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Research Article

Robust H_∞ Control for Nonlinear Uncertain Switched Descriptor Systems with Time Delay and Nonlinear Input: A Sliding Mode Approach

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This paper addresses the problem of sliding mode control (SMC) design for a class of uncertain switched descriptor systems with state delay and nonlinear input. An integral sliding function is designed and an adaptive sliding mode controller for the reaching motion is then synthesised such that the trajectories of the resulting closed-loop system can be driven onto a prescribed sliding surface and maintained there for all subsequent times. Moreover, based on a new Lyapunov-Krasovskii functional, a delay-dependent sufficient condition is established such that the admissibility as well as the H_∞ performance requirement of the sliding mode dynamics can be guaranteed in the presence of time delay, external disturbances, and nonlinear input which comprises dead-zones and/or sector nonlinearities. The major contributions of this paper of this approach include (i) the closed-loop system exhibiting strong robustness against nonlinear dynamics and (ii) the control scheme enjoying the chattering-free characteristic. Finally, two representative examples are given to illustrate the theoretical developments.

1. Introduction

A switched system is known as one important class of hybrid systems which consists of a family of continuous-time or discrete-time subsystems and a switching rule that orchestrates switching between them. Due to their capability to model a wide variety of complex dynamic systems, such as chemical processes, power electronic, and automatic highway systems, switched systems have been attracting considerable attention in recent years. A great number of interesting results related to switched systems have been reported in the literatures, such as [1] where the authors dealt with the problem of disturbance tolerance and rejection for discrete switched systems with time-varying delay and saturating actuator. The issue of D-stability with H_∞ performance for switched positive linear systems has been addressed in [2]. In [3], the adaptive fuzzy control design for a class of uncertain switched nonlinear systems with mismatched uncertainties and external disturbances has been studied. Based on the switched systems and the tracking state feedback control with a reference model, Lghani et al. have developed robust switched H_∞ controllers for a lateral vehicle control in [4].

To deal with problems of stability and stabilization for discrete and continuous switched systems, many efficient methods have been developed. The multiple Lyapunov functions were employed in [5]. For slow switching systems, the average dwell time technique was suggested in several papers (see [6, 7] and the references therein) to cope with the stability problem. The mode-dependent average dwell time was investigated in [8, 9].

In practice, there always exists a class of systems involving intrinsically time delays, such as chemistry and biological processes, electric power, and network control systems. Time delays, either constant or time-varying, can be the main cause of instability and performance degradation. The research on switched systems with delays is motivated by the following considerations: (1) various practical engineering systems exhibit fundamental characteristics of switching and delay between different system structures and (2) multicontroller switching is considered as an effective mechanism to cope with complex systems. Consequently, the study of switched delayed systems has gained much research attention and many relevant results have been reported [10–12].

As far as we know, many real-word processes always exhibit both dynamic and static constraints in their dynamic behaviour [13, 14]. These systems, called descriptor systems (also referred to as singular systems, differential-algebraic systems, or semistate systems), provide a more convenient and natural representation than standard state-space ones. It should be pointed that the class of switched descriptor system with time delay has been adopted to characterize systems in which their structure and parameters exhibit abrupt changes.

Based on the aforementioned approaches, numerous recent research works such as the stability analysis and the H_∞ control of switched singular systems have been developed in [15, 16]. The study of the problem of reliable dissipative control for a class of discrete-time switched singular systems with stochastic actuator failures has been presented in [17]. In [18], the passive control of switched singular systems via output feedback has been studied.

On the other hand, (SMC) theory has been recognized as one of the most effective tools for coping with model uncertainties and nonlinearities by taking advantage of the concepts of sliding mode surface design and equivalent control. This strategy uses a discontinuous control to force the system state trajectories to some prespecified sliding surfaces on which the system has desired properties such as stability and robustness. The key feature and advantages of (SMC) approach include (1) insensitivity to variations of uncertainties, (2) disturbance rejection capability, and (3) tracking ability. During the past decades, various (SMC) approaches have been successfully applied for solving many practical control problems (see, e.g., [19–23]).

Moreover, due to the physical limitation, the control input seems to have a nonlinear character, like sectors, saturation, dead-zone, and so on. These may naturally originate from actuators in system design in which the nonlinearities may deteriorate the system's stability and performances. So, their effects cannot be ignored in analysis design. In recent years, much attention has been paid to the input nonlinearity [24–29], but, to the best of our knowledge, the problems of (SMC) for switched descriptor systems subjected to time-varying delay and input nonlinearity have not been fully investigated and still remain open and challenging. This has motivated the present study compiled in this paper.

In light of the remarkable benefits previously mentioned, this paper will study the H_∞ adaptive (SMC) problem for a class of uncertain switched descriptor systems with mismatched uncertainties, time-varying delays, and input nonlinearity. The main contributions of the paper lie in the following aspects:

(i) Proposition of a new sliding function and establishment of a sufficient condition to ensure the robust admissibility as well as the H_∞ performance of the corresponding sliding mode dynamics through the feasibility of a convex optimization problem,

(ii) Design of a sliding mode controller for the reaching motion such that all trajectories of the resulting closed-loop system converge onto a prescribed sliding surface and maintain there for all subsequent times.

The remainder of this paper is organized as follows. System description and some preliminary knowledge are given in Section 2. The main results and (SMC) strategy are addressed in Section 3. Simulation results are developed in Section 4. The paper is concluded by some remarks and perspectives in Section 5.

Notations. The notations in this paper are quite standard except where otherwise stated. The superscript "T" stands for matrix transposition; $X \in \mathbb{R}^n$ denotes the n-dimensional Euclidean space, while $X \in \mathbb{R}^{n\times m}$ refers to the set of all $n \times m$ real matrices; $X > 0$ (resp., $X \geq 0$) means that matrix X is real symmetric positive definite (resp., positive semidefinite); $l_2[0,\infty)$ is the space of square summable vectors; I and 0 represent the identity matrix and a zero matrix with appropriate dimension, respectively; $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix, $\mathrm{sym}(X)$ stands for $X+X^T$; $\|\cdot\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. In symmetric block matrices or long matrix expressions, we use a star $*$ to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. System Description and Preliminaries

Consider a class of nonlinear switched descriptor systems with time-varying delay described by

$$\begin{aligned} E\dot{x}(t) &= A_{\sigma(t)}(t)\,x(t) + A_{h\sigma(t)}(t)\,x(t-h(t)) \\ &\quad + B_{\sigma(t)}\left(\phi_{\sigma(t)}(u) + f_{\sigma(t)}(t,x(t))\right) \\ &\quad + B_{w\sigma(t)}w(t) \\ z(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t) \\ x(t) &= \varphi(t), \\ &\qquad t \in [-h_M, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $\phi_\sigma(u)$ is a nonlinear function of u. $w(t) \in \mathbb{R}^w$ is the external disturbance input and $f_{\sigma(t)}(t,x(t))$ represents the system nonlinearity and any model uncertainties in the system including external disturbances. $z(t) \in \mathbb{R}^s$ is the controlled output. Delay $h(t)$ is time-varying and satisfies

$$\begin{aligned} h_m &\leq h(t) \leq h_M, \\ \dot{h}(t) &\leq h_d, \end{aligned} \tag{2}$$

where h_m and h_M are constants representing the bounds of the delay and h_d is a positive constant. $\varphi(t)$ is a compatible vector-valued initial function in $[-h_M, 0]$ representing the initial condition of the system. The system disturbance, $w(t)$, is assumed to belong to $L_2[0,\infty)$; that is, $\int_0^\infty w^T(t)w(t)dt < \infty$. This implies that the disturbance has finite energy.

$\sigma(t):[0,\infty)\rightarrow \mathbb{I}=\{1,2,\dots,N\}$ is the switching signal, which is a piecewise constant and right-continuous function. Matrix $E\in\mathbb{R}^{n\times n}$ may be singular with $\text{rank}(E)=q<n$. $A_i(t)=A_i+\Delta A_i(t)$ and $A_{hi}(t)=A_{hi}+\Delta A_{hi}(t)$ are time-varying system matrices. Matrices A_i, A_{hi}, B_i, B_{wi}, C_i, and D_i are constant with appropriate dimensions. Assume that uncertainties $\Delta A_i(t)$ and $\Delta A_{hi}(t)$ are of the form

$$\left[\Delta A_i(t)\ \ \Delta A_{hi}(t)\right]=M_iF(t)\left[N_i\ \ N_{hi}\right], \tag{3}$$

where M_i, N_i, and N_{hi} are known real constant matrices and $F(t)$ is an unknown time-varying matrix function satisfying $F^T(t)F(t)\leqslant I$.

