Research Article

Fluid Queue Driven by an $M/M/1$ Queue Subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption

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This paper deals with the stationary analysis of a fluid queue driven by an $M/M/1$ queueing model subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption. The model under consideration can be viewed as a quasi-birth and death process. The governing system of differential difference equations is solved using matrix-geometric method in the Laplacian domain. The resulting solutions are then inverted to obtain an explicit expression for the joint steady state probabilities of the content of the buffer and the state of the background queueing model. Numerical illustrations are added to depict the convergence of the stationary buffer content distribution to one subject to suitable stability conditions.

1. Introduction

In many real time situations, the server in the background queueing model may become unavailable for a random period of time to perform a secondary task, when there are no customers in the waiting line at the service completion epoch. Such period of server absence is termed as server vacation. Queueing models subject to various vacation policies are of interest to researchers in recent times owing to their widespread applicability.

There are different types of vacation queueing systems. In the single vacation scheme, the server takes a vacation of some random duration when the queue is empty. At the end of the vacation, the server returns to the queue. The server resumes service if there is at least one customer waiting upon his return from vacation. However, if the queue is empty on the server's return, the server waits to complete a busy period. In the multiple vacation scheme, if the server returns from a vacation and finds the queue empty, he immediately commences another vacation. If there is at least one waiting customer, then he will commence the service. Queueing models subject to single or multiple exponential vacation

are apt to model many practical scenarios [1–3]. However, a better modeling assumption would be to assume that the server works at a slower rate during vacation periods in comparison to that of a regular working period. Such models are classified as queues subject to working vacations [4–6]. In addition, the server can stop the vacation once some indices of the system, such as the number of customers, achieve a certain value in the vacation period. Certainly, it is possible for the server to take an interrupted vacation, so we call this policy vacation interruption. Li and Tian [7] studied the single server queueing model with working vacation and vacation interruption. The modulating queueing model considered in this paper is an $M/M/1$ queue wherein the server is subject to regular vacation with probability p or working vacation with probability $1 - p$. Further, the vacation duration of the server during working vacation epoch may be interrupted due to vacation interruption.

Fluid queues have become a fascinating area of research in recent years due to their widespread applicability in computer and communication systems [8, 9], manufacturing systems [10], and so forth. A stochastic fluid flow system is an input-output model where the input is modeled as a continuous

fluid that enters and leaves the storage device called a buffer, according to randomly varying rates. They are appropriate in a situation wherein the arrival is comprised of a discrete unit, but the interarrival time between successive arrivals is negligible. Therefore, the arrivals can be approximated by a continuous flow of fluid as individuals units have less impact on the performance of the system. In these models, a fluid buffer is either filled or depleted or both at rates determined by the current state of the background queueing model. Markov modulated fluid queues are a particular class of fluid models useful for modeling many physical phenomena and they often allow tractable analysis. In addition, fluid models are quite useful as approximate models for certain queueing and inventory systems where the flow consists of discrete entities, but the behavior of the individual is not important to identify the performance analysis. Certain interesting real world applications of Markov Modulated Fluid Flow models can be found in [11–14]. Besides, fluid queues also have successful applications in the field of congestion control [15] and risk processes [16]. More recently, Bosman and Nunez-Queija [17] considered a tandem fluid queue model to evaluate the performance of streaming media over an unreliable network.

For example, consider a production inventory model operating in a stochastic environment. The inventory level increases when the production rate exceeds the demand rate and decreases otherwise. The inventory level under continuous review can be viewed as a fluid process that fluctuates according to the evolution of the underlying background environment. For example, consider a machine shop with a single server. When the server is busy, items are produced continuously at a rate r and if he is idle, there is no production. However, for all practical reasons, the server might either take a vacation of some random duration with probability p or decide to provide service at a reduced rate with probability $1 - p$. Further, by offering service at a reduced rate, the server may continue to do so with probability q or due to certain unforeseen reasons, like a sudden increase in the demand, interrupt the vacation with probability $1 - q$, and continue the busy period. The demands are assumed to vary from time to time at the rate d_t independent of the state of the server. The level of inventory thus oscillates between $r - d_t$ and $-d_t$ depending on the busy or idle state of the server. Such scenario can be modeled as a fluid queue driven by an $M/M/1$ queue subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption.

The stationary analysis of fluid queueing models in a stochastic environment has been discussed by many authors. Fluid queues driven by an $M/M/1$ queueing model are extensively studied in the literature. Various techniques have been employed by researchers to obtain the stationary buffer content distribution. To mention a few, Adan and Resing [18] analyze the buffer content distribution by viewing the arrival process as an alternating renewal process and Virtamo and Norros [19] provide the buffer content distribution by finding the spectrum of the eigenvalue equation and explicit expressions for the corresponding eigenvectors in terms of Chebyshev polynomials of the second kind. Sericola and Tuffin [20] express the stationary distribution of the buffer

occupancy in terms of a sequence of recursively defined polynomials. Parthasarathy et al. [21] present an explicit expression for the buffer content distribution in terms of modified Bessel function of the first kind using continued fraction methodology.

Furthermore, fluid models driven by an $M/M/1$ queue subject to various vacation strategies were analyzed in steady state by Mao et al. [22] and Wang et al. [23]. The work was further extended to the stationary analysis of fluid queues driven by an $M/M/1$ queue with multiple exponential vacation and N policy [24]. Fluid model driven by an $M/M/1$ queue with working vacations and vacation interruption was studied by Xu et al. [25]. However, in most of the literature relating to fluid queues driven by vacation queueing models, the buffer content distribution is expressed in the Laplace domain. More recently, Vijayashree and Anjuka [26, 27] presented an explicit expression for the buffer content distribution of a fluid queueing model modulated by an $M/M/1$ queue subject to catastrophes and subsequent repair and a fluid queue driven by an $M/E_2/1$ queueing model, respectively. Also, Ammar [28] derives an explicit expression for the fluid queue driven by an $M/M/1$ queue with multiple exponential vacation using generating function methodology.

This paper presents an analytical solution for the fluid queue driven by an $M/M/1$ queue subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption in stationary regime. When the background queueing model is empty, the server will take either an ordinary vacation with probability p or a working vacation with probability $1 - p$. If the system is in a working vacation, upon completion of service, the server either ends the vacation and enters a regular busy period with probability $1 - q$ or continues the vacation with probability q . It is assumed that the fluid in the buffer content increases at a constant rate, σ , when there is one or more customers in the background queueing model and decreases at a constant rate, σ_0 , when the queue is empty. The system of equations governing the process is modeled in terms of quasi-birth and death process and solved using matrix-geometric method. The stationary distribution of the buffer content is thereby obtained in the Laplace domain and hence inverted to obtain explicit expressions for the joint steady state probabilities of the state of the background queueing model and content of the buffer. Closed form expressions help to gain a deeper insight into the model. As a special case, when $p = 0$, $q = 0$, and $p = 1$ the theoretical results, so obtained, are seen to coincide with the existing results of Xu et al. [25] and Wang et al. [23], respectively.

