Research Article

Delay-Induced Oscillations in a Competitor-Competitor-Mutualist Lotka-Volterra Model

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This paper deals with a competitor-competitor-mutualist Lotka-Volterra model. A series of sufficient criteria guaranteeing the stability and the occurrence of Hopf bifurcation for the model are obtained. Several concrete formulae determine the properties of bifurcating periodic solutions by applying the normal form theory and the center manifold principle. Computer simulations are given to support the theoretical predictions. At last, biological meaning and a conclusion are presented.

1. Introduction

The study of dynamical behaviors for tremendous predator-prey models has been a hot issue in population dynamics in the past few decades. Many results have been reported [1–11]. In the real world, any biological or environmental parameters are naturally subject to fluctuation in time. The effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Meanwhile, time delay due to gestation is a common example because, generally, the consumption of prey by the predator throughout its past history governs the present birth rate of the predator. Based on all the above point, Lv et al. [12] had investigated the periodic solution of the following competitor-competitor-mutualist Lotka-Volterra model by using Krasnoselskii’s fixed point theorem

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) [r_1(t) - a_{11}(t) x_1(t - \tau_{11}(t)) - a_{12}(t) x_2(t - \tau_{12}(t)) + a_{13}(t) x_3(t - \tau_{13}(t))], \\
\dot{x}_2(t) &= x_2(t) [r_2(t) - a_{21}(t) x_1(t - \tau_{21}(t)) - a_{22}(t) x_2(t - \tau_{22}(t)) + a_{23}(t) x_3(t - \tau_{23}(t))], \\
\dot{x}_3(t) &= x_3(t) [r_3(t) + a_{31}(t) x_1(t - \tau_{31}(t)) + a_{32}(t) x_2(t - \tau_{32}(t)) - a_{33}(t) x_3(t - \tau_{33}(t))],
\end{align*}
\]

where \(x_1(t)\) and \(x_2(t)\) denote the densities of competing species at time \(t\) and \(x_3(t)\) denotes the density of cooperating species at time \(t\). \(r_i, a_{ij} \in C(\mathbb{R}, [0, \infty))\) and \(\tau_{ij} \in C(\mathbb{R}, \mathbb{R})\) are \(\omega\)-periodic functions \((\omega > 0)\). The parameters \(\tau_{ij}(t) \geq 0 (i = 1, 2, 3; j = 1, 2, 3)\) are the feedback time delay of different species. In detail, one can see [12].

It is well known that the research on the Hopf bifurcation, especially on the stability of bifurcating periodic solutions and direction of Hopf bifurcation, is one of the most important themes on the predator-prey dynamics. There are a great deal of papers which deal with this topic [11, 13–22]. The purpose of this paper is to discuss the stability and the properties of Hopf bifurcation of model (1). To simplify the analysis for model (1), we make the following assumptions: all biological and environmental parameters are constants in time and only the feedback time delay of cooperating species \(x_3\) to the growth of the species itself exist and are the same. Then system (1) can be described as the form

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) [r_1 - a_{11} x_1(t - \tau) - a_{12} x_2(t) + a_{13} x_3(t)], \\
\dot{x}_2(t) &= x_2(t) [r_2 - a_{21} x_1(t) - a_{22} x_2(t - \tau) + a_{23} x_3(t)], \\
\dot{x}_3(t) &= x_3(t) [r_3 + a_{31} x_1(t) + a_{32} x_2(t - \tau) - a_{33} x_3(t - \tau)].
\end{align*}
\]
In this paper, we consider the effect of time delay $\tau$ on the dynamics of system (2). We not only give the conditions on the stability of the positive equilibrium of (2) and the existence of periodic solutions but also derive the formulae for determining the properties of a Hopf bifurcation.

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the positive equilibrium and the occurrence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Biological explanations and some main conclusions are drawn in Section 5.

**2. Stability of the Positive Equilibrium and Local Hopf Bifurcations**

Consider the realistic implication and actual application of biological system; in this section, we shall only study the stability of the positive equilibrium and the existence of local Hopf bifurcations. It is easy to see that system (2) has a unique positive equilibrium $E_0(x_1^*, x_2^*, x_3^*)$ if the condition

$$\text{sign} \left\{ \Delta \right\} = \text{sign} \left\{ \Delta_1 \right\} = \text{sign} \left\{ \Delta_2 \right\} = \text{sign} \left\{ \Delta_3 \right\}$$  \hspace{1cm} (H1)

holds, where

$$\Delta = \text{det} \begin{pmatrix} a_{11} & a_{12} & -a_{13} \\ a_{21} & a_{22} & -a_{23} \\ a_{31} & a_{32} & -a_{33} \end{pmatrix},$$

$$\Delta_1 = \text{det} \begin{pmatrix} r_1 & a_{12} & -a_{13} \\ r_2 & a_{22} & -a_{23} \\ -r_3 & a_{32} & -a_{33} \end{pmatrix},$$

$$\Delta_2 = \text{det} \begin{pmatrix} a_{11} & r_1 & -a_{13} \\ a_{21} & r_2 & -a_{23} \\ a_{31} & -r_3 & -a_{33} \end{pmatrix},$$

$$\Delta_3 = \text{det} \begin{pmatrix} a_{11} & a_{12} & r_1 \\ a_{21} & a_{22} & r_2 \\ a_{31} & a_{32} & -r_3 \end{pmatrix}.$$  \hspace{1cm} (3)