In addition, $\phi_i(u)=[\phi_{i1}(u_1),\phi_{i2}(u_2),\dots,\phi_{im}(u_m)]^T$, $\phi_i(0)=0$, is the nonlinear input vector which can be described by the following mathematical model:

$$\phi_{ij}(u_j)=\begin{cases}\phi_{ij}^{+}(u_j)(u_j-u_j^{+}), & u_j>u_j^{+}\\ 0, & u_j^{-}<u_j<u_j^{+}\\ \phi_{ij}^{-}(u_j)(u_j+u_j^{-}), & u_j<-u_j^{-},\end{cases}\qquad j=1,2,\dots,m. \tag{4}$$

Remark 1. The nonlinear input model (4), which contains both dead-zones and sector nonlinearities, is considered as more general form. However, if $u_j^+=u_j^-=0$ the nonlinear input $\phi_{ij}(u_j)$ will be reduced to a special sector nonlinear function.

In dealing with this study, the following assumptions, definitions, and lemmas are necessary for further development.

Assumption 2.

(1) The nonlinear input functions ϕ_{ij}^+ and ϕ_{ij}^- applied to the system satisfy

$$\begin{aligned}\alpha_j^+(u_j-u_j^+)^2&<\phi_{ij}^+(u_j)(u_j-u_j^+)<\beta_j^+(u_j-u_j^+)^2\\ \alpha_j^-(u_j+u_j^-)^2&<\phi_{ij}^-(u_j)(u_j+u_j^-)<\beta_j^-(u_j+u_j^-)^2,\end{aligned} \tag{5}$$

where $\alpha_j^+,\alpha_j^-,\beta_j^+$, and β_j^- are positive constants which are called gain reduction tolerances.

(2) Exogenous signal, $w(t)$, is bounded by upper bound $\overline{w}$:

$$\|w(t)\|\leq\overline{w}. \tag{6}$$

(3) Matched nonlinearity $f_i(t,x(t))$ satisfies the inequality

$$f_i(t,x(t))\leq\eta_i(t,x(t)), \tag{7}$$

where $\eta_i(t,x(t))$ is a positive known vector-valued function.

Definition 3 (see [30]). System $E\dot{x}(t)=A_ix(t)+A_{hi}x(t-h(t))$ (or the pair (E,A_i)) is said to be

(1) regular if $\det(sE-A_i)$ is not identically zero;

(2) impulse free if $\deg(\det(sE-A_i))=\text{rank}(E)$;

(3) admissible if it is regular, impulse free, and stable.

Lemma 4 (see [31]). *For any constant matrix $M>0$, any scalar h_m and h_M with $0<h_m<h_M$, and vector function $x(t):[-h_M,-h_m]\rightarrow\mathbb{R}^n$ such that the integrals concerned as well defined then the following holds:*

$$\begin{aligned}&-h_r\int_{t-h_M}^{t-h_m}x^T(s)Mx(s)\,ds\\ &\quad\leq-\int_{t-h_M}^{t-h_m}x^T(s)\,ds\,M\int_{t-h_M}^{t-h_m}x(s)\,ds\end{aligned} \tag{8}$$

with $h_r=h_M-h_m$.

Lemma 5 (see [32]). *Let M and N be real matrices of appropriate dimensions. Then, for any Δ matrix satisfying $\Delta^T\Delta\leq I$ and scalar $\varepsilon>0$,*

$$\text{sym}(M\Delta N)\leq\varepsilon MM^T+\varepsilon^{-1}N^TN. \tag{9}$$

Lemma 6 (see [33]). *For given real matrices $Q>0$, a, and b with appropriate dimensions, the following statements are equivalent:*

(1)

$$\begin{bmatrix}Q & a\\ a^T & 0\end{bmatrix}+\text{sym}\left\{\begin{bmatrix}F\\ G\end{bmatrix}\begin{bmatrix}b^T & -I\end{bmatrix}\right\}<0 \tag{10}$$

is feasible in variable F and G.

(2) *Q, a, and b satisfy*

$$Q+\text{sym}(ab^T)<0. \tag{11}$$

3. Main Results

In this section, an appropriate integral switching surface is designed such that the sliding mode dynamics restricted to the surface is admissible and satisfies the H_∞ performance. Furthermore, a sliding mode control law is synthesised to guarantee that the system state trajectories are globally driven onto the predefined sliding surface and then are maintained there for all subsequent time.

3.1. Integral Sliding Mode Surface. Let us define the following switching function:

$$s(t,i)=\mathbb{G}_iEx(t)+s_0(t,i), \tag{12}$$

where $s_0(t,i)$ is defined for each $i\in\mathbb{I}$ as

$$\begin{aligned}s_0(t,i)=-\mathbb{G}_i\Big(&Ex_0+\int_0^t(A_i+B_iK_i)x(\theta)\\ &+A_{hi}x(\theta-h(\theta))\,d\theta\Big)\end{aligned} \tag{13}$$

and $K_i \in \mathbb{R}^{m\times n}$ is a real matrix to be designed and $\mathbb{G}_i \in \mathbb{R}^{m\times n}$ is a constant matrix satisfying $\mathbb{G}_i B_i$ being nonsingular.

According to the sliding mode control theory, when the system trajectories reach onto the switching surface, we have $\dot{s}(t,i) = 0$. Then the equivalent control law can be obtained as follows:

$$\phi_e(u) = -K_i x(t) - f_i(x(t)) - (\mathbb{G}_i B_i)^{-1} \mathbb{G}_i B_{wi} w(t). \quad (14)$$

Substituting (14) into (1), we obtain the following sliding mode dynamics:

$$\begin{aligned} E\dot{x}(t) &= \overline{A}_i(t)x(t) + \overline{A}_{hi}(t)x(t-h(t)) + \overline{B}_{wi}w(t) \\ z(t) &= C_i x(t) + D_i w(t), \end{aligned} \quad (15)$$

where

$$\begin{aligned} \overline{\mathbb{G}}_i &= I - B_i(\mathbb{G}_i B_i)^{-1}\mathbb{G}_i, \\ \overline{A}_i(t) &= \overline{A}_i + \Delta\overline{A}_i(t), \\ \overline{A}_i &= A_i + B_i K_i, \\ \overline{A}_{hi}(t) &= A_{hi} + \Delta\overline{A}_{hi}(t), \\ \overline{B}_{wi} &= \overline{\mathbb{G}}_i B_{wi}, \\ \overline{M}_i &= \overline{\mathbb{G}}_i M_i, \\ \left[\Delta\overline{A}_i(t) \ \ \Delta\overline{A}_{hi}(t)\right] &= \overline{M}_i F(t)\left[N_i \ \ N_{hi}\right]. \end{aligned} \quad (16)$$

3.2. H_∞ Sliding Mode Dynamics Analysis. In this subsection, we will develop a sufficient delay-dependent condition that ensures for sliding mode dynamics (15) to be robustly admissible with H_∞ performance.