The rest of the paper is organized as follows: Section 2 gives a brief description of the background queueing model under consideration. Section 3 presents the system of differential difference equations that governs the fluid queueing model under steady state subject to suitable stability conditions. Section 4 gives detailed derivations of the closed form expressions for the joint steady state probabilities of the state of the background queueing model and also the buffer content distributions. Section 5 presents the numerical illustration of the stationary buffer content distribution for suitable choice of the parameter values.

2. Model Description

Consider an $M/M/1$ queueing model with infinite capacity. Let the customers arrive according to a Poisson process with parameter λ . The server provides service to the arriving customers according to an exponential distribution with parameter μ . When the system becomes empty, the server begins a vacation of random length and takes an ordinary vacation with probability p or a working vacation with probability $1 - p$, where $0 \leq p \leq 1$. In an ordinary vacation, the server will stop working even if there are new arrivals during the vacation period. In a working vacation, customers are served at a lower rate $\mu_v < \mu$. Further, in a working vacation, it is assumed that, at the instants of service completion, either the vacation is interrupted and the server resumes to a regular busy period with probability $1 - q$ or the server continues the vacation with probability q . When the vacation period ends and the system is nonempty, a new busy period starts. The ordinary vacation times and the working vacation times are also assumed to be exponentially distributed with parameters θ and θ_v , respectively. Let $N(t)$ denote the number of customers at time t . Define

$$J(t) = \begin{cases} 0, & \text{if the server is in a working vacation at time } t, \\ 1, & \text{if the server is in an ordinary vacation at time } t, \\ 2, & \text{if the server is in a regular busy period at time } t. \end{cases} \quad (1)$$

It is well known that the process $\{(N(t), J(t)), t \geq 0\}$ is a quasi-birth and death (QBD) process with state space given by

$$\Omega = \{(0, 0) \cup (0, 1)\} \cup (k, j), \quad k = 1, 2, \dots, \quad j = 0, 1, 2. \quad (2)$$

The state transition diagram of the background queueing model is given in Figure 1. Let

$$\pi_{k,j} = \lim_{t \rightarrow \infty} P\{N(t) = k, J(t) = j\}, \quad (k, j) \in \Omega, \quad (3)$$

represent the steady state probabilities of the background queueing model. Further, let $\pi_0 = (\pi_{0,0} \ \pi_{0,1})$ and $\pi_k = (\pi_{k,0} \ \pi_{k,1} \ \pi_{k,2})$ for $k \geq 1$. Then, the stationary probability vector is denoted by

$$\pi = (\pi_0, \pi_1, \pi_2, \dots). \quad (4)$$

It is readily seen that the system of equations governing the background queueing model under steady state can be written in the form of matrix as

$$\begin{aligned} \pi Q &= 0, \\ \pi_0 e_1 + \sum_{k=1}^{\infty} \pi_k e_2 &= 1, \end{aligned} \quad (5)$$

FIGURE 1: State transition diagram.

where $e_1 = (1, 1)^T$, $e_2 = (1, 1, 1)^T$, and $Q = \begin{pmatrix} B_0 & A_0 \\ C_0 & B \end{pmatrix}$.

Note that

$$\begin{aligned} B_0 &= \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \\ A_0 &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}, \\ C_0 &= \begin{pmatrix} \mu_v & 0 \\ 0 & 0 \\ (1-p)\mu & p\mu \end{pmatrix}, \\ A &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \\ B &= \begin{pmatrix} -(\lambda + \mu_v + \theta_v) & 0 & \theta_v \\ 0 & -(\lambda + \theta) & \theta \\ 0 & 0 & -(\lambda + \mu) \end{pmatrix}, \\ C &= \begin{pmatrix} q\mu_v & 0 & (1-q)\mu_v \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}. \end{aligned} \quad (6)$$

The steady state probabilities of the single server queueing models with Poisson arrival and exponentially distributed service times subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption were studied by Zhang and Shi [29].

3. Analysis of Fluid Queue

This section deals with the stationary analysis of a fluid queue modulated by an $M/M/1$ queueing model subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption. Let $C(t)$ be the content of the buffer at time t . Furthermore, it is assumed that the content of the buffer increases at the rate σ , when there are customers in the background

FIGURE 2: Interaction between fluid model and the background queueing model.

queueing model, while the buffer content decreases at the rate σ_0 , when the system is empty. The rate at which the content of the buffer varies with time is given by

$$\frac{dC(t)}{dt} = \begin{cases} 0, & (N(t), J(t)) = (0, 0), C(t) = 0 \\ \sigma_0, & (N(t), J(t)) = (0, 0), C(t) > 0 \\ \sigma_0, & (N(t), J(t)) = (0, 1), C(t) > 0 \\ \sigma, & (N(t), J(t)) = (k, j), k \geq 1, j = 0, 1, 2, \end{cases} \quad (7)$$

where $\sigma_0 < 0$ and $\sigma > 0$. Figure 2 depicts the interaction between the buffer content process and the background queueing model. It is seen that the content of the infinite capacity buffer decreases at the rate $\sigma_0 < 0$ when the background queueing model is empty with no waiting customers and it increases at the rate σ when $N(t) \neq 0$ irrespective of the states of $J(t)$.

Clearly the 3-dimensional process $\{(N(t), J(t), C(t)), t \geq 0\}$ represents a fluid queue driven by an $M/M/1$ queue with Bernoulli-Schedule-Controlled Vacation and Vacation Interruption. As the content of the buffer varies dynamically, it is necessary that the net effective rate of the fluid remains negative to ensure the stability of the process in a long run. Hence, the stability condition is given by

$$d = \sigma_0 (\pi_{0,0} + \pi_{0,1}) + \sigma \sum_{k=1}^{\infty} \sum_{j=0}^2 \pi_{k,j} < 0. \quad (8)$$

Define the joint probability distribution functions of the Markov process $\{(N(t), J(t), C(t)), t \geq 0\}$ at time t as

$$F_{k,j}(t, x) = \Pr \{N(t) = k, J(t) = j, C(t) \leq x\}, \quad (k, j) \in \Omega, x \geq 0. \quad (9)$$

When the process $\{(N(t), J(t), C(t)), t \geq 0\}$ is stable, its stationary random vector is denoted by (N, J, C) . Under steady state conditions, let

$$F_{k,j}(x) = \lim_{t \rightarrow +\infty} \Pr \{N(t) = k, J(t) = j, C(t) \leq x\}, \quad x > 0, (k, j) \in \Omega. \quad (10)$$