Let $\bar{x}_i(t) = x_i(t) - x_i^*$, $\bar{\bar{x}}_i(t) = x_i(t) - x_i^*$, and $\bar{\bar{\bar{x}}}_i(t) = x_i(t) - x_i^*$ and still denote $\bar{x}_i(t)$ ($i = 1, 2, 3$) by $x_i(t)$ ($i = 1, 2, 3$), and then (2) takes the form

$$\dot{x}_1(t) = b_1 x_1(t - r) + b_2 x_2(t) + b_3 x_3(t) + b_4 x_1(t - r)x_1(t) + b_5 x_1(t)x_2(t) + b_6 x_1(t)x_3(t),$$

$$\dot{x}_2(t) = c_1 x_1(t) + c_2 x_2(t - r) + c_3 x_3(t) + c_4 x_2(t)x_2(t) + c_5 x_2(t)x_3(t),$$

where $b_1 = -a_1 x_1^*$, $b_2 = -a_2 x_1^*$, $b_3 = a_{13} x_1^*$, $b_4 = -a_{11}$, $b_5 = -a_{12}$, $b_6 = a_{13}$, $c_1 = -a_{21} x_2^*$, $c_2 = -a_{22} x_2^*$, $c_3 = a_{23} x_2^*$, $c_4 = -a_{21}$, $c_5 = -a_{22}$, $c_6 = a_{23}$, $d_1 = a_{31} x_3^*$, $d_2 = a_{32} x_3^*$, $d_3 = -a_{33} x_3^*$, $d_4 = a_{31}$, $d_5 = a_{32}$, $d_6 = -a_{33}$.

The linearization of (6) near $(0, 0, 0)$ is given by

$$\dot{x}_1(t) = b_1 x_1(t - r) + b_2 x_2(t) + b_3 x_3(t),$$

$$\dot{x}_2(t) = c_1 x_1(t) + c_2 x_2(t - r) + c_3 x_3(t),$$

$$\dot{x}_3(t) = d_1 x_1(t) + d_2 x_2(t) + d_3 x_3(t - r),$$  \hspace{1cm} (6)

whose characteristic equation takes the form

$$\text{det} \begin{pmatrix} \lambda - b_1 e^{-\lambda r} & -b_2 & -b_3 \\ -c_1 & \lambda - c_2 e^{-\lambda r} & -c_3 \\ -d_1 & -d_2 & \lambda - d_3 e^{-\lambda r} \end{pmatrix} = 0.  \hspace{1cm} (7)$$

That is,

$$\lambda^3 + m_1 \lambda + m_2 + \left( m_3 \lambda^2 + m_4 \right) e^{-\lambda r} + m_5 \lambda e^{-2\lambda r} + m_6 e^{-3\lambda r} = 0,  \hspace{1cm} (8)$$

where $m_1, m_2, m_3, m_4, m_5, m_6$ are constants.
Lemma 1 (see [23]). For the transcendental equation
\[
P(\lambda, e^{-\lambda \tau}; \omega) = \lambda^n + P^{(0)}_1 \lambda^{n-1} + \cdots + P^{(0)}_{n-1} \lambda + P^{(0)}_n 
+ \left[ P^{(1)}_1 \lambda^{n-1} + \cdots + P^{(1)}_{n-1} \lambda + P^{(1)}_n \right] e^{-\lambda \tau} + \cdots 
+ \left[ P^{(m)}_1 \lambda^{n-1} + \cdots + P^{(m)}_{n-1} \lambda + P^{(m)}_n \right] e^{-m \lambda \tau} = 0,
\]
as \((\tau_1, \tau_2, \tau_3, \ldots, \tau_m)\) vary, the sum of orders of the zeros of \(P(\lambda, e^{-\lambda \tau}; \omega)\) in the open right half complex plane can change, only if a zero appears on or crosses the imaginary axis.

For \(\tau = 0\), (10) becomes
\[
\lambda^3 + m_3 \lambda^2 + (m_1 + m_5) \lambda + m_2 + m_4 + m_6 = 0.
\]
Obviously, \(m_3 > 0\). By the Routh-Hurwitz criteria, it follows that all eigenvalues of (12) have negative real parts if and only if the condition
\[
m_2 + m_4 + m_6 > 0,
\]
\[
m_3 (m_1 + m_5) > m_2 + m_4 + m_6
\]
is fulfilled.

