Research Article

Optimality Conditions and Duality of Three Kinds of Nonlinear Fractional Programming Problems

Xiaomin Zhang and Zezhong Wu

Department of Mathematics, Chengdu University of Information Technology, Sichuan 610225, China

Correspondence should be addressed to Xiaomin Zhang; zhangxiaomin228@163.com

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Some assumptions for the objective functions and constraint functions are given under the conditions of convex and generalized convex, which are based on the \( F \)-convex, \( \rho \)-convex, and \((F, \rho)\)-convex. The sufficiency of Kuhn-Tucker optimality conditions and appropriate duality results are proved involving \((F, \rho)\)-convex, \((F, \alpha, \rho, d)\)-convex, and generalized \((F, \alpha, \rho, d)\)-convex functions.

1. Introduction

Multiobjective optimization theory is a development of numerical optimization and related to many subjects, such as nonsmooth analysis, convex analysis, nonlinear analysis, and the theory of set value. It has a wide range of applications in the fields of industrial design, economics, engineering, military, management sciences, financial investment, transport, and so forth, and now it is an interdisciplinary science branch between applied mathematics and decision sciences. Convexity plays an important role in optimization theory, and it becomes an important theoretical basis and useful tool for mathematical programming and optimization theory.

Convex function theory can be traced back to the works of Holder, Jensen, and Minkowski in the beginning of this century, but the real work that caught the attention of people is the research on game theory and mathematical programming by von Neumann and Morgenstern [1], Dantzing, and Kuhn and Tucker in the forties to fifties, and people have done a lot of intensive research about convex functions from the fifties to sixties. In the middle of the sixties convex analysis was produced, and the concept of convex function is promoted in a variety of ways, and the notion of generalized convex is given.

Fractional programming has an important significance in the optimization problems; for instance, in order to measure the production or the efficiency of a system, we should minimize a ratio of functions between a given period of time and a utilized resource in engineering and economics.

Preda [2] has established the concept of \((F, \rho)\)-convex based on \( F \)-convex [3] and \( \rho \)-convex [4] and obtained some results, which are the expansion of \( F \)-convex and \( \rho \)-convex. Motivated by various concepts of convexity, Liang et al. [5] have put forward a generalized convexity, which was called \((F, \alpha, \rho, d)\)-convex, which extended \((F, \rho)\)-convex, and Liang et al. [6], Weir and Mond [7], Weir [8], Jeyakumar and Mond [9], Egudo [10], Preda [2], and Gulati and Islam [3] obtained some corresponding optimality conditions and applied these optimality conditions to define dual problems and derived duality theorems for single objective fractional problems and multiobjective problems. Then the definition of generalized \((F, \alpha, \rho, d)\)-convex is given under the condition of \((F, \alpha, \rho, d)\)-convex. However, in general, fractional programming problems are nonconvex and the Kuhn-Tucker optimality conditions are only necessary. Under what conditions are the Kuhn-Tucker conditions sufficient for the optimality of problems? This question appeals to the interests of many researchers, and those are what we should probe. Based on the former conclusions, by adding conditions to objective functions and constraint functions and by changing K-T conditions [11], the optimality conditions and dual are given involving weaker convexity conditions. The main results in this paper are based on convex and generalized convex functions and the properties of sublinear functions.
In this paper, we will discuss sufficient optimality conditions and dual problems for three kinds of nonlinear fractional programming problems, and the paper is organized as follows.

In Sections 3.1 and 3.2, we present the Kuhn-Tucker sufficient optimality conditions and dual for nonlinear fractional programming problem and multiobjective fractional programming problem based on generalized \((F, \alpha, \rho, d)\)-convex. Section 3.3 contains optimality conditions and dual for multiobjective fractional programming problem under \((F, \rho)\)-convex. In these sections, I present some assumptions for the objective functions and constraint functions such that the Kuhn-Tucker optimality conditions are sufficient and obtain the corresponding duality theorem.

### 2. Preliminaries

Let \(E^n\) be the \(n\)-dimensional real vector space, that is, \(n\)-dimensional Euclidean space, where \(y = (y_1, y_2, \ldots, y_n)^T \in E^n\), and provides as follows, (see [12]).

\[
y = z \iff y_i = z_i, \quad i = 1, 2, \ldots, n; \\
y > z \iff y_i > z_i, \quad i = 1, 2, \ldots, n; \\
y \geq z \iff y_i \geq z_i, \quad i = 1, 2, \ldots, n.
\]

**Definition 1** (see [12]). Suppose that \(x_0 \in X_0\); that is, if \(x \notin X_0\), such that \(f(x) \leq f(x_0)\), \(x_0\) is an efficient solution of multiobjective programming problem.

**Definition 2** (see [12]). Suppose that \(x_0 \in X_0\); that is, if \(x \notin X_0\), such that \(f(x) < f(x_0)\), \(x_0\) is a weakly efficient solution of multiobjective programming problem.

**Definition 3** (see [5]). Given an open set \(X_0 \subset \mathbb{R}^n\), a functional \(F: X_0 \times X_0 \times \mathbb{R}^n \rightarrow \mathbb{R}\) is called sublinear if, for any \(x, x_0 \in X_0\),

\[
F(x, x_0; a_1 + a_2) \leq F(x, x_0; a_1) + F(x, x_0; a_2), \\
\forall a_1, a_2 \in \mathbb{R}^n, \\
F(x, x_0; a a) = aF(x, x_0; a), \quad \forall a \in \mathbb{R}, \ \alpha \geq 0, \ \forall a \in \mathbb{R}^n.
\]  

(2)

It follows from the second equality that

\[
F(x, x_0; 0) = F(x, x_0; 0 \times a) = 0 \times F(x, x_0; a) = 0, \\
\text{for any } a \in \mathbb{R}^n.
\]  

(3)

Let \(F: X_0 \times X_0 \times \mathbb{R}^n \rightarrow \mathbb{R}\) be a sublinear function, and let \(\alpha: X_0 \times X_0 \rightarrow \mathbb{R}, \ \rho = (\rho_1, \rho_2, \ldots, \rho_n)^T, \ \rho \in \mathbb{R}, \ d: X_0 \times X_0 \rightarrow \mathbb{R}, \ \underline{d} = (\underline{d}_1, \underline{d}_2, \ldots, \underline{d}_n)^T \in \mathbb{R}^n\), the function \(f = (f_1, f_2, \ldots, f_m): X_0 \rightarrow \mathbb{R}^n\) is differentiable at \(x_0 \in X_0\).

