho_2)d^2(x,y). \end{aligned} \quad (19)$$

Since (y,λ,u,v) is the feasible solution of (FD), so

$$\overline{\lambda}^T\nabla\left(\frac{f(y)}{g(y)}\right) + u^T\nabla h(y) + v^T\nabla l(y) = 0. \quad (20)$$

Hence,

$$\begin{aligned} 0 = F\Bigg(x,y;\alpha(x,y)\Bigg(&\lambda^T\nabla\left(\frac{f(y)}{g(y)}\right) \\ &+u^T\nabla h(y) + v^T\nabla l(y)\Bigg)\Bigg) \\ &< -(\rho_1+\rho_2)d^2(x,y), \end{aligned} \quad (21)$$

which contradicts the known condition $\rho_1+\rho_2 \geqq 0$. The proof is complete. □

3.2. Nonlinear Multiobjective Fractional Programming Problem Involved Inequality and Equality Constraints Based on (F,α,ρ,d)-Convex and Generalized (F,α,ρ,d)-Convex. Consider the nonlinear multiobjective fractional programming problem (VFP)

$$\begin{aligned} \min \quad & M(x) = \frac{f(x)}{g(x)} = \left[\frac{f_1(x)}{g(x)}, \frac{f_2(x)}{g(x)}, \ldots, \frac{f_p(x)}{g(x)}\right] \\ \text{s.t.} \quad & h(x) \leqq 0,\ x \in X_0 \\ & l(x) = 0,\ x \in X_0, \end{aligned} \quad (22)$$

where X_0 is an open set of R^n, $f_i(x)$ $(i = 1,2,\ldots,p)$, $g(x)$, $h_j(x) : X_0 \to R$ $(j = 1,2,\ldots,m)$ are real-valued functions defined on X_0, $l_k(x) : X_0 \to R$ $(k = 1,2,\ldots,q)$ are real-valued functions defined also on X_0, $f(x) = (f_1(x), f_2(x),\ldots,f_p(x))^T$, $h(x) = (h_1(x), h_2(x),\ldots,h_m(x))^T$, and $l(x) = (l_1(x), l_2(x),\ldots,l_q(x))^T$.

Let

$$S = \{x \mid x \in X_0,\ h(x) \leqq 0,\ l(x) = 0\} \quad (23)$$

denotes the set of all feasible solutions of (VFP) and assume that $f_i(x)$ $(i = 1,2,\ldots,p)$, $g(x)$, $h_j(x)$ $(j = 1,2,\ldots,m)$, and $l_k(x)$ $(k = 1,2,\ldots,q)$ are continuously differentiable over X_0 and that $g(x) > 0$, for all $x \in X_0$.

Theorem 14. *Suppose that $f(x)/g(x)$ is weakly strictly (F,α,ρ_1,d)-pseudoconvex at $\overline{x} \in X_0$, $h_j(x)$ $(j = 1,2,\ldots,m)$ are (F,α,ρ_2,d)-quasiconvex with respect to $\overline{x} \in X_0$, $l_k(x)$ $(k = 1,2,\ldots,q)$ are (F,α,ρ_3,d)-convex with respect to $\overline{x} \in X_0$, and that there exists $\overline{\lambda} \in \wedge^{++}$ (or $\wedge^{+}$), $\overline{u} \in R^m_+$, $\overline{v} \in R^q_+$ satisfying*

$$\begin{aligned} &\sum_{i=1}^{p}\overline{\lambda}_i\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right) + \sum_{j=1}^{m}\overline{u}_j\nabla h_j(\overline{x}) + \sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x}) = 0, \\ &\overline{u}_j h_j(\overline{x}) = 0, \\ &h_j(x) \leqq 0, \quad j = 1,2,\ldots,m, \\ &l_k(x) = 0, \quad k = 1,2,\ldots,q \end{aligned} \quad (24)$$

and $\rho = \sum_{i=1}^{p}\overline{\lambda}_i\rho_1 + \sum_{j=1}^{m}\overline{u}_j\rho_2 + \sum_{k=1}^{q}\overline{v}_k\rho_3 \geqq 0$. Then, $\overline{x}$ is an efficient solution of (VFP), where