Using standard methods, the system of differential difference equations that governs the process $\{(N(t), J(t), C(t)), t \geq 0\}$ is given by

$$\begin{aligned} \sigma_0 \frac{dF_{0,0}(x)}{dx} &= -\lambda F_{0,0}(x) + \mu_v F_{1,0}(x) + (1-p) \mu F_{1,2}(x), \\ \sigma_0 \frac{dF_{0,1}(x)}{dx} &= -\lambda F_{0,1}(x) + p \mu F_{1,2}(x), \\ \sigma \frac{dF_{k,0}(x)}{dx} &= -(\lambda + \mu_v + \theta_v) F_{k,0}(x) + \lambda F_{k-1,0}(x) + q \mu_v F_{k+1,0}(x) \quad k \geq 1, \\ \sigma \frac{dF_{k,1}(x)}{dx} &= -(\lambda + \theta) F_{k,1}(x) + \lambda F_{k-1,1}(x) \quad k \geq 1, \\ \sigma \frac{dF_{1,2}(x)}{dx} &= -(\lambda + \mu) F_{1,2}(x) + \theta F_{1,1}(x) + \theta_v F_{1,0}(x) + (1-q) \mu_v F_{2,0}(x) + \mu F_{2,2}(x), \\ \sigma \frac{dF_{k,2}(x)}{dx} &= -(\lambda + \mu) F_{k,2}(x) + \theta F_{k,1}(x) + \theta_v F_{k,0}(x) + (1-q) \mu_v F_{k+1,0}(x) + \lambda F_{k-1,2}(x) + \mu F_{k+1,2}(x) \quad k \geq 2, \end{aligned} \quad (11)$$

with the boundary conditions

$$\begin{aligned} F_{0,0}(0) &= a_1, \\ F_{0,1}(0) &= a_2, \\ F_{k,j}(0) &= 0, \quad (k, j) \in \Omega \setminus \{(0,0) \cup (0,1)\}. \end{aligned} \quad (12)$$

The constants a_1 and a_2 are such that $0 < a_1 < 1$ and $0 < a_2 < 1$. Since we make an assumption that the content of the buffer increases at the rate σ when there is one or more customers in the background queueing model, it is impossible to have the buffer empty when the modulating process is in any of

the states other than $(0, 0)$ and $(0, 1)$. However, when there are no customers in the background queueing model, the buffer content depletes at rate $\sigma_0 < 0$ and hence with some positive probability, it is possible that the content of the buffer is empty. Therefore the boundary conditions given by (12) are valid.

4. Solution Methodology

The stationary distributions of the fluid process play a vital role as they give more information relating to quantities of interest for practical applications like tail probabilities, expected buffer content, traffic intensity, expected delay, and sojourn time. This section presents explicit expressions for the joint steady state probabilities of the background queueing model and the content of the buffer in terms of modified Bessel function of the first kind. The governing system of equations in the Laplace domain is expressed as a system of matrix equations. The minimal nonnegative solution of the matrix quadratic equation is determined. The stationary joint probability distributions are expressed in terms of this minimal nonnegative solution and are shown to satisfy the governing system of matrix equations. This section presents an explicit analytical solution to the governing system of equations represented by (11). In this sequel, define

$$F_0(x) = (F_{0,0}(x), F_{0,1}(x)), \quad (13)$$

$$F_k(x) = (F_{k,0}(x), F_{k,1}(x), F_{k,2}(x)) \quad k = 1, 2, \dots$$

Let $F(x) = (F_0(x), F_1(x), F_2(x), \dots)$. The system of equations represented by (11) can be written in matrix form as

$$F'(x) \Lambda = F(x) Q, \quad (14)$$

where

$$\Lambda = \begin{pmatrix} \Sigma' & & \\ & \Sigma & \\ & & \Sigma \\ & & & \ddots \end{pmatrix}, \quad (15)$$

$$\Sigma' = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}.$$

Let $\hat{F}(s) = (\hat{F}_0(s), \hat{F}_1(s), \hat{F}_2(s), \dots)$ represent the Laplace transform of $F(x)$. Then, the Laplace transform of (14) yields

$$\hat{F}(s) (Q - s\Lambda) = (-\bar{a} \ 0 \ 0 \ \dots \ 0), \quad (16)$$

where $\bar{a} = (\sigma_0 a_1 \ \sigma_0 a_2)$. The governing system of differential difference equations in the Laplace domain is then given by

$$\hat{F}_0(s) (B_0 - s\Sigma') + \hat{F}_1(s) C_0 = -\bar{a}, \quad (17)$$

$$\hat{F}_0(s) A_0 + \hat{F}_1(s) (B - s\Sigma) + \hat{F}_2(s) C = 0, \quad (18)$$

$$\hat{F}_{k-1}(s) A + \hat{F}_k(s) (B - s\Sigma) + \hat{F}_{k+1}(s) C = 0 \quad (19)$$

for $k = 2, 3, \dots$

Our objective is to solve the above system of matrix difference equations to obtain explicit expressions for the stationary probability distribution and hence determine the stationary buffer content distribution. Towards that end, we present a lemma below followed by a theorem.

Lemma 1. *The matrix quadratic equation*

$$A + R(s) (B - s\Sigma) + R^2(s) C = 0 \quad (20)$$

has the minimal nonnegative solution given by

$$R(s) = \begin{pmatrix} r(s) & 0 & \beta(s) \\ 0 & \psi(s) & \gamma(s) \\ 0 & 0 & z(s) \end{pmatrix}, \quad (21)$$

where

$$r(s) = \frac{(\lambda + \mu_v + \theta_v + s\sigma) - \sqrt{(\lambda + \mu_v + \theta_v + s\sigma)^2 - 4\lambda q \mu_v}}{2q \mu_v},$$

$$\beta(s) = \frac{r(s) z(s) [(1 - q) \mu_v r(s) + \theta_v]}{\lambda - \mu r(s) z(s)}, \quad (22)$$

$$\psi(s) = \frac{\lambda}{\lambda + \theta + s\sigma},$$

$$\gamma(s) = \frac{\theta z(s)}{\lambda + \theta + s\sigma - \mu z(s)},$$

$$z(s) = \frac{(\lambda + \mu + s\sigma) - \sqrt{(\lambda + \mu + s\sigma)^2 - 4\lambda \mu}}{2\mu}.$$

Proof. Since A , $(B - s\Sigma)$, and C are all upper triangular matrices, we can assume that the solution $R(s)$ has the same structure as

$$R(s) = \begin{pmatrix} r_{11}(s) & r_{12}(s) & r_{13}(s) \\ 0 & r_{22}(s) & r_{23}(s) \\ 0 & 0 & r_{33}(s) \end{pmatrix}. \quad (23)$$