**Definition 4** (see [3]). Let \(\phi(x)\) be a differentiable function defined on \(X_0 \subset \mathbb{R}^n\). The function \(\phi(x)\) is said to be \(F\)-convex on \(X_0\) with respect to \(F\), if \(\phi(x) - \phi(y) \geq F_{x,y}[\nabla \phi(y)]\).

**Definition 5** (see [4, 12]). Let \(f(x)\) be a real-valued function defined on the convex set \(X_0 \subset E^n\), if there exists a real number \(\rho \in \mathbb{R}\), such that

\[
f(\lambda x^1 + (1 - \lambda) x^2) \leq \lambda f(x^1) + (1 - \lambda) f(x^2) - \rho \lambda (1 - \lambda) \|x^1 - x^2\|^2,
\]  

(4)

for any \(x^1, x^2 \in X_0\) and any \(\lambda \in [0, 1]\), then the function \(f(x)\) is said to be \(\rho\)-convex on \(X_0\).

Especially, if \(\rho = 0\), then we obtain the definition of convex.

If \(\rho > 0\) (or \(\rho < 0\)) in the above definition, then we have strong convex (or weak convex).

**Definition 6** (see [2]). The function \(f_i: X_0 \rightarrow \mathbb{R}\) is said to be \((F, \rho)\)-convex at \(x_0 \in X_0\), if for any \(x_0 \in X_0, f_i(x)\) satisfies the following condition:

\[
f_i(x) - f_i(x_0) \geq F(x, x_0; \nabla f_i(x_0)) + \rho d^2(x, x_0).
\]  

(5)

**Definition 7** (see [5]). The function \(f_i\) is said to be \((F, \alpha, \rho, d)\)-convex at \(x_0 \in X_0\), if

\[
f_i(x) - f_i(x_0) \geq F(x, x_0; \alpha(x, x_0) \nabla f_i(x_0)) + \rho d^2(x, x_0), \\
\forall x \in X_0.
\]  

(6)

The function \(f\) is said to be \((F, \alpha, \rho, d)\)-convex at \(x_0\), if each component \(f_i\) of \(f\) is \((F, \alpha, \rho, d)\)-convex at \(x_0\).

The function \(f\) is said to be \((F, \alpha, \rho, d)\)-convex on \(X_0\), if it is \((F, \alpha, \rho, d)\)-convex at every point in \(X_0\).

**Definition 8**. The function \(f_i\) is said to be \((F, \alpha, \rho, d)\)-quasi-convex at \(x_0\), if \(f_i(x) \leq f_i(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0) \nabla f_i(x_0)) \leq -\rho d^2(x, x_0)\).

The function \(f\) is said to be \((F, \alpha, \rho, d)\)-quasiconvex at \(x_0\), if each component \(f_i\) of \(f\) is \((F, \alpha, \rho, d)\)-quasiconvex at \(x_0\).

**Definition 9**. The function \(f_i\) is said to be \((F, \alpha, \rho, d)\)-pseudoconvex at \(x_0\), if for all \(x \in X_0, f_i(x) < f_i(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0) \nabla f_i(x_0)) < -\rho d^2(x, x_0)\).

The function \(f\) is said to be \((F, \alpha, \rho, d)\)-pseudoconvex at \(x_0\), if each component \(f_i\) of \(f\) is \((F, \alpha, \rho, d)\)-pseudoconvex at \(x_0\).

**Definition 10**. The function \(f\) is said to be strictly \((F, \alpha, \rho, d)\)-pseudoconvex at \(x_0 \in X_0\), if \(f(x) - f(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0) \nabla f(x_0)) < -\rho d^2(x, x_0)\), where \(\nabla f(x_0; \alpha(x, x_0) \nabla f(x_0)) = (F(x, x_0; \alpha(x, x_0) \nabla f_1(x_0)), \ldots, F(x, x_0; \alpha(x, x_0) \nabla f_m(x_0)))\).

Further, \(f\) is said to be weakly strictly \((F, \alpha, \rho, d)\)-pseudoconvex at \(x_0 \in X_0\), if \(f(x) - f(x_0) \Rightarrow F(x, x_0; \alpha(x, x_0) \nabla f(x_0)) < -\rho d^2(x, x_0)\).

In order to prove our main result, we need a lemma which we present in this section.
Lemma 11 (see [13]). Suppose that differentiable real-valued functions \( h_j(x) \) \((j = 1, 2, \ldots, m)\) are \((F, \alpha, \rho, d)\)-quasiconvex at \( \bar{x} \in S \); then \( V^T h(x) \) is \((F, \alpha, \sum_{j=1}^m \nu_j \rho_j, d)\)-quasiconvex at \( x \in S \), where \( \rho_j \in \mathbb{R}, V \geq 0 \) and \( V^T \) denote the transpose of the \( m \)-dimensional column vector \( V \); that is, \( V^T = (v_1, v_2, \ldots, v_m) \).

3. Optimality Conditions and Duality

3.1. Nonlinear Fractional Programming Problem Involved Inequality and Equality Constraints Based on Generalized \((F,\alpha,\varphi,d)\)-Convex. Consider the nonlinear fractional programming problem (FP)

\[
\begin{align*}
\min & \quad \frac{f(x)}{g(x)} \\
\text{s.t.} & \quad h(x) \leq 0, \quad l(x) = 0, \quad x \in X_0,
\end{align*}
\]

where \( X_0 \) is an open set of \( \mathbb{R}^n \), \( f(x) \) and \( g(x) \) are real-valued functions defined on \( X_0 \), \( h(x) \) is an \( m \)-dimensional vector-valued function defined also on \( X_0 \), and \( l(x) \) a \( q \)-dimensional vector-valued function.

Let

\[
S = \{x \in X_0 | h(x) \leq 0, l(x) = 0\}
\]

denotes the set of all feasible solutions for (FP) and assume that \( f(x), g(x), h_j(x) (j = 1, 2, \ldots, m) \) and \( l_i(x) (i = 1, 2, \ldots, q) \) are continuously differentiable over \( X_0 \) and that \( f(x) \geq 0, g(x) > 0 \), for all \( x \in X_0 \).

