Research Article

A New Ranking Principle For Ordering Trapezoidal Intuitionistic Fuzzy Numbers

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Modelling real life (industrial) problems using intuitionistic fuzzy numbers is inevitable in the present scenario due to their efficiency in solving problems and their accuracy in the results. Particularly, trapezoidal intuitionistic fuzzy numbers (TrIFNs) are widely used in describing impreciseness and incompleteness of a data. Any intuitionistic fuzzy decision-making problem requires the ranking procedure for intuitionistic fuzzy numbers. Ranking trapezoidal intuitionistic fuzzy numbers play an important role in problems involving incomplete and uncertain information. The available intuitionistic fuzzy decision-making methods cannot perform well in all types of problems, due to the partial ordering on the set of intuitionistic fuzzy numbers. In this paper, a new total ordering on the class of TrIFNs using eight different score functions, namely, imprecise score, nonvague score, incomplete score, accuracy score, spread score, nonaccuracy score, left area score, and right area score, is achieved and our proposed method is validated using illustrative examples. Significance of our proposed method with familiar existing methods is discussed.

1. Introduction

Classical set theory cannot be the better choice for modelling problems involving qualitative or imprecise information. To model such problems, fuzzy number was introduced by Jain [1] and some operations on fuzzy numbers are defined in [2, 3]. Intuitionistic fuzzy numbers are comparatively better in modelling real life problems involving uncertainties and imprecise information. Particularly, trapezoidal intuitionistic fuzzy numbers are more effective in describing impreciseness and incompleteness of a data. To resolve the task of comparing trapezoidal intuitionistic fuzzy numbers, many authors have proposed different ranking methods but none of them yield a total order on the class of TrIFNs with finite number of scores. Different ranking methodologies on the class of intuitionistic fuzzy numbers are discussed in [4, 5].

Nehi and Maleki [6] generalised the idea of natural ordering on real numbers to the triangular intuitionistic fuzzy numbers (TIFNs) by adopting a statistical view point. Nehi [7] compared TIFNs using lexicographic technique. Li [2] developed the idea of value and ambiguity of a triangular intuitionistic fuzzy number and introduced the

new ranking method using the concept of the ratio of the value index to the ambiguity index. Ye [8] presented the new ranking method using expected value of a trapezoidal intuitionistic fuzzy number and solved the decision-making problem using weighted expected value of TrIFN. Dubey and Mehara [9] extended the concept of value and ambiguity to the slightly modified TIFN and proposed a new approach to solve intuitionistic fuzzy linear programming problem. Nehi [7] introduced the concept of characteristic values of membership and nonmembership functions of TrIFN and proposed a new ranking method for trapezoidal intuitionistic fuzzy numbers by using it. Zhang and Nan [10] developed a compromise ratio ranking method for fuzzy multiattribute decision-making (MADM) problem based on the concept that larger TIFN among other TIFNs will be closer to the maximum value index and it will be far away from the minimum ambiguity index simultaneously. Kumar and Kaur [11] proposed the ranking method for TrIFNs by modifying Nehi's [10] method. Zeng et al. [12] introduced a new ranking method for TrIFNs by extending the concept value and ambiguity of TIFN defined in Li [2]. Wan and Dong [13] introduced the concept of lower and upper weighted

possibility mean and possibility mean for a trapezoidal intuitionistic fuzzy numbers and proposed the new ranking method by use of it. Different ranking methods and their applications on multicriteria decision-making problem and other domains are studied in ([14–20]). Lakshmana Gomathi Nayagam et al. [19, 21, 22] have introduced a complete ranking procedure on the class of intuitionistic fuzzy numbers using countable number of parameter. In this paper, a new total ordering on the class of TrIFNs using finite number of score functions is achieved. Also the limitations and drawbacks of all the abovementioned methods are discussed and the efficiency of our proposed method is shown by comparing all existing methods.

This paper is organised in the following manner. After introduction, some important definitions on intuitionistic fuzzy numbers are given in Section 2. The different subclasses of TrIFNs are introduced and the new score functions on these subclasses are established in Section 3. In Section 4, a complete ranking on the class of trapezoidal intuitionistic fuzzy numbers by using score functions defined in Section 3 is explained. The ranking procedure is explained in detail with several examples and also our proposed method is compared with some other existing methods in the Section 5. Finally conclusions are given in Section 6.

2. Preliminaries

Here we give a brief review of some preliminaries.

Definition 1 (Atanassov [23]). Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A in X is defined by $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A(x) : X \rightarrow [0, 1]$ and $\nu_A(x) : X \rightarrow [0, 1]$, $x \in X$ with the conditions $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, $\forall x \in X$. The numbers $\mu_A(x), \nu_A(x) \in [0, 1]$ denote the degree of membership and nonmembership of x to lie in A , respectively. For each intuitionistic fuzzy subset A in X , $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called hesitancy degree of x to lie in A .

Definition 2 (Burillo et al., [24]). An intuitionistic fuzzy number $A = (\mu_A, \nu_A)$ in the set of real numbers R is defined as

$$\mu_A(x) = \begin{cases} f_A(x) & \text{if } a \leq x \leq b_1 \\ 1 & \text{if } b_1 \leq x \leq b_2 \\ g_A(x) & \text{if } b_2 \leq x \leq c \\ 0 & \text{Otherwise,} \end{cases} \quad (1)$$

$$\nu_A(x) = \begin{cases} h_A(x) & \text{if } e \leq x \leq f_1 \\ 0 & \text{if } f_1 \leq x \leq f_2 \\ k_A(x) & \text{if } f_2 \leq x \leq g \\ 1 & \text{Otherwise,} \end{cases}$$

where $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ and $a, b_1, b_2, c, e, f_1, f_2, g \in \mathfrak{R}$ such that $e \leq a, f_1 \leq b_1 \leq b_2 \leq f_2, c \leq g$, and four functions $f_A, g_A, h_A, k_A : \mathfrak{R} \rightarrow [0, 1]$ are the legs of membership

FIGURE 1: Intuitionistic fuzzy number.

function μ_A and nonmembership function ν_A . The functions f_A and k_A are nondecreasing continuous functions and the functions h_A and g_A are nonincreasing continuous functions.

An intuitionistic fuzzy number $\{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ with $(e, f_1, f_2, g) \leq (a, b_1, b_2, c)^c$ is shown in Figure 1.

Definition 3. A trapezoidal intuitionistic fuzzy number A with parameters $e \leq a, f_1 \leq b_1 \leq b_2 \leq f_2, c \leq g$ is denoted as $A = \{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ in the set of real numbers $\mathfrak{R}$ is an intuitionistic fuzzy number whose membership function and nonmembership function are given as

$$\mu_A(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4 - x}{a_4 - a_3} & \text{if } a_3 \leq x \leq a_4 \\ 0 & \text{Otherwise,} \end{cases} \quad (2)$$

$$\nu_A(x) = \begin{cases} \frac{x - c_2}{c_1 - c_2} & \text{if } c_1 \leq x \leq c_2 \\ 0 & \text{if } c_2 \leq x \leq c_3 \\ \frac{x - c_3}{c_4 - c_3} & \text{if } c_3 \leq x \leq c_4 \\ 1 & \text{Otherwise.} \end{cases}$$

If $a_2 = a_3$ (and $c_2 = c_3$) in a trapezoidal intuitionistic fuzzy number A , we have the triangular intuitionistic fuzzy numbers as special case of the trapezoidal intuitionistic fuzzy numbers.

A trapezoidal intuitionistic fuzzy number $A = \{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ with $f_1 \leq b_1, f_2 \geq b_2, e \leq a$, and $g \geq c$ is shown in Figure 2.

We note that the condition $(e, f_1, f_2, g) \leq (a, b_1, b_2, c)^c$ of the trapezoidal intuitionistic fuzzy number $A = \{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ whose membership and nonmembership fuzzy numbers of A are (a, b_1, b_2, c) and (e, f_1, f_2, g) implies

FIGURE 2: Trapezoidal intuitionistic fuzzy number.

$f_1 \leq b_1$, $f_2 \geq b_2$, $e \leq a$, and $g \geq c$ on the legs of trapezoidal intuitionistic fuzzy number.

Definition 4 (Atanassov and Gargov, [25]). Let $D[0, 1]$ be the set of all closed subintervals of the interval $[0, 1]$. An interval valued intuitionistic fuzzy set on a set $X \neq \emptyset$ is an expression given by $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$, where $\mu_A : X \rightarrow D[0, 1]$, $\nu_A : X \rightarrow D[0, 1]$ with the condition $0 < \sup_x \mu_A(x) + \sup_x \nu_A(x) \leq 1$.