$$\begin{aligned} &\wedge^{+} \\ &= \left\{\overline{\lambda} = \left(\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_p\right)^T \mid \overline{\lambda}_i \geqq 0, i = 1,2,\ldots,p \cdot \sum_{i=1}^{p}\overline{\lambda}_i = 1\right\}, \\ &\wedge^{++} \\ &= \left\{\overline{\lambda} = \left(\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_p\right)^T \mid \overline{\lambda}_i > 0, i = 1,2,\ldots,p \cdot \sum_{i=1}^{p}\overline{\lambda}_i = 1\right\}. \end{aligned} \quad (25)$$

Proof. Suppose that $\overline{x}$ is not an efficient solution of (VFP); then there exists a feasible solution $x \in X_0$ such that $M(x) \leqq M(\overline{x})$, that is, $f(x)/g(x) \leqq f(\overline{x})/g(\overline{x})$.

By the weakly strict (F,α,ρ_1,d)-pseudoconvexity of $f(x)/g(x)$ at $\overline{x} \in X_0$, we get

$$F\left(x,\overline{x};\alpha(x,\overline{x})\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right)\right) < -\rho_1 d^2(x,\overline{x}). \quad (26)$$

Using $\overline{\lambda} \in \wedge^{++}$ (or $\wedge^{+}$), then we have

$$\sum_{i=1}^{p}\overline{\lambda}_i F\left(x,\overline{x};\alpha(x,\overline{x})\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right)\right) < -\sum_{i=1}^{p}\overline{\lambda}_i\rho_1 d^2(x,\overline{x}). \quad (27)$$

Based on the sublinearity of F, we obtain

$$F\left(x,\overline{x};\alpha(x,\overline{x})\sum_{i=1}^{p}\overline{\lambda}_i\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right)\right)<-\sum_{i=1}^{p}\overline{\lambda}_i\rho_1 d^2(x,\overline{x}). \tag{28}$$

Since $\overline{u}^T h(\overline{x}) = 0$, $\overline{u} \geqq 0$ and $h_j(x) \leqq 0$, we have $\overline{u}^T h(x) - \overline{u}^T h(\overline{x}) \leqq 0$; that is, $\overline{u}^T h(x) \leqq \overline{u}^T h(\overline{x})$.

Since $h_j(x)$ $(j = 1, 2, \ldots, m)$ are (F, α, ρ_2, d)-quasiconvex at $\overline{x} \in X_0$, by Lemma 11, we have $\overline{u}^T h$ is $(F, \alpha, \rho_2 \sum_{j=1}^{m} \overline{u}_j, d)$-quasiconvex at $\overline{x} \in X_0$.

Hence, we obtain the following inequality:

$$F\left(x,\overline{x};\alpha(x,\overline{x})\sum_{j=1}^{m}\overline{u}_j\nabla h_j(\overline{x})\right)\leqq-\sum_{j=1}^{m}\overline{u}_j\rho_2 d^2(x,\overline{x}). \tag{29}$$

By the (F, α, ρ_3, d)-convexity of $l_k(x)$ at $\overline{x} \in X_0$, we have

$$l_k(x)-l_k(\overline{x})\geqq F\left(x,\overline{x};\alpha(x,\overline{x})\nabla l_k(\overline{x})\right)+\rho_3 d^2(x,\overline{x}). \tag{30}$$

Since $\overline{v} \geqq 0$, we have

$$\begin{aligned}0=\overline{v}l(x)-\overline{v}l(\overline{x})\geqq&\sum_{k=1}^{q}\overline{v}_k F\left(x,\overline{x};\alpha(x,\overline{x})\nabla l_k(\overline{x})\right)\\&+\sum_{k=1}^{q}\overline{v}_k\rho_3 d^2(x,\overline{x}).\end{aligned} \tag{31}$$

By the sublinearity of F, we obtain

$$F\left(x,\overline{x};\alpha(x,\overline{x})\sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x})\right)\leqq-\sum_{k=1}^{q}\overline{v}_k\rho_3 d^2(x,\overline{x}). \tag{32}$$