Note that the first row and second column elements of all the matrices of A , $(B - s\Sigma)$, and C are zero, so the element $r_{12}(s)$ in $R(s)$ is also zero. Therefore $r_{12}(s) = 0$. Substituting for $R(s)$ into (20) leads to

$$\lambda - (\lambda + \mu_v + \theta_v + s\sigma) r_{11}(s) + q \mu_v (r_{11}(s))^2 = 0, \quad (24)$$

$$\lambda - (\lambda + \mu + s\sigma) r_{33}(s) + \mu (r_{33}(s))^2 = 0, \quad (25)$$

$$\lambda - (\lambda + \theta + s\sigma) r_{22}(s) = 0, \quad (26)$$

$$(1 - q) \mu_v (r_{11}(s))^2 + \mu r_{13}(s) r_{11}(s) + \mu r_{13}(s) r_{33}(s) + \theta_v r_{11}(s) - (\lambda + \mu + s\sigma) r_{13}(s) = 0, \quad (27)$$

$$\theta r_{22}(s) - (\lambda + \mu + s\sigma) r_{23}(s) + \mu [r_{22}(s) r_{23}(s) + r_{23}(s) r_{33}(s)] = 0. \quad (28)$$

The solutions of (24) and (25) are given by

$$r_{11}(s) = \frac{(\lambda + \mu_v + \theta_v + s\sigma) \pm \sqrt{(\lambda + \mu_v + \theta_v + s\sigma)^2 - 4\lambda q\mu_v}}{2q\mu_v}, \quad (29)$$

$$r_{33}(s) = \frac{(\lambda + \mu + s\sigma) \pm \sqrt{(\lambda + \mu + s\sigma)^2 - 4\lambda\mu}}{2\mu}.$$

Let $r(s)$ ($r_1(s)$) and $z(s)$ ($z_1(s)$) denote the negative (positive) roots of $r_{11}(s)$ and $r_{33}(s)$, respectively. Considering the root that lies inside the unit circle, $r(s)$ and $z(s)$ are taken for further analysis. Further $z(s)$ satisfies the following relations:

$$s\sigma + \lambda + \mu(1 - z(s)) = \frac{\lambda}{z(s)} = \mu + \frac{s\sigma}{1 - z(s)}. \quad (30)$$

From (26), we get

$$r_{22}(s) = \psi(s) = \frac{\lambda}{\lambda + \theta + s\sigma}. \quad (31)$$

Substituting for $r_{11}(s)$ and $r_{33}(s)$ into (27) yields

$$r_{13}(s) = \beta(s) = \frac{r(s) [(1 - q)\mu_v r(s) + \theta_v]}{\lambda + \mu + s\sigma - \mu(r(s) + z(s))}. \quad (32)$$

Using the relation given by (30) in the above leads to

$$r_{13}(s) = \beta(s) = \frac{r(s) z(s) [(1 - q)\mu_v r(s) + \theta_v]}{\lambda - \mu r(s) z(s)}. \quad (33)$$

Again substituting for $r_{22}(s)$ and $r_{33}(s)$ into (28) leads to

$$r_{23}(s) = \gamma(s) = \frac{\lambda\theta}{(s\sigma + \lambda + \mu)(s\sigma + \lambda + \theta) - \lambda\mu - \mu z(s)(s\sigma + \theta + \lambda)}. \quad (34)$$

Using the relation given by (30) in the above equation yields

$$r_{23}(s) = \gamma(s) = \frac{\theta z(s)}{\lambda + \theta + s\sigma - \mu z(s)}. \quad (35)$$

This completes the proof. Note that $R^k(s)$ for $k = 1, 2, 3, \dots$ can be simplified as

$$R^k(s) = \begin{pmatrix} r^k(s) & 0 & \beta(s) \sum_{i=1}^k r^{k-i}(s) z^{i-1}(s) \\ 0 & \psi^k(s) & \gamma(s) \sum_{j=1}^k \psi^{k-j}(s) z^{j-1}(s) \\ 0 & 0 & z^k(s) \end{pmatrix}. \quad (36)$$

Also $R(0) = R$, where R is given by

$$R = \begin{pmatrix} r & 0 & \frac{\lambda - \mu_v r}{\mu} \\ 0 & \frac{\lambda}{\lambda + \theta} & \rho \\ 0 & 0 & \rho \end{pmatrix}, \quad (37)$$

with $\rho = \lambda/\mu$ and $r = ((\lambda + \mu_v + \theta_v) - \sqrt{(\lambda + \mu_v + \theta_v)^2 - 4\lambda q\mu_v})/2q\mu_v$. $\square$

Theorem 2. The stationary joint probability distributions of the content of the buffer and the state of the background queueing model in Laplace domain are given by

$$\hat{F}_k(s) = \hat{F}_0(s) eR^k(s) \quad \text{for } k = 1, 2, 3, \dots, \quad (38)$$

$$\hat{F}_0(s) = \frac{\bar{a}}{s\Sigma' - B_0 - eR(s)C_0}, \quad (39)$$

where $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Proof. Assume that

$$\hat{F}_k(s) = \hat{F}_{k-1}(s) R(s) \quad \text{for } k = 1, 2, 3, \dots \quad (40)$$

Then, it can be recursively written as

$$\hat{F}_k(s) = \hat{F}_0(s) eR^k(s) \quad \text{for } k = 1, 2, 3, \dots \quad (41)$$

Below, we verify that (40) satisfies (18) and (19). Substituting (41) into (18) leads to

$$\begin{aligned} & \hat{F}_0(s) A_0 + \hat{F}_1(s) (B - s\Sigma) + \hat{F}_2(s) C \\ &= \hat{F}_0(s) eA + \hat{F}_0(s) eR(s) (B - s\Sigma) \\ & \quad + \hat{F}_0(s) eR^2(s) C \\ &= \hat{F}_0(s) e [A + R(s) (B - s\Sigma) + R^2(s) C] \\ &= 0 \quad (\text{by Lemma 1}). \end{aligned} \quad (42)$$

Similarly, substituting (40) into (19) leads to

$$\begin{aligned} & \hat{F}_{k-1}(s) A + \hat{F}_k(s) (B - s\Sigma) + \hat{F}_{k+1}(s) C \\ &= \hat{F}_{k-1}(s) A + \hat{F}_{k-1}(s) R(s) (B - s\Sigma) \\ & \quad + \hat{F}_{k-1}(s) R^2(s) C \\ &= \hat{F}_{k-1}(s) [A + R(s) (B - s\Sigma) + R^2(s) C] \\ &= 0 \quad (\text{by Lemma 1}). \end{aligned} \quad (43)$$

From (17), we get

$$\hat{F}_0(s) (B_0 - s\Sigma') + \hat{F}_0(s) eR(s) C_0 = -\bar{a}. \quad (44)$$