Theorem 7. *Let h_m, h_M, and h_d given positive scalars. The sliding mode dynamics of (15) is admissible with H_∞ performance γ, if there exist matrices S_i, $P_i > 0$, $Q_{1i} > 0$, $Z_{1i} > 0$, $Z_{2i} > 0$, $\begin{bmatrix} Q_{11i} & Q_{12i} \\ Q_{12i}^T & Q_{22i} \end{bmatrix} > 0$, $\begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}^T & W_{22i} \end{bmatrix} > 0$, and G_j, $(j = 0, 1, \ldots, 7)$, and scalars $\gamma > 0$, $\varepsilon > 0$ such that the following inequalities hold:*

$$\Psi_i\left(\overline{A}_i, A_{hi}\right) = \begin{bmatrix} \Phi_i & \Upsilon_{1i} & \Upsilon_{2i} & \epsilon\Gamma_{1i} & \epsilon\Gamma_{2i} \\ * & -\gamma I & 0 & 0 & 0 \\ * & * & -\gamma I & 0 & 0 \\ * & * & * & -\epsilon I & 0 \\ * & * & * & * & -\epsilon I \end{bmatrix} < 0, \quad (17)$$

where

$$\Phi_i = \begin{bmatrix} \Phi_{11i} & \Phi_{12i} & \Phi_{13i} & \Phi_{14i} & \Phi_{15i} & \Phi_{16i} & \Phi_{17i} \\ * & \Phi_{22i} & \Phi_{23i} & \Phi_{24i} & \Phi_{25i} & \Phi_{26i} & \Phi_{27i} \\ * & * & \Phi_{33i} & -Q_{12i} & \Phi_{35i} & 0 & -G_3 \\ * & * & * & \Phi_{44i} & 0 & 0 & -G_4 \\ * & * & * & * & \Phi_{55i} & -W_{12i} & -G_5 \\ * & * & * & * & * & \Phi_{66i} & -G_6 \\ * & * & * & * & * & * & \Phi_{77i} \end{bmatrix}$$

$$\begin{aligned} \Upsilon_{1i} &= \mathbf{G}\overline{B}_{wi} \\ \Gamma_{1i} &= \mathbf{G}M_i, \\ \Upsilon_{2i} &= \begin{bmatrix} C_i & C_{hi} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \Gamma_{2i} &= \begin{bmatrix} N_i & N_{hi} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ \mathbf{G} &= \begin{bmatrix} G_1^T & G_2^T & G_3^T & G_4^T & G_5^T & G_6^T & G_7^T \end{bmatrix}^T, \\ \Phi_{11i} &= Q_{11i} + W_{11i} + Q_1 + \mathrm{sym}\left(G_1\overline{A}_i\right) \\ \Phi_{12i} &= \overline{A}_i^T G_2^T + G_1 A_{hi} \\ \Phi_{13i} &= Q_{12i} + \overline{A}_i^T G_3^T \\ \Phi_{14i} &= \overline{A}_i^T G_4^T \\ \Phi_{15i} &= W_{12i} + \overline{A}_i^T G_5^T \\ \Phi_{16i} &= \overline{A}_i^T G_6^T \\ \Phi_{17i} &= P_i + S_i R^T - G_1 + \overline{A}_i^T G_7^T \\ \Phi_{22i} &= -\left(1-h_d\right)Q_1 - 2E^T Z_{2i} E + \mathrm{sym}\left(G_2 A_{hi}\right) \\ \Phi_{23i} &= A_{hi}^T G_3^T \\ \Phi_{24i} &= A_{hi}^T G_4^T + E^T Z_{2i} E \\ \Phi_{25i} &= A_{hi}^T G_5^T \\ \Phi_{26i} &= E^T Z_{2i} E + A_{hi}^T G_6^T \\ \Phi_{27i} &= -G_2 + A_{hi}^T G_7^T \\ \Phi_{33i} &= Q_{22i} - Q_{11i} - E^T Z_{1i} E \\ \Phi_{35i} &= E^T Z_{1i} E \\ \Phi_{44i} &= -Q_{22i} - E^T Z_{2i} E \\ \Phi_{55i} &= W_{22i} - W_{11i} - E^T Z_{1i} E \\ \Phi_{66i} &= -W_{22i} - E^T Z_{2i} E \\ \Phi_{77i} &= -\mathrm{sym}\left(G_7\right) + \frac{h_r^2}{4} Z_{1i} + h_r^2 Z_{2i}. \end{aligned} \quad (18)$$

Proof. The proof of this theorem is divided into two parts. The first one is concerned with the regularity and the impulse-free characterizations, and the second one treats the stability property of system (15).

First we consider the nominal case of (15) (that is, $\Delta\overline{A}_i(t) = 0$ and $\Delta\overline{A}_{hi}(t) = 0$).

Since $\mathrm{rank}(E) = q < n$, there always exist two nonsingular matrices $\mathbb{M}$ and $\mathbb{N} \in \mathbb{R}^{n\times n}$ such that

$$\mathbb{E} = \mathbb{M}E\mathbb{N} = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}. \quad (19)$$

Then R can be characterized as $R = \mathbb{M}^T \begin{bmatrix} 0 \\ \widehat{\Phi} \end{bmatrix}$, where $\widehat{\Phi} \in \mathbb{R}^{(n-q)\times(n-q)}$ is any nonsingular matrix.

We also define

$$\begin{aligned} \widehat{A}_i &= \mathbb{M}\overline{A}_i\mathbb{N} = \begin{bmatrix} \widehat{A}_{11i} & \widehat{A}_{12i} \\ \widehat{A}_{21i} & \widehat{A}_{22i} \end{bmatrix}, \\ \widehat{S}_i &= \mathbb{N}^T S_i = \begin{bmatrix} \widehat{S}_{11i} \\ \widehat{S}_{21i} \end{bmatrix}, \\ \widehat{P}_i &= \mathbb{M}^{-T} P_i \mathbb{M}^{-1} = \begin{bmatrix} \widehat{P}_{11i} & \widehat{P}_{12i} \\ \widehat{P}_{21i} & \widehat{P}_{22i} \end{bmatrix} \\ \widehat{Z}_{li} &= \mathbb{M}^{-T} Z_{li} \mathbb{M}^{-1} = \begin{bmatrix} \widehat{Z}_{11li} & \widehat{Z}_{12li} \\ \widehat{Z}_{21li} & \widehat{Z}_{22li} \end{bmatrix}, \quad l = 1, 2. \end{aligned} \tag{20}$$

It follows from (17) that

$$\begin{bmatrix} \phi_{11i} & \phi_{12i} \\ * & \phi_{22i} \end{bmatrix} < 0, \tag{21}$$

where

$$\begin{aligned} \phi_{11i} &= \mathrm{sym}\left(G_1 \overline{A}_i\right) \\ \phi_{12i} &= P_i + S_i R^T + \overline{A}_i^T G_7^T - G_1 \\ \phi_{22i} &= -\mathrm{sym}\left(G_7\right) \end{aligned} \tag{22}$$

Before and after multiplying (21) by $[I \overline{A}_i^T]$ and its transpose, respectively, we obtain

$$\mathrm{sym}\left(S_i R^T \overline{A}_i + P\overline{A}_i\right) < 0. \tag{23}$$

Before and after multiplying (23) by $\mathbb{N}^T$ and $\mathbb{N}$, respectively, and then using expressions (19) and (20) yields

$$\mathrm{sym}\left(\widehat{S}_{21i}\overline{\Phi}^T \widehat{A}_{22i}\right) < 0 \tag{24}$$

and $\widehat{A}_{22i}$ is thus nonsingular. So, according to Definition 3, singular time delay system (15) is regular and impulse free for any time delay $h(t)$ satisfying (2).

In the following, we will prove that system (15) is asymptotically stable. For this purpose, we construct a candidate Lyapunov functional as follows:

$$\begin{aligned} &\mathbf{V}(x(t), i) \\ &\quad = \mathbf{V}_1(x(t), i) + \mathbf{V}_2(x(t), i) + \mathbf{V}_3(x(t), i) \\ &\mathbf{V}_1(x(t), i) = x^T(t) P_i E x(t) \\ &\mathbf{V}_2(x(t), i) \\ &\quad = \int_{t-h_m/2}^{t} \eta_1^T(s) Q_i \eta_1(s)\, ds \\ &\qquad + \int_{t-h_M/2}^{t} \eta_2^T(s) W_i \eta_2^T(s)\, ds \\ &\qquad + \int_{t-h(t)}^{t} x^T(s) Q_{1i} x(s)\, ds \\ &\mathbf{V}_3(x(t), i) \\ &\quad = \frac{h_r}{2} \int_{-h_M/2}^{-h_m/2} \int_{t+\theta}^{t} \dot{x}^T(s) E^T Z_{1i} E \dot{x}(s)\, ds\, d\theta \\ &\qquad + h_r \int_{-h_M}^{-h_m} \int_{t+\theta}^{t} \dot{x}^T(s) E^T Z_{2i} E \dot{x}(s)\, ds\, d\theta, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \eta_1(t) &= \left[x^T(t) \;\; x^T\left(t - \frac{h_m}{2}\right)\right]^T, \\ \eta_2(t) &= \left[x^T(t) \;\; x^T\left(t - \frac{h_M}{2}\right)\right]^T. \end{aligned} \tag{26}$$