The intervals $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degree of belongingness and nonbelongingness of the element x to the set A . Thus for each $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals whose lower and upper end points are, respectively, denoted by $\mu_{A_L}(x)$, $\mu_{A_U}(x)$ and $\nu_{A_L}(x)$, $\nu_{A_U}(x)$. We denote $A = \{\langle x, [\mu_{A_L}(x), \mu_{A_U}(x)], [\nu_{A_L}(x), \nu_{A_U}(x)] \rangle : x \in X\}$, where $0 < \mu_{A_U}(x) + \nu_{A_U}(x) \leq 1$.

For each element $x \in X$, we can compute the unknown degree (hesitance degree) of belongingness $\pi_A(x)$ to A as $\pi_A(x) = 1 - \mu_{A_U}(x) - \nu_{A_U}(x) = [1 - \mu_{A_U}(x) - \nu_{A_U}(x), 1 - \mu_{A_L}(x) - \nu_{A_L}(x)]$. An intuitionistic fuzzy interval number (IFIN) is denoted by $A = ([a, b], [c, d])$ for convenience.

Definition 5. Let $\alpha, \beta \in (0, 1]$. An (α, β) -cut of a trapezoidal intuitionistic fuzzy number, denoted by $^{(\alpha, \beta)}A_I$, is defined as $^{(\alpha, \beta)}A_I = \{\langle \mu_{A_I}^\alpha, \nu_{A_I}^\beta \rangle\}$, where $\mu_{A_I}^\alpha$ and $\nu_{A_I}^\beta$ are α -cut and β -cut of membership and nonmembership trapezoidal fuzzy numbers.

The (α, β) -cut of trapezoidal intuitionistic fuzzy number $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ is given by $^{(\alpha, \beta)}A_I = \{[\mu_{A_I}^\alpha, \mu_{A_I}^\beta], [\nu_{A_I}^\beta, \nu_{A_I}^\alpha]\} = ([a_{1\mu} + \alpha(a_{2\mu} - a_{1\mu}), a_{4\mu} - \alpha(a_{4\mu} - a_{3\mu})], [a'_{1\nu} + \beta(a'_{2\nu} - a'_{1\nu}), a'_{4\nu} - \beta(a'_{4\nu} - a'_{3\nu})])$, where $\mu_{A_I}^\alpha$ and $\mu_{A_I}^\beta$ represents the lower and upper end points of the α -cut of the membership function of A_I and $\nu_{A_I}^\beta$ and $\nu_{A_I}^\alpha$ represent the lower and upper end points of the β -cut of the nonmembership function of A_I , respectively.

3. Different Classes of Trapezoidal Intuitionistic Fuzzy Numbers

In this section the entire class of TrIFNs is partitioned into eight subclasses and further score functions using different concepts are defined in order to give total order on the entire class of TrIFNs and some theorems related to these concepts are established.

In this paper, A_I and B_I always denote trapezoidal intuitionistic fuzzy number (TrIFN).

$A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ with conditions that $a'_{1\nu} \leq a_{1\mu}$ and $a'_{2\nu} \leq a_{2\mu}$, $a'_{3\nu} \geq a_{3\mu}$ and $a'_{4\nu} \geq a_{4\mu}$ and $B_I = \langle (b_{1\mu}, b_{2\mu}, b_{3\mu}, b_{4\mu}), (b'_{1\nu}, b'_{2\nu}, b'_{3\nu}, b'_{4\nu}) \rangle$ with the condition that $b'_{1\nu} \leq b_{1\mu}$ and $b'_{2\nu} \leq b_{2\mu}$, $b'_{3\nu} \geq b_{3\mu}$, and $b'_{4\nu} \geq b_{4\mu}$ unless otherwise specifically stated.

3.1. Imprecise Score of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, a new subclass of TrIFNs is introduced. The imprecise score function on this class of trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 6. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (b_{1\mu}, b_{2\mu}, b_{3\mu}, b_{4\mu}), (b'_{1\nu}, b'_{2\nu}, b'_{3\nu}, b'_{4\nu}) \rangle$ be two TrIFNs.

A special subclass C_1 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \geq b_{1\mu}$, $a_{2\mu} \geq b_{2\mu}$, $a_{3\mu} \leq b_{3\mu}$, $a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}$, $a'_{2\nu} \geq b'_{2\nu}$, $a'_{3\nu} \leq b'_{3\nu}$, $a'_{4\nu} \leq b'_{4\nu}$. The imprecise relation on C_1 is denoted as $\sqsubseteq$ and defined as follows: if $A_I, B_I \in C_1 \subset \text{TrIFN}$ such that $A_I \sqsubseteq B_I$ then $a_{1\mu} \geq b_{1\mu}$, $a_{2\mu} \geq b_{2\mu}$, $a_{3\mu} \leq b_{3\mu}$, $a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}$, $a'_{2\nu} \geq b'_{2\nu}$, $a'_{3\nu} \leq b'_{3\nu}$, $a'_{4\nu} \leq b'_{4\nu}$.

If $A_I \subset B_I$ then one of the above inequalities becomes strict inequality.

Note 1. By Definition 6, we note that any pair of members of C_1 are related under $\sqsubseteq$.

The score function which measures the preciseness is defined as follows.

Definition 7. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ be a trapezoidal intuitionistic fuzzy number with $a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}, a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu} \in [0, 1]$. Then the imprecise score of a trapezoidal intuitionistic fuzzy number A_I is defined by

$$\begin{aligned} J_I(A_I) &= \int_0^1 \frac{[\mu_{A_I}^\alpha - \mu_{A_I}^\beta]}{4} d\alpha + \int_0^1 \frac{[\nu_{A_I}^\beta - \nu_{A_I}^\alpha]}{4} d\beta \\ &= \int_0^1 \frac{(a_{1\mu} + \alpha(a_{2\mu} - a_{1\mu}))}{4} d\alpha \\ &\quad - \int_0^1 \frac{(a_{4\mu} - \alpha(a_{4\mu} - a_{3\mu}))}{4} d\alpha \end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \frac{(a'_{1\nu} + \beta(a'_{2\nu} - a'_{1\nu}))}{4} d\beta \\
& - \int_0^1 \frac{(a'_{4\nu} - \beta(a'_{4\nu} - a'_{3\nu}))}{4} d\beta \\
& = \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}.
\end{aligned} \tag{3}$$

The proofs of the following propositions are immediate from the above definition and hence they are omitted.

Proposition 8. For any real number $r \in [0, 1]$, $J_1(r) = 0$.

Proposition 9. If $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$ is a trapezoidal fuzzy number, then $J_1(A_I) = ((a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}))/4$.

Proposition 10. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then $J_1(A_I) = ((a_{1\mu} - a_{3\mu}) + (a'_{1\nu} - a'_{3\nu}))/4$.

Proposition 11. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then $J_1(A_I) = ((a_{1\mu} - a_{2\mu}) + (a'_{1\nu} - a'_{2\nu}))/2$.

Theorem 12. Let A_I and $B_I \in C_1$. If $A_I \sqsubset B_I$ then $J_1(A_I) > J_1(B_I)$.

Proof. Let $A_I \sqsubset B_I$. We claim that $J_1(A_I) - J_1(B_I) > 0$.
By Definition 6, we have

$$\begin{aligned}
a_{1\mu} & \geq b_{1\mu}, \\
a_{2\mu} & \geq b_{2\mu}, \\
a_{3\mu} & \leq b_{3\mu}, \\
a_{4\mu} & \leq b_{4\mu}, \\
a'_{1\nu} & \geq b'_{1\nu}, \\
a'_{2\nu} & \geq b'_{2\nu}, \\
a'_{3\nu} & \leq b'_{3\nu}, \\
a'_{4\nu} & \leq b'_{4\nu}.
\end{aligned} \tag{4}$$

Now,

$$\begin{aligned}
8(J_1(A_I) - J_1(B_I)) & = [(a_{1\mu} - b_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\
& + (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\
& + (a'_{2\nu} - b'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})].
\end{aligned} \tag{5}$$

From (4) it is very easy to see that all the terms in (5) are positive. Therefore their sum is also positive. From Definition 6, we know that at least one of the above inequalities in (4) becomes strict inequality and hence we get $J_1(A_I) - J_1(B_I) > 0$. $\square$

Theorem 13. Let A_I and $B_I \in C_1$ such that $J_1(A_I) = J_1(B_I)$; then $A_I = B_I$.

Proof. Let $J_1(A_I) = J_1(B_I)$. We claim that $A_I = B_I$.