For \(\omega > 0\), \(i \omega \) is a root of (10) if and only if
\[
(-\omega^3 i + im_1 \omega + m_2) e^{\omega \tau} + m_3 \omega^2 i + m_4 + im_5 \omega e^{-\omega \tau} 
+ m_6 e^{-2\omega \tau} = 0.
\]
Separating the real and imaginary parts, we get
\[
m_2 \cos \omega \tau - (m_1 \omega - \omega^3) \sin \omega \tau + m_4
+ m_5 \omega \sin \omega \tau = -m_6 \cos 2\omega \tau,
\]
\[
m_2 \sin \omega \tau + (m_1 \omega - \omega^3) \cos \omega \tau + m_3 \omega^2 
+ m_5 \omega \cos \omega \tau = m_6 \sin 2\omega \tau.
\]
It follows from (14) that
\[
\begin{align*}
&\left[ m_2 \cos \omega \tau - (m_1 \omega - \omega^3) \sin \omega \tau + m_4 
+ m_5 \omega \sin \omega \tau \right]^2 + \left[ m_2 \sin \omega \tau + (m_1 \omega - \omega^3) \cos \omega \tau \right]^2 
+ m_3 \omega^2 + m_5 \omega \cos \omega \tau]^2 = m_6^2.
\end{align*}
\]
According to \(\sin \omega \tau = \pm \sqrt{1 - \cos^2 \omega \tau}\), then (15) takes the form
\[
\begin{align*}
&m_2 \cos \omega \tau - (m_1 \omega - \omega^3) \left( \pm \sqrt{1 - \cos^2 \omega \tau} \right) + m_4 
+ m_5 \omega \left( \pm \sqrt{1 - \cos^2 \omega \tau} \right)^2 + \left[ m_2 \left( \pm \sqrt{1 - \cos^2 \omega \tau} \right) 
+ (m_1 \omega - \omega^3) \cos \omega \tau + m_3 \omega^2 + m_5 \omega \cos \omega \tau \right]^2 = m_6^2.
\end{align*}
\]
It is easy to see that (16) is equivalent to
\[
\begin{align*}
&q_1 \cos^4 \omega \tau + q_2 \cos^3 \omega \tau + q_3 \cos^2 \omega \tau + q_4 \cos \omega \tau + q_5 
= 0,
\end{align*}
\]
where
\[
\begin{align*}
q_1 &= k_1^2 + k_2^2, \\
q_2 &= 2 (k_2 k_4 + k_3 k_5), \\
q_3 &= k_2^2 + k_3^2 - 2 k_4 \left( m_6^2 - k_1 \right), \\
q_4 &= 2 k_2 \left( k_1 - m_6^2 \right) - 2 k_3 k_5, \\
q_5 &= \left( m_6^2 - k_1 \right)^2 - k_3^2,
\end{align*}
\]
where
\[
\begin{align*}
k_1 &= m_2^2 + (m_1 \omega - \omega^3)^2 + m_5^2 \omega^2 + m_4^2 + m_3^2 \omega^4, \\
k_2 &= 2 m_2 m_4 + 2 m_5 m_3 \omega + 2 m_3 m_5 \omega^3 
+ 2 m_5 \omega^2 \left( m_1 \omega - \omega^3 \right), \\
k_3 &= 2 m_4 m_5 \omega - 2 m_4 \left( m_1 \omega - \omega^3 \right) + 2 m_2 m_5 \omega^3, \\
k_4 &= 2 m_5 \omega \left( m_1 \omega - \omega^3 \right), \\
k_5 &= 2 m_2 m_5 \omega.
\end{align*}
\]
Let \(\cos \omega \tau = r \) and denote
\[
\begin{align*}
l(r) &= r^4 + \frac{q_2}{q_1} r^3 + \frac{q_3}{q_1} r^2 + \frac{q_4}{q_1} r + \frac{q_5}{q_1},
\end{align*}
\]
It is easy to obtain that \( l'(r) = 4r^3 + (3q_2/q_1)r^2 + (2q_3/q_1)r + q_4/q_1 \). Set
\[
4r^3 + \frac{3q_2}{q_1}r^2 + \frac{2q_3}{q_1}r + \frac{q_4}{q_1} = 0. \tag{21}
\]
Let \( y = r + q_2/4q_1 \). Then (21) becomes
\[
y^3 + y_1y + y_2 = 0, \tag{22}
\]
where \( y_1 = q_2/2q_1 - 3q_2^2/16q_1^2 \) and \( y_2 = q_2^3/32q_1^3 - q_2q_3/8q_1^2 + q_4/4q_1 \).

Define \( \beta_1 = (y_2/2)^3 + (y_1/3)^3 \), \( \beta_2 = (-1 + i \sqrt{3})/2 \). By (22), then we obtain
\[
y_1 = \sqrt{\frac{y_2}{2} + \sqrt{\beta_1}}, \quad y_2 = \sqrt{\frac{y_2}{2} + \sqrt{\beta_1}} + \sqrt{\frac{y_2}{2} - \sqrt{\beta_1}}, \quad y_3 = \sqrt{\frac{y_2}{2} + \sqrt{\beta_1}} + \sqrt{\frac{y_2}{2} - \sqrt{\beta_1}}. \tag{23}
\]

By the discussion above, we can obtain the expression of \( \cos \omega r \), say
\[
\cos \omega r = f_1(\omega), \tag{24}
\]
where \( f_1(\omega) \) is a function with respect to \( \omega \). Substitute (24) into (15); then we can easily get the expression of \( \sin \omega r \), say
\[
\sin \omega r = f_2(\omega), \tag{25}
\]
where \( f_2(\omega) \) is a function with respect to \( \omega \). Thus we obtain
\[
f_1^2(\omega) + f_2^2(\omega) = 1. \tag{26}
\]

If all the coefficients of system (2) are given, it is easy to use computer to calculate the roots of (26) (say \( \omega \)). Then from (24), we derive
\[
\tau_k = \frac{1}{\omega} \left[ \arccos f_1(\omega) + 2k\pi \right] \quad (k = 0, 1, 2, \ldots). \tag{27}
\]

Let \( \lambda(r) = \alpha(r) + i\omega(r) \) be a root of (10) near \( r = \tau_k \), \( \alpha(\tau_k) = 0 \), and \( \omega(\tau_k) = \omega_k \). Due to functional differential equation theory, for every \( \tau_k \), \( k = 0, 1, 2, 3, \ldots \), there exists \( \epsilon > 0 \) such that \( \lambda(r) \) is continuously differentiable in \( r \) for \( |r - \tau_k| < \epsilon \). Substituting \( \lambda(r) \) into the left hand side of (10) and taking derivative with respect to \( r \), we have
\[
\frac{d\lambda}{dr} \left. \right|_{r=\tau_k} = -\frac{\lambda(\lambda^3 + m_1)e^{-\lambda r} + m_2e^{-\lambda r} - 2m_3\lambda e^{-2\lambda r}}{\lambda(\lambda^3 + m_1)e^{-\lambda r} - m_2\lambda^2 e^{-\lambda r} - 2m_3\lambda e^{-2\lambda r}} \tag{28}
\]

Then
\[
\frac{d(\text{Re} \lambda(r))}{dr} \left. \right|_{r=\tau_k} = -\text{Re} \left[ \frac{(3\lambda^2 + m_1)e^{-\lambda r} + m_2\lambda + m_3 e^{-\lambda r}}{\lambda(\lambda^3 + m_1)e^{-\lambda r} - m_2\lambda^2 e^{-\lambda r} - 2m_3\lambda e^{-2\lambda r}} \right]. \tag{29}
\]

