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Research Article

A Mathematical Model for Fuzzy p-Median Problem with Fuzzy Weights and Variables

Fatemeh Taleshian and Jafar Fathali

Department of Mathematics, Shahrood University of Technology, University Boulevard, Shahrood 3619995161, Iran

Correspondence should be addressed to Jafar Fathali; fathali@shahroodut.ac.ir

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We investigate the p-median problem with fuzzy variables and weights of vertices. The fuzzy equalities and inequalities transform to crisp cases by using some technique used in fuzzy linear programming. We show that the fuzzy objective function also can be replaced by crisp functions. Therefore an auxiliary linear programming model is obtained for the fuzzy p-median problem. The results are compared with two previously proposed methods.

1. Introduction

Location theory is an important topic in the fields of transportation and communication. The p-median problem is a classic problem in this line of investigation which consists of locating p facilities to cover the given demands such that the total transportation cost is minimized.

In the graph version of p-median problem it is shown that there exists an optimal solution that all the facilities are located at p vertices of the graph and the demand of each vertex will be totally covered by the nearest facility. The p-median problem for arbitrary p in general graphs is NP-hard [1]. For more information about location problems on networks see [2].

There are many situations in real world that can be modeled using p-median problem. In actual cases the amounts of parameters are seldom determined precisely. Hence the parameters are determined with some degree of uncertainty. On the other hand fuzzy set theory is the best tool to illustrate this uncertainty. That is, the amounts of parameters are considered as fuzzy numbers. In the p-median problem, the weight of each point represents the amounts of corresponding customers demand and the aim is to find the p best places for locating p facility center which provide customers demand. Therefore in the problem with ambiguous and uncertain demands, providing the exact amount of customer's need by facility centers is far from reality. Therefore it is expected that the value of objective function and the amounts of variables be in fuzzy form. However in last researches the exact amounts for objective value and variables were yielded. Thus in this paper we overcome this shortcoming and consider the variables as fuzzy variables.

The concept of decision making in fuzzy environment is presented by Bellman and Zadeh [3]. Many authors applied this concept for solving fuzzy linear programming problems. Lai and Hwang [4] provided an auxiliary multiple objective linear programming model to solve a linear programming problem with fuzzy constraint coefficients of objective functions. Recently Allahviranloo et al. [5] solved a full fuzzy linear programming using ranking function and Lotfi et al. [6] solved this kind of models by lexicography method and fuzzy approximate solution. Kumar et al. [7] proposed a new method for finding the fuzzy optimal solution of full fuzzy linear programming with equality constraints. Nasseri et al. [8] considered the case that constraints are in inequality forms and presented a new fuzzy solution for solving full fuzzy linear programming.

Many researchers consider the fuzzy location problems. Canós et al. [9] considered the fuzzy p-median problem. They presented a fuzzy formulation to combine the standard

minimization of transport costs with an acceptable reduction of the covered demand. Their algorithm considered only slight modifications of the total demand that should be covered and the optimal transport cost associated with it. In [10] the same method applied in a global sense of fuzzy p-median problem. For this problem Canós et al. [11] introduced some marginal analysis techniques to study how solutions depend on membership functions. Moreno-Perez et al. [12] considered some location problems with fuzzy weights and lengths and presented methods to solve them. Kutangila-Mayoya and Verdegay [13] proposed a formulation to find optimal solution for the p-median problem in a fuzzy environment when data related to the node demands and the edge distances are imprecise and uncertain. Many other fuzzy location problems are studied by authors (e.g., see [14, 15]).

In this paper we consider the p-median problem where the demands of clients and variables are fuzzy numbers and fuzzy variables, respectively. We show the fuzzy model can be transformed to a crisp linear programming. Our method is the extended method of Lai and Hwang [4]. Their model was linear programming and we extend the method for mixed integer programming problem and apply it for the p-median problem.

In what follows in this paper the model of crisp p-median and some basic definitions and arithmetics between two triangular fuzzy numbers are reviewed in Section 2. In Section 3 the fuzzy model is converted to a linear programming model. To illustrate the proposed method, numerical examples are solved and the obtained results are discussed and compared with two other methods in Section 4.

2. Preliminaries

In this section the crisp model of p-median problem and some necessary notions of fuzzy set theory are reviewed.

2.1. The Crisp p-Median Model. Let $V = \{v_1, \ldots, v_n\}$ be n existing points. Each point v_j has a nonnegative weight w_j, usually called the demand at v_j, and d_{ij} is the distance between points v_i and v_j. The p-median problem asks to select p facilities of these n points such that the sum of the weighted distances of the existing points to the closest facility is minimized.

The first integer linear programming for the p-median problem was presented by ReVelle and Swain [16]. A general integer programming for this problem can be written as follows.

Let x_{ij} be the demand of customer in point v_j which provide by facility in v_i and

$$y_i = \begin{cases} 1 & \text{if point } v_i \text{ is selected as a facility} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and then the model can be written as follows:

$$\begin{aligned} (P_1)\ \min \quad & z = \sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_{ij} = w_j \quad j = 1, \ldots, n \\ & \sum_{i=1}^{n} y_i = p \\ & x_{ij} \le y_i w_j \quad i = 1, \ldots, n \ j = 1, \ldots, n \\ & y_i \in \{0, 1\} \quad i = 1, \ldots, n \\ & x_{ij} \ge 0 \quad i = 1, \ldots, n \ j = 1, \ldots, n. \end{aligned} \tag{2}$$

2.2. Fuzzy Backgrounds and Notations

Definition 1 (see [17]). Let $\widetilde{A} = \langle a_1, a_2, a_3 \rangle$ and $\widetilde{B} = \langle b_1, b_2, b_3 \rangle$ be two triangular fuzzy numbers and then the fuzzy operators are defined as follows:

(1) $\widetilde{A} \oplus \widetilde{B} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$.

(2) $\widetilde{A} \simeq \widetilde{B}$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

(3) $\widetilde{A} \gtrsim 0$ if and only if $a_1 \ge 0$.

Definition 2 (see [18]). Let $F(R)$ be a set of fuzzy numbers defined on set of real numbers. A ranking function is a function $\mathbb{R} : F(R) \to R$ which maps each fuzzy number into the real line, where a natural order exists.

In this paper we use the following ranking function:

$$\mathbb{R}(a, b, c) = \frac{1}{4}(a + 2b + c). \tag{3}$$

Remark 3 (see [18]). An inequality of fuzzy numbers $\widetilde{X} \lesssim \widetilde{A}$ can be transformed to equality, by adding fuzzy numbers $\widetilde{S}$ and $\widetilde{T}$ to the left and right sides of inequality, respectively; that is,

$$\widetilde{X} \oplus \widetilde{S} = \widetilde{A} \oplus \widetilde{T}, \tag{4}$$

where $\mathbb{R}(\widetilde{S}) - \mathbb{R}(\widetilde{T}) \ge 0$.