Evaluating the derivative of $\mathbf{V}(x(t), i)$ along the solutions of system (15), it yields

$$\begin{aligned} &\dot{\mathbf{V}}_1(x(t), i) = 2x^T(t) P_i E \dot{x}(t) \\ &\dot{\mathbf{V}}_2(x(t), i) \\ &\leq \begin{bmatrix} x(t) \\ x\left(t - \frac{h_m}{2}\right) \end{bmatrix}^T \begin{bmatrix} Q_{11i} & Q_{12i} \\ Q_{12i}^T & Q_{22i} \end{bmatrix} \begin{bmatrix} x(t) \\ x\left(t - \frac{h_m}{2}\right) \end{bmatrix} \\ &\quad - \begin{bmatrix} x\left(t - \frac{h_m}{2}\right) \\ x(t - h_m) \end{bmatrix}^T \begin{bmatrix} Q_{11i} & Q_{12i} \\ Q_{12i}^T & Q_{22i} \end{bmatrix} \begin{bmatrix} x\left(t - \frac{h_m}{2}\right) \\ x(t - h_m) \end{bmatrix} \\ &\quad + \begin{bmatrix} x(t) \\ x\left(t - \frac{h_M}{2}\right) \end{bmatrix}^T \begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}^T & W_{22i} \end{bmatrix} \begin{bmatrix} x(t) \\ x\left(t - \frac{h_M}{2}\right) \end{bmatrix} \\ &\quad - \begin{bmatrix} x\left(t - \frac{h_M}{2}\right) \\ x(t - h_M) \end{bmatrix}^T \begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}^T & W_{22i} \end{bmatrix} \begin{bmatrix} x\left(t - \frac{h_M}{2}\right) \\ x(t - h_M) \end{bmatrix} \\ &\quad + x^T(t) Q_1 x(t) \\ &\quad - \left(1 - h_d\right) x^T(t - h(t)) Q_1 x(t - h(t)) \\ &\dot{\mathbf{V}}_3(x(t), i) \\ &= \frac{h_r^2}{4} \dot{x}^T(t) E^T Z_{1i} E \dot{x}(t) + h_r^2 \dot{x}^T(t) E^T Z_{2i} E \dot{x}(t) \\ &\quad - \frac{h_r}{2} \int_{t-h_M/2}^{t-h_m/2} \dot{x}^T(s) E^T Z_{1i} E \dot{x}(s)\, ds \\ &\quad - h_r \int_{t-h_M}^{t-h_m} \dot{x}^T(s) E^T Z_{2i} E \dot{x}(s)\, ds. \end{aligned} \tag{27}$$

From Lemma 4, one can obtain

$$
\begin{aligned}
&-\frac{h_r}{2}\int_{t-h_M/2}^{t-h_m/2}\dot{x}^T(s)E^TZ_{1i}E\dot{x}(s)\,ds\\
&\quad\le-\left[x\left(t-\frac{h_m}{2}\right)-x\left(t-\frac{h_M}{2}\right)\right]^T\\
&\quad\cdot E^TZ_{1i}E\left[x\left(t-\frac{h_m}{2}\right)-x\left(t-\frac{h_M}{2}\right)\right],\\
&-h_r\int_{t-h_M}^{t-h_m}\dot{x}^T(s)E^TZ_{2i}E\dot{x}(s)\,ds\\
&\quad=-h_r\int_{t-h_M}^{t-h(t)}\dot{x}^T(s)E^TZ_{2i}E\dot{x}(s)\,ds\\
&\quad-h_r\int_{t-h(t)}^{t-h_m}\dot{x}^T(s)E^TZ_{2i}E\dot{x}(s)\,ds\\
&\quad\le-\frac{h_r}{h_M-h(t)}\left[x(t-h(t))-x(t-h_M)\right]^T\\
&\quad\cdot E^TZ_{2i}E\left[x(t-h(t))-x(t-h_M)\right]\\
&\quad-\frac{h_r}{h(t)-h_m}\left[x(t-h_m)-x(t-h(t))\right]^T\\
&\quad\cdot E^TZ_{2i}E\left[x(t-h_m)-x(t-h(t))\right]\\
&\quad\le-\left[x(t-h(t))-x(t-h_M)\right]^T\\
&\quad\cdot E^TZ_{2i}E\left[x(t-h(t))-x(t-h_M)\right]\\
&\quad-\left[x(t-h_m)-x(t-h(t))\right]^T\\
&\quad\cdot E^TZ_{2i}E\left[x(t-h_m)-x(t-h(t))\right].
\end{aligned}
\tag{28}
$$

Define

$$
\xi(t)=\left[x^T(t)\;\; x^T(t-h(t))\;\; x^T\left(t-\frac{h_m}{2}\right)\;\; x^T(t-h_m)\;\; x^T\left(t-\frac{h_M}{2}\right)\;\; x^T(t-h_M)\;\; \dot{x}^T(t)E^T\right]^T \tag{29}
$$

From (15), the following equation holds for any matrices G_j, $j=1,2,\ldots,7$, with the appropriate dimensions:

$$2\xi^T(t)\mathbf{G}\left[-E\dot{x}(t)+\overline{A}_ix(t)+\overline{A}_{hi}x(t-h(t))\right]=0. \tag{30}$$

Furthermore, noting $E^TR=0$, we can deduce that

$$2\dot{x}^T(t)E^TRS_i^Tx(t)=0. \tag{31}$$

Combining (27), (28), (30), and (31) yields

$$\dot{\mathbf{V}}(x(t),i)\le\xi^T(t)\Phi_i\xi(t). \tag{32}$$

Hence, $\dot{\mathbf{V}}(x(t),i)\le-\lambda\|\xi(t)\|^2$ which implies that nominal singular system (15), with $w(t)=0$, is asymptotically stable.

Let us now analyse the H_∞ performance of the system (15). Consider the following performance index:

$$J_{zw}=\int_0^\infty\left(\frac{1}{\gamma}z^T(t)z(t)-\gamma w^T(t)w(t)\right)dt. \tag{33}$$

Note that

$$
\begin{aligned}
J_{zw}&=\int_0^\infty\left(\frac{1}{\gamma}z^T(t)z(t)-\gamma w^T(t)w(t)\right.\\
&\quad\left.+\dot{\mathbf{V}}(x(t),i)\right)dt-\int_0^\infty\dot{\mathbf{V}}(x(t),i)\,dt\\
&=\int_0^\infty\left(\frac{1}{\gamma}z^T(t)z(t)-\gamma w^T(t)w(t)\right.\\
&\quad\left.+\dot{\mathbf{V}}(x(t),i)\right)dt-\lim_{t\to\infty}\mathbf{V}(x(t),i)+\mathbf{V}(x(0),\\
&\quad i).
\end{aligned}
\tag{34}
$$

For any zero initial condition, we have

$$J_{zw}\le\int_0^\infty\xi^T(t)\overline{\Lambda}_i\xi(t)\,dt-\lim_{t\to\infty}\mathbf{V}(x(t),i), \tag{35}$$

where

$$
\begin{aligned}
\overline{\Lambda}_i&=\begin{bmatrix}\Phi_i & \Upsilon_{1i}\\ * & -\gamma I\end{bmatrix}+\gamma^{-1}\begin{bmatrix}\Upsilon_{2i}\\ 0\end{bmatrix}\begin{bmatrix}\Upsilon_{2i}\\ 0\end{bmatrix}^T,\\
\zeta(t)&=\left[\xi^T(t)\;\; w^T(t)\right]^T.
\end{aligned}
\tag{36}
$$

Since $\overline{\Lambda}_i<0$, resulting from (17) by Schur complement, and $\mathbf{V}(x(t),i)>0$, we get $J_{zw}<0$. Therefore, for any $0\ne w(t)\in L_2[0,\infty)$ we have

$$\|z(t)\|_2<\gamma\|w(t)\|_2. \tag{37}$$

Consider now the uncertain case. By following the same procedure as used above it is easy to verify that

$$\begin{bmatrix}\Phi_i & \Upsilon_{1i} & \Upsilon_{2i}\\ * & -\gamma I & 0\\ * & * & -\gamma I\end{bmatrix}+\operatorname{sym}\left(\Gamma_{1i}F(t)\Gamma_{2i}^T\right)<0. \tag{38}$$

Then, according to Lemma 5, inequality (17) holds using the Schur complement. This completes the proof. □

3.3. H_∞ Sliding Mode Dynamics Synthesis. Here, our goal is to design gains K_i in (12) such that sliding mode dynamics (15) is robustly admissible with H_∞ norm bound γ.