For convenience, let \( \mathfrak{X}(\tau) = x(\tau) \) (i = 1, 2, 3) and \( \tau = \tau_k + \mu \), where \( \tau_k \) is defined by (27) and \( \mu \in \mathbb{R} \), drop the bar for the simplification of notations, and then system (4) can be written as an FDE in \( C = C([-1,0]), R^3 \)
\[
\dot{u}(t) = L(u_t) + F(\mu, u_t), \quad (31)
\]
where \( u(t) = (x_1(t), x_2(t), x_3(t))^T \in C \) and \( u_1(\theta) = u(t + \theta) = (x_1(t + \theta), x_2(t + \theta), x_3(t + \theta))^T \in C \), and \( L_\mu : C \to R \) and \( F : R \times C \to R \) are given by

\[
L_\mu \phi = (\tau_k + \mu) \begin{pmatrix} 0 & b_2 & b_3 \\ c_1 & 0 & c_3 \\ d_1 & d_2 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\
+ (\tau_k + \mu) \begin{pmatrix} b_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} \phi_1(1) \\ \phi_2(1) \\ \phi_3(1) \end{pmatrix},
\]

respectively, where \( \phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C \) and

\[
f(\mu, \phi) = (\tau_k + \mu) \begin{pmatrix} f_1(\mu, \phi) \\ f_2(\mu, \phi) \\ f_3(\mu, \phi) \end{pmatrix},
\]

for \( \phi \in C([-1, 0], R^3) \), define

\[
A(\mu) \phi = \begin{pmatrix} -\frac{d\phi(\theta)}{d\theta} \\ 0 \\ 0 \end{pmatrix}, \quad -1 \leq \theta < 0,
\]

\[
R \phi = \begin{pmatrix} 0 \\ 0 \\ f(\mu, \phi) \end{pmatrix}, \quad -1 \leq \theta < 0.
\]

Then (31) is equivalent to the abstract differential equation

\[
\dot{u}_t = A(\mu) u_t + R(\mu) u_t,
\]

where \( u_t(\theta) = u(t + \theta), \theta \in [-1, 0] \).

For \( \psi \in C([-1, 0], (R^3)^*), \) define

\[
A^* \psi(s) = \begin{pmatrix} \frac{d\psi(s)}{ds} \\ \int_{-1}^s dt \frac{d\eta(t, 0)}{ds} \psi(t, 0) \end{pmatrix}, \quad s \in (0, 1],
\]

where \( \eta(\theta, 0) = A(0) \), and \( A^* \) are adjoint operators. By the discussions in Section 2, we know that \( \pm i\omega_k \tau_k \) are eigenvalues of \( A(0) \), and they are also eigenvalues of \( A^* \).

Suppose that \( q(\theta) = (1, \alpha, \beta)^T e^{i\omega_k \tau_k \theta} \) is the eigenvector of \( A(0) \) corresponding to \( i\omega_k \tau_k \); then \( A(0) q(\theta) = i\omega_k \tau_k q(\theta) \). It follows from definition \( A(0) \) and (32) and (35) that

\[
\tau_k \begin{pmatrix} i\omega_k - b_1 e^{-i\omega_k \tau_k} & -b_2 \\ -c_1 & i\omega_k - c_2 e^{i\omega_k \tau_k} \\ -d_1 & -d_2 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Then we can obtain

\[
\alpha = \frac{c_1 \left( b_1 e^{-i\omega_k \tau_k} - i\omega_k \right) + b_2 c_3}{b_1 c_3 - b_3 (i\omega_k - c_2 e^{i\omega_k \tau_k})},
\]

\[
\beta = \frac{b_1 e^{i\omega_k \tau_k} - i\omega_k (i\omega_k - c_2 e^{i\omega_k \tau_k})}{b_1 c_3 - b_3 (i\omega_k - c_2 e^{i\omega_k \tau_k})}.
\]

Similarly, let \( q^*(s) = D(1, \alpha^*, \beta^*) e^{i\omega_k \tau_k s} \) be the eigenvector of \( A^* \) corresponding to \( -i\omega_k \tau_k^{(l)} \); then \( A^* q^*(\theta) = -i\omega_k \tau_k q^*(\theta) \). It follows from the definition \( A^* \) that
\[ \begin{pmatrix} \omega_k + i b_1 e^{-i \omega_k \tau_1} & c_1 & d_1 \\ b_2 & \omega_k + c_2 e^{-i \omega_k \tau_1} & d_2 \\ b_3 & c_3 & \omega_k + d_3 e^{-i \omega_k \tau_1} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{43} \]

Hence
\[
\alpha^* = \frac{b_2 c_1 - d_2 (\omega_k + b_1 e^{-i \omega_k \tau_1})}{c_1 d_2 - d_1 (\omega_k + c_2 e^{-i \omega_k \tau_1})}, \quad \beta^* = -\frac{(\omega_k + c_2 e^{-i \omega_k \tau_1}) (\omega_k + b_1 e^{-i \omega_k \tau_1}) - b_2 c_1}{c_1 d_2 - d_1 (\omega_k + c_2 e^{-i \omega_k \tau_1})}. \tag{44} \]