3. Fuzzy Models

Let $\widetilde{x}_{ij} = \langle x_{ij1}, x_{ij2}, x_{ij3} \rangle$ and $\widetilde{w}_{ij} = \langle w_{ij1}, w_{ij2}, w_{ij3} \rangle$ be triangular fuzzy numbers corresponding to x_{ij} and w_{ij}, respectively. Then the fuzzy model of (P_1) can be written as follows:

$$\begin{aligned} (P_2)\ \min \quad & \sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij} \left\langle x_{ij1}, x_{ij2}, x_{ij3} \right\rangle \\ \text{s.t.} \quad & \sum_{i=1}^{n} \left\langle x_{ij1}, x_{ij2}, x_{ij3} \right\rangle \simeq \left\langle w_{ij1}, w_{ij2}, w_{ij3} \right\rangle \\ & j = 1, \ldots, n \end{aligned} \tag{5}$$

$$\begin{aligned}
&\sum_{i=1}^{n} y_i = p \\
&\left\langle x_{ij1}, x_{ij2}, x_{ij3} \right\rangle \lesssim y_i \left\langle w_{ij1}, w_{ij2}, w_{ij3} \right\rangle \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&y_i \in \{0, 1\} \quad i = 1, \ldots, n \\
&x_{ij1} \geq 0 \quad i = 1, \ldots, n \;\; j = 1, \ldots, n.
\end{aligned} \tag{6}$$

Let $\widetilde{X} = [\widetilde{x}_{ij}]_{n\times n}$ and $Y = [y_i]_{n\times n}$. Then $[\widetilde{X}, Y]$ is the optimal solution of problem (P_2) if it satisfies the following characteristics:

(1) $\widetilde{X}$ is a nonnegative fuzzy number.

(2) $\widetilde{X}$ and Y satisfy conditions (6).

(3) If there exist any nonnegative fuzzy number $\widetilde{X}'$ and a vector Y' such that they satisfy conditions (6), then

$$\begin{aligned}
&\mathbb{R}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij} \left\langle x_{ij1}, x_{ij2}, x_{ij3} \right\rangle\right) \\
&\quad \leq \mathbb{R}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij} \left\langle x'_{ij1}, x'_{ij2}, x'_{ij3} \right\rangle\right).
\end{aligned} \tag{7}$$

Model (P_2) can be transformed to a crisp model by the following steps.

Step 1. Let $z_{ijk} = d_{ij}x_{ijk}$ and $v_{ijk} = y_i w_{jk}$ for $i, j = 1, \ldots, n$ and $k = 1, 2, 3$; then model (P_2) is converted to the following model:

$$(P_3) \;\; \min \quad \sum_{j=1}^{n}\sum_{i=1}^{n} \left\langle z_{ij1}, z_{ij2}, z_{ij3} \right\rangle \tag{8}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \left\langle x_{ij1}, x_{ij2}, x_{ij3} \right\rangle \simeq \left\langle w_{ij1}, w_{ij2}, w_{ij3} \right\rangle \qquad j = 1, \ldots, n \tag{9}$$

$$\sum_{i=1}^{n} y_i = p \tag{10}$$

$$\left\langle x_{ij1}, x_{ij2}, x_{ij3} \right\rangle \lesssim \left\langle v_{ij1}, v_{ij2}, v_{ij3} \right\rangle \qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \tag{11}$$

$$y_i \in \{0, 1\} \quad i = 1, \ldots, n. \tag{12}$$

$$x_{ij1} \geq 0 \quad i = 1, \ldots, n \;\; j = 1, \ldots, n. \tag{13}$$

Step 2. Using Remark 3, by adding fuzzy variables $\widetilde{s}_{ij} = \langle s_{ij1}, s_{ij2}, s_{ij3} \rangle$ and $\widetilde{t}_{ij} = \langle t_{ij1}, t_{ij2}, t_{ij3} \rangle$ for $i = 1, \ldots, n$ $j = 1, \ldots, n$ inequality (11) can be replaced by the following equalities:

$$\begin{aligned}
&\left\langle x_{ij1}, x_{ij2}, x_{ij3} \right\rangle \oplus \left\langle s_{ij1}, s_{ij2}, s_{ij3} \right\rangle \\
&\quad = \left\langle v_{ij1}, v_{ij2}, v_{ij3} \right\rangle \oplus \left\langle t_{ij1}, t_{ij2}, t_{ij3} \right\rangle \\
&\mathbb{R}\left(\widetilde{s}_{ij}\right) - \mathbb{R}\left(\widetilde{t}_{ij}\right) \geq 0.
\end{aligned} \tag{14}$$

Step 3. With definition of fuzzy operators, the fuzzy equalities can be replaced by crisp equalities. Then the model can be written as follows:

$$\begin{aligned}
(P_4) \;\; \min \quad & \sum_{j=1}^{n}\sum_{i=1}^{n} \left\langle z_{ij1}, z_{ij2}, z_{ij3} \right\rangle \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_{ij1} = w_{ij1} \quad j = 1, \ldots, n \\
& \sum_{i=1}^{n} x_{ij2} = w_{ij2} \quad j = 1, \ldots, n \\
& \sum_{i=1}^{n} x_{ij3} = w_{ij3} \quad j = 1, \ldots, n \\
& \sum_{i=1}^{n} y_i = p \\
& x_{ij1} + s_{ij1} = v_{ij1} + t_{ij1} \\
& \qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
& x_{ij2} + s_{ij2} = v_{ij2} + t_{ij2} \\
& \qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
& x_{ij3} + s_{ij3} = v_{ij3} + t_{ij3} \\
& \qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
& \left(s_{ij1} + 2s_{ij2} + s_{ij3}\right) - \left(t_{ij1} + 2t_{ij2} + t_{ij3}\right) \\
& \geq 0 \\
& \qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
& x_{ij1} \geq 0, \\
& x_{ij2} - x_{ij1} \geq 0, \\
& x_{ij3} - x_{ij2} \geq 0 \\
& \qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
& s_{ij2} - s_{ij1} \geq 0, \\
& s_{ij3} - s_{ij2} \geq 0 \\
& \qquad i = 1, \ldots, n \;\; j = 1, \ldots, n
\end{aligned}$$

$$
\begin{aligned}
&t_{ij2} - t_{ij1} \geq 0, \\
&t_{ij3} - t_{ij2} \geq 0 \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&y_i \in \{0, 1\} \quad i = 1, \ldots, n.
\end{aligned} \tag{15}
$$

In the following steps we consider the objective function of model (P_4) and convert model to a crisp model by using the method of Lai and Hwang [4].

Step 4. Consider the following objective function:

$$\min \quad \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle z_{ij1}, z_{ij2}, z_{ij3} \right\rangle. \tag{16}$$

We have

$$
\begin{aligned}
&\sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle z_{ij1}, z_{ij2}, z_{ij3} \right\rangle \\
&\quad = \left\langle \sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij1}, \sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij2}, \sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij3} \right\rangle
\end{aligned} \tag{17}
$$

in which $\sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij1}$ is the most possible value and $\sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij2}$ and $\sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij3}$ are the least possible values. This fuzzy objective is fully defined by three corner points geometrically. Thus, minimizing the fuzzy objective can be obtained by pushing these three critical points in the direction of the left-hand side (see Figure 1). Therefore the objective function can be replaced by the following auxiliary functions:

$$
\begin{aligned}
z_1 &= \max \sum_{j=1}^{n} \sum_{i=1}^{n} \left(z_{ij2} - z_{ij1} \right), \\
z_2 &= \min \sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij2}, \\
z_3 &= \min \sum_{j=1}^{n} \sum_{i=1}^{n} \left(z_{ij3} - z_{ij2} \right).
\end{aligned} \tag{18}
$$

The three new objectives also guarantee the previous argument of pushing the triangular possibility distribution in the direction of the left-hand side.