If \( \bar{x} \in X_0 \) is a solution for problem (FP) and if a constraint qualification [14] holds, then the Kuhn-Tucker necessary conditions are given below: there exists \( V_0 \in \mathbb{R}^m \) and \( W_0 \in \mathbb{R}^q \) such that

\[
\begin{align*}
\nabla \left( \frac{f(\bar{x})}{g(\bar{x})} \right) + \nabla h(\bar{x}) V_0 + \nabla l(\bar{x}) W_0 &= 0, \\
V_0^T h(\bar{x}) &= 0, \\
V_0 &\geq 0, \\
l(\bar{x}) &\leq 0.
\end{align*}
\]

Theorem 12. Suppose that \( \bar{x} \) is a feasible solution of (FP), that the Kuhn-Tucker conditions hold at \( \bar{x} \), that \( f(x)/g(x) \) in problem (FP) is \((F, \alpha, \rho, d)\)-pseudoconvex on \( S \), \( h_j(x) \) \((j = 1, 2, \ldots, m)\) are \((F, \alpha, \rho, d)\)-quasiconvex on \( S \), and that \( l_i(x) \) \((i = 1, 2, \ldots, q)\) are \((F, \alpha, \rho, d)\)-quasiconvex over \( S \) and \( \rho \), \( \rho_1, \rho_2 \in \mathbb{R}, \rho \geq \rho_1, \rho + V_0^T \rho' + W_0^T \rho'' \geq 0 \), where \( V_0^T \rho' \) is the inner product about \( V_0 \) and \( \rho' \) and \( W_0^T \rho'' \) the inner product about \( W_0 \) and \( \rho'' \). Then, \( \bar{x} \) is an optimality solution for problem (FP).

Proof. Suppose that \( \bar{x} \) is not an optimality solution of (FP). Then, there exists a feasible solution \( x \in S \) such that \( f(x)/g(x) < f(\bar{x})/g(\bar{x}) \).

By the \((F, \alpha, \rho, d)\)-pseudoconvexity assumption of \( f(x)/g(x) \), we have

\[
F \left( x, \bar{x}, \alpha (x, \bar{x}) \nabla \left( \frac{f(\bar{x})}{g(\bar{x})} \right) \right) < -\rho d^2 (x, \bar{x}).
\]

For each \( j (j = 1, 2, \ldots, m) \), by the \((F, \alpha, \rho_j, d)\)-quasiconvexity assumption of \( h_j(x) \) and Lemma II, we have that \( V_0^T h(x) \) is \((F, \alpha, \sum_{j=1}^m \nu_j \rho_j, d)\)-quasiconvex on \( S \). Therefore,

\[
F(x, \bar{x}, \alpha (x, \bar{x}) \nabla h(\bar{x}) V_0) \leq -\sum_{j=1}^m \nu_j \rho_j d^2 (x, \bar{x}).
\]

is that, \( F(x, \bar{x}, \alpha (x, \bar{x}) \nabla h(\bar{x}) V_0) \leq -V_0^T \rho' d^2 (x, \bar{x}) \).

By the \((F, \alpha, \rho_j, d)\)-quasiconvexity of \( l_i(x) \) \((i = 1, 2, \ldots, q)\) over \( S \), we have that \( W_0^T l(x) \) is \((F, \alpha, \sum_{j=1}^q \nu_j \rho_j, d)\)-quasiconvex on \( S \). Then we obtain

\[
F(x, \bar{x}, \alpha(x, \bar{x}) \nabla l(\bar{x}) W_0) \leq -\sum_{i=1}^q \nu_i \omega_i d^2 (x, \bar{x}),
\]

is that, \( F(x, \bar{x}, \alpha(x, \bar{x}) \nabla l(\bar{x}) W_0) \leq -W_0^T \rho'' d^2 (x, \bar{x}) \).

 By (10), (11), and (12), and based on the sublinearity of \( F \), we have

\[
F \left( x, \bar{x}, \alpha (x, \bar{x}) \nabla (\frac{f(\bar{x})}{g(\bar{x})} + \nabla h(\bar{x}) V_0 + \nabla l(\bar{x}) W_0) \right) \leq - \left( \rho + V_0^T \rho' + W_0^T \rho'' \right) d^2 (x, \bar{x}).
\]

By (10), (11), and (12), and based on the sublinearity of \( F \), we have

\[
F \left( x, \bar{x}, \alpha (x, \bar{x}) \nabla (\frac{f(\bar{x})}{g(\bar{x})} + \nabla h(\bar{x}) V_0 + \nabla l(\bar{x}) W_0) \right) < 0.
\]

By the K-T conditions, we have \( V_l f(\bar{x})/g(\bar{x}) + \nabla h(\bar{x}) V_0 + \nabla l(\bar{x}) W_0 = 0 \).

Hence, based on the sublinearity of \( F \), we obtain

\[
F \left( x, \bar{x}, \alpha (x, \bar{x}) \nabla (\frac{f(\bar{x})}{g(\bar{x})} + \nabla h(\bar{x}) V_0 + \nabla l(\bar{x}) W_0) \right) = 0,
\]

which contradicts (14). The proof is complete. \( \square \)

Consider the dual problem of (FP):

\[
\begin{align*}
\max & \quad \frac{f(y)}{g(y)} + u^T h(y) + v^T l(y) \\
\text{s.t.} & \quad \lambda^T \nabla \left( \frac{f(y)}{g(y)} \right) + u^T \nabla h(y) + v^T \nabla l(y) = 0, \\
& \quad u^T h(y) \geq 0, \\
& \quad v^T l(y) = 0, \\
& \quad u \geq 0, \quad v \geq 0.
\end{align*}
\]

Theorem 13. Suppose that \( \lambda^T (f(y)/g(y)) \) is \((F, \alpha, \rho_1, d)\)-pseudoconvex at \( y, u^T h(y) + v^T l(y) \) is \((F, \alpha, \rho_2, d)\)-quasiconvex at \( y \) in problem (FP) and (FD), and that \( \rho_1 + \rho_2 \geq 0; \) then \( \lambda^T (f(x)/g(x)) \geq \lambda^T (f(y)/g(y)) \), for any feasible solution \( x \) of (FP) and \( (y, \lambda, u, v) \) of (FD).
Proof. Assume that the conclusion is not true; that is, \( \lambda^T \left( \frac{f(x)}{g(x)} \right) < \lambda^T \left( \frac{f(y)}{g(y)} \right) \).