By the known conditions, we have

$$\begin{aligned}F\Bigg(x,\overline{x};\alpha(x,\overline{x})\Bigg(&\sum_{i=1}^{p}\overline{\lambda}_i\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right)+\sum_{j=1}^{m}\overline{u}_j\nabla h_j(\overline{x})\\&+\sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x})\Bigg)\Bigg)=0.\end{aligned} \tag{33}$$

By (28), (29), and (32) and by the sublinearity of F, we obtain

$$\begin{aligned}&F\Bigg(x,\overline{x};\alpha(x,\overline{x})\Bigg(\sum_{i=1}^{p}\overline{\lambda}_i\nabla\left(\frac{f(\overline{x})}{g(\overline{x})}\right)+\sum_{j=1}^{m}\overline{u}_j\nabla h_j(\overline{x})\\&\qquad+\sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x})\Bigg)\Bigg)\\&<-\left(\sum_{i=1}^{p}\overline{\lambda}_i\rho_1+\sum_{j=1}^{m}\overline{u}_j\rho_2+\sum_{k=1}^{q}\overline{v}_k\rho_3\right)d^2(x,\overline{x})\\&=-\rho d^2(x,\overline{x})\leqq 0\end{aligned} \tag{34}$$

which contradicts the fact of (33). Therefore, $\overline{x}$ is an efficient solution of (VFP). The proof is complete. □

Consider the dual problem of (VFP)

$$\begin{aligned}\max\quad&\left(\frac{f_1(y)}{g(y)}+u^T h(y)+v^T l(y),\ldots,\right.\\&\left.\frac{f_p(y)}{g(y)}+u^T h(y)+v^T l(y)\right)\\\text{s.t.}\quad&\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(y)}{g(y)}\right)+\sum_{j=1}^{m}u_j\nabla h_j(y)\\&+\sum_{k=1}^{q}v_k\nabla l_k(y)=0\\&\lambda\in\wedge^{++},\quad u\in R_+^m,\quad v\in R_+^q.\end{aligned} \tag{35}$$

Theorem 15. *(f_i/g) $(i = 1, 2, \ldots, p)$ is (F, α, ρ_1, d)-convex at y, h_j $(j = 1, 2, \ldots, m)$ is (F, α, ρ_2, d)-convex at y, l_k $(k = 1, 2, \ldots, q)$ is (F, α, ρ_3, d)-convex at y, and $\sum_{i=1}^{p} \lambda_i \rho_1 + \sum_{j=1}^{m} u_j \rho_2 + \sum_{k=1}^{q} v_k \rho_3 \geqq 0$, then*

$$\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)}{g(x)}\right)\geqq\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(y)}{g(y)}\right)+u^T h(y)+v^T l(y). \tag{36}$$

Proof. By the (F, α, ρ_1, d)-convex of f_i/g at y, we get

$$\begin{aligned}\frac{f_i(x)}{g(x)}-\frac{f_i(y)}{g(y)}\geqq&F\left(x,y;\alpha(x,y)\nabla\left(\frac{f_i(y)}{g(y)}\right)\right)\\&+\rho_1 d^2(x,y),\quad i=1,2,\ldots,p.\end{aligned} \tag{37}$$

By the (F, α, ρ_2, d)-convex of h_j at y, we get

$$\begin{aligned}h_j(x)-h_j(y)\geqq F\left(x,y;\alpha(x,y)\nabla h_j(y)\right)+\rho_2 d^2(x,y),\\j=1,2,\ldots,m.\end{aligned} \tag{38}$$

By the (F, α, ρ_3, d)-convex of l_k at y, we get

$$\begin{aligned}l_k(x)-l_k(y)\geqq F\left(x,y;\alpha(x,y)\nabla l_k(y)\right)+\rho_3 d^2(x,y),\\k=1,2,\ldots,q.\end{aligned} \tag{39}$$