Therefore

$$\hat{F}_0(s) = \frac{\bar{a}}{s\Sigma' - B_0 - eR(s)C_0}, \quad (45)$$

which upon simplification leads to

$$\begin{aligned} \hat{F}_{0,0}(s) &= \sum_{k=0}^{\infty} (\delta(s))^k \left[\frac{a_1\sigma_0}{s\sigma_0 + \lambda - \mu_v r(s)} \right. \\ & \quad \left. + a_2\sigma_0(1 - p)\mu\gamma(s)D(s) \right], \end{aligned} \quad (46)$$

$$\begin{aligned} \hat{F}_{0,1}(s) &= \sum_{k=0}^{\infty} (\delta(s))^k \\ & \quad \cdot \left[(a_1\sigma_0 p\mu - a_2\sigma(1 - p)\mu)\beta(s)D(s) \right. \\ & \quad \left. + \frac{a_2\sigma_0}{s\sigma_0 + \lambda - p\mu\gamma(s)} \right], \end{aligned} \quad (47)$$

where

$$\begin{aligned}\delta(s) &= \frac{\beta(s)(1-p)\mu(s\sigma_0 + \lambda)}{(s\sigma_0 + \lambda - p\mu\gamma(s))(s\sigma_0 + \lambda - \mu_v r(s))}, \\ D(s) &= \frac{1}{(s\sigma_0 + \lambda - p\mu\gamma(s))(s\sigma_0 + \lambda - \mu_v r(s))}.\end{aligned}\quad (48)$$

which can be written as

$$\begin{aligned}\hat{F}_{k,0}(s) &= r(s)^k \hat{F}_{0,0}(s), \\ \hat{F}_{k,1}(s) &= \psi(s)^k \hat{F}_{0,1}(s), \\ \hat{F}_{k,2}(s) &= \beta(s) \sum_{i=1}^k r^{k-i}(s) z^{i-1}(s) \hat{F}_{0,0}(s) \\ &\quad + \gamma(s) \sum_{j=1}^k \psi^{k-j}(s) z^{j-1}(s) \hat{F}_{0,1}(s).\end{aligned}\quad (50)$$

With $\alpha = 2\sqrt{\lambda\mu}/\sigma$ and $\beta = 2\sqrt{\lambda q\mu_v}/\sigma$, inversion of (46), (47), and (50) yields

$$\begin{aligned}F_{0,0}(x) &= \sum_{k=0}^{\infty} \delta(x)^{*k} * \left[a_1 \sum_{l=0}^{\infty} \left(\frac{\mu_v}{\sigma_0} \right)^l e^{-(\lambda/\sigma_0)x} \frac{x^l}{l!} \right. \\ &\quad * \frac{II_l(\beta x) \beta^l}{x} \left(\frac{\sigma}{2q\mu_v} \right)^l e^{-((\lambda+\mu_v+\theta_v)/\sigma)x} + a_2 \sigma_0 (1 \\ &\quad \left. - p) \mu \gamma(x) * D(x) \right],\end{aligned}\quad (51)$$

$$\begin{aligned}F_{0,1}(x) &= \sum_{k=0}^{\infty} \delta(x)^{*k} * \left[(a_1 \sigma_0 p \mu - a_2 \sigma (1-p) \mu) \right. \\ &\quad * \beta(x) * D(x) + a_2 \sum_{k=0}^{\infty} \left(\frac{p\mu}{\sigma_0} \right)^k e^{-(\lambda/\sigma_0)x} \frac{x^k}{k!} \\ &\quad \left. * \gamma(x)^{*k} \right],\end{aligned}\quad (52)$$

$$\begin{aligned}F_{k,0}(x) &= \frac{k I_k(\beta x) \beta^k}{x} \left(\frac{\sigma}{2q\mu_v} \right)^k e^{-((\lambda+\mu_v+\theta_v)/\sigma)x} \\ &\quad * F_{0,0}(x) \quad k = 1, 2, 3, \dots,\end{aligned}\quad (53)$$

$$\begin{aligned}F_{k,1}(x) &= \left(\frac{\lambda}{\sigma} \right)^k e^{-((\lambda+\theta)/\sigma)x} \frac{x^{k-1}}{(k-1)!} * F_{0,1}(x) \\ &\quad k = 1, 2, 3, \dots,\end{aligned}\quad (54)$$

Hence all the joint stationary probabilities, $\hat{F}_k(s)$, for $k = 1, 2, 3, \dots$ are in terms of $\hat{F}_0(s)$, where $\hat{F}_0(s)$ is given by (45). This completes the proof. $\square$

Having determined all the joint steady state probabilities in the Laplace domain, we now present the explicit analytical solution by inverting using transform techniques. From (41), we get

$$\begin{bmatrix} \hat{F}_{k,0}(s) & \hat{F}_{k,1}(s) & \hat{F}_{k,2}(s) \end{bmatrix} = \begin{bmatrix} \hat{F}_{0,0}(s) & \hat{F}_{0,1}(s) \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r^k(s) & 0 & \beta(s) \sum_{i=1}^k r^{k-i}(s) z^{i-1}(s) \\ 0 & \psi^k(s) & \gamma(s) \sum_{j=1}^k \psi^{k-j}(s) z^{j-1}(s) \\ 0 & 0 & z^k(s) \end{pmatrix}, \quad (49)$$

$$\begin{aligned}F_{k,2}(x) &= \left(\beta(x) * \sum_{i=1}^k \left(\frac{\sigma}{2q\mu_v} \right)^{k-i} \right. \\ &\quad * \frac{(k-i) I_{k-i}(\beta x) \beta^{k-i}}{x} e^{-((\lambda+\mu_v+\theta_v)/\sigma)x} * \left(\frac{\sigma}{2\mu} \right)^{i-1} \\ &\quad * \frac{(i-1) I_{i-1}(\alpha x) \alpha^{i-1}}{x} e^{-((\lambda+\mu)/\sigma)x} \left. \right) + \left(\gamma(x) \right. \\ &\quad * \sum_{j=1}^k \left(\frac{\lambda}{\sigma} \right)^{k-j} e^{-((\lambda+\theta)/\sigma)x} \frac{x^{k-j-1}}{(k-j-1)!} * \left(\frac{\sigma}{2\mu} \right)^{j-1} \\ &\quad \left. * \frac{(j-1) I_{j-1}(\alpha x) \alpha^{j-1}}{x} e^{-((\lambda+\mu)/\sigma)x} * F_{0,1}(x) \right) \\ &\quad k \geq 1,\end{aligned}\quad (55)$$

where

$$\begin{aligned}\delta(x) &= (1-p) \mu \beta(x) * \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(p\mu)^i \mu_v^j}{\sigma_0^{i+j+1}} e^{-(\lambda/\sigma_0)x} \\ &\quad * \frac{x^{i+j}}{(i+j)!} * \gamma(x)^{*i} * \frac{j I_j(\beta x) \beta^j}{x} \left(\frac{\sigma}{2q\mu_v} \right)^j \\ &\quad * e^{-((\lambda+\mu_v+\theta_v)/\sigma)x} \\ D(x) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(p\mu)^i \mu_v^j}{\sigma_0^{i+j+2}} e^{-((\lambda+\mu_v+\theta_v)/\sigma)x} \frac{x^{i+j+1}}{(i+j+1)!} * \gamma(x)^{*i} \\ &\quad * \frac{j I_j(\beta x) \beta^j}{x} \left(\frac{\sigma}{2q\mu_v} \right)^j e^{-((\lambda+\mu_v+\theta_v)/\sigma)x},\end{aligned}$$