By Definition 6, without loss of generality, we have

$$\begin{aligned}
a_{1\mu} & \geq b_{1\mu}, \\
a_{2\mu} & \geq b_{2\mu}, \\
a_{3\mu} & \leq b_{3\mu}, \\
a_{4\mu} & \leq b_{4\mu}, \\
a'_{1\nu} & \geq b'_{1\nu}, \\
a'_{2\nu} & \geq b'_{2\nu}, \\
a'_{3\nu} & \leq b'_{3\nu}, \\
a'_{4\nu} & \leq b'_{4\nu}.
\end{aligned} \tag{6}$$

Now,

$$\begin{aligned}
8(J_1(A_I) - J_1(B_I)) & = [(a_{1\mu} - b_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\
& + (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\
& + (a'_{2\nu} - b'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})] = 0.
\end{aligned} \tag{7}$$

Therefore from (6) and (7), it is clear that all the terms in (7) are positive and their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}$, $a_{2\mu} = b_{2\mu}$, $a_{3\mu} = b_{3\mu}$, $a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}$, $a'_{2\nu} = b'_{2\nu}$, $a'_{3\nu} = b'_{3\nu}$, $a'_{4\nu} = b'_{4\nu}$, hence the proof. $\square$

Note 2. The imprecise score can be calculated to any TriFN but J_1 gives total order in C_1 .

Definition 14. If $J_1(A_I) > J_1(B_I)$ ($J_1(A_I) < J_1(B_I)$), then $A_I > B_I$ ($A_I < B_I$).

The following example explains the ranking procedure introduced in Definition 14.

Example 15. Let $A_I = \langle (0.1, 0.2, 0.35, 0.5), (0, 0.15, 0.45, 0.6) \rangle$ and $B_I = \langle (0.15, 0.2, 0.35, 0.45), (0.1, 0.1, 0.5, 0.55) \rangle \in C_1$. Now $J_1(A_I) = -0.18125$ and $J_1(B_I) = -0.1625$, $J_1(A_I) < J_1(B_I)$.

Hence $A_I < B_I$.

Example 16. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} + \epsilon, a_{3\mu} - \epsilon, a_{4\mu} - \epsilon), (a'_{1\nu} - \epsilon, a'_{2\nu} - \epsilon, a'_{3\nu} + \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TriFN}$ but not in C_1 , where $\epsilon \in [0, 1]$.

Now $J_1(A_I) = J_1(B_I) = ((a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu}))/8$. But $A_I \neq B_I$. Pictorial representation of this example is given in Figure 3.

Example 16 shows that J_1 is not enough cover the entire class of TriFNs. Therefore it is needed for us to define another

FIGURE 3: Pictorial representation of Example 16.

score function that can cover some subclass of TrIFNs which cannot be covered by J_1 .

3.2. Nonvague Score of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, another subclass of TrIFNs is introduced using nonvague relation. The nonvague score function on this class of trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 17. Let A_I and B_I be two TrIFNs.

A special subclass C_2 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \geq b_{1\mu}, a_{2\mu} \geq b_{2\mu}, a_{3\mu} \leq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$. The nonvague relation on C_2 is denoted as $\sqsupseteq$ and defined as follows: if $A_I, B_I \in C_2 \subset \text{TrIFN}$ such that $A_I \sqsupseteq B_I$ then $a_{1\mu} \geq b_{1\mu}, a_{2\mu} \geq b_{2\mu}, a_{3\mu} \leq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$.

If $A_I \sqsupset B_I$ then one of the above inequalities becomes strict inequality.

Note 3. By Definition 17, we note that any pair of members of C_2 are related under $\sqsupseteq$.

The score function which measures the nonvagueness is defined as follows.

Definition 18. Let A_I be a trapezoidal intuitionistic fuzzy number. Then the nonvague score of a trapezoidal intuitionistic fuzzy number A_I is defined by

$$\begin{aligned} J_2(A_I) &= \int_0^1 \frac{[\alpha \mu_{A_{IL}} - \alpha \mu_{A_{IU}}]}{4} d\alpha - \int_0^1 \frac{[\beta \nu_{A_{IL}} - \beta \nu_{A_{IU}}]}{4} d\beta \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}. \end{aligned} \quad (8)$$

The proofs of the following propositions are immediate from Definition 18 and hence they are omitted.

Proposition 19. For any real number $r \in [0, 1]$, $J_2(r) = 0$.

Proposition 20. If $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$ is a trapezoidal fuzzy number, then $J_2(A_I) = 0$.

Proposition 21. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then $J_2(A_I) = ((a_{1\mu} - a_{3\mu}) + (-a'_{1\nu} + a'_{3\nu}))/8$.

Proposition 22. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then $J_2(A_I) = ((a_{1\mu} - a_{2\mu}) + (-a'_{1\nu} + a'_{2\nu}))/4$.

Theorem 23. Let A_I and $B_I \in C_2$. If $A_I \sqsupset B_I$ then $J_2(A_I) > J_2(B_I)$.

Proof. Let us assume $A_I \sqsupset B_I$. We claim that $J_2(A_I) - J_2(B_I) > 0$.

By Definition 17, we have

$$\begin{aligned} a_{1\mu} &\geq b_{1\mu}, \\ a_{2\mu} &\geq b_{2\mu}, \\ a_{3\mu} &\leq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\leq b'_{1\nu}, \\ a'_{2\nu} &\leq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\geq b'_{4\nu}. \end{aligned} \quad (9)$$

Now,

$$\begin{aligned} 8(J_2(A_I) - J_2(B_I)) &= [(a_{1\mu} - b_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\ &\quad + (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\ &\quad + (b'_{2\nu} - a'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})]. \end{aligned} \quad (10)$$

From (9) it is very easy to see that all the terms in (10) are positive. Therefore their sum is also positive. From Definition 17, we know that at least one of the above inequalities in (9) becomes strict inequality and hence we get $J_2(A_I) - J_2(B_I) > 0$. $\square$

Theorem 24. Let A_I and $B_I \in C_2$. If $J_2(A_I) = J_2(B_I)$ then $A_I = B_I$.

Proof. Let us assume $J_2(A_I) = J_2(B_I)$. We claim that $A_I = B_I$.

By Definition 17, without loss of generality, we have

$$\begin{aligned}
 a_{1\mu} &\geq b_{1\mu}, \\
 a_{2\mu} &\geq b_{2\mu}, \\
 a_{3\mu} &\leq b_{3\mu}, \\
 a_{4\mu} &\leq b_{4\mu}, \\
 a'_{1\nu} &\leq b'_{1\nu}, \\
 a'_{2\nu} &\leq b'_{2\nu}, \\
 a'_{3\nu} &\geq b'_{3\nu}, \\
 a'_{4\nu} &\geq b'_{4\nu}.
 \end{aligned} \tag{11}$$

Now,

$$\begin{aligned}
 8(J_2(A_I) - J_2(B_I)) &= [(a_{1\mu} - b_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\
 &+ (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\
 &+ (b'_{2\nu} - a'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})] = 0.
 \end{aligned} \tag{12}$$

Therefore from (11) and (12), it is clear that all the terms in (12) are positive; therefore their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}, a_{2\mu} = b_{2\mu}, a_{3\mu} = b_{3\mu}, a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}, a'_{2\nu} = b'_{2\nu}, a'_{3\nu} = b'_{3\nu}, a'_{4\nu} = b'_{4\nu}$. Hence $A_I = B_I$. $\square$

Note 4. The nonvague score can be calculated to any TrIFN. But J_2 gives total order on C_2 .

Definition 25. If $J_2(A_I) > J_2(B_I)$ ($J_2(A_I) < J_2(B_I)$), then $A_I > B_I$ ($A_I < B_I$).

Ranking relation defined above is explained in the following example.

Example 26. Let $A_I = \langle (0.15, 0.25, 0.35, 0.45), (0.1, 0.2, 0.4, 0.5) \rangle$ and $B_I = \langle (0.18, 0.25, 0.35, 0.4), (0, 0.15, 0.45, 0.55) \rangle \in C_2$. Now $J_2(A_I) = 0.025$ and $J_2(B_I) = 0.06625, J_2(B_I) > J_2(A_I)$.

Hence $B_I > A_I$.

Example 27 shows the inefficiency of J_1 in comparing any two arbitrary TrIFNs and the importance of defining new score function J_2 .

Example 27. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} + \epsilon, a_{3\mu} - \epsilon, a_{4\mu} - \epsilon), (a'_{1\nu} - \epsilon, a'_{2\nu} - \epsilon, a'_{3\nu} + \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$.

Now

$$\begin{aligned}
 J_1(A_I) &= J_1(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8} \implies \tag{13}
 \end{aligned}$$

$$A_I = B_I.$$

FIGURE 4: Pictorial representation of Example 28.

But

$$\begin{aligned}
 J_2(A_I) &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_2(B_I) &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8} \\
 &+ \epsilon.
 \end{aligned} \tag{14}$$

Therefore $J_2(B_I) > J_2(A_I)$; hence $B_I > A_I$.

Example 28. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} + \epsilon, a_{3\mu} + \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} + \epsilon/2, a'_{2\nu} + \epsilon/2, a'_{3\nu} + \epsilon/2, a'_{4\nu} + \epsilon/2) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$ but not in C_2 . Now

$$\begin{aligned}
 J_1(A_I) &= J_1(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_2(A_I) &= J_2(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}.
 \end{aligned} \tag{15}$$

But $A_I \neq B_I$.