Step 5. Let F be the set of feasible solutions of model (P_4). Consider the Positive Ideal Solutions (PIS) and Negative Ideal Solutions (NIS) of the three objective functions as follows:

$$
\begin{aligned}
z_1^{\mathrm{PIS}} &= \max \sum_{j=1}^{n} \sum_{i=1}^{n} \left(z_{ij2} - z_{ij1} \right), \quad \left[\widetilde{X}, Y \right] \in F, \\
z_1^{\mathrm{NIS}} &= \min \sum_{j=1}^{n} \sum_{i=1}^{n} \left(z_{ij2} - z_{ij1} \right), \quad \left[\widetilde{X}, Y \right] \in F,
\end{aligned}
$$

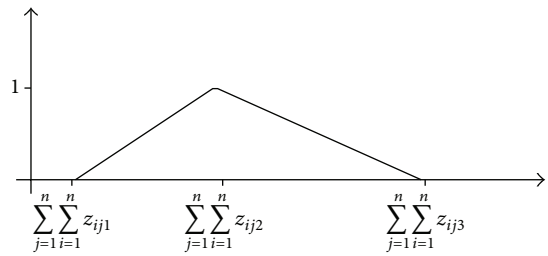

Figure 1: Objective function in Step 4.

$$
\begin{aligned}
z_2^{\mathrm{PIS}} &= \min \sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij2}, \quad \left[\widetilde{X}, Y \right] \in F, \\
z_2^{\mathrm{NIS}} &= \max \sum_{j=1}^{n} \sum_{i=1}^{n} z_{ij2}, \quad \left[\widetilde{X}, Y \right] \in F, \\
z_3^{\mathrm{PIS}} &= \min \sum_{j=1}^{n} \sum_{i=1}^{n} \left(z_{ij3} - z_{ij2} \right), \quad \left[\widetilde{X}, Y \right] \in F, \\
z_3^{\mathrm{NIS}} &= \max \sum_{j=1}^{n} \sum_{i=1}^{n} \left(z_{ij3} - z_{ij2} \right), \quad \left[\widetilde{X}, Y \right] \in F.
\end{aligned} \tag{19}
$$

Then the linear membership function of the objective functions can be computed as follows:

$$
\mu_{z1} = \begin{cases} 1 & \text{if } z_1 > z_1^{\mathrm{PIS}} \\ \dfrac{z_1^{\mathrm{NIS}} - z_1}{z_1^{\mathrm{NIS}} - z_1^{\mathrm{PIS}}} & \text{if } z_1^{\mathrm{NIS}} \leq z_1 \leq z_1^{\mathrm{PIS}} \\ 0 & \text{if } z_1 < z_1^{\mathrm{NIS}}, \end{cases} \tag{20}
$$

and for $k = 2, 3$,

$$
\mu_{zk} = \begin{cases} 1 & \text{if } z_k < z_k^{\mathrm{PIS}} \\ \dfrac{z_k^{\mathrm{NIS}} - z_k}{z_k^{\mathrm{NIS}} - z_k^{\mathrm{PIS}}} & \text{if } z_k^{\mathrm{PIS}} \leq z_k \leq z_k^{\mathrm{NIS}} \\ 0 & \text{if } z_k > z_k^{\mathrm{NIS}}. \end{cases} \tag{21}
$$

Now we solve the following equivalent single-objective linear programming model introduced by Zimmermann [19]:

$$
\begin{aligned}
(P_5) \;\; \max \quad & \alpha \\
\text{s.t.} \quad & \mu_{zk} \geq \alpha \quad k = 1, 2, 3 \\
& \sum_{i=1}^{n} x_{ij1} = w_{ij1} \quad j = 1, \ldots, n \\
& \sum_{i=1}^{n} x_{ij2} = w_{ij2} \quad j = 1, \ldots, n
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n} x_{ij3} = w_{ij3} \quad j = 1, \ldots, n \\
&\sum_{i=1}^{n} y_i = p \\
&x_{ij1} + s_{ij1} = v_{ij1} + t_{ij1} \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&x_{ij2} + s_{ij2} = v_{ij2} + t_{ij2} \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&x_{ij3} + s_{ij3} = v_{ij3} + t_{ij3} \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&\left(s_{ij1} + 2s_{ij2} + s_{ij3}\right) - \left(t_{ij1} + 2t_{ij2} + t_{ij3}\right) \\
&\geq 0 \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&x_{ij1} \geq 0, \\
&x_{ij2} - x_{ij1} \geq 0, \\
&x_{ij3} - x_{ij2} \geq 0 \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&s_{ij2} - s_{ij1} \geq 0, \\
&s_{ij3} - s_{ij2} \geq 0 \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&t_{ij2} - t_{ij1} \geq 0, \\
&t_{ij3} - t_{ij2} \geq 0 \\
&\qquad\qquad i = 1, \ldots, n \;\; j = 1, \ldots, n \\
&y_i \in \{0, 1\} \quad i = 1, \ldots, n.
\end{aligned}
\tag{22}
$$

4. Comparing with Other Methods and Numerical Examples

In this section we compare our method with the methods of Nasseri et al. [8] and Moreno-Perez et al. [12] and use our method to solve some test problems.

First to compare our method with Nasseri et al. [8], we should mention that they used the following ranking for defuzzification fuzzy constraints:

$$
\begin{aligned}
&\langle b_1, b_2, b_3 \rangle \leq \langle a_1, a_2, a_3 \rangle \\
&\qquad\qquad \text{if } b_1 \leq a_1,\; b_2 \leq a_2,\; b_3 \leq a_3.
\end{aligned}
\tag{23}
$$

Their method is not always able to presented correct ranking. It is clear that their method is not able to rank the following two fuzzy numbers: $A = \langle 1, 2, 3 \rangle$ and $B = \langle 0.99, 200, 201 \rangle$. In this paper we deal with this problem. By adding fuzzy variable, each fuzzy inequality constraint reduces to equality form. Then each fuzzy equality constraint transforms to three constraints with constant coefficients as follows:

$$
\begin{aligned}
&\langle b_1, b_2, b_3 \rangle \oplus \left\langle s_{ij1}, s_{ij2}, s_{ij3} \right\rangle \\
&\quad = \langle a_1, a_2, a_3 \rangle \oplus \left\langle t_{ij1}, t_{ij2}, t_{ij3} \right\rangle,
\end{aligned}
\tag{24}
$$

where $R(\tilde{s}_{ij}) - R(\tilde{t}_{ij}) \leq 0$.

Second to compare our method with method of Moreno-Perez et al. [12], in the following example, first we solve the crisp case and then explain our method to solve the problem with fuzzy weights and variables. Finally we solve the fuzzy model by method of Moreno-Perez et al. [12] and compare their results with those obtained by our method.

Table 1: The coordinates and demands of existing facilities for Example 1.

j	a_{j1}	a_{j2}	w_j
1	1	1	5
2	3	7	8
3	5	5	7
4	7	10	12
5	10	3	15

Example 1. Consider a problem of finding two medians of five existing points on the plane. The coordinates of existing facilities and corresponding demands are given in Table 1.

Using the Euclidean distance the distance matrix of these points is obtained as follows:

$$
D = \begin{bmatrix}
0 & 6.324 & 5.656 & 10.816 & 9.219 \\
6.324 & 0 & 2.828 & 5 & 8.062 \\
5.656 & 2.828 & 0 & 5.385 & 5.385 \\
10.816 & 5 & 5.385 & 0 & 7.615 \\
9.219 & 8.062 & 5.385 & 7.615 & 0
\end{bmatrix}.
\tag{25}
$$

Thus the problem can be formulated as follows:

$$
\begin{aligned}
\min \quad z &= \sum_{j=1}^{5} \sum_{i=1}^{5} d_{ij} x_{ij} \\
&= 6.324x_{12} + 5.656x_{13} + 10.816x_{14} \\
&\quad + 9.219x_{15} + 6.324x_{21} + 2.828x_{23} + 5x_{24} \\
&\quad + 8.062x_{25} + 5.656x_{31} + 2.828x_{32} \\
&\quad + 5.385x_{34} + 5.385x_{35} + 10.816x_{41} + 5x_{42} \\
&\quad + 5.385x_{43} + 7.615x_{45} + 9.219x_{51} \\
&\quad + 8.062x_{52} + 5.385x_{53} + 7.615x_{54}
\end{aligned}
$$