Based on Theorem 7, we suggest the following result.

Theorem 8. *Let h_m, h_M, and h_d be given positive scalars and K_{si} given matrices with appropriate dimensions. The sliding mode dynamics of (15) is admissible with H_∞ performance γ, if there exist matrices X_i, S_i, $P_i > 0$, $Q_{1i} > 0$, $Z_{1i} > 0$, $Z_{2i} > 0$, $\begin{bmatrix} Q_{11i} & Q_{12i} \\ Q_{12i}^T & Q_{22i} \end{bmatrix} > 0$, $\begin{bmatrix} W_{11i} & W_{12i} \\ W_{12i}^T & W_{22i} \end{bmatrix} > 0$, Y_i, and G_j, $(j = 0, 1, \ldots, 7)$, and scalars $\gamma > 0$, $\varepsilon > 0$ such that the following inequalities hold:*

$$\Xi_i = \begin{bmatrix} \Psi_i\left(\overline{A}_{si}, A_{hi}\right) & \mathfrak{B}_{Gi} \\ * & 0 \end{bmatrix} + \operatorname{sym}\left(\mathcal{I}\mathcal{Y}_i\right) < 0, \tag{39}$$

where $\overline{A}_{si} = A_i + B_i K_{si}$, $\mathfrak{B}_{Gi} = \mathbf{G} B_i$, and

$$\begin{aligned} \mathcal{Y}_i &= \begin{bmatrix} Y_i - X_i K_{si} & 0 & 0 & 0 & 0 & 0 & 0 & -X_i \end{bmatrix} \\ \mathcal{I} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}^T. \end{aligned} \tag{40}$$

The gain matrices are given by

$$K_i = X_i^{-1} Y_i. \tag{41}$$

Proof. Under the conditions of Theorem 8, a feasible solution satisfies condition $-\operatorname{sym}(X_i) < 0$. This implies that X_i is nonsingular.

By introducing auxiliary variables K_{si}, sliding mode dynamics (15) can be written as

$$\begin{aligned} E\dot{x}(t) &= \left(\overline{A}_i(t) + B_i K_{si} - B_i K_{si}\right) x(t) \\ &\quad + \overline{A}_{hi}(t)\, x(t - h(t)) + \overline{B}_{wi} w(t) \\ z(t) &= C_i x(t) + C_{hi} x(t - h(t)). \end{aligned} \tag{42}$$

Applying Theorem 7 to system (42) yields

$$\Psi_i\left(\overline{A}_{si}, A_{hi}\right) + \operatorname{sym}\left(\mathfrak{B}_{Gi}\mathcal{K}_i\right) < 0, \tag{43}$$

where

$$\mathcal{K}_i = \begin{bmatrix} \left(K_i - K_{si}\right) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

Set $\mathcal{X}_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_i \end{bmatrix}^T$. According to Lemma 6 we get

$$\Xi_i = \begin{bmatrix} \Psi_i\left(\overline{A}_{si}, A_{hi}\right) & \mathfrak{B}_{Gi} \\ * & 0 \end{bmatrix} + \operatorname{sym}\left(\mathcal{X}_i\mathcal{K}_i\right) < 0. \tag{45}$$

Then condition (39) holds by setting $Y_i = X_i K_i$. This completes the proof. □

Remark 9. To obtain the best H_∞ performance, the minimum allowed γ satisfying the LMIs in Theorem 8 can be computed by solving the following optimization problem:

$$\begin{aligned} &\text{minimise} \quad \gamma \\ &\text{subject to} \quad \text{LMI (39)}. \end{aligned} \tag{46}$$

3.4. Adaptive SMC Law Synthesis. After establishing the appropriate switching surface (12), an adaptive SMC law will be designed to guarantee the reachability of the specified sliding surface $s(t, i) = 0$ even though uncertainties and input nonlinearity are present.

Let $\widehat{\alpha}(t)$ represent the estimate of $\alpha(t)$. The adaptive SMC that achieves the control objective can be designed as

$$u_j = \begin{cases} -\widehat{\alpha}(t)\,\Psi_i(t)\,\dfrac{s_j(t,i)}{\|s(t,i)\|} - u_j^-, & s_j(i,t) > 0 \\ 0, & s_j(i,t) = 0 \\ -\widehat{\alpha}(t)\,\Psi_i(t)\,\dfrac{s_j(t,i)}{\|s(t,i)\|} + u_j^+, & s_j(i,t) < 0, \end{cases} \qquad j = 1, 2, \ldots, m, \tag{47}$$

where $\Psi_i(t) = \psi_i(t) + \chi_i$ and

$$\begin{aligned} \psi_i(t) &= \left\|\mathbb{G}_i\right\| \left(\left\|M_i\right\| \left\|N_i\right\| \|x(t)\| \right. \\ &\quad \left. + \left\|M_i\right\| \left\|N_{hi}\right\| \|x(t - h(t))\| + \left\|B_{wi}\right\| \overline{w}\right) \\ &\quad + \left\|K_i x(t)\right\| + \eta_i(t, x(t)) \end{aligned} \tag{48}$$

and $\chi_i > 0$ is a small constant.

The adaptive law is given by

$$\dot{\widehat{\alpha}}(t) = \kappa\widehat{\alpha}^3 \psi_i(t)\, \|s(t,i)\| \tag{49}$$

with $\widehat{\alpha}(0) = \alpha_0$, where α_0 is a bounded positive initial value of $\widehat{\alpha}(t)$ and κ is a positive constant.

This proposed control scheme will drive the state to reach the sliding surface $s(t, i) = 0$. This fact is stated in Theorem 10.

Theorem 10. *If the adaptive control input $u(t)$ is designed as (47), with adaptive law (49) then the trajectory of the system (15) converges to the sliding surface $s(t, i) = 0$.*

Proof. Consider the following Lyapunov function:

$$V_s(t,i) = \frac{1}{2} s^T(t,i)\, s(t,i) + \frac{1}{2\kappa}\widetilde{\alpha}^2(t), \tag{50}$$

where $\widetilde{\alpha}(t) = \widehat{\alpha}^{-1}(t) - \alpha$.

According to (12), we get

$$\begin{aligned} \dot{s}(t,i) &= \mathbb{G}_i \left\{ \left(\Delta A_i(t) - B_i K_i\right) x(t) \right. \\ &\quad + \Delta A_{hi}(t)\, x(t - h(t)) + B_{wi}(t)\, w(t) \\ &\quad \left. + B_i\left(\phi_i(u(t)) + f_i(t, x(t))\right) \right\}. \end{aligned} \tag{51}$$

Without loss of generality, we can choose $\mathbb{G}_i = B_i^+ = (B_i^T B_i)^{-1} B_i^T$. So $\mathbb{G}_i B_i$ is nonsingular. By taking the derivative of $V_s(t,i)$ we get

$$\begin{aligned}\dot{V}_s(t,i) &= s^T(t,i)\,\dot{s}(t,i) - \frac{1}{\kappa}\tilde{\alpha}(t)\frac{\dot{\hat{\alpha}}(t)}{\hat{\alpha}^2(t)} = s^T(t,i) \\ &\cdot \mathbb{G}_i\{\Delta A_i(t)x(t) + \Delta A_{hi}(t)x(t-h(t)) + B_{wi}(t) \\ &\cdot w(t)\} + s^T(t,i)\left(\phi_i(u(t)) + \left(f_i(t,x(t))\right.\right. \\ &\left.\left. - K_i x(t)\right)\right) - \hat{\alpha}(t)\tilde{\alpha}(t)\psi_i(t)\|s(t,i)\| \le \|s(t,i)\| \\ &\cdot \{\|\mathbb{G}_i\|(\|M_i\|\|N_i\|\|x(t)\| \\ &+ \|M_i\|\|N_{hi}\|\|x(t-h(t))\| + \|B_{wi}\|\overline{w}) + \|K_i x(t)\| \\ &+ \eta_i(t,x(t)) - \hat{\alpha}(t)\tilde{\alpha}(t)\psi_i(t)\} + s^T(t,i)\phi_i(u(t)).\end{aligned} \tag{52}$$

Considering (4) and (5), when $u_j < -u_j^-$ for $s_j(i,t) > 0$,

$$\begin{aligned}\phi_{ij}^-(u_j)(u_j + u_j^-) &= -\hat{\alpha}(t)\Psi_i(t)\frac{s_j(t,i)}{\|s(t,i)\|}\phi_{ij}^- \\ &\ge \alpha_j^-\hat{\alpha}^2(t)\Psi_i^2(t)\frac{s_j^2(t,i)}{\|s(t,i)\|^2}\end{aligned} \tag{53}$$

and when $u_j > u_j^+$ for $s_j(i,t) < 0$,

$$\begin{aligned}\phi_{ij}^+(u_j)(u_j - u_j^+) &= -\hat{\alpha}(t)\Psi_i(t)\frac{s_j(t,i)}{\|s(t,i)\|}\phi_{ij}^+ \\ &\ge \alpha_j^+\hat{\alpha}^2(t)\Psi_i^2(t)\frac{s_j^2(t,i)}{\|s(t,i)\|^2}.\end{aligned} \tag{54}$$

From (53) and (54), we can obtain

$$s_j(t,i)\phi_{ij} \le -\alpha\hat{\alpha}(t)\Psi_i(t)\frac{s_j^2(t,i)}{\|s(t,i)\|}, \tag{55}$$

where $\alpha \le \min\{\alpha_j^+, \alpha_j^-\}$.