Pictorial representation of this example is given in Figure 4.

From Example 28 it is easy to see that J_1 and J_2 are not enough to cover the entire class of TrIFNs. Therefore in the next subsection we are introducing a new score function which covers some more subclasses of TrIFNs that cannot be covered by J_1 and J_2 .

3.3. Incomplete Score of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, another class of TrIFNs is introduced. The incomplete score function on this class of

trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 29. Let A_I and B_I be two TrIFNs.

A special subclass C_3 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \leq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}, a'_{2\nu} \geq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$. The incomplete relation on C_3 is denoted as $\subseteq$ and defined as follows: if $A_I, B_I \in C_3 \subset \text{TrIFN}$ such that $A_I \subseteq B_I$ then $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \leq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}, a'_{2\nu} \geq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$.

If $A_I \subset B_I$ then one of the above inequalities becomes strict inequality.

Note 5. By Definition 29, we note that any pair of members of C_3 are related under $\subset$.

The incomplete score function which measures the completeness is defined as follows.

Definition 30. Let A_I be a trapezoidal intuitionistic fuzzy number. Then the incomplete score of A_I is defined by

$$\begin{aligned} J_3(A_I) &= \int_0^1 \frac{[\alpha \mu_{A_{IL}} + \alpha \mu_{A_{IU}}]}{4} d\alpha \\ &+ \int_0^1 \frac{[\beta \nu_{A_{IL}} + \beta \nu_{A_{IU}}]}{4} d\beta \\ &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}. \end{aligned} \quad (16)$$

The proofs of the following propositions are immediate from the above definition and hence they are omitted.

Proposition 31. For any real number $r \in [0, 1]$, $J_3(r) = 0$.

Proposition 32. If $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$ is a trapezoidal fuzzy number, then $J_3(A_I) = 0$.

Proposition 33. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then

$$J_3(A_I) = \frac{(a'_{1\nu} + 2a'_{2\nu} + a'_{3\nu}) - (a_{1\mu} + 2a_{2\mu} + a_{3\mu})}{8}. \quad (17)$$

Proposition 34. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then

$$J_3(A_I) = \frac{(a'_{1\nu} + a'_{2\nu}) - (a_{1\mu} + a_{2\mu})}{4}. \quad (18)$$

Theorem 35. Let $A_I, B_I \in C_3$. If $A_I \subset B_I$ then $J_3(A_I) > J_3(B_I)$.

Proof. Let us assume that $A_I \subset B_I$. We claim that $J_3(A_I) - J_3(B_I) > 0$.

By Definition 29, we have

$$\begin{aligned} a_{1\mu} &\leq b_{1\mu}, \\ a_{2\mu} &\leq b_{2\mu}, \\ a_{3\mu} &\leq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\geq b'_{1\nu}, \\ a'_{2\nu} &\geq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\geq b'_{4\nu}. \end{aligned} \quad (19)$$

Now,

$$\begin{aligned} 8(J_3(A_I) - J_3(B_I)) &= [(b_{1\mu} - a_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &+ (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\ &+ (a'_{2\nu} - b'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})]. \end{aligned} \quad (20)$$

From (19) it is very easy to see that all the terms in (20) are positive. Therefore their sum is also positive. From Definition 29, we know that at least one of the above inequalities in (19) becomes strict inequality and hence we get $J_3(A_I) - J_3(B_I) > 0$, hence the proof. $\square$

Theorem 36. Let $A_I, B_I \in C_3$. If $J_3(A_I) = J_3(B_I)$ then $A_I = B_I$.

Proof. Let us assume $J_3(A_I) = J_3(B_I)$. We prove that $A_I = B_I$. By Definition 29, without loss of generality, we have

$$\begin{aligned} a_{1\mu} &\leq b_{1\mu}, \\ a_{2\mu} &\leq b_{2\mu}, \\ a_{3\mu} &\leq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\geq b'_{1\nu}, \\ a'_{2\nu} &\geq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\geq b'_{4\nu}. \end{aligned} \quad (21)$$

Now,

$$\begin{aligned} 8(J_3(A_I) - J_3(B_I)) &= [(b_{1\mu} - a_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &+ (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\ &+ (a'_{2\nu} - b'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})] = 0. \end{aligned} \quad (22)$$

Therefore from (21) and (22), it is clear that all the terms in (22) are positive and therefore their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}, a_{2\mu} = b_{2\mu}, a_{3\mu} = b_{3\mu}, a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}, a'_{2\nu} = b'_{2\nu}, a'_{3\nu} = b'_{3\nu}, a'_{4\nu} = b'_{4\nu}$. Hence $A_I = B_I$. $\square$

Note 6. The incomplete score can be calculated to any TrIFN. But J_3 gives total order on C_3 .

Definition 37. If $J_3(A_I) > J_3(B_I)$ ($J_3(A_I) < J_3(B_I)$), then $A_I > B_I$ ($A_I < B_I$).

The following example is used to explain the ranking procedure defined in Definition 37.

Example 38. Let $A_I = \langle (0.15, 0.3, 0.4, 0.45), (0.1, 0.25, 0.45, 0.6) \rangle$ and $B_I = \langle (0.2, 0.4, 0.45, 0.55), (0, 0.15, 0.45, 0.55) \rangle \in C_3$. Now $J_3(A_I) = 0.0125$ and $J_3(B_I) = -0.05625$, $J_3(A_I) > J_3(B_I)$.

Hence $A_I > B_I$.

The inefficiency of the score functions J_1 and J_2 in discriminating any two arbitrary TrIFNs and the ability of J_3 in comparing arbitrary TrIFNs is shown in the following example.

Example 39. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} + \epsilon, a_{3\mu} + \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} + \epsilon/2, a'_{2\nu} + \epsilon/2, a'_{3\nu} + \epsilon/2, a'_{4\nu} + \epsilon/2) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$.

Now

$$\begin{aligned} J_1(A_I) &= J_1(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \end{aligned} \quad (23)$$

$$\begin{aligned} J_2(A_I) &= J_2(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}. \end{aligned}$$

But

$$\begin{aligned} J_3(A_I) &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \\ J_3(B_I) &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8} \end{aligned} \quad (24)$$

$$+ \frac{\epsilon}{4} \Rightarrow$$

$$J_3(B_I) > J_3(A_I).$$

Hence $B_I > A_I$.

The inefficiency of the score functions J_1 to J_3 in the task of comparing TrIFNs is explained in the following example.

FIGURE 5: Pictorial representation of Example 40.

Example 40. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} + \epsilon, a_{3\mu} + \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} + \epsilon, a'_{2\nu} + \epsilon, a'_{3\nu} + \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$ but not in C_3 .

Now

$$\begin{aligned} J_1(A_I) &= J_1(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_2(A_I) &= J_2(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_3(A_I) &= J_3(B_I) \\ &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}. \end{aligned} \quad (25)$$

But $A_I \neq B_I$.

Pictorial representation of this example is given in Figure 5.

Example 40 shows that J_1, J_2 , and J_3 cannot be sufficient to cover the entire class of TrIFNs and the class of TrIFNs in the above example excite us to define new score function that can fill the subclass of TrIFNs which cannot be covered by J_1, J_2 , and J_3 .

3.4. Accuracy Score of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, a new subclass of TrIFNs using accuracy relation is introduced and the accuracy score function on this class of trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 41. Let A_I and B_I be two TrIFNs.

A special subclass C_4 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \leq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \leq b'_{3\nu}, a'_{4\nu} \leq b'_{4\nu}$. The accuracy relation on C_4 is denoted as $\leq$ and defined as follows: if $A_I, B_I \in C_4 \subset \text{TrIFN}$ such that $A_I \leq B_I$ then $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \leq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \leq b'_{3\nu}, a'_{4\nu} \leq b'_{4\nu}$.

If $A_I < B_I$ then one of the above inequalities becomes strict inequality.

Note 7. By Definition 41, we note that any pair of members of C_4 are related under $<$.

The score function which measures the accuracy is defined as follows.

Definition 42. Let A_I be a trapezoidal intuitionistic fuzzy number. Then the accuracy score of A_I is defined by

$$\begin{aligned} J_4(A_I) &= \int_0^1 \frac{[\alpha \mu_{A_{IL}} + \alpha \mu_{A_{IU}}]}{4} d\alpha + \int_0^1 \frac{[\beta \nu_{A_{IL}} + \beta \nu_{A_{IU}}]}{4} d\beta \quad (26) \\ &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}. \end{aligned}$$

The proofs of the following propositions are immediate from Definition 42 and hence they are omitted.

Proposition 43. For any real number $r \in [0, 1]$, $J_4(r) = r$.

Proposition 44. For any trapezoidal fuzzy number $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$, $J_4(A_I) = (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})/4$.