Table 2: The fuzzy weights of demand points.

j	1	2	3	4	5
$\widetilde{w}_j = \langle w_{1j}, w_{2j}, w_{3j} \rangle$	$\langle 4, 5, 6 \rangle$	$\langle 6.5, 8, 10 \rangle$	$\langle 5, 7, 8 \rangle$	$\langle 11.5, 12, 13 \rangle$	$\langle 14, 15, 15.5 \rangle$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{i=1}^{5} x_{i1} = 5 \\
& \sum_{i=1}^{5} x_{i2} = 8 \\
& \sum_{i=1}^{5} x_{i3} = 7 \\
& \sum_{i=1}^{5} x_{i4} = 12 \\
& \sum_{i=1}^{5} x_{i5} = 15 \\
& \sum_{i=1}^{5} y_i = 1 \\
& x_{ij} \le y_i w_j \quad i = 1, \ldots, 5 \;\; j = 1, \ldots, 5 \\
& y_i \in \{0, 1\} \quad i = 1, \ldots, 5.
\end{aligned} \tag{26}
$$

By solving this problem the value of objective function and the amounts of variables are obtained as follows:

$$
\begin{aligned}
& Z^* = 111.422, \\
& y_2 = y_5 = 1, \\
& x_{12} = 5, \\
& x_{22} = 8, \\
& x_{32} = 7, \\
& x_{42} = 12, \\
& x_{55} = 15.
\end{aligned} \tag{27}
$$

Now suppose that the corresponding demands are all triangular fuzzy numbers as shown in Table 2.

The fuzzy model is as follows:

$$
\begin{aligned}
\min \quad z = & \sum_{j=1}^{5} \sum_{i=1}^{5} d_{ij} \left\langle x_{1ij}, x_{2ij}, x_{3ij} \right\rangle \\
= & \; 6.324 \left\langle x_{112}, x_{212}, x_{312} \right\rangle \\
& + 5.656 \left\langle x_{113}, x_{213}, x_{313} \right\rangle \\
& + 10.816 \left\langle x_{114}, x_{214}, x_{314} \right\rangle \\
& + 9.219 \left\langle x_{115}, x_{215}, x_{315} \right\rangle \\
& + 6.324 \left\langle x_{121}, x_{221}, x_{321} \right\rangle \\
& + 2.828 \left\langle x_{123}, x_{223}, x_{323} \right\rangle \\
& + 5 \left\langle x_{124}, x_{224}, x_{324} \right\rangle \\
& + 8.062 \left\langle x_{125}, x_{225}, x_{325} \right\rangle \\
& + 5.656 \left\langle x_{131}, x_{231}, x_{331} \right\rangle \\
& + 2.828 \left\langle x_{132}, x_{232}, x_{332} \right\rangle \\
& + 5.385 \left\langle x_{134}, x_{234}, x_{334} \right\rangle \\
& + 5.385 \left\langle x_{135}, x_{235}, x_{335} \right\rangle \\
& + 10.816 \left\langle x_{141}, x_{241}, x_{341} \right\rangle \\
& + 5 \left\langle x_{142}, x_{242}, x_{342} \right\rangle \\
& + 5.385 \left\langle x_{143}, x_{243}, x_{343} \right\rangle \\
& + 7.615 \left\langle x_{145}, x_{245}, x_{345} \right\rangle \\
& + 9.219 \left\langle x_{151}, x_{251}, x_{351} \right\rangle \\
& + 8.062 \left\langle x_{152}, x_{252}, x_{352} \right\rangle \\
& + 5.385 \left\langle x_{153}, x_{253}, x_{353} \right\rangle \\
& + 7.615 \left\langle x_{154}, x_{254}, x_{354} \right\rangle \\
\text{s.t.} \quad & \sum_{i=1}^{5} \left\langle x_{1i1}, x_{2i1}, x_{3i1} \right\rangle = \langle 4, 5, 6 \rangle, \\
& \sum_{i=1}^{5} \left\langle x_{1i2}, x_{2i2}, x_{3i2} \right\rangle = \langle 6.5, 8, 10 \rangle \\
& \sum_{i=1}^{5} \left\langle x_{1i3}, x_{2i3}, x_{3i3} \right\rangle = \langle 5, 7, 8 \rangle, \\
& \sum_{i=1}^{5} \left\langle x_{1i4}, x_{2i4}, x_{3i4} \right\rangle = \langle 11.5, 12, 13 \rangle \\
& \sum_{i=1}^{5} \left\langle x_{1i5}, x_{2i5}, x_{3i5} \right\rangle = \langle 14, 15, 15.5 \rangle, \\
& \sum_{i=1}^{5} y_i = 1 \\
& \left\langle x_{1ij}, x_{2ij}, x_{3ij} \right\rangle \le y_i \left\langle w_{1j}, w_{2j}, w_{3j} \right\rangle \\
& \qquad\qquad i = 1, \ldots, 5 \;\; j = 1, \ldots, 5 \\
& y_i \in \{0, 1\} \quad i = 1, \ldots, 5.
\end{aligned} \tag{28}
$$

By adding the fuzzy variables $\langle s_{1ij}, s_{2ij}, s_{3ij}\rangle$ and $\langle t_{1ij}, t_{2ij}, t_{3ij}\rangle$ for $i, j = 1, \ldots, 5$ and replacing $y_i\langle w_{1j}, w_{2j}, w_{3j}\rangle$ by $\langle v_{1ij}, v_{2ij}, v_{3ij}\rangle$ for $i, j = 1, \ldots, 5$, the constraints are reduced to the following:

$$\begin{aligned}
&\sum_{i=1}^{5} x_{1i1} = 4,\\
&\sum_{i=1}^{5} x_{2i1} = 5,\\
&\sum_{i=1}^{5} x_{3i1} = 6,\\
&\sum_{i=1}^{5} x_{1i2} = 6.5,\\
&\sum_{i=1}^{5} x_{2i2} = 8,\\
&\sum_{i=1}^{5} x_{3i2} = 10,\\
&\sum_{i=1}^{5} x_{1i3} = 5,\\
&\sum_{i=1}^{5} x_{2i3} = 7,\\
&\sum_{i=1}^{5} x_{3i3} = 8,\\
&\sum_{i=1}^{5} x_{1i4} = 11.5,\\
&\sum_{i=1}^{5} x_{2i4} = 12,\\
&\sum_{i=1}^{5} x_{3i4} = 13,\\
&\sum_{i=1}^{5} x_{1i5} = 14,\\
&\sum_{i=1}^{5} x_{2i5} = 15,\\
&\sum_{i=1}^{5} x_{3i5} = 15.5,\\
&\sum_{i=1}^{5} y_i = 1,\\
&x_{1ij} + s_{1ij} = v_{1ij} + t_{1ij}, \quad i, j = 1, \ldots, 5\\
&x_{2ij} + s_{2ij} = v_{2ij} + t_{2ij}, \quad i, j = 1, \ldots, 5\\
&x_{3ij} + s_{3ij} = v_{3ij} + t_{3ij}, \quad i, j = 1, \ldots, 5\\
&\left(s_{1ij}, s_{2ij}, s_{3ij}\right) - \left(t_{1ij}, t_{2ij}, t_{3ij}\right) \geq 0, \quad i, j = 1, \ldots, 5\\
&x_{1ij} \geq 0,\\
&x_{2ij} - x_{1ij} \geq 0,\\
&x_{3ij} - x_{2ij} \geq 0, \quad i, j = 1, \ldots, 5\\
&s_{2ij} - s_{1ij} \geq 0,\\
&s_{3ij} - s_{2ij} \geq 0, \quad i, j = 1, \ldots, 5\\
&t_{2ij} - t_{1ij} \geq 0,\\
&t_{3ij} - t_{2ij} \geq 0, \quad i, j = 1, \ldots, 5\\
&y_i \in \{0, 1\} \quad i = 1, \ldots, 5.
\end{aligned} \tag{29}$$