Hence, for the overall sliding surface

$$\begin{aligned}s^T(t,i)\phi_i(u) &= -\sum_{j=1}^{m}\alpha\hat{\alpha}(t)\Psi_i(t)\frac{s_j^2(t,i)}{\|s(t,i)\|} \\ &\le -\alpha\hat{\alpha}(t)\Psi_i(t)\|s(t,i)\|.\end{aligned} \tag{56}$$

Substituting (56) into (52), we obtain

$$\begin{aligned}&\dot{V}_s(t,i) \\ &\quad = \left(\psi_i(t) - \hat{\alpha}(t)\tilde{\alpha}(t)\psi_i(t) - \alpha\hat{\alpha}(t)\left(\psi_i(t) + \chi_i\right)\right) \\ &\quad \cdot \|s(t,i)\| < 0, \quad \forall\|s(t,i)\| \ne 0.\end{aligned} \tag{57}$$

Noting that $\alpha\hat{\alpha}(t) + \hat{\alpha}(t)\tilde{\alpha}(t) = 1$ and $\hat{\alpha}(t) > 0$, it is easy to verify that

$$\dot{V}_s(t,i) < 0, \quad \forall t > 0. \tag{58}$$

This means that the system trajectories converge to the predefined sliding surface and are restricted to the surface for all subsequent time, thereby completing the proof. □

Remark 11. It is noted that the sliding mode controller (47) contains term $s(t,i)/\|s(t,i)\|$ which is ill-defined when $s(t,i) = 0$. In order to avoid this problem and reduce the effect of chattering caused by the discontinuous controller, a sigmoid-like function $s(t,i)/(\mu + \|s(t,i)\|)$ can be introduced to replace $s(t,i)/\|s(t,i)\|$. μ is a small positive scalar value.

4. Examples

Now, we demonstrate the applicability of the analysis by means exposing the main results from two simulation examples.

Example 1. Consider the switched systems (1) with two modes and parameters as follows:

(i) Subsystem 1

$$\begin{aligned}E &= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \\ A_1 &= \begin{bmatrix}-1 & 3 & 2\\ -2 & 3 & 0\\ 1 & 2 & -1.5\end{bmatrix}, \\ A_{h1} &= \begin{bmatrix}-0.1 & 0 & 0.1\\ 0.1 & -0.2 & 0.1\\ 0.1 & 0.3 & -0.2\end{bmatrix}, \\ B_1 &= \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, \\ B_{w1} &= \begin{bmatrix}0.1\\ 0.1\\ 0\end{bmatrix}, \\ C_1 &= \begin{bmatrix}1 & 1 & 0\end{bmatrix}, \\ D_1 &= 0.1.\end{aligned} \tag{59}$$

(ii) Subsystem 2

$$\begin{aligned}E &= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \\ A_2 &= \begin{bmatrix}2 & 1 & 1\\ 1 & -2 & 0\\ 2 & 4 & -3\end{bmatrix}, \\ A_{h2} &= \begin{bmatrix}0 & 0.2 & 0.1\\ -0.1 & 0.2 & 0\\ 0.2 & 0.6 & -0.38\end{bmatrix},\end{aligned}$$

$$B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$B_{w2} = \begin{bmatrix} 0.1 \\ -0.1 \\ 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$

$$D_2 = 0.1 \tag{60}$$

The time-varying delay is given as $h(t) = 0.1 + 0.4e^{-0.5t}$. A straightforward calculation gives $h_m = 0.1$, $h_M = 0.5$, and $h_d = 0.2$. Assume that the uncertain matrices are as follows:

$$\begin{aligned} M_i &= \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T, \\ N_i &= \begin{bmatrix} 0.1 & 0.15 & 0.1 \end{bmatrix}, \\ N_{hi} &= \begin{bmatrix} 0.1 & 0.1 & 0 \end{bmatrix} \\ &\qquad i = 1, 2. \end{aligned} \tag{61}$$

Our aim is to design an SMC in the form of (48) such that the sliding mode dynamics is robustly admissible with a guaranteed H_∞ noise attenuation level.

In the simulation, the design parameters are chosen as

$$\begin{aligned} \mathbb{G}_1 &= \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}, \\ \mathbb{G}_2 &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \\ \mathbb{K}_{s1} &= \begin{bmatrix} -1 & -5 & -3 \end{bmatrix}, \\ \mathbb{K}_{s2} &= \begin{bmatrix} -5 & -3 & 1 \end{bmatrix}. \end{aligned} \tag{62}$$

By solving problem (46) we can obtain a feasible solution with the following parameters:

$$\begin{aligned} P_1 &= \begin{bmatrix} 94.611 & 40.452 & -29.925 \\ 40.452 & 53.431 & 6.4609 \\ -29.925 & 6.4609 & 189.11 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 109.4 & 42.744 & -27.113 \\ 42.744 & 42.949 & -12.698 \\ -27.113 & -12.698 & 173.48 \end{bmatrix} \\ S_1 &= \begin{bmatrix} 28.563 \\ -170.18 \\ -106.7 \end{bmatrix}, \\ S_2 &= \begin{bmatrix} -19.982 \\ -193.97 \\ -22.911 \end{bmatrix}. \end{aligned} \tag{63}$$

The minimum allowed $\gamma^* = 0.7$ and the associate controller gains are

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.28105 & -4.9305 & -2.0821 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -5.2486 & -1.085 & 1.2996 \end{bmatrix}. \end{aligned} \tag{64}$$

The existence of a feasible solution shows that there exists a desire sliding surface in (12) such that the resulting sliding mode dynamics in (15) is admissible with H_∞ performance. The remaining task is to design a sliding mode controller such that the system trajectories can be driven onto the predefined sliding surface and maintained there for all subsequent time. For simulation purposes, we take exogenous input $\omega(t) = \cos(2.5t)e^{-0.5t}$, uncertain matrix function $F(t) = 0.5 + 0.5\sin(2t)$, and

$$\begin{aligned} f_1(x) &= \sqrt{|x_1 x_2|}, \\ f_2(x) &= -2x_1^2 \cos(x_1 x_2). \end{aligned} \tag{65}$$

The nonlinear model actuator is described by

$$\begin{aligned} \Phi_1(u) &= \begin{cases} (1 - 0.3\cos(u))(u - 0.5), & u > 0.5 \\ 0, & -0.5 \le u \le 0.5 \\ (1 - 0.3\cos(u))(u + 0.5), & u < -0.5 \end{cases} \\ \Phi_2(u) &= \begin{cases} (0.7 + 0.3\cos(u))(u - 1), & u > 1 \\ 0, & -1 \le u \le 1 \\ (0.7 + 0.3\cos(u))(u + 1), & u < -1. \end{cases} \end{aligned} \tag{66}$$

With $\chi_1 = 0.05$, $\chi_2 = 0.1$, and $\kappa = 0.0001$, the adaptive SMC law can be designed according to (48)-(49).