Since $\lambda \in \wedge^{++}$, $u \in R_+^m$, $v \in R_+^q$, and by the previous three inequalities, we have that

$$\begin{aligned}&\left(\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)}{g(x)}\right)+u^Th(x)+v^Tl(x)\right)\\&-\left(\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(y)}{g(y)}\right)+u^Th(y)+v^Tl(y)\right)\\&\geqq\sum_{i=1}^{p}\lambda_iF\left(x,y;\alpha(x,y)\nabla\left(\frac{f_i(y)}{g(y)}\right)\right)\\&+\sum_{j=1}^{m}u_jF\left(x,y;\alpha(x,y)\nabla h_j(y)\right)\\&+\sum_{k=1}^{q}v_kF\left(x,y;\alpha(x,y)\nabla l_k(y)\right)\\&+\left(\sum_{i=1}^{p}\lambda_i\rho_1+\sum_{j=1}^{m}u_j\rho_2+\sum_{k=1}^{q}v_k\rho_3\right)d^2(x,y).\end{aligned}\tag{40}$$

By the sublinearity of F, we obtain

$$\begin{aligned}&\left(\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)}{g(x)}\right)+u^Th(x)+v^Tl(x)\right)\\&-\left(\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(y)}{g(y)}\right)+u^Th(y)+v^Tl(y)\right)\\&\geqq F\left(x,y;\alpha(x,y)\left(\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(y)}{g(y)}\right)+\sum_{j=1}^{m}u_j\nabla h_j(y)\right.\right.\\&\qquad\left.\left.+\sum_{k=1}^{q}v_k\nabla l_k(y)\right)\right)\\&+\left(\sum_{i=1}^{p}\lambda_i\rho_1+\sum_{j=1}^{m}u_j\rho_2+\sum_{k=1}^{q}v_k\rho_3\right)d^2(x,y).\end{aligned}\tag{41}$$

By the feasibility of (y,λ,u,v), we have

$$\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(y)}{g(y)}\right)+\sum_{j=1}^{m}u_j\nabla h_j(y)+\sum_{k=1}^{q}v_k\nabla l_k(y)=0.\tag{42}$$

Since $\sum_{i=1}^{p}\lambda_i\rho_1+\sum_{j=1}^{m}u_j\rho_2+\sum_{k=1}^{q}v_k\rho_3\geqq 0$ and by (41), we get

$$\begin{aligned}&\left(\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)}{g(x)}\right)+u^Th(x)+v^Tl(x)\right)\\&-\left(\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(y)}{g(y)}\right)+u^Th(y)+v^Tl(y)\right)\geqq 0.\end{aligned}\tag{43}$$

Since $u^Th(x)\leqq 0$, $v^Tl(x)=0$, we obtain

$$\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(x)}{g(x)}\right)\geqq\sum_{i=1}^{p}\lambda_i\left(\frac{f_i(y)}{g(y)}\right)+u^Th(y)+v^Tl(y).\tag{44}$$

The proof is complete. □

3.3. Nonlinear Multiobjective Fractional Programming Problem Involved Inequality and Equality Constraints under (F,ρ)-Convex. Consider the multiobjective fractional programming problem (MFP)

$$\begin{aligned}\min\quad&\left(\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\ldots,\frac{f_p(x)}{g_p(x)}\right)\\\text{s.t.}\quad&h_j(x)\leqq 0,\ j=1,2,\ldots,m,\ x\in X_0,\\&l_k(x)=0,\ k=1,2,\ldots,q,\ x\in X_0,\end{aligned}\tag{45}$$

where X_0 is an open set of R^n, $f_i(x)$ $(i=1,2,\ldots,p):X_0\to R$, $f_i(x)\geqq 0$, $g_i(x)$ $(i=1,2,\ldots,p):X_0\to R$, $g_i(x)>0$, and $h_j(x)$ $(j=1,2,\ldots,m):X_0\to R$, $l_k(x)$ $(k=1,2,\ldots,q):X_0\to R$ are continuously differentiable over X_0.

Denote by G the set of all feasible solutions for (MFP); that is,

$$\begin{aligned}G=\{&x\in X_0\mid h_j(x)\leqq 0,\ j=1,2,\ldots,m;\\&l_k(x)=0,\ k=1,2,\ldots,q\}\end{aligned}\tag{46}$$

and let $\varphi_i(x)=f_i(x)/g_i(x)$, $\varphi(x)=(\varphi_1(x),\varphi_2(x),\ldots,\varphi_p(x))$.