The objective function also can be transformed to the following tree crisp objective functions:

$$\begin{aligned}
\max \quad z_1 &= \sum_{j=1}^{5}\sum_{i=1}^{5}\left(z_{2ij} - z_{1ij}\right),\\
\min \quad z_2 &= \sum_{j=1}^{5}\sum_{i=1}^{5} z_{2ij},\\
\min \quad z_3 &= \sum_{j=1}^{5}\sum_{i=1}^{5}\left(z_{3ij} - z_{2ij}\right),
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
z_1 = {} & 6.324\left(x_{212} - x_{112}\right) + 5.656\left(x_{213} - x_{113}\right)\\
& + 10.816\left(x_{214} - x_{114}\right) + 9.219\left(x_{215} - x_{115}\right)\\
& + 6.324\left(x_{221} - x_{121}\right) + 2.828\left(x_{223} - x_{123}\right)\\
& + 5\left(x_{224} - x_{124}\right) + 8.062\left(x_{225} - x_{125}\right)
\end{aligned}$$

Table 3: The demands of existing facilities for Example 2.

$W_1 = \langle 23, 24, 25\rangle$	$W_2 = \langle 9, 10, 10.5\rangle$	$W_3 = \langle 16.5, 17.5, 18\rangle$
$W_4 = \langle 19, 20, 21\rangle$	$W_5 = \langle 8, 9, 9.5\rangle$	$W_6 = \langle 18.5, 20, 21\rangle$
$W_7 = \langle 11, 11.5, 12\rangle$	$W_8 = \langle 18, 19, 20\rangle$	$W_9 = \langle 14, 15, 16.5\rangle$
$W_{10} = \langle 4, 5, 6\rangle$	$W_{11} = \langle 21, 22, 23\rangle$	$W_{12} = \langle 21, 22, 23\rangle$
$W_{13} = \langle 17.5, 18.5, 19.5\rangle$	$W_{14} = \langle 8, 9, 10\rangle$	$W_{15} = \langle 25, 26, 27\rangle$
$W_{16} = \langle 11, 11.5, 12.5\rangle$	$W_{17} = \langle 6, 7, 8\rangle$	$W_{18} = \langle 19, 20, 21\rangle$
$W_{19} = \langle 4.5, 5.5, 6\rangle$	$W_{20} = \langle 4.5, 5.5, 6.5\rangle$	$W_{21} = \langle 26, 27, 28\rangle$
$W_{22} = \langle 9, 10, 11\rangle$	$W_{23} = \langle 19, 20, 21\rangle$	$W_{24} = \langle 9, 10, 11\rangle$
$W_{25} = \langle 11, 12, 13\rangle$	$W_{26} = \langle 3, 4, 4.5\rangle$	$W_{27} = \langle 22, 23, 24\rangle$
$W_{28} = \langle 14.5, 15, 16\rangle$	$W_{29} = \langle 7, 7.5, 9\rangle$	$W_{30} = \langle 27, 28, 29\rangle$
$W_{31} = \langle 19.5, 20, 21\rangle$	$W_{32} = \langle 14, 15, 16\rangle$	$W_{33} = \langle 9.5, 10, 11\rangle$
$W_{34} = \langle 5, 6, 7\rangle$	$W_{35} = \langle 12, 13, 14\rangle$	$W_{36} = \langle 29, 30, 31\rangle$
$W_{37} = \langle 25, 26, 27\rangle$	$W_{38} = \langle 13, 14, 15\rangle$	$W_{39} = \langle 6.5, 7.5, 8.5\rangle$
$W_{40} = \langle 9, 10, 11\rangle$	$W_{41} = \langle 19, 20, 21\rangle$	$W_{42} = \langle 7.5, 8.5, 9.5\rangle$
$W_{43} = \langle 12, 13, 14\rangle$	$W_{44} = \langle 21, 22, 23\rangle$	$W_{45} = \langle 18, 19, 20\rangle$
$W_{46} = \langle 6, 7, 8\rangle$	$W_{47} = \langle 30.5, 31, 32\rangle$	$W_{48} = \langle 10.5, 12, 13\rangle$
$W_{49} = \langle 9.5, 11, 12\rangle$	$W_{50} = \langle 9.5, 10, 11\rangle$	$W_{51} = \langle 24, 25, 26\rangle$
$W_{52} = \langle 19.5, 20, 21\rangle$	$W_{53} = \langle 18.5, 19.5, 20.5\rangle$	$W_{54} = \langle 12, 13, 14\rangle$
$W_{55} = \langle 4.5, 5, 5.5\rangle$	$W_{56} = \langle 16.5, 17, 18\rangle$	$W_{57} = \langle 19.5, 20, 21\rangle$
$W_{58} = \langle 7.5, 9, 10.5\rangle$	$W_{59} = \langle 26, 27, 28\rangle$	$W_{60} = \langle 10, 11, 12\rangle$
$W_{61} = \langle 8.5, 9, 9.5\rangle$	$W_{62} = \langle 22, 23, 24\rangle$	$W_{63} = \langle 19, 20, 21\rangle$
$W_{64} = \langle 30, 31, 32\rangle$	$W_{65} = \langle 29.5, 30, 31\rangle$	$W_{66} = \langle 12, 13, 14\rangle$
$W_{67} = \langle 4, 5, 6\rangle$	$W_{68} = \langle 14, 15, 16\rangle$	$W_{69} = \langle 14, 15, 16\rangle$
$W_{70} = \langle 10, 11, 12\rangle$	$W_{71} = \langle 20, 21, 22\rangle$	$W_{72} = \langle 27.5, 28.5, 29\rangle$
$W_{73} = \langle 9, 10, 11\rangle$	$W_{74} = \langle 14, 15, 15.5\rangle$	$W_{75} = \langle 30, 31, 32.5\rangle$
$W_{76} = \langle 8.5, 9, 10\rangle$	$W_{77} = \langle 19, 20, 21\rangle$	$W_{78} = \langle 6.5, 7, 8.5\rangle$
$W_{79} = \langle 11, 12, 13\rangle$	$W_{80} = \langle 20, 21, 22\rangle$	$W_{81} = \langle 8.5, 9.5, 10\rangle$
$W_{82} = \langle 14.5, 15, 16.5\rangle$	$W_{83} = \langle 25, 26, 27\rangle$	$W_{84} = \langle 23, 24, 25\rangle$
$W_{85} = \langle 20, 21, 22\rangle$	$W_{86} = \langle 20.5, 11, 12\rangle$	$W_{87} = \langle 16.5, 17.5, 19\rangle$
$W_{88} = \langle 14, 15, 16.5\rangle$	$W_{89} = \langle 30, 31, 32\rangle$	$W_{90} = \langle 27, 28, 29\rangle$
$W_{91} = \langle 9.5, 10.5, 11.5\rangle$	$W_{92} = \langle 9.5, 7.5, 8\rangle$	$W_{93} = \langle 16, 17, 18\rangle$
$W_{94} = \langle 18, 19, 20\rangle$	$W_{95} = \langle 8.5, 9.5, 10\rangle$	$W_{96} = \langle 12.5, 13, 14\rangle$
$W_{97} = \langle 4, 5, 6\rangle$	$W_{98} = \langle 28, 29, 30\rangle$	$W_{99} = \langle 18.5, 19, 20\rangle$
$W_{100} = \langle 7.5, 8.5, 9\rangle$		