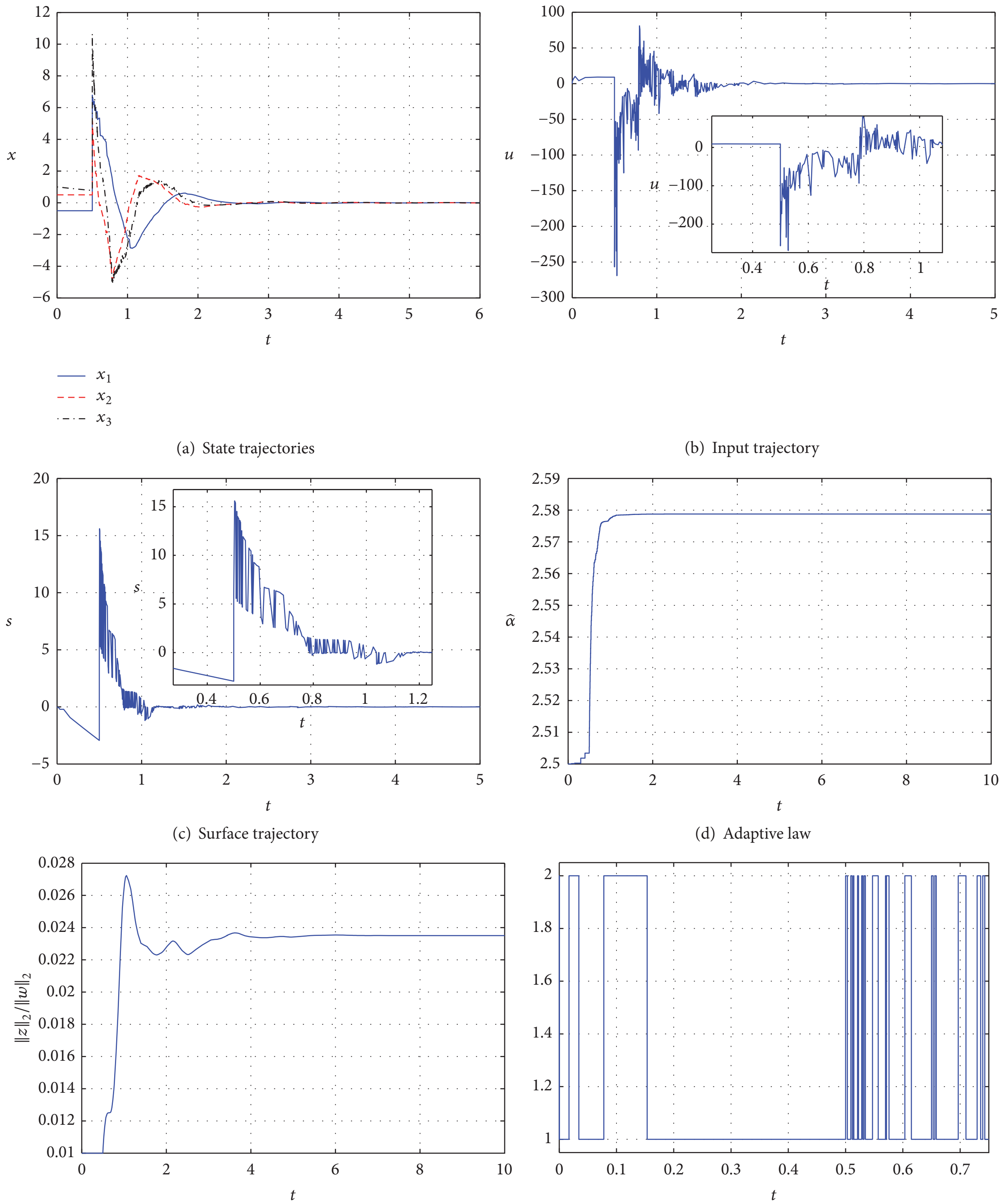
To prevent the control signal from chattering, we replace $s(t,i)/\|s(t,i)\|$ with $s(t,i)/(0.05 + \|s(t,i)\|)$.

The simulation results depicted in Figure 1 show that

(i) for initial condition $\varphi(t) = [-0.5, 0.5, e^{-0.5t}]^T$, $t = -0.5, \ldots, 0$, Figures 1(a)–1(d) depict, respectively, the system state trajectories, the control input, the resulting sliding surface, and the adaptive law when the SMC is applied. We observe that the system is stable despite the presence of actuator nonlinearity, parameter uncertainties, and external disturbances;

(ii) from Figure 1(e), the ratio of $\|z(t)\|_2/\|w(t)\|_2$ is about 0.0235 under zero initial condition, which reveals that the H_∞ disturbance attenuation level is less than required $\gamma^* = 0.7$;

(iii) the proposed scheme can obtain better convergence performance by driving the system trajectories to the specified sliding surface asymptotically instead of to some neighbour of the surface.



(a) State trajectories
(b) Input trajectory
(c) Surface trajectory
(d) Adaptive law
(e) Ratio trajectory
(f) Switching signal

Figure 1: Simulation results for Example 1.

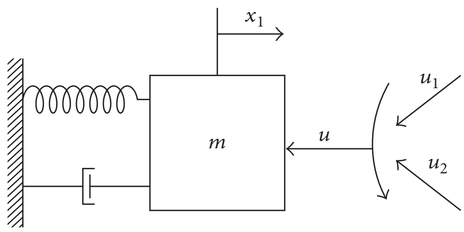


Figure 2: Mass-spring-damper system with controller switching.

Example 2. Consider a mass-spring-damper system with controller switching depicted in Figure 2. This mechanical system with nonlinear stiffness and damping is described by equations of the form

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ 0 &= -x_3 - \frac{1}{m}g(x_1) - \frac{1}{m}h(x_2) + \frac{1}{m}u(t),\end{aligned} \tag{67}$$

where $m > 0$ is an unknown constant, $g(0) = 0$, and $h(0) = 0$. Suppose that we are only allowed to apply two prespecified candidate controllers, $u_k = -\Delta g_k(x) + \nu_k$, $k = 1, 2$, to the system (67) and switch between them. This results in the following switched nonlinear uncertain system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ 0 &= -x_3 - \frac{1}{m}(g(x_1) + \Delta g_\sigma(x)) - \frac{1}{m}h(x_2)\\ &\quad + \frac{1}{m}\nu_\sigma(t).\end{aligned} \tag{68}$$

This system can be considered as a mechanical system with changing stiffness and damper, or changing environment where $g(x_1) = 2x_1^2$, $h(x_2) = x_2^2\cos(x_2)$, $\Delta g_1 = x_1^3\sin(x_1x_2)$, and $\Delta g_2 = x_2\cos(x_1^2) - 2x_1$and the function $\sigma : [0,\infty) \to \mathbb{I} = \{1, 2\}$ is a switching signal which is assumed to be a piecewise continuous function of time.

Taking the parameter uncertain and external disturbance into account, the resulting system (67) can be written in the form of (1) with the following parameters:

(i) Subsystem 1

$$\begin{aligned}E &= \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\\ A_1 &= (1-\lambda)\begin{bmatrix}0&1&0\\0&0&1\\1&0&-1\end{bmatrix},\\ A_{h1} &= \lambda\begin{bmatrix}0&1&0\\0&0&1\\1&0&-1\end{bmatrix},\\ B_1 &= \begin{bmatrix}0\\0\\1\end{bmatrix},\\ B_{w1} &= \begin{bmatrix}0\\0\\0.1\end{bmatrix},\\ C_1 &= \begin{bmatrix}1&1&0\end{bmatrix},\\ D_1 &= 0.1.\end{aligned} \tag{69}$$