Theorem 16. *Assume that there exists $(\overline{x},\overline{\alpha},\overline{\lambda},\overline{v})$ and $\overline{\alpha}=(\overline{\alpha}_1,\overline{\alpha}_2,\ldots,\overline{\alpha}_p)\in R_+^p$, $\overline{\lambda}=(\overline{\lambda}_1,\overline{\lambda}_2,\ldots,\overline{\lambda}_m)$, $\overline{v}=(\overline{v}_1,\overline{v}_2,\ldots,\overline{v}_q)$ such that*

(i) $\sum_{i=1}^{p}\overline{\alpha}_iT_i(\overline{x})+\sum_{j=1}^{m}\overline{\lambda}_j\nabla h_j(\overline{x})+\sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x})=0$, $\overline{\lambda}_jh_j(\overline{x})=0$, $j=1,2,\ldots,m$, $h_j(\overline{x})\leqq 0$, $j=1,2,\ldots,m$, $l_k(\overline{x})=0$, $k=1,2,\ldots,q$, *where* $T_i(\overline{x})=(1/g_i(\overline{x}))[\nabla f_i(\overline{x})-\varphi_i(\overline{x})\nabla g_i(\overline{x})]$;

(ii) *f_i and $-g_i$ $(i=1,2,\ldots,p)$ are (F,ρ)-convex at $\overline{x}$, and $\rho>0$;*

(iii) *h_j are (F,ρ)-convex at $\overline{x}$ for all j, $j=1,2,\ldots,m$, and $\rho>0$;*

(iv) *l_k are (F,ρ)-convex at $\overline{x}$ for all k, $k=1,2,\ldots,q$, and $\rho>0$.*

Then $\overline{x}$ is a Pareto optimality solution of (MFP).

Proof. Suppose that $\overline{x}$ is not a Pareto optimality solution of (MFP); then there exists a feasible solution $x\in G$ such that $f_i(x)/g_i(x)\leqq f_i(\overline{x})/g_i(\overline{x})$, $i=1,2,\ldots,p$, that is, $f_i(x)-(f_i(\overline{x})g_i(x))/g_i(\overline{x})\leqq 0$, that is, $f_i(x)-(f_i(\overline{x})/g_i(\overline{x}))g_i(x)\leqq f_i(\overline{x})-(f_i(\overline{x})/g_i(\overline{x}))g_i(\overline{x})$; it follows that

$$f_i(x)-\varphi_i(\overline{x})g_i(x)\leqq f_i(\overline{x})-\varphi_i(\overline{x})g_i(\overline{x}).\tag{47}$$

By the (F,ρ)-convexity of f_i and $-g_i$, $i=1,2,\ldots,p$, we have

$$f_i(x)-f_i(\overline{x})\geqq F(x,\overline{x};\nabla f_i(\overline{x}))+\rho d^2(x,\overline{x}),\quad \forall x\in X_0,$$
$$-g_i(x)+g_i(\overline{x})\geqq F(x,\overline{x};-\nabla g_i(\overline{x}))+\rho d^2(x,\overline{x}),\quad \forall x\in X_0. \tag{48}$$

Using the conditions $f_i(\overline{x})\geqq 0$, $g_i(\overline{x})>0$, we see that $\varphi_i(\overline{x})=f_i(\overline{x})/g_i(\overline{x})\geqq 0$.

By the properties of the sublinear functional F, we obtain

$$\begin{aligned}&-\varphi_i(\overline{x})g_i(x)+\varphi_i(\overline{x})g_i(\overline{x})\\&\quad\geqq \varphi_i(\overline{x})F(x,\overline{x};-\nabla g_i(\overline{x}))+\varphi_i(\overline{x})\rho d^2(x,\overline{x})\\&\quad= F(x,\overline{x};-\varphi_i(\overline{x})\nabla g_i(\overline{x}))+\varphi_i(\overline{x})\rho d^2(x,\overline{x}).\end{aligned} \tag{49}$$