$$
\begin{aligned}
&+ 5.656\left(x_{231} - x_{131}\right) + 2.828\left(x_{232} - x_{132}\right)\\
&+ 5.385\left(x_{234} - x_{134}\right) + 5.385\left(x_{235} - x_{135}\right)\\
&+ 10.816\left(x_{241} - x_{141}\right) + 5\left(x_{242} - x_{142}\right)\\
&+ 5.385\left(x_{243} - x_{143}\right) + 7.615\left(x_{245} - x_{145}\right)\\
&+ 9.219\left(x_{251} - x_{151}\right) + 8.062\left(x_{252} - x_{152}\right)\\
&+ 5.385\left(x_{253} - x_{153}\right) + 7.615\left(x_{254} - x_{154}\right),\\
z_2 = {} & 6.324x_{212} + 5.656x_{213} + 10.816x_{214} + 9.219x_{215}\\
&+ 6.324x_{221} + 2.828x_{223} + 5x_{224} + 8.062x_{225}\\
&+ 5.656x_{231} + 2.828x_{232} + 5.385x_{234}\\
&+ 5.385x_{235} + 10.816x_{241} + 5x_{242}\\
&+ 5.385x_{243} + 7.615x_{245} + 9.219x_{251}\\
&+ 8.062x_{252} + 5.385x_{253} + 7.615x_{254},\\
z_3 = {} & 6.324\left(x_{312} - x_{212}\right) + 5.656\left(x_{313} - x_{213}\right)\\
&+ 10.816\left(x_{314} - x_{214}\right) + 9.219\left(x_{315} - x_{215}\right)\\
&+ 6.324\left(x_{321} - x_{221}\right) + 2.828\left(x_{323} - x_{223}\right)\\
&+ 5\left(x_{324} - x_{224}\right) + 8.062\left(x_{325} - x_{225}\right)\\
&+ 5.656\left(x_{331} - x_{231}\right) + 2.828\left(x_{332} - x_{232}\right)\\
&+ 5.385\left(x_{334} - x_{234}\right) + 5.385\left(x_{335} - x_{235}\right)\\
&+ 10.816\left(x_{341} - x_{241}\right) + 5\left(x_{342} - x_{242}\right)\\
&+ 5.385\left(x_{343} - x_{243}\right) + 7.615\left(x_{345} - x_{245}\right)\\
&+ 9.219\left(x_{351} - x_{251}\right) + 8.062\left(x_{352} - x_{252}\right)\\
&+ 5.385\left(x_{353} - x_{253}\right) + 7.615\left(x_{354} - x_{254}\right).
\end{aligned}
\tag{31}
$$

Table 4: Fuzzy numbers of assignment facility 4 to the customers for Example 2.

Facility number 4			
Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$	Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$
2	$\langle 9, 9, 9.5 \rangle$	3	$\langle 16.5, 16.5, 16.5 \rangle$
4	$\langle 19, 19, 20 \rangle$	5	$\langle 8, 8, 8 \rangle$
6	$\langle 18.5, 18.5, 18.5 \rangle$	7	$\langle 5.95, 5.95, 5.95 \rangle$
14	$\langle 8, 8, 8 \rangle$	15	$\langle 25, 25, 25 \rangle$
16	$\langle 11, 11, 11 \rangle$	19	$\langle 4.5, 4.5, 4.5 \rangle$
20	$\langle 4.5, 4.5, 4.5 \rangle$	24	$\langle 0, 0, 1 \rangle$
26	$\langle 3, 3, 3 \rangle$	27	$\langle 22, 22, 23 \rangle$
28	$\langle 14.5, 14.5, 15.5 \rangle$	31	$\langle 0, 0, 1 \rangle$
33	$\langle 9.5, 9.5, 10.5 \rangle$	34	$\langle 5, 5, 6 \rangle$
35	$\langle 12, 12, 13 \rangle$	37	$\langle 0, 0, 1 \rangle$
45	$\langle 18, 18, 19 \rangle$	46	$\langle 0, 0, 1 \rangle$
48	$\langle 0, 0, 1 \rangle$	49	$\langle 0, 0, 1 \rangle$
50	$\langle 9.5, 9.5, 10.5 \rangle$	52	$\langle 0, 0, 1 \rangle$
58	$\langle 0, 0, 1.5 \rangle$	59	$\langle 26, 26, 27 \rangle$
60	$\langle 10, 10, 11 \rangle$	62	$\langle 22, 22, 23 \rangle$
63	$\langle 19, 19, 19 \rangle$	65	$\langle 29.5, 29.5, 29.5 \rangle$
67	$\langle 4, 4, 5 \rangle$	68	$\langle 14, 14, 15 \rangle$
71	$\langle 0, 0, 1 \rangle$	72	$\langle 27.5, 27.5, 28 \rangle$
73	$\langle 9, 9, 10 \rangle$	76	$\langle 8.5, 8.5, 9.5 \rangle$
77	$\langle 19, 19, 19 \rangle$	78	$\langle 6.5, 6.5, 8 \rangle$
79	$\langle 11, 11, 12 \rangle$	80	$\langle 20, 20, 20 \rangle$
84	$\langle 0, 0, 1 \rangle$	87	$\langle 16.5, 16.5, 16.5 \rangle$
88	$\langle 14, 14, 15.5 \rangle$	89	$\langle 30, 30, 31 \rangle$
90	$\langle 27, 27, 28 \rangle$	91	$\langle 9.5, 9.5, 10.5 \rangle$
92	$\langle 6.5, 6.5, 7 \rangle$	94	$\langle 18, 18, 19 \rangle$
96	$\langle 12.5, 12.5, 13.5 \rangle$	97	$\langle 4, 4, 5 \rangle$
98	$\langle 0, 0, 1 \rangle$	99	$\langle 0, 0, 1 \rangle$

Let S be the set of feasible solutions for the recent model. The Positive Ideal Solution and the Negative Ideal Solution for each of these functions are obtained the same as Step 5, for $n = 5$.

By solving the obtained problems, we will find the following values:

$$\begin{aligned} Z_1^{\mathrm{NIS}} &= 12.32, \\ Z_1^{\mathrm{PIS}} &= 47.255, \\ Z_2^{\mathrm{NIS}} &= 418, \\ Z_2^{\mathrm{PIS}} &= 111.422, \\ Z_3^{\mathrm{NIS}} &= 46.427, \\ Z_3^{\mathrm{PIS}} &= 11.86. \end{aligned} \tag{32}$$

Then the linear membership functions of them are obtained as follows:

$$\begin{aligned} \mu_1(v) &= \begin{cases} 1 & \text{if } z_1 > 47.255 \\ \dfrac{z_1 - 12.32}{34.935} & \text{if } 12.32 \le z_1 \le 47.255 \\ 0 & \text{if } z_1 < 12.32, \end{cases} \\ \mu_2(v) &= \begin{cases} 1 & \text{if } z_2 < 111.422 \\ \dfrac{418 - z_2}{306.578} & \text{if } 111.422 \le z_2 \le 418 \\ 0 & \text{if } z_2 > 418, \end{cases} \\ \mu_3(v) &= \begin{cases} 1 & \text{if } z_3 < 11.86 \\ \dfrac{46.627 - z_3}{34.567} & \text{if } 11.86 \le z_3 \le 46.427 \\ 0 & \text{if } z_3 > 46.427. \end{cases} \end{aligned} \tag{33}$$

Table 5: Fuzzy numbers of assignment facility 8 to the customers for Example 2.