In order to assure \( \langle q^* (s), q (\theta) \rangle = 1 \) and \( \langle q^*_* (s), \bar{q} (\theta) \rangle = 0 \), we need to determine the value of \( D \). From (40), we have
\[
\langle q^* (s), q (\theta) \rangle = D_\theta (1, \alpha^*, \beta^*)^T (1, \alpha, \beta)^T - \int_0^\theta D_\eta (1, \alpha^*, \beta^*) e^{-i \omega_k \tau_1 (t-\theta)} d \eta (\theta) (1, \alpha, \beta)^T \]
\[
\cdot e^{i \omega_k \tau_1} d \xi = D_\theta \left\{ 1 + a \alpha^* + \beta \beta^* - \int_0^\theta (1, \alpha^*, \beta^*) \right\} \]
\[
\cdot \theta e^{i \omega_k \tau_1} d \eta (\theta) (1, \alpha, \beta)^T = D_\theta \left( 1 + a \alpha^* + \beta \beta^* \right) \]
\[
\cdot \tau_k e^{-i \omega_k \tau_1} \left( b_1 + c_2 \alpha^* \alpha + d_3 \beta^* \beta \right). \tag{45} \]

Thus we can choose
\[
D = \frac{1}{1 + a \alpha^* + \beta \beta^* + \tau_k e^{-i \omega_k \tau_1} \left( b_1 + c_2 \alpha^* \alpha + d_3 \beta^* \beta \right)}. \tag{46} \]

Next, we use the same notations as those in Hassard et al. [24] and we first compute the coordinates to describe the center manifold \( C_0 \) at \( \mu = 0 \). Let \( u_i \) be the solution of (31) when \( \mu = 0 \). Define
\[
z (t) = \langle q^* , u_i \rangle, \]
\[
W (t, \theta) = u_i (\theta) - 2 \text{ Re } [z (t) q (\theta)], \tag{47} \]
on the center manifold \( C_0 \), and we have
\[
W (t, \theta) = W (z (t), \bar{z} (t), \theta), \tag{48} \]
where
\[
W (z (t), \bar{z} (t), \theta) = W (z, \bar{z}) \]
\[
= W_{20} \frac{z^2}{2} + W_{11} z \bar{z} + W_{02} \frac{\bar{z}^2}{2} + \cdots, \tag{49} \]
and \( z \) and \( \bar{z} \) are local coordinates for center manifold \( C_0 \) in the direction of \( q^* \) and \( \bar{q}^* \). Noting that \( W \) is also real if \( u_i \) is real, we consider only real solutions. For solutions \( u_i \in C_0 \) of (31)
\[
\dot{z} (t) = \omega_k \tau_k z + \bar{q}^* (\theta) f (0, W (z, \bar{z}, \theta)) \]
\[
+ 2 \text{ Re } \left[ z q (\theta) \right] \text{ def } = \omega_k \tau_k z + \bar{q}^* (0) f_0, \tag{50} \]
which we write in abbreviated form as
\[
\dot{z} (t) = \omega_k \tau_k z + g (z, \bar{z}), \tag{51} \]
Hence we have
\[
g (z, \bar{z}) = g_2 z^2 + g_1 z \bar{z} + g_0 z + \bar{z} + g_2 z + \bar{z} \cdots. \tag{52} \]

where
\[
f_1 (x_{11}, x_{22}, x_{33}) = b_4 x_{11} (0) x_{11} (-1) + b_3 x_{11} (0) x_{22} (0) + b_2 x_{11} (0) x_{33} (0), \]
\[
f_2 (x_{11}, x_{22}, x_{33}) = c_4 x_{11} (0) x_{22} (0) + c_3 x_{22} (0) x_{33} (0) + c_2 x_{33} (0) x_{33} (0), \]
\[
f_3 (x_{11}, x_{22}, x_{33}) = d_4 x_{11} (0) x_{33} (0) + d_3 x_{22} (0) x_{33} (0) + d_2 x_{33} (0) x_{33} (0), \]
\[
H_1 = b_4 e^{-i \omega_k \tau_1} + b_3 a + b_2 \beta, \]
\[
H_2 = c_4 a + c_3 a e^{-i \omega_k \tau_1} + c_2 a \beta, \]
\[
H_3 = d_4 \beta + d_3 a \beta + d_2 a e^{-i \omega_k \tau_1}, \]
\[
P_1 = b_4 \text{ Re } \left[ e^{i \omega_k \tau_1} \right] + b_3 \text{ Re } [a] + b_2 \text{ Re } [\beta], \]
\[
P_2 = c_4 \text{ Re } [a] + c_3 [a]^2 \text{ Re } [a] + c_2 \text{ Re } [\alpha \beta], \]
\[ P_3 = d_4 \text{Re}\{be^{i\omega \tau}x\} + d_3 \text{Re}\{aF\} + d_6 |\beta|^2 \text{Re}\{e^{i\omega \tau}x\}, \]
\[ K_1 = b_4 \text{Re}\{e^{i\omega \tau}x\} + b_5 \{a\} + b_6 \{b\}, \]
\[ K_2 = c_4 \alpha \alpha + c_5 \alpha \alpha e^{i\omega \tau}x + c_6 \text{Re}\{\alpha b\}, \]
\[ K_3 = d_4 \beta + d_5 \{\alpha b\} + d_6 \beta^2 e^{i\omega \tau}x, \]
\[ M_1 = b_5 \left[ W_{11}^{(1)}(-1) + \frac{1}{2} W_{20}^{(1)}(-1) + W_{11}^{(1)}(0) e^{-i\omega \tau}x \right. \]
\[ + \frac{1}{2} W_{20}^{(0)}(0) e^{-i\omega \tau}x + b_5 \left[ \frac{1}{2} W_{20}^{(0)}(0) + \frac{1}{2} W_{20}^{(1)}(0) \right] \]
\[ + W_{11}^{(2)}(0) + W_{11}^{(1)}(0) \right] + b_5 \left[ W_{11}^{(3)}(0) + \frac{1}{2} W_{20}^{(0)}(0) \right. \]
\[ + \frac{1}{2} \alpha W_{20}^{(0)}(0) + W_{11}^{(1)}(0) \left. \right], \]
\[ M_2 = c_4 \left[ W_{11}^{(2)}(0) + \frac{1}{2} W_{20}^{(2)}(0) + \frac{1}{2} \alpha W_{20}^{(0)}(0) \right. \]
\[ + W_{11}^{(1)}(0) \right] + c_5 \left[ a W_{11}^{(2)}(-1) + \frac{1}{2} \alpha W_{20}^{(0)}(-1) \right. \]
\[ + \alpha e^{-i\omega \tau}x W_{11}^{(2)}(0) + \frac{1}{2} \alpha W_{20}^{(0)}(0) \right] \]
\[ + c_5 \left[ a W_{20}^{(3)}(0) + \frac{1}{2} \alpha W_{20}^{(0)}(0) + \frac{1}{2} \alpha W_{20}^{(2)}(0) \right. \]
\[ + \beta W_{11}^{(0)}(0), \]
\[ M_3 = d_4 \left[ W_{20}^{(3)}(0) + \frac{1}{2} W_{20}^{(0)}(0) + \frac{1}{2} \alpha W_{20}^{(0)}(0) \right. \]
\[ + \beta W_{20}^{(2)}(0) \right] + d_5 \left[ W_{20}^{(5)}(0) \alpha + \frac{1}{2} W_{20}^{(3)}(0) \alpha \right. \]
\[ + \frac{1}{2} W_{20}^{(2)}(0) \beta + \beta W_{20}^{(2)}(0) \right] + d_6 \left[ W_{20}^{(5)}(-1) \beta \right. \]
\[ + \frac{1}{2} W_{20}^{(3)}(-1) \beta + \beta e^{-i\omega \tau}x W_{11}^{(3)}(0) \]
\[ + \alpha e^{-i\omega \tau}x W_{11}^{(1)}(0) \right]. \]