Proposition 45. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then $J_4(A_I) = ((a_{1\mu} + 2a_{2\mu} + a_{3\mu}) + (a'_{1\nu} + 2a'_{2\nu} + a'_{3\nu}))/8$.

Proposition 46. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then $J_4(A_I) = ((a_{1\mu} + a_{2\mu}) + (a'_{1\nu} + a'_{2\nu}))/4$.

Note 8. The accuracy score can be calculated to any TrIFN. But J_4 gives total order on C_4 which is proved in Theorems 47 and 48.

Theorem 47. Let $A_I, B_I \in C_4$. If $A_I < B_I$ then $J_4(A_I) < J_4(B_I)$.

Proof. Let us assume that $A_I < B_I$. We claim that $J_4(B_I) - J_4(A_I) > 0$.

By Definition 41, we have

$$\begin{aligned} a_{1\mu} &\leq b_{1\mu}, \\ a_{2\mu} &\leq b_{2\mu}, \\ a_{3\mu} &\leq b_{3\mu}, \end{aligned}$$

$$\begin{aligned} a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\leq b'_{1\nu}, \\ a'_{2\nu} &\leq b'_{2\nu}, \\ a'_{3\nu} &\leq b'_{3\nu}, \\ a'_{4\nu} &\leq b'_{4\nu}. \end{aligned} \quad (27)$$

Now,

$$\begin{aligned} 8(J_4(B_I) - J_4(A_I)) &= [(b_{1\mu} - a_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &\quad + (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\ &\quad + (b'_{2\nu} - a'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})]. \end{aligned} \quad (28)$$

From (27) it is very easy to see that all the terms in (28) are positive. Therefore their sum is also positive. From Definition 41, we know that at least one of the above inequalities in (27) becomes strict inequality and hence we get $J_4(B_I) - J_4(A_I) > 0$. $\square$

Theorem 48. Let $A_I, B_I \in C_4$ such that $J_4(A_I) = J_4(B_I)$; then $A_I = B_I$.

Proof. Let us assume $J_4(A_I) = J_4(B_I)$. We claim that $A_I = B_I$. By Definition 41, without loss of generality, we have

$$\begin{aligned} a_{1\mu} &\leq b_{1\mu}, \\ a_{2\mu} &\leq b_{2\mu}, \\ a_{3\mu} &\leq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\geq b'_{1\nu}, \\ a'_{2\nu} &\geq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\geq b'_{4\nu}. \end{aligned} \quad (29)$$

Now,

$$\begin{aligned} 8(J_4(B_I) - J_4(A_I)) &= [(b_{1\mu} - a_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &\quad + (b_{3\mu} - a_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\ &\quad + (b'_{2\nu} - a'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})] = 0. \end{aligned} \quad (30)$$

Therefore from (29) and (30), it is clear that all the terms in (30) are positive and therefore their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}, a_{2\mu} = b_{2\mu}, a_{3\mu} = b_{3\mu}, a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}, a'_{2\nu} = b'_{2\nu}, a'_{3\nu} = b'_{3\nu}, a'_{4\nu} = b'_{4\nu}$. Hence $A_I = B_I$. $\square$

Definition 49. If $J_4(A_I) > J_4(B_I)$ ($J_4(A_I) < J_4(B_I)$) then $A_I > B_I$ ($A_I < B_I$).

Ranking procedure defined in Definition 49 is illustrated in the following example.

Example 50. Let $A_I = \langle (0.1, 0.25, 0.37, 0.49), (0.1, 0.2, 0.4, 0.6) \rangle$ and $B_I = \langle (0.2, 0.37, 0.47, 0.6), (0.15, 0.32, 0.52, 0.7) \rangle \in C_4$. Now $J_4(A_I) = 0.31375$ and $J_4(B_I) = 0.41625$, $J_4(A_I) < J_4(B_I)$.

Hence $A_I < B_I$.

The relative importance of the score function J_4 in discriminating two different TrIFNs when J_1, J_2 , and J_3 fail to discriminate them is explained by Example 51.

Example 51. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} + \epsilon, a_{3\mu} + \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} + \epsilon, a'_{2\nu} + \epsilon, a'_{3\nu} + \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$, and $\epsilon \in [0, 1]$.

Now

$$\begin{aligned} J_1(A_I) &= J_1(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_2(A_I) &= J_2(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_3(A_I) &= J_3(B_I) \\ &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}. \end{aligned} \quad (31)$$

But

$$\begin{aligned} J_4(A_I) &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}, \\ J_4(B_I) &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8} \\ &\quad + \epsilon \implies \\ J_4(A_I) &< J_4(B_I). \end{aligned} \quad (32)$$

Hence $A_I < B_I$.

Example 52. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} + \epsilon, a_{4\mu} - \epsilon), (a'_{1\nu} + \epsilon, a'_{2\nu} - \epsilon, a'_{3\nu} + \epsilon, a'_{4\nu} - \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$ but not in C_4 .

FIGURE 6: Pictorial representation of Example 52.

Now

$$\begin{aligned} J_1(A_I) &= J_1(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_2(A_I) &= J_2(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_3(A_I) &= J_3(B_I) \\ &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \\ J_4(A_I) &= J_4(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}. \end{aligned} \quad (33)$$

But $A_I \neq B_I$.

Pictorial representation of this example is given in Figure 6.

Example 52 shows that even J_1, J_2, J_3 , and J_4 altogether are not enough to cover the entire class of TrIFNs; the class of TrIFNs in the above example excite us to define new score function that can cover the class of TrIFNs which cannot be covered by J_1, J_2, J_3 , and J_4 .

3.5. Spread Score of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, another subclass of TrIFNs is introduced. The spread score function on this class of trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 53. Let A_I and B_I be two TrIFNs.

A special subclass C_5 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \geq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \leq b'_{4\nu}$. The preference relation on C_5 is denoted as

$\trianglelefteq$ and defined as follows: if $A_I, B_I \in C_5 \subset \text{TrIFN}$ such that $A_I \trianglelefteq B_I$ then $a_{1\mu} \geq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \leq b'_{4\nu}$. If $A_I \triangleleft B_I$ then one of the above inequalities becomes strict inequality.

Note 9. By Definition 53, we note that any pair of members of C_5 are related under $\triangleleft$.

The score function which measures the spread is defined as follows.

Definition 54. Let A_I be a trapezoidal intuitionistic fuzzy number. Then the spread score of A_I is defined by

$$J_5(A_I) = \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) - (a'_{2\nu} + a'_{4\nu}) + (a'_{1\nu} + a'_{3\nu})}{8} \quad (34)$$

The proofs of the following propositions are immediate from Definition 54 and hence they are omitted.

Proposition 55. For any real number $r \in [0, 1]$, $J_5(r) = 0$.

Proposition 56. If $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$ is a trapezoidal fuzzy number, then $J_5(A_I) = ((a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}))/4$.

Proposition 57. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then $J_5(A_I) = -(a_{3\mu} - a_{1\mu}) - (a'_{3\nu} - a'_{1\nu})/8$.

Proposition 58. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then $J_5(A_I) = 0$.

Theorem 59. Let $A_I, B_I \in C_5$. If $A_I \triangleleft B_I$ then $J_5(A_I) > J_5(B_I)$.

Proof. Let us assume that $A_I \triangleleft B_I$. We claim that $J_5(A_I) - J_5(B_I) > 0$.

By Definition 53, we have

$$\begin{aligned} a_{1\mu} &\geq b_{1\mu}, \\ a_{2\mu} &\leq b_{2\mu}, \\ a_{3\mu} &\geq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\geq b'_{1\nu}, \\ a'_{2\nu} &\leq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\leq b'_{4\nu}. \end{aligned} \quad (35)$$

Now,

$$\begin{aligned} 8(J_5(A_I) - J_5(B_I)) &= [(a_{1\mu} - b_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &+ (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\ &+ (b'_{2\nu} - a'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})]. \end{aligned} \quad (36)$$

From (35) it is very easy to see that all the terms in (36) are positive. Therefore their sum is also positive. From Definition 53, we know that at least one of the above inequalities in (35) becomes strict inequality and hence we get $J_5(A_I) - J_5(B_I) > 0$. $\square$

Theorem 60. Let $A_I, B_I \in C_5$. If $J_5(A_I) = J_5(B_I)$ then $A_I = B_I$.