By (48) and (49) and based on the sublinearity of F, we have

$$\begin{aligned}&f_i(x)-\varphi_i(\overline{x})g_i(x)-[f_i(\overline{x})-\varphi_i(\overline{x})g_i(\overline{x})]\\&\quad\geqq F(x,\overline{x};\nabla f_i(\overline{x})-\varphi_i(\overline{x})\nabla g_i(\overline{x}))\\&\qquad+[1+\varphi_i(\overline{x})]\rho d^2(x,\overline{x}).\end{aligned} \tag{50}$$

By (47), we have

$$F(x,\overline{x};\nabla f_i(\overline{x})-\varphi_i(\overline{x})\nabla g_i(\overline{x}))+[1+\varphi_i(\overline{x})]\rho d^2(x,\overline{x})\leqq 0. \tag{51}$$

If we sum up after multiplying by $\overline{\alpha}_i(1/g_i(\overline{x}))\geqq 0$ $(i=1,2,\ldots,p)$ in the above inequality and by using the sublinearity of F, we have

$$\begin{aligned}&F\left(x,\overline{x};\sum_{i=1}^{p}\overline{\alpha}_i\left(\frac{1}{g_i(\overline{x})}\right)[\nabla f_i(\overline{x})-\varphi_i(\overline{x})\nabla g_i(\overline{x})]\right)\\&\quad+\sum_{i=1}^{p}\overline{\alpha}_i\left(\frac{1}{g_i(\overline{x})}\right)[1+\varphi_i(\overline{x})]\rho d^2(x,\overline{x})\leqq 0.\end{aligned} \tag{52}$$

Since $T_i(\overline{x})=(1/g_i(\overline{x}))[\nabla f_i(\overline{x})-\varphi_i(\overline{x})\nabla g_i(\overline{x})]$, we get

$$\begin{aligned}&F\left(x,\overline{x};\sum_{i=1}^{p}\overline{\alpha}_iT_i(\overline{x})\right)\\&\quad+\sum_{i=1}^{p}\overline{\alpha}_i\left(\frac{1}{g_i(\overline{x})}\right)[1+\varphi_i(\overline{x})]\rho d^2(x,\overline{x})\leqq 0.\end{aligned} \tag{53}$$

On the other hand, for $j=1,2,\ldots,m$, by the (F,ρ)-convexity of h_j at $\overline{x}$, we have

$$h_j(x)-h_j(\overline{x})\geqq F(x,\overline{x};\nabla h_j(\overline{x}))+\rho d^2(x,\overline{x}),\quad \forall x\in X_0. \tag{54}$$

On multiplying the inequality (54) by $\overline{\lambda}_j\geq 0$ and using the sublinearity of F, we have

$$\overline{\lambda}_jh_j(x)-\overline{\lambda}_jh_j(\overline{x})\geqq F(x,\overline{x};\overline{\lambda}_j\nabla h_j(\overline{x}))+\overline{\lambda}_j\rho d^2(x,\overline{x}), \tag{55}$$

which together with $\overline{\lambda}_jh_j(x)\leqq 0$ and $\overline{\lambda}_jh_j(\overline{x})=0$ yields

$$F(x,\overline{x};\overline{\lambda}_j\nabla h_j(\overline{x}))+\overline{\lambda}_j\rho d^2(x,\overline{x})\leqq 0. \tag{56}$$

By accumulating the inequality (56) with j, we have

$$\sum_{j=1}^{m}F(x,\overline{x};\overline{\lambda}_j\nabla h_j(\overline{x}))+\sum_{j=1}^{m}\overline{\lambda}_j\rho d^2(x,\overline{x})\leqq 0, \tag{57}$$

that is,

$$F\left(x,\overline{x};\sum_{j=1}^{m}\overline{\lambda}_j\nabla h_j(\overline{x})\right)+\sum_{j=1}^{m}\overline{\lambda}_j\rho d^2(x,\overline{x})\leqq 0. \tag{58}$$

For $k=1,2,\ldots,q$, by the (F,ρ)-convexity of l_k at $\overline{x}$, we have that

$$l_k(x)-l_k(\overline{x})\geqq F(x,\overline{x};\nabla l_k(\overline{x}))+\rho d^2(x,\overline{x}),\quad \forall x\in X_0. \tag{59}$$