Thus we obtain
\[ g_{20} = 2\overline{\tau}_k \left( H_1 + \alpha^2 H_2 + \beta^2 H_3 \right), \]
\[ g_{11} = 2\overline{\tau}_k \left( P_1 + \alpha^2 P_2 + \beta^2 P_3 \right), \]
\[ g_{02} = 2\overline{\tau}_k \left( K_1 + \alpha^2 K_2 + \beta^2 K_3 \right), \]
\[ g_{21} = 2\overline{\tau}_k \left( M_1 + \alpha^2 M_2 + \beta^2 M_3 \right). \]

For unknown \( W_{20}^{(0)}(0), W_{20}^{(1)}(0), W_{20}^{(1)}(-1), W_{11}^{(1)}(-1) \), \( i = 1, 2, 3 \) in \( g_{21} \), we still need to compute them.

In view of (38) and (47), we have
\[ W = \begin{cases} \text{AW} - 2 \text{Re}\{\overline{q}^*(0) \overline{f} q(\theta)\}, & -1 \leq \theta < 0, \\ \text{AW} - 2 \text{Re}\{\overline{q}^*(0) \overline{f} q(\theta)\} + \overline{f}, & \theta = 0 \end{cases} \]

where
\[ H(\theta) = H_{20}(\theta) \frac{\vartheta^2}{2} + H_{11}(\theta) \overline{z} \overline{\tau} + H_{02}(\theta) \frac{\vartheta^2}{2} \]
\[ + \cdots. \]

Comparing the coefficients, we obtain
\[ A - 2i\omega \overline{k} \overline{a}_k W_{20} = -H_{20}(\theta), \]
\[ \text{AW}_{11}(\theta) = -H_{11}(\theta). \]

We know that for \( \theta \in [-1, 0) \),
\[ H(\theta) = -\overline{q}^*(0) f_0 q(\theta) - q^*(0) \overline{f}_0 \overline{q}(\theta) \]
\[ = -g(\theta, z) q(\theta) - \overline{g}(z, \overline{z}) \overline{q}(\theta). \]

Comparing the coefficients of (60) with (57) gives that
\[ H_{20}(\theta) = -g_{20} q(\theta) - \overline{g}_0 \overline{q}(\theta), \]
\[ H_{11}(\theta) = -g_{11} q(\theta) - \overline{g}_1 \overline{q}(\theta). \]

From (58), (61), and the definition of \( A \), we get
\[ W_{20}(\theta) = 2i\omega \overline{k} \overline{a}_k W_{20}(\theta) + g_{20} q(\theta) + \overline{g}_0 \overline{q}(\theta). \]

Noting that \( q(\theta) = q(0)e^{i\omega \tau_\theta} \), we have
\[ W_{20}(\theta) = \frac{i\overline{g}_0}{\omega \overline{k}_k} q(0) e^{i\omega \tau_\theta} + \frac{i\overline{g}_0}{3\omega \overline{k}_k} \overline{q}(0) e^{-i\omega \tau_\theta} \]
\[ + E_1 e^{i\omega \tau_\theta}, \]
where \( E_1 = (E_{11}^{(1)}, E_{12}^{(2)}, E_{13}^{(3)})^T \in R^3 \) is a constant vector.

Similarly, from (59), (62), and the definition of \( A \), we have
\[ \text{W}_{11}(\theta) = g_{11} q(\theta) + \overline{g}_1 \overline{q}(\theta), \]
\[ W_{11}(\theta) = -\frac{i\overline{g}_1}{\omega \overline{k}_k} q(0) e^{i\omega \tau_\theta} + \frac{i\overline{g}_1}{\omega \overline{k}_k} \overline{q}(0) e^{-i\omega \tau_\theta} \]
\[ + E_2, \]
where \( E_2 = (E_{11}^{(1)}, E_{12}^{(2)}, E_{13}^{(3)})^T \in R^3 \) is a constant vector.