By the \((F, \alpha, \rho, d)\)-pseudoconvex of \( \lambda^T \left( \frac{f(x)}{g(x)} \right) \) at \( y \), we get

\[
F \left( x, y; \alpha \left( x, y \right) \lambda^T \left( \frac{f(y)}{g(y)} \right) \right) < -\rho_1 d^2 \left( x, y \right). \tag{16}
\]

Using \( u^T h(x) \leq 0, v^T l(x) = 0, u^T h(y) \geq 0, v^T l(y) = 0 \), we have

\[
u^T h(x) + v^T l(x) \leq u^T h(y) + v^T l(y). \tag{17}
\]

By the \((F, \alpha, \rho, d)\)-quasiconvex of \( u^T h(y) + v^T l(y) \), we get

\[
F \left( x, y; \alpha \left( x, y \right) \left( u^T \nabla h(y) + v^T \nabla l(y) \right) \right) \leq -\rho_2 d^2 \left( x, y \right). \tag{18}
\]

By (16) and (18) and based on the sublinearity of \( F \), we have

\[
F \left( x, y; \alpha \left( x, y \right) \left( \lambda^T \left( \frac{f(y)}{g(y)} \right) + u^T \nabla h(y) + v^T \nabla l(y) \right) \right) < -\left( \rho_1 + \rho_2 \right) d^2 \left( x, y \right). \tag{19}
\]

Since \((y, \lambda, u, v)\) is the feasible solution of (FD), so

\[
\lambda^T \left( \frac{f(y)}{g(y)} \right) + u^T \nabla h(y) + v^T \nabla l(y) = 0. \tag{20}
\]

Hence,

\[
0 = F \left( x, y; \alpha \left( x, y \right) \left( \lambda^T \left( \frac{f(y)}{g(y)} \right) + u^T \nabla h(y) + v^T \nabla l(y) \right) \right) < -\left( \rho_1 + \rho_2 \right) d^2 \left( x, y \right), \tag{21}
\]

which contradicts the known condition \( \rho_1 + \rho_2 \geq 0 \). The proof is complete. \( \square \)

### 3.2. Nonlinear Multiobjective Fractional Programming Problem Involved Inequality and Equality Constraints Based on \((F, \alpha, \rho, d)\)-Convex and Generalized \((F, \alpha, \rho, d)\)-Convex

Consider the nonlinear multiobjective fractional programming problem (VFP)

\[
\begin{array}{ll}
\text{min} & M(x) = \frac{f(x)}{g(x)} = \left[ \frac{f_1(x)}{g(x)}, \frac{f_2(x)}{g(x)}, \ldots, \frac{f_p(x)}{g(x)} \right] \\
\text{s.t.} & h(x) \leq 0, \; x \in X_0 \\
& l(x) = 0, \; x \in X_0,
\end{array} \tag{22}
\]

where \( X_0 \) is an open set of \( \mathbb{R}^n \), \( f_i(x) (i = 1, 2, \ldots, p) \), \( g(x), h_i(x) : X_0 \to \mathbb{R} (j = 1, 2, \ldots, m) \) are real-valued functions defined on \( X_0 \), \( l_k(x) : X_0 \to \mathbb{R} (k = 1, 2, \ldots, q) \) are real-valued functions defined also on \( X_0 \).

Proof. Let \( S = \{ x \mid x \in X_0, h(x) \leq 0, l(x) = 0 \} \) denotes the set of all feasible solutions of (VFP) and assume that \( f_i(x) (i = 1, 2, \ldots, p) \), \( g(x), h_i(x) (j = 1, 2, \ldots, m) \), and \( l_k(x) (k = 1, 2, \ldots, q) \) are continuously differentiable over \( X_0 \) and that \( g(x) > 0 \) for all \( x \in X_0 \).

**Theorem 14.** Suppose that \( f(x)/g(x) \) is weakly strictly \((F, \alpha, \rho, d)\)-pseudoconvex at \( \bar{x} \in X_0 \), \( h_j(x) (j = 1, 2, \ldots, m) \) are \((F, \alpha, \rho, d)\)-quasiconvex with respect to \( x \in X_0 \), \( l_k(x) (k = 1, 2, \ldots, q) \) are \((F, \alpha, \rho, d)\)-convex with respect to \( x \in X_0 \), and that there exists \( \lambda^+ \) (or \( \lambda^+^+ \)), \( \bar{u} \in \mathbb{R}^m_+ \), \( \bar{v} \in \mathbb{R}^q_+ \) satisfying

\[
\begin{align*}
\sum_{i=1}^p \lambda_i \nabla \left( \frac{f_i(\bar{x})}{g(\bar{x})} \right) + \sum_{j=1}^m \bar{u}_j \nabla h_j(\bar{x}) + \sum_{k=1}^q \bar{v}_k \nabla l_k(\bar{x}) &= 0, \\
\bar{u}_j h_j(\bar{x}) &= 0, \quad j = 1, 2, \ldots, m, \\
l_k(\bar{x}) &= 0, \quad k = 1, 2, \ldots, q.
\end{align*}
\]

(24)

and \( \rho = \sum_{i=1}^p \lambda_i \rho_1 + \sum_{j=1}^m \bar{u}_j \rho_2 + \sum_{k=1}^q \bar{v}_k \rho_3 \geq 0 \). Then, \( \bar{x} \) is an efficient solution of (VFP), where

\[
\lambda^+ = \left\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T \mid \lambda_i \geq 0, i = 1, 2, \ldots, p \right\}, \tag{25}
\]

\[
\lambda^{++} = \left\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)^T \mid \lambda_i > 0, i = 1, 2, \ldots, p \right\}.
\]

**Proof.** Suppose that \( \bar{x} \) is not an efficient solution of (VFP); then there exists a feasible solution \( x \in X_0 \) such that \( M(x) \leq M(\bar{x}) \), that is, \( f(x)/g(x) \leq f(\bar{x})/g(\bar{x}) \).