$$\begin{aligned}\beta(x) &= (1-q) \\ &\quad * \left(\mu_v \sum_{k=0}^{\infty} \frac{\mu^k}{\lambda^{k+1}} \frac{(k+2) I_{k+2}(\beta x) \beta^{k+2}}{x} \left(\frac{\sigma}{2q\mu_v} \right)^{k+2} \right.\end{aligned}$$

$$\begin{aligned}
& \cdot e^{-(\lambda+\mu_v+\theta_v)/\sigma)x} * \left(\frac{\sigma}{2\mu} \right)^{k+1} \\
& \cdot \frac{(k+1) I_{k+1}(\alpha x) \alpha^{k+1}}{x} e^{-((\lambda+\mu)/\sigma)x} \Bigg) \\
& + \theta_v \left(\sum_{k=0}^{\infty} \frac{\mu^k}{\lambda^{k+1}} \frac{(k+1) I_{k+1}(\beta x) \beta^{k+1}}{x} \left(\frac{\sigma}{2q\mu_v} \right)^{k+1} \right. \\
& \cdot e^{-(\lambda+\mu_v+\theta_v)/\sigma)x} * \left(\frac{\sigma}{2\mu} \right)^{k+1} \\
& \cdot \frac{(k+1) I_{k+1}(\alpha x) \alpha^{k+1}}{x} e^{-((\lambda+\mu)/\sigma)x} \Bigg), \\
\gamma(x) &= \theta \sum_{k=0}^{\infty} \frac{\mu^k}{\sigma^{k+1}} e^{-((\lambda+\theta)/\sigma)x} \frac{x^k}{k!} * \left(\frac{\sigma}{2\mu} \right)^{k+1} \\
& \cdot \frac{(k+1) I_{k+1}(\alpha x) \alpha^{k+1}}{x} e^{-((\lambda+\mu)/\sigma)x}.
\end{aligned} \tag{56}$$

Thus all the joint steady state probabilities of the state of the system and the content of the buffer are explicitly obtained in terms of modified Bessel function of the first kind.

Remark 3. When $p = 0$ and $q = 0$, the model under consideration reduces to a fluid queue driven by an $M/M/1$ queue with working vacation and vacation interruption discussed by Xu et al. [25]. With $a_2 = 0$, the expression for $\hat{F}_{0,0}(s)$ from (39) yields

$$\hat{F}_{0,0}(s) = \frac{a_1 \sigma_0}{\lambda + s\sigma_0 - \mu_v r(s) - \mu \beta(s)}, \tag{57}$$

where

$$\begin{aligned}
r(s) &= \frac{\lambda}{\lambda + \mu_v + \theta_v + s\sigma}, \\
\beta(s) &= \frac{r(s) [\mu_v r(s) + \theta_v]}{\mu [z_1(s) - r(s)]}, \\
z_1(s) &= \frac{(\lambda + \mu + s\sigma) + \sqrt{(\lambda + \mu + s\sigma)^2 - 4\lambda\mu}}{2\mu}.
\end{aligned} \tag{58}$$

Observe that (57) is seen to coincide with the first expression in (6) of [25].

Remark 4. When $p = 1$, the model under consideration reduces to a fluid queue modulated by an $M/M/1$ queue with multiple exponential vacation discussed by Wang et al. [23].

Let $a_1 = 0$ and $z(s) = (\lambda/\mu)z_0(s)$, where $z_0(s) = ((\lambda + \mu + sr) - \sqrt{(\lambda + \mu + sr)^2 - 4\lambda\mu})/2\lambda$. Then the expression for $\hat{F}_{0,1}(s)$ from (39) becomes

$$\hat{F}_{0,1}(s) = \frac{a_2 \sigma_0}{\lambda + s\sigma_0 - \mu \gamma(s)}, \tag{59}$$

where $\gamma(s) = \lambda \theta z_0(s)/\mu(\lambda + \theta + s\sigma - \lambda z_0(s))$. Substituting $\gamma(s)$ in the above equation and after certain simplification, we obtain

$$\begin{aligned}
& \hat{F}_{0,1}(s) \\
&= \frac{a_2 \sigma_0 [s\sigma + \lambda(1 - z_0(s)) + \theta]}{(s\sigma_0 + \lambda)(s\sigma + \lambda(1 - z_0(s)) + \theta) - \lambda \theta z_0(s)}.
\end{aligned} \tag{60}$$

Observe that (60) is seen to coincide with (16) of [23].

Buffer Content Distribution. The stationary buffer content distribution of the fluid model under consideration is given by

$$\begin{aligned}
F(x) &= P\{C \leq x\} \\
&= \sum_{k=0}^{\infty} F_{k,0}(x) + \sum_{k=0}^{\infty} F_{k,1}(x) + \sum_{k=1}^{\infty} F_{k,2}(x).
\end{aligned} \tag{61}$$

Taking Laplace transform of the above equation yields

$$\hat{F}(s) = \sum_{k=0}^{\infty} \hat{F}_{k,0}(s) + \sum_{k=0}^{\infty} \hat{F}_{k,1}(s) + \sum_{k=1}^{\infty} \hat{F}_{k,2}(s). \tag{62}$$

In matrix notation, the above equation can be rewritten as

$$\hat{F}(s) = \hat{F}_0(s) e_1 + \sum_{k=1}^{\infty} \hat{F}_k(s) e_2. \tag{63}$$

Then,

$$\begin{aligned}
\hat{F}(s) &= \hat{F}_0(s) e_1 + \hat{F}_0(s) e \sum_{k=1}^{\infty} (R(s))^k e_2 \\
&\quad \text{(from (41))}
\end{aligned} \tag{64}$$

$$= \hat{F}_0(s) e_1 + \hat{F}_0(s) e [I - R(s)]^{-1} e_2,$$

where

$$\begin{aligned}
& [I - R(s)]^{-1} \\
&= \begin{pmatrix} \frac{1}{1-r(s)} & 0 & \frac{\beta(s)}{(1-r(s))(1-z(s))} \\ 0 & \frac{s\sigma + \lambda + \theta}{s\sigma + \theta} & \frac{\gamma(s)(s\sigma + \lambda + \theta)}{(1-z(s))(s\sigma + \theta)} \\ 0 & 0 & \frac{1}{1-z(s)} \end{pmatrix}.
\end{aligned} \tag{65}$$