In what follows, we shall seek appropriate \( E_1 \) and \( E_2 \) in (64) and (66), respectively. It follows from the definition of \( A, (61), \) and (62) that
\[ \int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega \overline{k} \overline{a}_k W_{20}(0) - H_{20}(0), \]
\[ \int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \]
where \( \eta(\theta) = \eta(0, \theta) \). From (58), we have

\[
H_{20}(0) = -g_{20}q(0) - \frac{\gamma_0}{\tau} \bar{q}(0) + 2\tau_k (H_1, H_2, H_3)^T, \quad (69)
\]

\[
H_{11}(0) = -g_{11}q(0) - \frac{\gamma_1}{\tau} \bar{q}(0) + 2\tau_k (P_1, P_2, P_3)^T. \quad (70)
\]

Noting that

\[
\left( i\omega_k \tau_k I - \int_{-1}^{0} e^{2\omega_k \tau_k \theta} \eta(\theta) \right) q(0) = 0,
\]

\[
\left( -i\omega_k \tau_k I - \int_{-1}^{0} e^{2\omega_k \tau_k \theta} \eta(\theta) \right) \bar{q}(0) = 0,
\]

and substituting (64) and (69) into (67), we have

\[
\left( 2i\omega_k \tau_k I - \int_{-1}^{0} e^{2\omega_k \tau_k \theta} \eta(\theta) \right) E_1 = 2\tau_k (H_1, H_2, H_3)^T. \quad (72)
\]

That is,

\[
\begin{pmatrix}
2i\omega_k \tau_k I - \int_{-1}^{0} e^{2\omega_k \tau_k \theta} \eta(\theta)
\end{pmatrix}
\begin{pmatrix}
-b_2 \\
-b_3 \\
-c_3 \\
-\tau d_1 \\
-\tau d_2 \\
-2i\omega_k \tau_k - d e^{-2i\omega_k \tau_k}
\end{pmatrix}
\begin{pmatrix}
E_1
\end{pmatrix}
\]

\[
= 2 \begin{pmatrix}
H_1 \\
H_2 \\
H_3
\end{pmatrix}.
\]

It follows that

\[
E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},
\]

\[
E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},
\]

\[
E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1},
\]

where

\[
\Delta_1 = \det \begin{pmatrix}
2i\omega_k \tau_k - b_1 e^{-2i\omega_k \tau_k} & -b_2 & -b_3 \\
-c_1 & 2i\omega_k - c_2 e^{-2i\omega_k \tau_k} & -c_3 \\
-\tau d_1 & -\tau d_2 & 2i\omega_k \tau_k - d e^{-2i\omega_k \tau_k}
\end{pmatrix}.
\]

\[
\Delta_{11} = 2 \det \begin{pmatrix}
H_1 & -b_2 & -b_3 \\
H_2 & 2i\omega_k - c_2 e^{-2i\omega_k \tau_k} & -c_3 \\
H_3 & -\tau d_1 & -\tau d_2 & 2i\omega_k \tau_k - d e^{-2i\omega_k \tau_k}
\end{pmatrix},
\]

\[
\Delta_{12} = 2 \det \begin{pmatrix}
2i\omega_k - b_1 e^{-2i\omega_k \tau_k} & H_1 & -b_3 \\
-c_1 & H_2 & -c_3 \\
-\tau d_1 & H_3 & 2i\omega_k - d e^{-2i\omega_k \tau_k}
\end{pmatrix},
\]

\[
\Delta_{13} = 2 \det \begin{pmatrix}
2i\omega_k - b_1 e^{-2i\omega_k \tau_k} & -b_2 & H_1 \\
-c_1 & 2i\omega_k - c_2 e^{-2i\omega_k \tau_k} & H_3 \\
-\tau d_1 & -\tau d_2 & H_3
\end{pmatrix}.
\]

Similarly, substituting (65) and (70) into (68), we have

\[
\left( \int_{-1}^{0} d\eta(\theta) \right) E_2 = 2\tau_k (P_1, P_2, P_3)^T. \quad (76)
\]

That is,

\[
\begin{pmatrix}
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
d_1 & d_2 & d_3
\end{pmatrix} E_2 = 2 \begin{pmatrix}
-P_1 \\
P_2 \\
P_3
\end{pmatrix}.
\]

It follows that

\[
E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},
\]

\[
E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},
\]

\[
E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},
\]

where

\[
\Delta_2 = \det \begin{pmatrix}
\begin{pmatrix}b_1 & b_2 & b_3 \end{pmatrix} & \begin{pmatrix}c_1 & c_2 & c_3 \end{pmatrix} & \begin{pmatrix}d_1 & d_2 & d_3 \end{pmatrix}
\end{pmatrix},
\]

\[
\Delta_{21} = - \det \begin{pmatrix}P_1 & b_2 & b_3 \\
c_1 & P_2 & c_3 \\
P_3 & d_2 & d_3
\end{pmatrix},
\]

\[
\Delta_{22} = - \det \begin{pmatrix}b_1 & P_1 & b_3 \\
c_1 & P_2 & c_3 \\
-d_1 & P_3 & d_3
\end{pmatrix},
\]

\[
\Delta_{23} = - \det \begin{pmatrix}b_1 & b_2 & P_1 \\
c_1 & c_2 & P_2 \\
d_1 & d_2 & P_3
\end{pmatrix}.
\]

From (64), (66), (74), and (78), we can calculate \( g_{21} \) and derive the following values:

\[
c_1(0) = \frac{i}{2\omega_k \tau_k} \left( g_{20} g_{11} - 2 |g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2},
\]

\[
\mu_2 = - \frac{\text{Re} [c_1(0)]}{\text{Re} \left( \lambda (\tau_k) \right)},
\]

\[
\beta_2 = 2 \text{Re} (c_1(0)),
\]

\[
T_2 = - \frac{\text{Im} [c_1(0)] + \mu_2 \text{Im} \left( \lambda' (\tau_k) \right)}{\omega_k \tau_k}.
\]

These formulae give a description of the Hopf bifurcation periodic solutions of (31) at \( \tau = \tau_k \) (\( k = 0, 1, 2, 3, \ldots \) on the center manifold. From the discussion above, we have the following result.
Figure 1: Dynamical behavior of system (81) with $\tau = 0.26 < \tau_0 \approx 0.265$. The positive equilibrium $E_0(1.9170, 0.9170, 2.9251)$ is asymptotically stable. The initial value $(x_1(0), x_2(0), x_3(0)) = (1.9, 0.85, 2.3)$.