By the weakly strict \((F, \alpha, \rho, d)\)-pseudoconvexity of \( f(x)/g(x) \) at \( \bar{x} \in X_0 \), we get

\[
F \left( x, \bar{x}; \lambda \left( x, \bar{x} \right) \nabla \left( \frac{f(x)}{g(x)} \right) \right) < -\rho_1 d^2 \left( x, \bar{x} \right). \tag{26}
\]

Using \( \lambda \in \lambda^+ \) (or \( \lambda^{++} \)), then we have

\[
\sum_{i=1}^p \lambda_i F \left( x, \bar{x}; \alpha \left( x, \bar{x} \right) \nabla \left( \frac{f(x)}{g(x)} \right) \right) < -\rho_1 d^2 \left( x, \bar{x} \right). \tag{27}
\]
Based on the sublinearity of \( F \), we obtain
\[
F \left( x, \alpha (x, x) \sum_{i=1}^{p} \lambda_i \nabla \left( \frac{f(x)}{g(x)} \right) \right) < - \sum_{i=1}^{p} \lambda_i p_i d^2 (x, x).
\] (28)

Since \( \overline{u}^T h(x) = 0 \), \( \overline{u} \geq 0 \) and \( h_j(x) \leq 0 \), we have \( \overline{u}^T h(x) - \overline{u}^T h(x) \leq 0 \); that is, \( \overline{u}^T h(x) \leq \overline{u}^T h(x) \).

Since \( h_j(x) \) (\( j = 1, 2, \ldots, m \)) are \( (F, \alpha, \rho_2, d) \)-quasiconvex at \( \overline{x} \in X_0 \), by Lemma II, we have \( \overline{u}^T h \) is \( (F, \alpha, \rho_2 \sum_{j=1}^{m} \overline{u}_j, d) \)-quasiconvex at \( \overline{x} \in X_0 \).

Hence, we obtain the following inequality:
\[
F \left( x, \alpha (x, x) \sum_{j=1}^{m} \overline{u}_j \nabla h_j (x) \right) \leq - \sum_{j=1}^{m} \overline{u}_j p_j d^2 (x, x).
\] (29)

By the \((F, \alpha, \rho_3, d)\)-convexity of \( l_k (x) \) at \( \overline{x} \in X_0 \), we have
\[
l_k (x) - l_k (\overline{x}) \geq F \left( x, \alpha (x, x) \sum_{k=1}^{q} \overline{v}_k \nabla l_k (x) \right) + \rho_3 d^2 (x, x) \geq 0. \] (30)

Since \( \overline{v} \geq 0 \), we have
\[
0 = \overline{v} (x) - \overline{v} (\overline{x}) \geq \sum_{k=1}^{q} \overline{v}_k F \left( x, \alpha (x, x) \sum_{k=1}^{q} \overline{v}_k \nabla l_k (x) \right) + \rho_3 d^2 (x, x).
\] (31)

By the sublinearity of \( F \), we obtain
\[
F \left( x, \alpha (x, x) \sum_{k=1}^{q} \overline{v}_k \nabla l_k (x) \right) \leq - \sum_{k=1}^{q} \overline{v}_k p_k d^2 (x, x). \] (32)

By the known conditions, we have
\[
F \left( x, \alpha (x, x) \left( \sum_{i=1}^{p} \lambda_i \nabla \left( \frac{f(x)}{g(x)} \right) + \sum_{j=1}^{m} \overline{u}_j \nabla h_j (x) \right) \right) = 0. \] (33)

By (28), (29), and (32) and by the sublinearity of \( F \), we obtain
\[
F \left( x, \alpha (x, x) \left( \sum_{i=1}^{p} \lambda_i \nabla \left( \frac{f(x)}{g(x)} \right) + \sum_{j=1}^{m} \overline{u}_j \nabla h_j (x) \right) \right) \leq \left( \sum_{i=1}^{p} \lambda_i p_i + \sum_{j=1}^{m} \overline{u}_j p_j + \sum_{k=1}^{q} \overline{v}_k p_k \right) d^2 (x, x) \leq 0
\] (34)

which contradicts the fact of (33). Therefore, \( \overline{x} \) is an efficient solution of (VFP). The proof is complete. □

Consider the dual problem of (VFP)
\[
\begin{align*}
\text{max} & \quad \left( \frac{f_i (y)}{g (y)} \right) + u^T h (y) + v^T l (y), \ldots, \\
\text{s.t.} & \quad \sum_{i=1}^{p} \lambda_i \nabla \left( \frac{f_i (y)}{g (y)} \right) + \sum_{j=1}^{m} u_j \nabla h_j (y) + \sum_{k=1}^{q} v_k \nabla l_k (y) = 0, \\
& \quad \lambda \in \Lambda^+, \quad u \in R^m, \quad v \in R^q.
\end{align*}
\] (35)

Theorem 15. \((f_i / g) (i = 1, 2, \ldots, p)\) is \((F, \alpha, \rho_1, d)\)-convex at \( y \), \( h_j (j = 1, 2, \ldots, m)\) is \((F, \alpha, \rho_2, d)\)-convex at \( y \), \( l_k (k = 1, 2, \ldots, q)\) is \((F, \alpha, \rho_3, d)\)-convex at \( y \), and \( \sum_{i=1}^{p} \lambda_i \rho_1 + \sum_{j=1}^{m} u_j \rho_2 + \sum_{k=1}^{q} v_k \rho_3 \geq 0 \), then
\[
\sum_{i=1}^{p} \lambda_i \left( \frac{f_i (x)}{g (x)} \right) \geq \sum_{i=1}^{p} \lambda_i \left( \frac{f_i (y)}{g (y)} \right) + u^T h (y) + v^T l (y).
\] (36)

Proof. By the \((F, \alpha, \rho_1, d)\)-convex of \( f_i / g \) at \( y \), we get
\[
\frac{f_i (x)}{g (x)} - \frac{f_i (y)}{g (y)} \geq F \left( x, y; \alpha (x, y) \nabla \left( \frac{f_i (y)}{g (y)} \right) \right) + \rho_1 d^2 (x, y), \quad i = 1, 2, \ldots, p.
\] (37)

By the \((F, \alpha, \rho_2, d)\)-convex of \( h_j \) at \( y \), we get
\[
h_j (x) - h_j (y) \geq F \left( x, y; \alpha (x, y) \nabla h_j (y) \right) + \rho_2 d^2 (x, y), \quad j = 1, 2, \ldots, m.
\] (38)