(ii) Subsystem 2

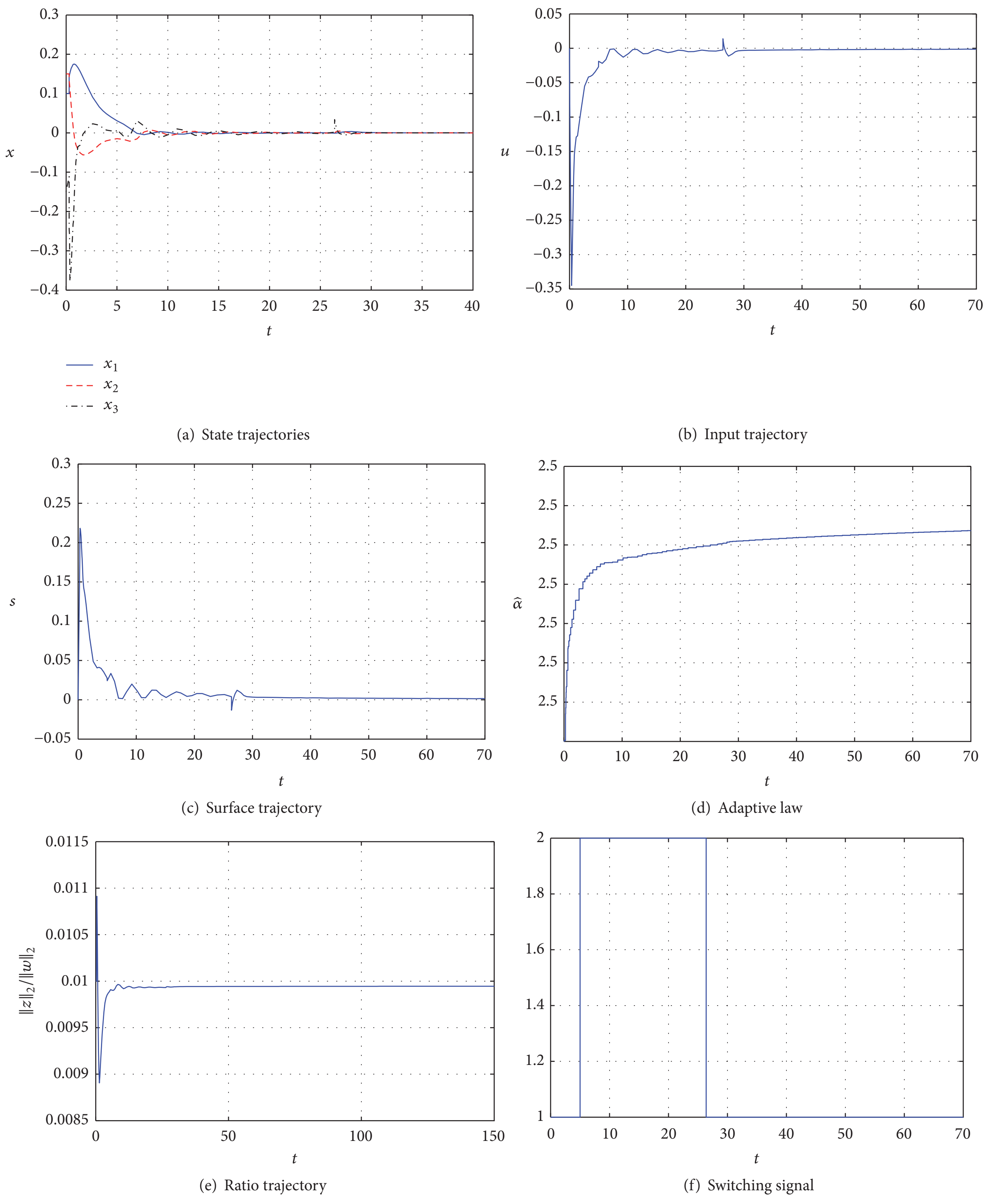
$$\begin{aligned}E &= \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\\ A_2 &= (1-\lambda)\begin{bmatrix}0&1&0\\0&0&1\\0&0&-1\end{bmatrix},\\ A_{h2} &= \lambda\begin{bmatrix}0&1&0\\0&0&1\\0&0&-1\end{bmatrix},\\ B_2 &= \begin{bmatrix}0\\0\\2\end{bmatrix},\\ B_{w2} &= \begin{bmatrix}0\\0\\0.2\end{bmatrix},\\ C_2 &= \begin{bmatrix}1&1&0\end{bmatrix},\\ D_2 &= 0.1\\ f_1(x) &= -2x_1^2 - x_2^2\cos(x_2) - x_1^3\sin(x_1x_2),\\ f_2(x) &= -2\left(2x_1^2 + x_2^2\cos(x_2) + x_2\cos\left(x_1^2\right)\right).\end{aligned} \tag{70}$$

The nonlinear model actuator is described by

$$\begin{aligned}\Phi_1(u) &= (0.9 + 0.4\sin(u))\,u\\ \Phi_2(u) &= (1.1 + 0.5\cos(2u))\,u.\end{aligned} \tag{71}$$

Assume the uncertain parameter matrix function $F(t) = 0.5+0.5\sin(2t)$ and the constant matrices

$$\begin{aligned}M_i &= \begin{bmatrix}0&0&1\end{bmatrix}^T,\\ N_i &= \begin{bmatrix}0.1&0.15&0.1\end{bmatrix},\end{aligned}$$



(a) State trajectories

(b) Input trajectory

(c) Surface trajectory

(d) Adaptive law

(e) Ratio trajectory

(f) Switching signal

FIGURE 3: Simulation results for Example 2.

$$N_{hi} = \begin{bmatrix} 0.1 & 0.1 & 0 \end{bmatrix}$$

$$i = 1, 2. \tag{72}$$

Set $\lambda = 0.9$, $\mathbb{G}_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $\mathbb{G}_2 = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}$, $\mathbb{K}_{s1} = \begin{bmatrix} -1.5 & -4.5 & -2 \end{bmatrix}$, and $\mathbb{K}_{s2} = \begin{bmatrix} -2 & -1.5 & -1 \end{bmatrix}$

Using the convex optimization problem (46) (solved using the Yalmip toolbox and the SeDuMi solver), the obtained matrices for the Lyapunov functions and the feedback gain matrices are

$$\begin{gathered} P_1 = \begin{bmatrix} 3.8025 & 1.3736 & -0.19201 \\ 1.3736 & 2.9732 & -1.7904 \\ -0.19201 & -1.7904 & 2.0436 \end{bmatrix}, \\ P_2 = \begin{bmatrix} 4.0025 & 1.0117 & -0.20765 \\ 1.0117 & 2.949 & -1.8876 \\ -0.20765 & -1.8876 & 2.101 \end{bmatrix} \\ S_1 = \begin{bmatrix} 1.2044 \\ 3.9242 \\ -1.0938 \end{bmatrix}, \\ S_2 = \begin{bmatrix} 1.6588 \\ 3.524 \\ -1.1682 \end{bmatrix}. \end{gathered} \tag{73}$$

The minimum allowed $\gamma^* = 0.366$ and the associate controller gains are

$$\begin{gathered} K_1 = \begin{bmatrix} -1.5357 & -4.897 & -1.2069 \end{bmatrix}, \\ K_2 = \begin{bmatrix} -2.3601 & -2.3175 & -0.27275 \end{bmatrix}. \end{gathered} \tag{74}$$

From Theorem 10, the robust adaptive SMC can be designed with $\chi_1 = 0.3$, $\chi_2 = 0.05$, and $\kappa = 0.00001$.

Let time-varying delay $h(t) = 0.2 + 0.1\sin(t)$ and $\sigma = 0.98$. Figure 3(a) shows the behaviour of the state response corresponding to the obtained solution, where the initial-state $\varphi(t) = [0.1, 0.15, -0.15e^{-t}]^T$, $t = -0.3, \ldots, 0$, and the exogenous noise is $w(t) = 1/(t+5)$. This figure illustrates that all the states converge to zero. Furthermore, Figures 3(b)–3(d) depict, respectively, the control input, the resulting sliding surface, and the adaptive law when the SMC is applied. It is observed that all state trajectories of the system converge asymptotically to the origin, the sliding variable is driven to converge to zero, and the control signal is chattering free.

To explain the relevance of the obtained H_∞ performance, Figure 3(e) depicts the value of $\|z(t)\|_2/\|w(t)\|_2$ for the zero initial condition. As shown in Figure 3(e), the value of converges to a constant value (about 0.01), which is certainly less than 0.366.

5. Conclusions

A novel adaptive sliding model controller is proposed in this paper for stabilizing uncertain nonlinear descriptor systems with state delay and nonlinear input containing sector nonlinearities and/or dead-zones. An integral sliding function is proposed and a delay-dependent sufficient condition is derived to guarantee that the sliding mode dynamics is robustly admissible with H_∞ disturbance rejection performance. Moreover, an adaptive (SMC) law is designed such that the trajectories of the resulting closed-loop system can be driven onto a prescribed sliding surface and maintained there for all subsequent time.

Simulation studies, developed through two examples, have well demonstrated the effectiveness of the proposed (SMC) strategy. The suggested approach would have great potentials in applications to complex industrial processes subject to time delay and high nonlinearities. It should be emphasized that the computational simplicity of the suggested method can be another prominent feature of this work.

In the future, the network and fault tolerant based SMC for this class of systems will be interesting research fields.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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