Facility number 8			
Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$	Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$
1	$\langle 23, 23, 24 \rangle$	3	$\langle 0, 0, 0.5 \rangle$
5	$\langle 0, 0, 0.5 \rangle$	6	$\langle 0, 0, 1 \rangle$
7	$\langle 5.05, 5.05, 5.05 \rangle$	8	$\langle 18, 18, 19 \rangle$
9	$\langle 14, 14, 14 \rangle$	14	$\langle 0, 0, 1 \rangle$
16	$\langle 0, 0, 1 \rangle$	18	$\langle 19, 19, 20 \rangle$
19	$\langle 0, 0, 0.5 \rangle$	21	$\langle 0, 0, 1 \rangle$
26	$\langle 3, 3, 3 \rangle$	27	$\langle 26, 26, 26 \rangle$
22	$\langle 9, 9, 10 \rangle$	26	$\langle 0, 0, 0.5 \rangle$
29	$\langle 7, 7, 8.5 \rangle$	32	$\langle 14, 14, 15 \rangle$
38	$\langle 13, 13, 13 \rangle$	39	$\langle 6.5, 6.5, 7.5 \rangle$
48	$\langle 10.5, 10.5, 10.5 \rangle$	49	$\langle 9.5, 9.5, 9.5 \rangle$
51	$\langle 24, 24, 24 \rangle$	52	$\langle 19.5, 19.5, 19.5 \rangle$
53	$\langle 18.5, 18.5, 19.5 \rangle$	61	$\langle 8.5, 8.5, 8.5 \rangle$
63	$\langle 0, 0, 1 \rangle$	64	$\langle 30, 30, 31 \rangle$
65	$\langle 0, 0, 1 \rangle$	66	$\langle 12, 12, 13 \rangle$
71	$\langle 20, 20, 20 \rangle$	74	$\langle 14, 14, 14.5 \rangle$
75	$\langle 30, 30, 30 \rangle$	80	$\langle 0, 0, 1 \rangle$
82	$\langle 14.5, 14.5, 16 \rangle$	83	$\langle 25.5, 25.5, 26 \rangle$
86	$\langle 0, 0, 1 \rangle$	93	$\langle 16, 16, 16 \rangle$
95	$\langle 8.5, 8.5, 8.5 \rangle$	98	$\langle 28, 28, 28 \rangle$
99	$\langle 18.5, 18.5, 18.5 \rangle$		

Using model (P_5) it follows that the minimum degree of satisfaction α for this example is equal to 0.904 and $y_2 = y_5 = 1$. So the facilities should be located at points 2 and 5. The fuzzy optimal solution $\widetilde{X}$ obtained by solving the model is as follows:

$$\begin{aligned}
&\widetilde{x}_{11} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{12} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{13} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{14} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{15} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{21} = \langle 4, 4, 6 \rangle, \\
&\widetilde{x}_{22} = \langle 6.5, 6.5, 8.5 \rangle, \\
&\widetilde{x}_{23} = \langle 5, 5, 6 \rangle, \\
&\widetilde{x}_{24} = \langle 11.5, 11.52, 12.52 \rangle, \\
&\widetilde{x}_{25} = \langle 0, 1, 1 \rangle, \\
&\widetilde{x}_{31} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{32} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{33} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{34} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{35} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{41} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{42} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{43} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{44} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{45} = \langle 0, 0, 0 \rangle, \\
&\widetilde{x}_{51} = \langle 0, 1, 1 \rangle, \\
&\widetilde{x}_{52} = \langle 0, 1.5, 1.5 \rangle, \\
&\widetilde{x}_{53} = \langle 0, 2, 2 \rangle, \\
&\widetilde{x}_{54} = \langle 0, 0.48, 0.48 \rangle, \\
&\widetilde{x}_{55} = \langle 14, 14, 14.5 \rangle.
\end{aligned} \tag{34}$$

Now suppose that the above problem is solved using the method of Moreno-Perez et al. [12]. Based on their method the fuzzy weights are transformed to crisp weights

Table 6: Fuzzy numbers of assignment facility 10 to the customers for Example 2.

	Facility number 10		
Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$	Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$
4	$\langle 0, 1, 1 \rangle$	5	$\langle 0, 1, 1 \rangle$
6	$\langle 0, 1.5, 1.5 \rangle$	7	$\langle 0, 0.5, 0.5 \rangle$
9	$\langle 0, 0, 1.5 \rangle$	10	$\langle 4, 4, 5 \rangle$
11	$\langle 21, 21, 22 \rangle$	15	$\langle 0, 0, 1 \rangle$
16	$\langle 0, 0.5, 0.5 \rangle$	17	$\langle 0, 1, 1 \rangle$
18	$\langle 0, 1, 1 \rangle$	19	$\langle 0, 1, 1 \rangle$
20	$\langle 0, 1, 1 \rangle$	21	$\langle 0, 1, 1 \rangle$
22	$\langle 0, 1, 1 \rangle$	23	$\langle 0, 1, 1 \rangle$
24	$\langle 0, 1, 1 \rangle$	25	$\langle 0, 1, 1 \rangle$
26	$\langle 0, 1, 1 \rangle$	27	$\langle 0, 1, 1 \rangle$
28	$\langle 0, 0.5, 0.5 \rangle$	30	$\langle 0, 1, 1 \rangle$
32	$\langle 0, 1, 1 \rangle$	33	$\langle 0, 0.5, 0.5 \rangle$
34	$\langle 0, 1, 1 \rangle$	35	$\langle 0, 1, 1 \rangle$
36	$\langle 29, 29, 29 \rangle$	37	$\langle 25, 25, 25 \rangle$
38	$\langle 0, 1, 1 \rangle$	39	$\langle 0, 1, 1 \rangle$
40	$\langle 0, 1, 1 \rangle$	41	$\langle 0, 1, 1 \rangle$
43	$\langle 0, 1, 1 \rangle$	45	$\langle 0, 1, 1 \rangle$
46	$\langle 0, 1, 1 \rangle$	47	$\langle 0, 0.5, 0.5 \rangle$
52	$\langle 0, 0.5, 0.5 \rangle$	53	$\langle 0, 1, 1 \rangle$
54	$\langle 0, 1, 1 \rangle$	55	$\langle 0, 0.5, 0.5 \rangle$
56	$\langle 0, 0.5, 0.5 \rangle$	57	$\langle 0, 0.5, 0.5 \rangle$
58	$\langle 0, 1.5, 1.5 \rangle$	59	$\langle 0, 1, 1 \rangle$
60	$\langle 0, 1, 1 \rangle$	61	$\langle 0, 0.5, 0.5 \rangle$
62	$\langle 0, 1, 1 \rangle$	63	$\langle 0, 1, 1 \rangle$
64	$\langle 0, 1, 1 \rangle$	65	$\langle 0, 0.5, 0.5 \rangle$
66	$\langle 0, 1, 1 \rangle$	71	$\langle 0, 1, 1 \rangle$
72	$\langle 0, 1, 1 \rangle$	74	$\langle 0, 1, 1 \rangle$
75	$\langle 0, 1, 1 \rangle$	76	$\langle 0, 0.5, 0.5 \rangle$
77	$\langle 0, 1, 1 \rangle$	78	$\langle 0, 0.5, 0.5 \rangle$
80	$\langle 0, 1, 1 \rangle$	81	$\langle 0, 1, 1 \rangle$
82	$\langle 0, 0.5, 0.5 \rangle$	83	$\langle 00, 0.5, 0.5 \rangle$
86	$\langle 0, 0.5, 0.5 \rangle$	87	$\langle 0, 1, 1 \rangle$
88	$\langle 0, 1, 1 \rangle$	89	$\langle 0, 1, 1 \rangle$
90	$\langle 0, 1, 1 \rangle$	91	$\langle 0, 1, 1 \rangle$
92	$\langle 0, 1, 1 \rangle$	93	$\langle 0, 1, 1 \rangle$
94	$\langle 0, 1, 1 \rangle$	95	$\langle 0, 1, 1 \rangle$
96	$\langle 0, 0.5, 0.5 \rangle$	97	$\langle 0, 1, 1 \rangle$
98	$\langle 0, 1, 1 \rangle$		

by applying third index of Yager; that is, the triangular fuzzy weight $\widetilde{w}_j = \langle w_{1j}, w_{2j}, w_{3j} \rangle$ is transformed to $(1/4)(w_{1j} + 2w_{2j} + w_{3j})$. Therefore the crisp weights are as follows:

$$\begin{aligned} w_1' &= 5, \\ w_2' &= 8.125, \\ w_3' &= 6.75, \\ w_4' &= 12.125, \\ w_5' &= 14.875. \end{aligned} \tag{35}$$