Proof. Let us assume $J_5(A_I) = J_5(B_I)$. We claim that $A_I = B_I$. By Definition 53, without loss of generality, we have

$$\begin{aligned} a_{1\mu} &\geq b_{1\mu}, \\ a_{2\mu} &\leq b_{2\mu}, \\ a_{3\mu} &\geq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\geq b'_{1\nu}, \\ a'_{2\nu} &\leq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\leq b'_{4\nu}. \end{aligned} \quad (37)$$

Now,

$$\begin{aligned} 8(J_5(A_I) - J_5(B_I)) &= [(a_{1\mu} - b_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &+ (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\ &+ (b'_{2\nu} - a'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})] = 0. \end{aligned} \quad (38)$$

Therefore from (37) and (38), it is clear that all the terms in (38) are positive and therefore their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}, a_{2\mu} = b_{2\mu}, a_{3\mu} = b_{3\mu}, a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}, a'_{2\nu} = b'_{2\nu}, a'_{3\nu} = b'_{3\nu}, a'_{4\nu} = b'_{4\nu}$. Hence $A_I = B_I$. $\square$

Note 10. The spread score can be calculated to any TrIFN. But J_5 gives total order on C_5 which is seen from Theorems 59 and 60.

Definition 61. If $J_5(A_I) > J_5(B_I)$ ($J_5(A_I) < J_5(B_I)$) then $A_I > B_I$ ($A_I < B_I$).

The following example is used to explain the ranking procedure defined on C_5 .

Example 62. Let $A_I = \langle (0.15, 0.23, 0.31, 0.45), (0.1, 0.16, 0.35, 0.5) \rangle$ and $B_I = \langle (0.1, 0.25, 0.25, 0.5), (0, 0.2, 0.3, 0.6) \rangle \in C_5$. Now $J_5(A_I) = -0.05375$ and $J_5(B_I) = -0.1125$, $J_5(A_I) > J_5(B_I)$.

Hence $A_I > B_I$.

Example 63. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} + \epsilon, a_{4\mu} - \epsilon), (a'_{1\nu} + \epsilon, a'_{2\nu} - \epsilon, a'_{3\nu} + \epsilon, a'_{4\nu} - \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$.

Now

$$\begin{aligned} J_1(A_I) &= J_1(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_2(A_I) &= J_2(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \end{aligned} \quad (39)$$

$$\begin{aligned} J_3(A_I) &= J_3(B_I) \\ &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \end{aligned}$$

$$\begin{aligned} J_4(A_I) &= J_4(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}. \end{aligned}$$

But

$$\begin{aligned} J_5(A_I) &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}, \\ J_5(B_I) &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8} \end{aligned} \quad (40)$$

$+ \epsilon \implies$

$$J_5(B_I) > J_5(A_I).$$

Hence $B_I > A_I$. In this example the importance of J_5 in ranking arbitrary TrIFNs is shown.

The inability of J_5 in comparing any two TrIFNs is shown in the following example.

Example 64. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} + \epsilon, a_{4\mu} - \epsilon), (a'_{1\nu} - \epsilon, a'_{2\nu} + \epsilon, a'_{3\nu} - \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$ but not in C_5 .

FIGURE 7: Pictorial representation of Example 64.

Now

$$\begin{aligned} J_1(A_I) &= J_1(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_2(A_I) &= J_2(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\ J_3(A_I) &= J_3(B_I) \\ &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \end{aligned} \quad (41)$$

$$\begin{aligned} J_4(A_I) &= J_4(B_I) \\ &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}, \\ J_5(A_I) &= J_5(B_I) \\ &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}. \end{aligned}$$

But $A_I \neq B_I$.

Pictorial representation of this example is given in Figure 7.

Example 64 shows that all the above defined scores are not enough to cover the entire class of TrIFNs; therefore we are introducing another score function in the next subsection.

3.6. Nonaccuracy Score of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, a new subclass of TrIFNs is introduced. The nonaccuracy score function on this class of trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 65. Let A_I and B_I be two TrIFNs.

A special subclass C_6 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \geq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \geq b'_{2\nu}, a'_{3\nu} \leq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$. The exact relation on C_6 is denoted as $\triangleright$ and defined as follows: if $A_I, B_I \in C_6 \subset \text{TrIFN}$ such that $A_I \triangleright B_I$ then $a_{1\mu} \geq b_{1\mu}, a_{2\mu} \leq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \geq b'_{2\nu}, a'_{3\nu} \leq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$. If $A_I \triangleright B_I$ then one of the above inequalities becomes strict inequality.

Note 11. By Definition 65, we note that any pair of members of C_6 are related under $\triangleright$.

The score function which measures the nonaccuracy is defined as follows.

Definition 66. Let A_I be a trapezoidal intuitionistic fuzzy number. Then the nonaccuracy score of A_I is defined by

$$J_6(A_I) = \frac{(a_{2\mu} + a_{4\mu}) - (a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}. \quad (42)$$

The proofs of the following propositions are easy and hence they are omitted.

Proposition 67. For any real number $r \in [0, 1]$, $J_6(r) = 0$.

Proposition 68. If $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$ is a trapezoidal fuzzy number, then $J_6(A_I) = 0$.

Proposition 69. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then $J_6(A_I) = ((a'_{1\nu} - a'_{3\nu}) - (a_{1\mu} - a_{3\mu}))/8$.

Proposition 70. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then $J_6(A_I) = 0$.

Note 12. The nonaccuracy score can be calculated to any TrIFN. But J_6 gives total order on C_6 which is proved in Theorems 71 and 72.

Theorem 71. Let $A_I, B_I \in C_6$. If $A_I \triangleright B_I$ then $J_6(A_I) < J_6(B_I)$.

Proof. Let us assume that $A_I \triangleright B_I$. We claim that $J_6(B_I) - J_6(A_I) > 0$.

By Definition 65, we have

$$\begin{aligned} a_{1\mu} &\geq b_{1\mu}, \\ a_{2\mu} &\leq b_{2\mu}, \\ a_{3\mu} &\geq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\leq b'_{1\nu}, \\ a'_{2\nu} &\geq b'_{2\nu}, \end{aligned}$$

$$a'_{3\nu} \leq b'_{3\nu},$$

$$a'_{4\nu} \geq b'_{4\nu}$$

(43)

and at least one of these inequalities becomes strict.

Now,

$$\begin{aligned} 8(J_6(B_I) - J_6(A_I)) &= [(a_{1\mu} - b_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &\quad + (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\ &\quad + (a'_{2\nu} - b'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})]. \end{aligned} \quad (44)$$

From (43) it is very easy to see that all the terms in (44) are positive and therefore their sum is also positive. From Definition 65, we know that at least one of the above inequalities in (43) becomes strict inequality and hence we get $J_6(B_I) - J_6(A_I) > 0$. $\square$

Theorem 72. Let $A_I, B_I \in C_6$. If $J_6(A_I) = J_6(B_I)$ then $A_I = B_I$.

Proof. Let us assume $J_6(A_I) = J_6(B_I)$. We claim that $A_I = B_I$. By Definition 65, without loss of generality, we have

$$a_{1\mu} \geq b_{1\mu},$$

$$a_{2\mu} \leq b_{2\mu},$$

$$a_{3\mu} \geq b_{3\mu},$$

$$a_{4\mu} \leq b_{4\mu},$$

$$a'_{1\nu} \leq b'_{1\nu},$$

$$a'_{2\nu} \geq b'_{2\nu},$$

$$a'_{3\nu} \leq b'_{3\nu},$$

$$a'_{4\nu} \geq b'_{4\nu}.$$

(45)

Now,

$$\begin{aligned} 8(J_6(B_I) - J_6(A_I)) &= [(a_{1\mu} - b_{1\mu}) + (b_{2\mu} - a_{2\mu}) \\ &\quad + (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\ &\quad + (a'_{2\nu} - b'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})] = 0. \end{aligned} \quad (46)$$

Therefore from (45) and (46), it is clear that all the terms in (46) are positive and therefore their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}, a_{2\mu} = b_{2\mu}, a_{3\mu} = b_{3\mu}, a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}, a'_{2\nu} = b'_{2\nu}, a'_{3\nu} = b'_{3\nu}, a'_{4\nu} = b'_{4\nu}$. Hence $A_I = B_I$. $\square$

Definition 73. If $J_6(A_I) > J_6(B_I)$ ($J_6(A_I) < J_6(B_I)$), then $A_I > B_I$ ($A_I < B_I$).

Ranking procedure introduced in Definition 73 is explained in Example 74.

Example 74. Let $A_I = \langle (0.23, 0.27, 0.34, 0.51), (0.17, 0.23, 0.4, 0.6) \rangle$ and $B_I = \langle (0.2, 0.3, 0.3, 0.55), (0.2, 0.2, 0.5, 0.55) \rangle \in C_6$. Now $J_6(A_I) = -0.00625$ and $J_6(B_I) = 0.0375$, $J_6(A_I) < J_6(B_I)$.

Hence $A_I < B_I$.

The efficiency of J_6 in ranking TrIFNs is shown in the following example.

Example 75. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} + \epsilon, a_{4\mu} - \epsilon), (a'_{1\nu} - \epsilon, a'_{2\nu} + \epsilon, a'_{3\nu} - \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$.