By the \((F, \alpha, \rho_3, d)\)-convex of \( l_k \) at \( y \), we get
\[
l_k (x) - l_k (y) \geq F \left( x, y; \alpha (x, y) \nabla l_k (y) \right) + \rho_3 d^2 (x, y), \quad k = 1, 2, \ldots, q.
\] (39)
Since $\lambda \in \Lambda^+$, $u \in R^n$, $v \in R^d$, and by the previous three inequalities, we have that
\[
\left( \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(x)}{g(x)} \right) + u^T h(x) + v^T l(x) \right) 
- \left( \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(y)}{g(y)} \right) + u^T h(y) + v^T l(y) \right) 
\geq \sum_{i=1}^{p} \lambda_i F \left( x, y; \alpha(x, y) \nabla \left( \frac{f_i(y)}{g(y)} \right) \right) 
\quad + \sum_{j=1}^{m} u_j \nabla h_j(y) 
\quad + \sum_{k=1}^{q} v_k \nabla l_k(y) 
\quad + \left( \sum_{i=1}^{p} \lambda_i \rho_1 + \sum_{j=1}^{m} u_j \rho_2 + \sum_{k=1}^{q} v_k \rho_3 \right) d^2(x, y). 
\]
\[
(40) 
\]
By the sublinearity of $F$, we obtain
\[
\left( \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(x)}{g(x)} \right) + u^T h(x) + v^T l(x) \right) 
- \left( \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(y)}{g(y)} \right) + u^T h(y) + v^T l(y) \right) 
\geq F \left( x, y; \alpha(x, y) \right) \left( \sum_{i=1}^{p} \lambda_i \nabla \left( \frac{f_i(y)}{g(y)} \right) \right) 
\quad + \sum_{j=1}^{m} u_j \nabla h_j(y) 
\quad + \sum_{k=1}^{q} v_k \nabla l_k(y) 
\quad + \left( \sum_{i=1}^{p} \lambda_i \rho_1 + \sum_{j=1}^{m} u_j \rho_2 + \sum_{k=1}^{q} v_k \rho_3 \right) d^2(x, y). 
\]
\[
(41) 
\]
By the feasibility of $(y, \lambda, u, v)$, we have
\[
\sum_{j=1}^{m} u_j \nabla h_j(y) + \sum_{k=1}^{q} v_k \nabla l_k(y) = 0. 
\]
\[
(42) 
\]
Since $\sum_{i=1}^{p} \lambda_i \rho_1 + \sum_{j=1}^{m} u_j \rho_2 + \sum_{k=1}^{q} v_k \rho_3 \geq 0$ and by (41), we get
\[
\left( \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(x)}{g(x)} \right) + u^T h(x) + v^T l(x) \right) 
- \left( \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(y)}{g(y)} \right) + u^T h(y) + v^T l(y) \right) 
\geq 0. 
\]
\[
(43) 
\]
Since $u^T h(x) \leq 0$, $v^T l(x) = 0$, we obtain
\[
\sum_{i=1}^{p} \lambda_i \left( \frac{f_i(x)}{g(x)} \right) \geq \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(y)}{g(y)} \right) + u^T h(y) + v^T l(y). 
\]
\[
(44) 
\]
The proof is complete. \hfill \Box

3.3. Nonlinear Multiobjective Fractional Programming Problem Involved Inequality and Equality Constraints under $(F, \rho)$-Convex. Consider the multiobjective fractional programming problem (MFP)
\[
\begin{align*}
\min & \quad \left( \frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right) \\
\text{s.t.} & \quad h_j(x) \leq 0, \quad j = 1, 2, \ldots, m, \quad x \in X_0, \\
& \quad l_k(x) = 0, \quad k = 1, 2, \ldots, q, \quad x \in X_0, 
\end{align*}
\]
where $X_0$ is an open set of $R^n$, $f_i(x) (i = 1, 2, \ldots, p) : X_0 \rightarrow R, f_i(x) \geq 0$, $g_i(x) (i = 1, 2, \ldots, p) : X_0 \rightarrow R, g_i(x) > 0$, and $h_j(x) (j = 1, 2, \ldots, m) : X_0 \rightarrow R, l_k(x) (k = 1, 2, \ldots, q)$ are continuously differentiable over $X_0$.

Denote by $G$ the set of all feasible solutions for (MFP); that is,
\[
G = \{ x \in X_0 \mid h_j(x) \leq 0, \quad j = 1, 2, \ldots, m; \}
\quad l_k(x) = 0, \quad k = 1, 2, \ldots, q \}
\]
and let $\phi_0(x) = f_j(x)/g_j(x)$, $\phi_0(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_p(x))$.

**Theorem 16.** Assume that there exists $(\bar{x}, \bar{x}, \bar{\lambda}, \bar{v})$ and $\bar{x} = (\bar{x}, \bar{x}_2, \ldots, \bar{x}_p) \in R^n$, $\bar{l} = (\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_m)$, $\bar{v} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_q)$ such that
\[
(ii) \quad f_j \text{ and } -g_j \text{ are } (F, \rho)-\text{convex at } \bar{x}, \text{ and } \rho > 0; \\
(iii) \quad h_j \text{ are } (F, \rho)-\text{convex at } \bar{x} \text{ for all } j, \quad j = 1, 2, \ldots, m, \text{ and } \rho > 0; \\
(iv) \quad l_k \text{ are } (F, \rho)-\text{convex at } \bar{x} \text{ for all } k, \quad k = 1, 2, \ldots, q, \text{ and } \rho > 0.
\]
Then $\bar{x}$ is a Pareto optimality solution of (MFP).

**Proof.** Suppose that $\bar{x}$ is not a Pareto optimality solution of (MFP); then there exists a feasible solution $x \in G$ such that $f_j(x)/g_j(x) \leq f_j(\bar{x})/g_j(\bar{x})$, $i = 1, 2, \ldots, p$, that is, $f_j(x) - (f_j(\bar{x})/g_j(\bar{x}))g_j(x) \leq 0$, that is, $f_j(x) - (f_j(\bar{x})/g_j(\bar{x}))g_j(x) \leq f_j(x) - (f_j(\bar{x})/g_j(\bar{x}))g_j(x)$; it follows that
\[
f_j(x) - \phi_j(x) g_j(x) \leq f_j(x) - \phi_j(x) g_j(x). \]
By the \((F, \rho)\)-convexity of \(f_i\) and \(-g_i, i = 1, 2, \ldots, p\), we have
\[
f_i(x) - f_i(\overline{x}) \geq F(x, \overline{x}; \nabla f_i(\overline{x})) + \rho d^2(x, \overline{x}), \quad \forall x \in X_o,
\]
\[
-g_i(x) + g_i(\overline{x}) \geq F(x, \overline{x}; -\nabla g_i(\overline{x})) + \rho d^2(x, \overline{x}), \quad \forall x \in X_o.
\] (48)

Using the conditions \(f_i(\overline{x}) \geq 0, g_i(\overline{x}) > 0\), we see that \(\varphi_i(\overline{x}) = f_i(\overline{x})/g_i(\overline{x}) \geq 0\).