After solving the problem with these new weights the value of objective function and the amounts of variables will be obtained as follows:

$$\begin{aligned} z^* &= 111.340, \\ y_2 &= 1, \\ y_5 &= 1, \\ x_{12} &= 5, \\ x_{22} &= 8.125, \end{aligned}$$

Table 7: Fuzzy numbers of assignment facility 23 to the customers for Example 2.

Facility number 23			
Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$	Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$
1	$\langle 0, 1, 1 \rangle$	2	$\langle 0, 1, 1 \rangle$
2	$\langle 0, 1, 1 \rangle$	3	$\langle 0, 1, 1 \rangle$
8	$\langle 0, 1, 1 \rangle$	9	$\langle 0, 1, 1 \rangle$
10	$\langle 0, 1, 1 \rangle$	11	$\langle 0, 1, 1 \rangle$
12	$\langle 0, 1, 1 \rangle$	13	$\langle 0, 1, 1 \rangle$
14	$\langle 0, 1, 1 \rangle$	15	$\langle 0, 1, 1 \rangle$
17	$\langle 0, 0, 0.126 \rangle$	21	$\langle 0, 0, 1 \rangle$
23	$\langle 19, 19, 20 \rangle$	24	$\langle 9, 9, 9 \rangle$
25	$\langle 11, 11, 12 \rangle$	26	$\langle 0, 1, 1 \rangle$
29	$\langle 0, 0.5, 0.5 \rangle$	31	$\langle 0, 0.5, 0.5 \rangle$
36	$\langle 0, 1, 1 \rangle$	37	$\langle 0, 1, 1 \rangle$
42	$\langle 0, 1, 1 \rangle$	44	$\langle 0, 1, 1 \rangle$
48	$\langle 0, 1.5, 1.5 \rangle$	49	$\langle 0, 1.5, 1.5 \rangle$
50	$\langle 0, 0.5, 0.5 \rangle$	51	$\langle 0, 1, 1 \rangle$
67	$\langle 0, 1, 1 \rangle$	68	$\langle 0, 1, 1 \rangle$
69	$\langle 0, 1, 1 \rangle$	61	$\langle 0, 0.5, 0.5 \rangle$
70	$\langle 0, 1, 1 \rangle$	73	$\langle 0, 1, 1 \rangle$
75	$\langle 0, 0, 1.5 \rangle$	79	$\langle 0, 1, 1 \rangle$
84	$\langle 0, 1, 1 \rangle$	85	$\langle 0, 1, 1 \rangle$
99	$\langle 0, 0.5, 0.5 \rangle$	100	$\langle 0, 1, 1 \rangle$

Table 8: Fuzzy numbers of assignment facility 42 to the customers for Example 2.

Facility number 42			
Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$	Customers	$x_{ij} = \langle x_{1ij}, x_{2ij}, x_{3ij} \rangle$
7	$\langle 0, 0, 0.5 \rangle$	12	$\langle 21, 21, 21 \rangle$
13	$\langle 17.5, 17.5, 18.5 \rangle$	17	$\langle 6, 6, 6.874 \rangle$
8	$\langle 0, 1, 1 \rangle$	9	$\langle 0, 1, 1 \rangle$
20	$\langle 0, 0, 1 \rangle$	30	$\langle 27, 27, 28 \rangle$
31	$\langle 19.5, 19.5, 19.5 \rangle$	36	$\langle 0, 0, 1 \rangle$
38	$\langle 0, 0, 1 \rangle$	40	$\langle 9, 9, 10 \rangle$
41	$\langle 19, 19, 20 \rangle$	42	$\langle 7.5, 7.5, 8.5 \rangle$
43	$\langle 12, 12, 13 \rangle$	44	$\langle 21, 21, 22 \rangle$
46	$\langle 6, 6, 6 \rangle$	47	$\langle 30.5, 30.5, 31.5 \rangle$
51	$\langle 0, 0, 1 \rangle$	54	$\langle 12, 12, 13 \rangle$
55	$\langle 4.5, 4.5, 5 \rangle$	56	$\langle 16.5, 16.5, 17.5 \rangle$
57	$\langle 19.5, 19.5, 20 \rangle$	58	$\langle 7.5, 7.5, 7.5 \rangle$
61	$\langle 0, 0, 0.5 \rangle$	69	$\langle 14, 14, 15 \rangle$
70	$\langle 10, 10, 11 \rangle$	77	$\langle 0, 0, 1 \rangle$
81	$\langle 8.5, 8.5, 9 \rangle$	84	$\langle 23, 23, 23 \rangle$
85	$\langle 20, 20, 21 \rangle$	86	$\langle 10.5, 10.5, 10.5 \rangle$
87	$\langle 0, 0, 1.5 \rangle$	93	$\langle 0, 0, 1 \rangle$
95	$\langle 0, 0, 0.5 \rangle$		

$$x_{32} = 6.75,$$
$$x_{42} = 12.125,$$
$$x_{55} = 14.875. \tag{36}$$

As it is clear based on their method a lot of information of problem is lost. Also they obtained crisp amount for objective value and variables. It is very unrealistic that the optimal value of fuzzy problem is obtained as real constant amount. However based on our method the fuzzy objective value and fuzzy amount for variables are yielded. Obviously our method

presented realistic and effectiveness results. Therefore the decision maker is able to get the best decision based on the possibilities and circumstances.

Example 2. As the second example we consider the first test problem (p-median instances) from the ORLIB library which is composed of a graph with 100 nodes and 200 edges; see Beasley [20]. A number of medians are supposed to be $p = 5$ and the fuzzy demands of vertices are given in Table 3.

By applying our new proposed method it yields that $\alpha = 0.911$ and $y_4 = y_8 = y_{10} = y_{23} = y_{42} = 1$, which means the vertices 4, 8, 10, 23, and 42 should be chosen as the location of facilities. Also the fuzzy objective is obtained as $\langle 125944, 170330, 180999.56 \rangle$. Tables 4–8 presented how the facility centers are provided the demands of customers.

5. Conclusion

One of the most important facility location problems which is used to model real situations is p-median problem. However in real world problems it is not easy to determine the parameters and data precisely and in some cases it is not possible at all. Hence most of the times the parameters and data are considered by some degree of uncertainty. As fuzzy set theory is a very useful tool to illustrate this ambiguity and uncertainty, in this paper the fuzzy numbers are used to determining uncertain parameters. In this paper the p-median problem with fuzzy data is considered. Since the parameters of the presented model are known as the demands of costumers and variables are the amounts of demands that each customer served by facility centers, it is far from reality that the facilities cover constant amounts of customer's demands. Thus in this paper the variables are considered as fuzzy variables. Using some technique in fuzzy linear programming the fuzzy model is transformed to a mathematical linear model. By solving the obtained linear programming, fuzzy objective value and fuzzy variables are obtained. Some numerical examples are solved to illustrate the proposed method.

Competing Interests

The authors declare that they have no competing interests.

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