Now

$$\begin{aligned}
 J_1(A_I) &= J_1(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_2(A_I) &= J_2(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_3(A_I) &= J_3(B_I) \\
 &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \\
 J_4(A_I) &= J_4(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_5(A_I) &= J_5(B_I) \\
 &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}.
 \end{aligned} \tag{47}$$

But

$$\begin{aligned}
 J_6(A_I) &= \frac{(a_{2\mu} + a_{4\mu}) - (a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}, \\
 J_6(B_I) &= \frac{(a_{2\mu} + a_{4\mu}) - (a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8} \\
 &\quad + \epsilon \implies \\
 J_6(A_I) &< J_6(B_I).
 \end{aligned} \tag{48}$$

Hence $A_I < B_I$.

FIGURE 8: Pictorial representation of Example 76.

Example 76. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} - \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} - \epsilon, a'_{2\nu} + \epsilon, a'_{3\nu} - \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$ but not in C_6 , where $\epsilon \in [0, 1]$. Now

$$\begin{aligned}
 J_1(A_I) &= J_1(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_2(A_I) &= J_2(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_3(A_I) &= J_3(B_I) \\
 &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \\
 J_4(A_I) &= J_4(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_5(A_I) &= J_5(B_I) \\
 &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}.
 \end{aligned} \tag{49}$$

But

$$\begin{aligned}
 J_6(A_I) &= J_6(B_I) \\
 &= \frac{(a_{2\mu} + a_{4\mu}) - (a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}.
 \end{aligned} \tag{50}$$

But $A_I \neq B_I$.

Pictorial representation of this example is given in Figure 8.

Example 76 shows that the score functions J_1 to J_6 are not sufficient to cover the entire class of TrIFNs and hence

the class of TrIFNs in the above example excite us to define new score function that can cover a subclass of TrIFNs which cannot be covered by J_1 to J_6 .

3.7. Left Area of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, another subclass of TrIFNs is introduced and the left area score function on this class of trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 77. Let A_I and B_I be two TrIFNs.

A special subclass C_7 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \geq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \leq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$. The nonaccurate relation on C_7 is denoted as $\leq$ and defined as follows: if $A_I, B_I \in C_7 \subset \text{TrIFN}$ such that $A_I \leq B_I$ then $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \geq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \geq b'_{1\nu}, a'_{2\nu} \leq b'_{2\nu}, a'_{3\nu} \leq b'_{3\nu}, a'_{4\nu} \geq b'_{4\nu}$.

If $A_I \ll B_I$ then one of the above inequalities becomes strict inequality.

Note 13. By Definition 77, we note that any pair of members of C_7 are related under $\leq$.

The score function which measures the left area is defined as follows.

Definition 78. Let A_I be a trapezoidal intuitionistic fuzzy number. Then the left area of A_I is defined by

$$J_7(A_I) = \frac{(a'_{3\nu} - a'_{1\nu}) + (a'_{2\nu} - a'_{4\nu}) - (a_{3\mu} - a_{1\mu}) - (a_{2\mu} - a_{4\mu})}{8} \quad (51)$$

The proofs of the following propositions are immediate application of Definition 78 and hence they are omitted.

Proposition 79. For any real number $r \in [0, 1]$, $J_7(r) = 0$.

Proposition 80. If $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$ is a trapezoidal fuzzy number, then $J_7(A_I) = 0$.

Proposition 81. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then $J_7(A_I) = ((a_{1\mu} + a_{3\mu} - 2a_{2\mu}) + (2a'_{2\nu} - a'_{1\nu} - a'_{3\nu}))/8$.

Proposition 82. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then $J_7(A_I) = 0$.

Theorem 83. Let $A_I, B_I \in C_7$. If $A_I \ll B_I$ then $J_7(A_I) < J_7(B_I)$.

Proof. Let us assume that $A_I \ll B_I$.

We claim that $J_7(B_I) - J_7(A_I) > 0$.

By Definition 77, we have

$$a_{1\mu} \leq b_{1\mu},$$

$$a_{2\mu} \geq b_{2\mu},$$

$$a_{3\mu} \geq b_{3\mu},$$

$$a_{4\mu} \leq b_{4\mu},$$

$$a'_{1\nu} \geq b'_{1\nu},$$

$$a'_{2\nu} \leq b'_{2\nu},$$

$$a'_{3\nu} \leq b'_{3\nu},$$

$$a'_{4\nu} \geq b'_{4\nu}.$$

(52)

Now,

$$\begin{aligned} 8(J_7(B_I) - J_7(A_I)) &= [(b_{1\mu} - a_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\ &+ (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\ &+ (b'_{2\nu} - a'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})]. \end{aligned} \quad (53)$$

From (52) it is very easy to see that all the terms in (53) are positive. Therefore their sum is also positive. From Definition 77, we know that at least one of the above inequalities in (52) becomes strict inequality and hence we get $J_7(B_I) - J_7(A_I) > 0$, hence the proof. $\square$

Theorem 84. Let $A_I, B_I \in C_7$. If $J_7(A_I) = J_7(B_I)$ then $A_I = B_I$.

Proof. Let us assume $J_7(A_I) = J_7(B_I)$. We claim that $A_I = B_I$.

By Definition 77, without loss of generality, we have

$$a_{1\mu} \leq b_{1\mu},$$

$$a_{2\mu} \geq b_{2\mu},$$

$$a_{3\mu} \geq b_{3\mu},$$

$$a_{4\mu} \leq b_{4\mu},$$

$$a'_{1\nu} \geq b'_{1\nu},$$

$$a'_{2\nu} \leq b'_{2\nu},$$

$$a'_{3\nu} \leq b'_{3\nu},$$

$$a'_{4\nu} \geq b'_{4\nu}.$$

(54)

Now,

$$\begin{aligned} 8(J_7(B_I) - J_7(A_I)) &= [(b_{1\mu} - a_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\ &+ (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(a'_{1\nu} - b'_{1\nu}) \\ &+ (b'_{2\nu} - a'_{2\nu}) + (b'_{3\nu} - a'_{3\nu}) + (a'_{4\nu} - b'_{4\nu})] = 0. \end{aligned} \quad (55)$$

Therefore from (54) and (55), it is clear that all the terms in (55) are positive and therefore their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}, a_{2\mu} = b_{2\mu}, a_{3\mu} = b_{3\mu}, a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}, a'_{2\nu} = b'_{2\nu}, a'_{3\nu} = b'_{3\nu}, a'_{4\nu} = b'_{4\nu}$. Hence $A_I = B_I$. $\square$

Note 14. The left area score can be calculated to any TrIFN. But J_7 gives total order on C_7 which is seen from Theorems 83 and 84.

Definition 85. If $J_7(A_I) > J_7(B_I)$ ($J_7(A_I) < J_7(B_I)$) then $A_I > B_I$ ($A_I < B_I$).

Ranking relation defined above is explained in the following example.

Example 86. Let $A_I = \langle (0.17, 0.23, 0.34, 0.49), (0.1, 0.15, 0.4, 0.6) \rangle$ and $B_I = \langle (0.2, 0.2, 0.3, 0.55), (0, 0.18, 0.47, 0.58) \rangle \in C_7$. Now $J_7(A_I) = -0.0075$ and $J_7(B_I) = 0.04$, $J_7(B_I) > J_7(A_I)$.

Hence $B_I > A_I$.

The relative importance of the score function J_7 is explained in the following example.

Example 87. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} - \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} - \epsilon, a'_{2\nu} + \epsilon, a'_{3\nu} + \epsilon, a'_{4\nu} - \epsilon) \rangle \in \text{TrIFN}$ and $\epsilon \in [0, 1]$.

Now

$$\begin{aligned}
 J_1(A_I) &= J_1(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_2(A_I) &= J_2(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_3(A_I) &= J_3(B_I) \\
 &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \\
 J_4(A_I) &= J_4(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_5(A_I) &= J_5(B_I) \\
 &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}, \\
 J_6(A_I) &= J_6(B_I) \\
 &= \frac{(a_{2\mu} + a_{4\mu}) - (a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}.
 \end{aligned} \tag{56}$$

But

$$\begin{aligned}
 J_7(A_I) &= \frac{(a'_{3\nu} - a'_{1\nu}) + (a'_{2\nu} - a'_{4\nu}) - (a_{3\mu} - a_{1\mu}) - (a_{2\mu} - a_{4\mu})}{8}, \\
 J_7(B_I) &= \frac{(a'_{3\nu} - a'_{1\nu}) + (a'_{2\nu} - a'_{4\nu}) - (a_{3\mu} - a_{1\mu}) - (a_{2\mu} - a_{4\mu})}{8} \\
 &\quad - \epsilon \implies \\
 J_7(A_I) &> J_7(B_I).
 \end{aligned} \tag{57}$$

Hence $A_I > B_I$.