Upon simplification, we get

$$\begin{aligned}
\hat{F}(s) &= \frac{1}{1-r(s)} \hat{F}_{0,0}(s) \\
&+ \frac{\beta(s)}{(1-r(s))(1-z(s))} \hat{F}_{0,0}(s) \\
&+ \frac{(s\sigma + \lambda + \theta)}{(s\sigma + \theta)} \hat{F}_{0,1}(s) \\
&- \frac{\gamma(s)(s\sigma + \lambda + \theta)}{(1-z(s))(s\sigma + \theta)} \hat{F}_{0,1}(s) \\
&= \sum_{k=0}^{\infty} (r(s))^k \hat{F}_{0,0}(s)
\end{aligned}$$

$$\begin{aligned}
& + \beta(s) \sum_{k=0}^{\infty} (r(s))^k \sum_{j=0}^{\infty} (z(s))^j \hat{F}_{0,0}(s) + \hat{F}_{0,1}(s) \\
& + \frac{\lambda}{s\sigma + \theta} \hat{F}_{0,1}(s) + \gamma(s) \sum_{k=0}^{\infty} (z(s))^k \hat{F}_{0,1}(s) \\
& + \gamma(s) \sum_{k=0}^{\infty} (z(s))^k \frac{\lambda}{s\sigma + \theta} \hat{F}_{0,1}(s)
\end{aligned} \tag{66}$$

which on inversion leads to

$$\begin{aligned}
F(x) &= \sum_{k=0}^{\infty} \left(\frac{\sigma}{2q\mu_v} \right)^k \frac{kI_k(\beta x) \beta^k}{x} e^{-((\lambda+\mu_v+\theta_v)/\sigma)x} \\
& * F_{0,0}(x) + \beta(x) \\
& * \sum_{k=0}^{\infty} \left(\frac{\sigma}{2q\mu_v} \right)^k \frac{kI_k(\beta x) \beta^k}{x} e^{-((\lambda+\mu_v+\theta_v)/\sigma)x} \\
& * \sum_{j=0}^{\infty} \left(\frac{\sigma}{2\mu} \right)^j \frac{jI_j(\alpha x) \alpha^j}{x} e^{-((\lambda+\mu)/\sigma)x} * F_{0,0}(x) \\
& + F_{0,1}(x) + \frac{\lambda}{\sigma} e^{-(\theta/\sigma)x} * F_{0,1}(x) + \gamma(x) \\
& * \sum_{k=0}^{\infty} \left(\frac{\sigma}{2\mu} \right)^k \frac{kI_k(\alpha x) \alpha^k}{x} e^{-((\lambda+\mu)/\sigma)x} * F_{0,1}(x) \\
& + \gamma(x) * \sum_{k=0}^{\infty} \left(\frac{\sigma}{2\mu} \right)^k \frac{kI_k(\alpha x) \alpha^k}{x} e^{-((\lambda+\mu)/\sigma)x} \\
& * \frac{\lambda}{\sigma} e^{-(\theta/\sigma)x} * F_{0,1}(x),
\end{aligned} \tag{67}$$

where $F_{0,0}(x)$ and $F_{0,1}(x)$ are given by (51) and (52), respectively. Thus all the joint steady state probabilities and the buffer content distribution of the fluid queue driven by an $M/M/1$ queue subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption are explicitly obtained under steady state. Explicit analytical expressions help the practitioner to better understand the behavior of any quantity of interest, like the mean buffer content, for varying values of the parameters involved in the model.

5. Numerical Illustrations

This section illustrates the variation of the buffer content distribution against the content of the buffer for $\lambda = 1, \mu = 2, \mu_v = 1.1, \theta = 0.9, \theta_v = 0.6, p = 0.5, q = 0.5, \sigma = 1$ and varying values of σ_0 . The choice of σ_0 is relatively high as compared to σ because of our assumptions that σ happens when the background queueing model is nonempty and σ_0 happens otherwise. To compensate for the rarity in the occurrence of σ_0 , it is assumed to be larger.

Figure 3 depicts the behavior of the buffer content distribution, $F(x)$, against x for the above choice of the parameter values with $\sigma_0 = -450.9$. For this choice of the parameters, it is seen that $d = -140.86 < 0$. Therefore, the stability condition is satisfied. It is seen that $F(x)$ increases

TABLE 1: Convergence of stationary buffer content distribution for varying values of σ_0 .

x	$F(x)$			
	$\sigma_0 = -140.9$	$\sigma_0 = -250.9$	$\sigma_0 = 350.9$	$\sigma_0 = -450.9$
0	0.3091	0.3112	0.3120	0.3124
0.5	0.4408	0.4434	0.4444	0.4449
1.0	0.5373	0.5403	0.5414	0.5421
1.5	0.6116	0.6148	0.616	0.6167
2.0	0.6706	0.674	0.6752	0.6759
2.5	0.7185	0.7219	0.7232	0.7238
3.0	0.7580	0.7614	0.7626	0.7633
3.5	0.7909	0.7942	0.7955	0.7961
4.0	0.8185	0.8218	0.823	0.8237
4.5	0.8419	0.8452	0.8463	0.847
5.0	0.8618	0.865	0.8662	0.8668
6.0	0.8936	0.8966	0.8977	0.8983
11.0	0.9670	0.9694	0.9703	0.9708
14.0	0.9819	0.9842	0.9850	0.9855
17.0	0.9892	0.9914	0.9922	0.9926
20.0	0.9928	0.9950	0.9958	0.9962
25.0	0.9954	0.9976	0.9983	0.9987
30.0	0.9962	0.9984	0.9992	0.9996
34.0	0.9964	0.9987	0.9994	0.9998
36.0	0.9964	0.9988	0.9995	0.9999

FIGURE 3: Variations of buffer content distribution against x .

with increase in the value of x and converges to 1 as x tends to infinity. Observe that

$$\begin{aligned}
& \lim_{x \rightarrow \infty} P(C < x) \\
&= \lim_{x \rightarrow \infty} \left(F_{0,0}(x) + F_{0,1}(x) + \sum_{k=1}^{\infty} \sum_{j=0}^2 F_{k,j}(x) \right) \\
&= \pi_{0,0} + \pi_{0,1} + \sum_{k=1}^{\infty} \sum_{j=0}^2 \pi_{k,j} = 1.
\end{aligned} \tag{68}$$

Furthermore, as the value of σ_0 greatly affects buffer content distribution, its variation against x for a different value of σ_0 is presented in Table 1.

6. Conclusion

Markov Modulated Fluid Flows (MMFF) are a class of fluid models wherein the rates at which the content of the fluid buffer varies are modulated by the Markov process evolving in the background. This paper studies a fluid model driven by an $M/M/1$ queue subject to Bernoulli-Schedule-Controlled Vacation and Vacation Interruption. The study of such models provides greater flexibility to the design and control of input and output rates of fluid flow thereby adapting the fluid models to wider application background. The governing system of infinite differential difference equations is explicitly solved using Laplace transform and matrix-geometric methodology. Most of the existing results in the literature pertaining to MMFF have presented the solution to the buffer content distribution in the Laplace domain. However, closed form analytical solutions help to gain a deeper insight into the model and other related performance measures. The current findings can be thought of as one of the key contributions to the theoretical development of MMFF rather than the practical context. The theoretical results so obtained are verified with the existing results in the literature as a special case. The variations of the stationary buffer content distribution against the content of the buffer for varying values of σ_0 are numerically illustrated.