On multiplying the inequality (59) with $\overline{v}_k\geqq 0$, we get

$$\overline{v}_kl_k(x)-\overline{v}_kl_k(\overline{x})\geqq F(x,\overline{x};\overline{v}_k\nabla l_k(\overline{x}))+\overline{v}_k\rho d^2(x,\overline{x}) \tag{60}$$

which together with $\overline{v}_kl_k(x)=0$ and $\overline{v}_kl_k(\overline{x})=0$ yields

$$F(x,\overline{x};\overline{v}_k\nabla l_k(\overline{x}))+\overline{v}_k\rho d^2(x,\overline{x})\leqq 0. \tag{61}$$

By accumulating the inequality (61) with k, we have

$$\sum_{k=1}^{q}F(x,\overline{x};\overline{v}_k\nabla l_k(\overline{x}))+\sum_{k=1}^{q}\overline{v}_k\rho d^2(x,\overline{x})\leqq 0. \tag{62}$$

The inequality (62) along with the sublinearity of F implies

$$F\left(x,\overline{x};\sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x})\right)+\sum_{k=1}^{q}\overline{v}_k\rho d^2(x,\overline{x})\leqq 0. \tag{63}$$

The sublinearity of F, (53), (58), and (63) yields

$$\begin{aligned}&F\left(x,\overline{x};\sum_{i=1}^{p}\overline{\alpha}_iT_i(\overline{x})+\sum_{j=1}^{m}\overline{\lambda}_j\nabla h_j(\overline{x})+\sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x})\right)\\&\quad+\sum_{i=1}^{p}\overline{\alpha}_i\left(\frac{1}{g_i(\overline{x})}\right)[1+\varphi_i(\overline{x})]\rho d^2(x,\overline{x})\\&\quad+\sum_{j=1}^{m}\overline{\lambda}_j\rho d^2(x,\overline{x})+\sum_{k=1}^{q}\overline{v}_k\rho d^2(x,\overline{x})\leqq 0.\end{aligned} \tag{64}$$

According to the assumption and the sublinearity of F, we obtain

$$\begin{aligned}&F\left(x,\overline{x};\sum_{i=1}^{p}\overline{\alpha}_iT_i(\overline{x})+\sum_{j=1}^{m}\overline{\lambda}_j\nabla h_j(\overline{x})+\sum_{k=1}^{q}\overline{v}_k\nabla l_k(\overline{x})\right)\\&\quad+\sum_{i=1}^{p}\overline{\alpha}_i\left(\frac{1}{g_i(\overline{x})}\right)[1+\varphi_i(\overline{x})]\rho d^2(x,\overline{x})+\sum_{j=1}^{m}\overline{\lambda}_j\rho d^2(x,\overline{x})\\&\quad+\sum_{k=1}^{q}\overline{v}_k\rho d^2(x,\overline{x})>0,\end{aligned} \tag{65}$$

which contradicts (64) obviously.

Therefore, $\overline{x}$ is a Pareto optimality solution of (MFP). The proof is complete. □

Consider the dual problem of (MFP)

$$\begin{aligned}
\max \quad & \left(\frac{f_1(y)}{g_1(y)} + u^T h(y) + v^T l(y), \ldots, \frac{f_p(y)}{g_p(y)} + u^T h(y) + v^T l(y)\right) \\
\text{s.t.} \quad & \sum_{i=1}^{p} \lambda_i \nabla\left(\frac{f_i(y)}{g_i(y)}\right) + \sum_{j=1}^{m} u_j \nabla h_j(y) + \sum_{k=1}^{q} v_k \nabla l_k(y) = 0, \\
& u^T h(y) \geqq 0, \\
& v^T l(y) = 0, \\
& u \geqq 0, \quad v \geqq 0.
\end{aligned} \tag{66}$$

Theorem 17. *$f_i(y)/g_i(y)$ $(i = 1, 2, \ldots, p)$ is (F, ρ)-convex at y, $h_j(y)$ $(j = 1, 2, \ldots, m)$ is (F, ρ)-convex at y, $l_k(y)$ $(k = 1, 2, \ldots, q)$ is (F, ρ)-convex at y, and $\lambda \in \wedge^{++}$, $u \in R_+^m$, $v \in R_+^q$, $\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} u_j \rho_j + \sum_{k=1}^{q} v_k \rho_k \geqq 0$, then $\lambda^T(f_i(x)/g_i(x)) \geqq \lambda^T(f_i(y)/g_i(y))$.*