By the properties of the sublinear functional \(F\), we obtain
\[
-\varphi_i(\overline{x}) g_i(x) + \varphi_i(\overline{x}) g_i(\overline{x}) \geq \varphi_i(\overline{x}) F(x, \overline{x}; -\nabla g_i(\overline{x})) + \varphi_i(\overline{x}) \rho d^2(x, \overline{x}) \geq F(x, \overline{x}; -\nabla g_i(\overline{x})) + \varphi_i(\overline{x}) \rho d^2(x, \overline{x}).
\] (49)

By (48) and (49) and based on the sublinearity of \(F\), we have
\[
f_i(x) - \varphi_i(\overline{x}) g_i(x) \geq F(x, \overline{x}; -\nabla f_i(\overline{x})) + \varphi_i(\overline{x}) \rho d^2(x, \overline{x}) \geq F(x, \overline{x}; -\nabla f_i(\overline{x})) + \varphi_i(\overline{x}) \rho d^2(x, \overline{x}).
\] (50)

By (47), we have
\[
F(x, \overline{x}; Vf_i(\overline{x}) - \varphi_i(\overline{x}) Vg_i(\overline{x})) + \rho d^2(x, \overline{x}) \leq 0.
\] (51)

If we sum up after multiplying by \(\overline{\alpha}_i(1/\overline{g}_i(\overline{x})) \geq 0\) \(i = 1, 2, \ldots, p\) in the above inequality and by using the sublinearity of \(F\), we have
\[
F\left(x, \overline{x}; \sum_{i=1}^{p} \overline{\alpha}_i \left(\frac{1}{\overline{g}_i(\overline{x})}\right) \left[\nabla f_i(\overline{x}) - \varphi_i(\overline{x}) \nabla g_i(\overline{x})\right]\right)
+ \sum_{i=1}^{p} \overline{\alpha}_i \left(\frac{1}{\overline{g}_i(\overline{x})}\right) [1 + \varphi_i(\overline{x})] \rho d^2(x, \overline{x}) \leq 0.
\] (52)

Since \(T_i(\overline{x}) = (1/\overline{g}_i(\overline{x}))(Vf_i(\overline{x}) - \varphi_i(\overline{x}) Vg_i(\overline{x}))\), we get
\[
F\left(x, \overline{x}; \sum_{i=1}^{p} \overline{\alpha}_i T_i(\overline{x})\right)
+ \sum_{i=1}^{p} \overline{\alpha}_i \left(\frac{1}{\overline{g}_i(\overline{x})}\right) [1 + \varphi_i(\overline{x})] \rho d^2(x, \overline{x}) \leq 0.
\] (53)

On the other hand, for \(j = 1, 2, \ldots, m\), by the \((F, \rho)\)-convexity of \(h_j\) at \(\overline{x}\), we have
\[
h_j(x) - h_j(\overline{x}) \geq F(x, \overline{x}; \nabla h_j(\overline{x})) + \rho d^2(x, \overline{x}), \quad \forall x \in X_o.
\] (54)

On multiplying the inequality (54) by \(\overline{\lambda}_j \geq 0\) and using the sublinearity of \(F\), we have
\[
\overline{\lambda}_j h_j(x) - \overline{\lambda}_j h_j(\overline{x}) \geq F\left(x, \overline{x}; \overline{\lambda}_j \nabla h_j(\overline{x})\right) + \overline{\lambda}_j \rho d^2(x, \overline{x}),
\] (55)

which together with \(\overline{\lambda}_j h_j(x) \leq 0\) and \(\overline{\lambda}_j h_j(\overline{x}) = 0\) yields
\[
F\left(x, \overline{x}; \overline{\lambda}_j \nabla h_j(\overline{x})\right) + \overline{\lambda}_j \rho d^2(x, \overline{x}) \leq 0.
\] (56)

By accumulating the inequality (56) with \(j\), we have
\[
\sum_{j=1}^{m} F\left(x, \overline{x}; \overline{\lambda}_j \nabla h_j(\overline{x})\right) + \sum_{j=1}^{m} \overline{\lambda}_j \rho d^2(x, \overline{x}) \leq 0,
\] (57)

that is,
\[
F\left(x, \overline{x}; \sum_{j=1}^{m} \overline{\lambda}_j \nabla h_j(\overline{x})\right) + \sum_{j=1}^{m} \overline{\lambda}_j \rho d^2(x, \overline{x}) \leq 0.
\] (58)

For \(k = 1, 2, \ldots, q\), by the \((F, \rho)\)-convexity of \(l_k\) at \(\overline{x}\), we have that
\[
l_k(x) - l_k(\overline{x}) \geq F(x, \overline{x}; Vl_k(\overline{x})) + \rho d^2(x, \overline{x}), \quad \forall x \in X_o.
\] (59)

On multiplying the inequality (59) with \(\overline{\nu}_k \geq 0\), we get
\[
\overline{\nu}_k l_k(x) - l_k(\overline{x}) \geq F(x, \overline{x}; \overline{\nu}_k Vl_k(\overline{x})) + \overline{\nu}_k \rho d^2(x, \overline{x})
\] (60)

which together with \(\overline{\nu}_k l_k(x) = 0\) and \(\overline{\nu}_k l_k(\overline{x}) = 0\) yields
\[
F(x, \overline{x}; \overline{\nu}_k Vl_k(\overline{x})) + \overline{\nu}_k \rho d^2(x, \overline{x}) \leq 0.
\] (61)

By accumulating the inequality (61) with \(k\), we have
\[
\sum_{k=1}^{q} F(x, \overline{x}; \overline{\nu}_k Vl_k(\overline{x})) + \sum_{k=1}^{q} \overline{\nu}_k \rho d^2(x, \overline{x}) \leq 0.
\] (62)