Competing Interests

The authors declare that they have no competing interests.

References

- [1] Z. Huo, S. Jin, and N. Tian, "Performance analysis and evaluation for connection-oriented networks based on discrete time vacation queueing model," *Quality Technology & Quantitative Management*, vol. 5, no. 1, pp. 51–62, 2008.
- [2] V. C. Narayanan, T. G. Deepak, A. Krishnamoorthy, and B. Krishnakumar, "On an (s, S) inventory policy with service time, vacation to server and correlated lead time," *Quality Technology & Quantitative Management*, vol. 5, no. 2, pp. 129–143, 2008.
- [3] M. Jain and S. Upadhyaya, "Threshold N -policy for degraded machining system with multiple type spares and multiple vacations," *Quality Technology & Quantitative Management*, vol. 6, no. 2, pp. 185–203, 2009.
- [4] S. Gao and J. Wang, "Discrete-time $Geo^X/G/1$ retrial queue with general retrial times, working vacations and vacation interruption," *Quality Technology and Quantitative Management*, vol. 10, no. 4, pp. 495–512, 2013.
- [5] S. Gao and C. Yin, "Discrete-time $Geo^X/G/1$ queue with geometrically working vacations and vacation interruption," *Quality Technology and Quantitative Management*, vol. 10, no. 4, pp. 423–442, 2013.
- [6] P. Vijaya Laxmi, V. Goswami, and K. Jyothsna, "Analysis of discrete-time single server queue with balking and multiple working vacations," *Quality Technology and Quantitative Management*, vol. 10, no. 4, pp. 443–456, 2013.
- [7] J. Li and N. Tian, "The $M/M/1$ queue with working vacations and vacation interruptions," *Journal of Systems Science and Systems Engineering*, vol. 16, no. 1, pp. 121–127, 2007.
- [8] R. Bekker and M. Mandjes, "A fluid model for a relay node in an ad hoc network: the case of heavy-tailed input," *Mathematical Methods of Operations Research*, vol. 70, no. 2, pp. 357–384, 2009.
- [9] G. Latouche and P. G. Taylor, "A stochastic fluid model for an ad hoc mobile network," *Queueing Systems*, vol. 63, no. 1–4, pp. 109–129, 2009.
- [10] D. Mitra, "Stochastic theory of a fluid model of producers and consumers coupled by a buffer," *Advances in Applied Probability*, vol. 20, no. 3, pp. 646–676, 1988.
- [11] V. Aggarwal, N. Gautam, S. R. T. Kumara, and M. Greaves, "Stochastic fluid flow models for determining optimal switching thresholds," *Performance Evaluation*, vol. 59, no. 1, pp. 19–46, 2005.
- [12] V. G. Kulkarni and K. Yan, "A fluid model with upward jumps at the boundary," *Queueing Systems*, vol. 56, no. 2, pp. 103–117, 2007.
- [13] E. I. Tzenova, I. J. Adan, and V. G. Kulkarni, "Fluid models with jumps," *Stochastic Models*, vol. 21, no. 1, pp. 37–55, 2005.
- [14] K. Yan and V. G. Kulkarni, "Optimal inventory policies under stochastic production and demand rates," *Stochastic Models*, vol. 24, no. 2, pp. 173–190, 2008.
- [15] M. H. Veatch and J. R. Senning, "Fluid analysis of an input control problem," *Queueing Systems*, vol. 61, no. 2–3, pp. 87–112, 2009.
- [16] J. Ren, L. Breuer, D. A. Stanford, and K. Yu, "Perturbed risk processes analyzed as fluid flows," *Stochastic Models*, vol. 25, no. 3, pp. 522–544, 2009.
- [17] J. W. Bosman and R. Nunez-Queija, "A spectral theory approach for extreme value analysis in a tandem of fluid queues," *Queueing Systems. Theory and Applications*, vol. 78, no. 2, pp. 121–154, 2014.
- [18] I. Adan and J. Resing, "Simple analysis of a fluid queue driven by an $M/M/1$ queue," *Queueing Systems*, vol. 22, no. 1–2, pp. 171–174, 1996.
- [19] J. Virtamo and I. Norros, "Fluid queue driven by an $M/M/1$ queue," *Queueing Systems*, vol. 16, no. 3–4, pp. 373–386, 1994.
- [20] B. Sericola and B. Tuffin, "A fluid queue driven by a Markovian queue," *Queueing Systems*, vol. 31, no. 3–4, pp. 253–264, 1999.
- [21] P. R. Parthasarathy, K. V. Vijayashree, and R. B. Lenin, "An $M/M/1$ driven fluid queue—continued fraction approach," *Queueing Systems*, vol. 42, no. 2, pp. 189–199, 2002.
- [22] B. Mao, F. Wang, and N. Tian, "Fluid model driven by an $M/M/1$ queue with single exponential vacation," in *Proceedings of the 2nd International Conference on Information Technology and Computer Science*, pp. 539–542, June 2010.
- [23] F. Wang, B. Mao, and N. Tian, "Fluid model driven by an $M/M/1$ queue with multiple exponential vacations," in *Proceedings of the IEEE International Conference on Advanced Computer Control (ICACC '10)*, vol. 3, pp. 112–115, March 2010.
- [24] B.-W. Mao, F.-W. Wang, and N.-S. Tian, "Fluid model driven by an $M/M/1$ queue with multiple vacations and N -policy," *Journal of Applied Mathematics and Computing*, vol. 38, no. 1–2, pp. 119–131, 2012.
- [25] X. Xu, H. Guo, Y. Zhao, and J. Geng, "The fluid model driven by the $M/M/1$ queue with working vacations and vacation interruption," *Journal of Computational Information Systems*, vol. 8, no. 18, pp. 7643–7651, 2012.
- [26] K. V. Vijayashree and A. Anjuka, "Stationary analysis of an $M/M/1$ driven fluid queue subject to catastrophes and subsequent repair," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 4, pp. 238–241, 2013.

- [27] K. V. Vijayashree and A. Anjuka, "Fluid queue driven by an $M/E_2/1$ queueing model," in *Computational Intelligence, Cyber Security and Computational Models*, M. Senthilkumar, V. Ramasamy, S. Sheen, C. Veeramani, A. Bonato, and L. Batten, Eds., vol. 412 of *Advances in Intelligent Systems and Computing*, pp. 493–504, Springer India, New Delhi, India, 2016.
- [28] S. I. Ammar, "Analysis of an $M/M/1$ driven fluid queue with multiple exponential vacations," *Applied Mathematics and Computation*, vol. 227, pp. 329–334, 2014.
- [29] H. Zhang and D. Shi, "The $M/M/1$ queue with Bernoulli-schedule-controlled vacation and vacation interruption," *International Journal of Information and Management Sciences*, vol. 20, no. 4, pp. 579–587, 2009.

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