Proof. By the (F, ρ)-convexity of $f_i(y)/g_i(y)$, $h_j(y)$, and $l_k(y)$, the sublinearity of F, and since $\lambda \in \wedge^{++}$, $u \in R_+^m$, $v \in R_+^q$, we have that

$$\begin{aligned}
& \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(x)}{g_i(x)}\right) - \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(y)}{g_i(y)}\right) \\
& \quad \geqq F\left(x, y; \sum_{i=1}^{p} \lambda_i \nabla\left(\frac{f_i(y)}{g_i(y)}\right)\right) + \sum_{i=1}^{p} \lambda_i \rho_i d^2(x, y), \\
& \sum_{j=1}^{m} u_j h_j(x) - \sum_{j=1}^{m} u_j h_j(y) \\
& \quad \geqq F\left(x, y; \sum_{j=1}^{m} u_j \nabla h_j(y)\right) + \sum_{j=1}^{m} u_j \rho_j d^2(x, y), \\
& \sum_{k=1}^{q} v_k l_k(x) - \sum_{k=1}^{q} v_k l_k(y) \\
& \quad \geqq F\left(x, y; \sum_{k=1}^{q} v_k \nabla l_k(y)\right) + \sum_{k=1}^{q} v_k \rho_k d^2(x, y).
\end{aligned} \tag{67}$$

By (67) and based on the sublinearity of F, we have

$$\begin{aligned}
& \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(x)}{g_i(x)}\right) - \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(y)}{g_i(y)}\right) + \sum_{j=1}^{m} u_j h_j(x) \\
& - \sum_{j=1}^{m} u_j h_j(y) + \sum_{k=1}^{q} v_k l_k(x) - \sum_{k=1}^{q} v_k l_k(y) \\
& \quad \geqq F\left(x, y; \left(\sum_{i=1}^{p} \lambda_i \nabla\left(\frac{f_i(y)}{g_i(y)}\right) + \sum_{j=1}^{m} u_j \nabla h_j(y) + \sum_{k=1}^{q} v_k \nabla l_k(y)\right)\right) \\
& \quad + \left(\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} u_j \rho_j + \sum_{k=1}^{q} v_k \rho_k\right) d^2(x, y).
\end{aligned} \tag{68}$$

Since $\sum_{i=1}^{p} \lambda_i \nabla(f_i(y)/g_i(y)) + \sum_{j=1}^{m} u_j \nabla h_j(y) + \sum_{k=1}^{q} v_k \nabla l_k(y) = 0$, we have

$$\begin{aligned}
& \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(x)}{g_i(x)}\right) - \sum_{i=1}^{p} \lambda_i \left(\frac{f_i(y)}{g_i(y)}\right) \\
& \quad \geqq \sum_{j=1}^{m} u_j h_j(y) - \sum_{j=1}^{m} u_j h_j(x) + \sum_{k=1}^{q} v_k l_k(y) - \sum_{k=1}^{q} v_k l_k(x) \\
& \quad + \left(\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} u_j \rho_j + \sum_{k=1}^{q} v_k \rho_k\right) d^2(x, y).
\end{aligned} \tag{69}$$

Since $h_j(x) \leqq 0$, $(j = 1, 2, \ldots, m)$, $l_k(x) = 0$, $(k = 1, 2, \ldots, q)$, $u^T h(y) \geqq 0$, $v^T l(y) = 0$, $\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} u_j \rho_j + \sum_{k=1}^{q} v_k \rho_k \geqq 0$, we have $\sum_{i=1}^{p} \lambda_i (f_i(x)/g_i(x)) \geqq \sum_{i=1}^{p} \lambda_i (f_i(y)/g_i(y))$. That is, $\lambda^T(f_i(x)/g_i(x)) \geqq \lambda^T(f_i(y)/g_i(y))$. □

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