The inequality (62) along with the sublinearity of \(F\) implies
\[
F\left(x, \overline{x}; \sum_{k=1}^{q} \overline{\nu}_k Vl_k(\overline{x})\right) + \sum_{k=1}^{q} \overline{\nu}_k \rho d^2(x, \overline{x}) \leq 0.
\] (63)

The sublinearity of \(F\), (53), (58), and (63) yields
\[
F\left(x, \overline{x}; \sum_{i=1}^{p} \overline{\alpha}_i T_i(\overline{x})\right) + \sum_{j=1}^{m} \overline{\lambda}_j \nabla h_j(\overline{x}) + \sum_{k=1}^{q} \overline{\nu}_k Vl_k(\overline{x})
+ \sum_{i=1}^{p} \overline{\alpha}_i \left(\frac{1}{\overline{g}_i(\overline{x})}\right) [1 + \varphi_i(\overline{x})] \rho d^2(x, \overline{x})
+ \sum_{j=1}^{m} \overline{\lambda}_j \rho d^2(x, \overline{x}) + \sum_{k=1}^{q} \overline{\nu}_k \rho d^2(x, \overline{x}) \leq 0.
\] (64)

According to the assumption and the sublinearity of \(F\), we obtain
\[
F\left(x, \overline{x}; \sum_{i=1}^{p} \overline{\alpha}_i T_i(\overline{x})\right) + \sum_{j=1}^{m} \overline{\lambda}_j \nabla h_j(\overline{x}) + \sum_{k=1}^{q} \overline{\nu}_k Vl_k(\overline{x})
+ \sum_{i=1}^{p} \overline{\alpha}_i \left(\frac{1}{\overline{g}_i(\overline{x})}\right) [1 + \varphi_i(\overline{x})] \rho d^2(x, \overline{x})
+ \sum_{j=1}^{m} \overline{\lambda}_j \rho d^2(x, \overline{x}) + \sum_{k=1}^{q} \overline{\nu}_k \rho d^2(x, \overline{x}) > 0,
\] (65)

which contradicts (64) obviously.
Therefore, \( \bar{x} \) is a Pareto optimality solution of (MFP). The proof is complete.

Consider the dual problem of (MFP)

\[
\begin{align*}
\text{max} & \quad \left( \frac{f_1(y)}{g_1(y)} + u^T h(y) + v^T l(y) \right), \\
\text{s.t.} & \quad \sum_{i=1}^{p} \lambda_i \nabla \left( \frac{f_i(y)}{g_i(y)} \right) + \sum_{j=1}^{m} u_j \nabla h_j(y) \\
& \quad + \sum_{k=1}^{q} v_k l_k(y) = 0, \\
& \quad u^T h(y) \geq 0, \\
& \quad v^T l(y) = 0, \\
& \quad u \geq 0, \quad v \geq 0.
\end{align*}
\]

(66)

**Theorem 17.** \( f_i(y)/g_i(y) \) (i = 1, 2, ..., p) is (F, \( \rho \))-convex at \( y \), \( h_j(y) \) (j = 1, 2, ..., m) is (F, \( \rho \))-convex at \( y \), \( l_k(y) \) (k = 1, 2, ..., q) is (F, \( \rho \))-convex at \( y \), and \( \lambda \in \mathbb{R}^+ \), \( u \in \mathbb{R}^m \), \( v \in \mathbb{R}^q \), \( \sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} u_j \rho_j + \sum_{k=1}^{q} v_k \rho_k \geq 0 \), then \( \lambda^T (f_i(y)/g_i(y)) \geq \lambda^T (f_i(y)/g_i(y)) \).

**Proof.** By the (F, \( \rho \))-convexity of \( f_i(y)/g_i(y), h_j(y), \) and \( l_k(y) \), the sublinearity of \( F \), and since \( \lambda \in \mathbb{R}^+ \), \( u \in \mathbb{R}^m \), \( v \in \mathbb{R}^q \), we have that

\[
\begin{align*}
\sum_{i=1}^{p} \lambda_i \left( \frac{f_i(x)}{g_i(x)} \right) - \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(y)}{g_i(y)} \right) \\
\geq F \left( x, y; \sum_{i=1}^{m} u_i \nabla h_i(y) \right) + \sum_{j=1}^{m} u_j \rho_j d^2(x, y), \\
\sum_{j=1}^{m} u_j h_j(x) - \sum_{j=1}^{m} u_j h_j(y) \\
\geq F \left( x, y; \sum_{j=1}^{m} u_j \nabla h_j(y) \right) + \sum_{j=1}^{m} u_j \rho_j d^2(x, y), \\
\sum_{k=1}^{q} v_k l_k(x) - \sum_{k=1}^{q} v_k l_k(y) \\
\geq F \left( x, y; \sum_{k=1}^{q} v_k \nabla l_k(y) \right) + \sum_{k=1}^{q} v_k \rho_k d^2(x, y).
\end{align*}
\]

(67)

By (67) and based on the sublinearity of \( F \), we have

\[
\begin{align*}
\sum_{i=1}^{p} \lambda_i \left( \frac{f_i(x)}{g_i(x)} \right) - \sum_{i=1}^{p} \lambda_i \left( \frac{f_i(y)}{g_i(y)} \right) & \geq \sum_{i=1}^{m} u_i h_i(x) - \sum_{i=1}^{m} u_i h_i(y) \\
& \geq F \left( x, y; \sum_{i=1}^{m} u_i \nabla h_i(y) \right) + \sum_{j=1}^{m} u_j \rho_j d^2(x, y), \\
\sum_{k=1}^{q} v_k l_k(x) - \sum_{k=1}^{q} v_k l_k(y) & \geq F \left( x, y; \sum_{k=1}^{q} v_k \nabla l_k(y) \right) + \sum_{k=1}^{q} v_k \rho_k d^2(x, y).
\end{align*}
\]

Since \( h_j(x) \leq 0, (j = 1, 2, ..., m), l_k(x) = 0, (k = 1, 2, ..., q) \), \( n h(y) \geq 0, n l(y) = 0 \), \( \sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} u_j \rho_j + \sum_{k=1}^{q} v_k \rho_k \geq 0 \), we have \( \sum_{i=1}^{p} \lambda_i (f_i(x)/g_i(x)) \geq \sum_{i=1}^{p} \lambda_i (f_i(y)/g_i(y)) \).

\( \lambda^T (f_i(x)/g_i(x)) \geq \lambda^T (f_i(y)/g_i(y)) \).

\( \Box \)

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**References**