**Theorem 3.** The direction of the Hopf bifurcation is forward (backward) if $\mu_2 > 0$ ($\mu_2 < 0$); the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); the periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).

### 4. Numerical Examples

In this section, numerical simulations are carried out to investigate the effect of the time delay on dynamical behaviors of system (2) as well as illustrate our theoretical results. For convenience, we assume that some parametric values of
system (2) are kept as \( r_1 = 7, r_2 = 6, r_3 = 5, a_{11} = 3, a_{12} = 2, a_{13} = 0.2, a_{21} = 2, a_{22} = 3, a_{31} = 0.3, a_{32} = 0.3, \) and \( a_{33} = 2. \) Then system (2) takes the form

\[
\begin{align*}
\dot{x}_1(t) & = x_1(t) \left[ 7 - 3x_1(t - \tau) - 2x_2(t) + 0.2x_3(t) \right], \\
\dot{x}_2(t) & = x_2(t) \left[ 6 - 2x_1(t) - 3x_2(t - \tau) + 0.2x_3(t) \right], \\
\dot{x}_3(t) & = x_3(t) \left[ 5 + 0.3x_1(t) + 0.3x_2(t) - 2x_3(t - \tau) \right], \\
\end{align*}
\]

(81)

which has a positive equilibrium \( E_0(1.9170,0.9170,2.9251) \) and all the conditions given in Theorem 2 are satisfied. When \( \tau = 0, \) the positive equilibrium \( E_0(1.9170,0.9170,2.9251) \) is asymptotically stable. Fix \( k = 0 \) and by means of Matlab 7.0, we get \( \omega_0 \approx 0.9775, \) \( \tau_0 \approx 0.265, \) and \( \lambda(\tau_0) \approx 0.6505 - 8.4322i. \) Thus we can derive the following values: \( c_1(0) \approx -2.3628 - 4.6636i, \) \( \mu_2 \approx 3.6323, \) \( \beta_2 \approx -4.7256, \) and \( T_2 \approx 136.2424. \) Furthermore, it follows that \( \mu_2 > 0 \) and \( \beta_2 < 0. \)
Thus the positive equilibrium $E_0(1.9170, 0.9170, 2.9251)$ is stable when $\tau < \tau_0$. When $\tau = 0.26 < \tau_0 = 0.265$, we can easily plot the time series of every population and phase portrait of the system and find that the solution of (8) with initial value $x_1(0) = 1.9, x_2(0) = 0.85,$ and $x_3(0) = 2.3$ would tend to a positive equilibrium $E_0(1.9170, 0.9170, 2.9251)$ (see Figures 1(a)–1(j), in detail); that is, the positive equilibrium $E_0(1.9170, 0.9170, 2.9251)$ is asymptotically stable. When $\tau$ passes through the critical value $\tau_0 \approx 0.265$, the positive equilibrium $E_0(1.9170, 0.9170, 2.9251)$ loses its stability and a Hopf bifurcation occurs; that is, a family of periodic solutions bifurcate from the positive equilibrium $E_0(1.9170, 0.9170, 2.9251)$. Figures 2(a)–2(j) are plotted by fixing the time delay $\tau = 1$ and initial value $x_1(0) = 1.9, x_2(0) = 0.85$, and $x_3(0) = 2.3$. It is shown that a Hopf bifurcation occurs from the positive equilibrium $E_0(1.9170, 0.9170, 2.9251)$. In view of $\beta_2 > 0$ and $\beta_3 < 0$, we find that the direction of the Hopf bifurcation is $\tau > \tau_0 \approx 0.265$ and these bifurcating periodic solutions from $E_0(1.9170, 0.9170, 2.9251)$ at $\tau_0 = 0.265$ are stable.

5. Biological Meanings and Conclusions

In this paper, the dynamics including the local stability of the positive equilibrium $E_0(x_1^*, x_2^*, x_3^*)$ and the local Hopf bifurcation of a competitor–competitor–mutualist Lotka–Volterra model are investigated. It is shown that the positive equilibrium $E_0(x_1^*, x_2^*, x_3^*)$ of system (2) is asymptotically stable for all $\tau \in [0, \tau_0]$ when the conditions (H1)–(H3) are satisfied. This reveals that the densities of competing species $x_1, x_2$ and cooperating species $x_3$ tend to stabilization; in other words, the densities of competing species $x_1, x_2$ and cooperating species will tend to $x_1^*, x_2^*, x_3^*$, respectively, and this situation does not vary with the delay $\tau \in [0, \tau_0]$. When the delay $\tau$ increases, the positive equilibrium $E_0(x_1^*, x_2^*, x_3^*)$ loses its stability and a sequence of Hopf bifurcations occur from the positive equilibrium $E_0(x_1^*, x_2^*, x_3^*)$; that is, a family of periodic orbits bifurcate from the positive equilibrium $E_0(x_1^*, x_2^*, x_3^*)$. From this we can find that competing species $x_1, x_2$ and cooperating species $x_3$ can coexist and keep in periodic mode; that is, the densities of competing species $x_1, x_2$ and cooperating species $x_3$ will oscillate in the vicinity of $x_1^*, x_2^*, x_3^*$, respectively. Finally, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem. A numerical example is also included to verify our theoretical findings.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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