Example 88. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} - \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} + \epsilon, a'_{2\nu} - \epsilon, a'_{3\nu} - \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$ but not in C_7 , where $\epsilon \in [0, 1]$.

Now

$$\begin{aligned}
 J_1(A_I) &= J_1(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_2(A_I) &= J_2(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_3(A_I) &= J_3(B_I) \\
 &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \\
 J_4(A_I) &= J_4(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_5(A_I) &= J_5(B_I) \\
 &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}, \\
 J_6(A_I) &= J_6(B_I) \\
 &= \frac{(a_{2\mu} + a_{4\mu}) - (a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}, \\
 J_7(A_I) &= J_7(B_I) \\
 &= \frac{(a'_{3\nu} - a'_{1\nu}) + (a'_{2\nu} - a'_{4\nu}) - (a_{3\mu} - a_{1\mu}) - (a_{2\mu} - a_{4\mu})}{8}.
 \end{aligned} \tag{58}$$

But $A_I \neq B_I$.

Pictorial representation of this example is given in Figure 9.

Example 87 shows that all the above defined scores are not enough to cover the entire class of TrIFNs. Hence the new score function is introduced in the next subsection in order to give total ordering on the entire class of TrIFNs.

3.8. Right Area Score of a Trapezoidal Intuitionistic Fuzzy Number. In this subsection, a new subclass of TrIFNs is introduced and the right area score function on this class of trapezoidal intuitionistic fuzzy numbers is defined and some of its properties are studied using illustrative examples.

Definition 89. Let A_I and B_I be two TrIFNs.

A special subclass C_8 of the set of TrIFNs consist of TrIFNs for which every pair of A_I and B_I are with $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \geq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and $a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \geq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \leq b'_{4\nu}$. The definite relation on C_8 is denoted as $\gg$ and defined as follows: if $A_I, B_I \in C_8 \subset \text{TrIFN}$ such that $A_I \gg B_I$ then $a_{1\mu} \leq b_{1\mu}, a_{2\mu} \geq b_{2\mu}, a_{3\mu} \geq b_{3\mu}, a_{4\mu} \leq b_{4\mu}$ and

FIGURE 9: Pictorial representation of Example 88.

$a'_{1\nu} \leq b'_{1\nu}, a'_{2\nu} \geq b'_{2\nu}, a'_{3\nu} \geq b'_{3\nu}, a'_{4\nu} \leq b'_{4\nu}$. If $A_I \gg B_I$ then one of the above inequalities becomes strict inequality.

Note 15. By Definition 89, we note that any pair of members of C_8 are related under $\gg$.

The score function which measures the right area is defined as follows.

Definition 90. Let A_I be a trapezoidal intuitionistic fuzzy number. Then the right area score of A_I is defined by

$$J_8(A_I) = \frac{(a_{1\mu} - a_{3\mu}) + (a_{4\mu} - a_{2\mu}) + (a'_{1\nu} - a_{3\nu}) + (a'_{4\nu} - a_{2\nu})}{8} \quad (59)$$

The proofs of the following propositions are immediate from Definition 90 and hence they are omitted.

Proposition 91. For any real number $r \in [0, 1]$, $J_8(r) = 0$.

Proposition 92. If $A_I = (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu})$ is a trapezoidal fuzzy number, then $J_8(A_I) = ((a_{1\mu} - a_{3\mu}) + (a_{4\mu} - a_{2\mu}))/4$.

Proposition 93. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}) \rangle$ be a triangular intuitionistic fuzzy number. Then $J_8(A_I) = ((a_{1\mu} + a_{3\mu} - 2a_{2\mu}) + (a'_{1\nu} + a'_{3\nu} - 2a'_{2\nu}))/8$.

Proposition 94. Let $A_I = ([a_{1\mu}, a_{2\mu}], [a'_{1\nu}, a'_{2\nu}])$ be an interval valued intuitionistic fuzzy number. Then $J_8(A_I) = 0$.

Note 16. The right area score can be calculated to any TrIFN. But J_8 gives total order on C_8 which is proved in the following theorems.

Theorem 95. Let $A_I, B_I \in C_8$. If $A_I \gg B_I$ then $J_8(A_I) < J_8(B_I)$.

Proof. Let us assume that $A_I \gg B_I$. We claim that $J_8(B_I) - J_8(A_I) > 0$.

By Definition 89, we have

$$\begin{aligned} a_{1\mu} &\leq b_{1\mu}, \\ a_{2\mu} &\geq b_{2\mu}, \\ a_{3\mu} &\geq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\leq b'_{1\nu}, \\ a'_{2\nu} &\geq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\leq b'_{4\nu}. \end{aligned} \quad (60)$$

Now,

$$\begin{aligned} 8(J_8(B_I) - J_8(A_I)) &= [(b_{1\mu} - a_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\ &\quad + (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\ &\quad + (a'_{2\nu} - b'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})]. \end{aligned} \quad (61)$$

From (60) it is very easy to see that all the terms in (61) are positive. Therefore their sum is also positive. From Definition 89, we know that at least one of the above inequalities in (60) becomes strict inequality and hence we get $J_8(B_I) - J_8(A_I) > 0$. $\square$

Theorem 96. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (b_{1\mu}, b_{2\mu}, b_{3\mu}, b_{4\mu}), (b'_{1\nu}, b'_{2\nu}, b'_{3\nu}, b'_{4\nu}) \rangle \in C_8$ such that $J_8(A_I) = J_8(B_I)$; then $A_I = B_I$.

Proof. Let us assume $J_8(A_I) = J_8(B_I)$. We claim that $A_I = B_I$. By Definition 89, without loss of generality, we have

$$\begin{aligned} a_{1\mu} &\leq b_{1\mu}, \\ a_{2\mu} &\geq b_{2\mu}, \\ a_{3\mu} &\geq b_{3\mu}, \\ a_{4\mu} &\leq b_{4\mu}, \\ a'_{1\nu} &\leq b'_{1\nu}, \\ a'_{2\nu} &\geq b'_{2\nu}, \\ a'_{3\nu} &\geq b'_{3\nu}, \\ a'_{4\nu} &\leq b'_{4\nu}. \end{aligned} \quad (62)$$

Now,

$$\begin{aligned} 8(J_1(B_I) - J_1(A_I)) &= [(b_{1\mu} - a_{1\mu}) + (a_{2\mu} - b_{2\mu}) \\ &\quad + (a_{3\mu} - b_{3\mu}) + (b_{4\mu} - a_{4\mu})] + [(b'_{1\nu} - a'_{1\nu}) \\ &\quad + (a'_{2\nu} - b'_{2\nu}) + (a'_{3\nu} - b'_{3\nu}) + (b'_{4\nu} - a'_{4\nu})] = 0. \end{aligned} \quad (63)$$

Therefore from (62) and (63), it is clear that all the terms in (63) are positive and therefore their sum gives zero only when each term is equal to zero. Hence $a_{1\mu} = b_{1\mu}, a_{2\mu} = b_{2\mu}, a_{3\mu} = b_{3\mu}, a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}, a'_{2\nu} = b'_{2\nu}, a'_{3\nu} = b'_{3\nu}, a'_{4\nu} = b'_{4\nu}$, hence the proof. $\square$

Definition 97. If $J_8(A_I) > J_8(B_I)$ ($J_8(A_I) < J_8(B_I)$) then $A_I > B_I$ ($A_I < B_I$).

Ranking procedure introduced in Definition 97 is demonstrated in Example 98.

Example 98. Let $A_I = \langle (0.1, 0.2, 0.3, 0.45), (0, 0.15, 0.35, 0.5) \rangle$ and $B_I = \langle (0.15, 0.15, 0.2, 0.45), (0.1, 0.1, 0.3, 0.45) \rangle \in C_8$. Now $J_8(A_I) = 0.00625$ and $J_8(B_I) = 0.05, J_8(B_I) > J_8(A_I)$.

Hence $B_I > A_I$.

The relative importance of the score function J_8 in ranking arbitrary TrIFNs is explained mathematically in the following example.

Example 99. Let $A_I = \langle (a_{1\mu}, a_{2\mu}, a_{3\mu}, a_{4\mu}), (a'_{1\nu}, a'_{2\nu}, a'_{3\nu}, a'_{4\nu}) \rangle$ and $B_I = \langle (a_{1\mu} + \epsilon, a_{2\mu} - \epsilon, a_{3\mu} - \epsilon, a_{4\mu} + \epsilon), (a'_{1\nu} + \epsilon, a'_{2\nu} - \epsilon, a'_{3\nu} - \epsilon, a'_{4\nu} + \epsilon) \rangle \in \text{TrIFN}$ but not in C_7 , where $\epsilon \in [0, 1]$.

Now

$$\begin{aligned}
 J_1(A_I) &= J_1(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu}) - (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_2(A_I) &= J_2(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu}) - (a_{3\mu} + a_{4\mu}) - (a'_{1\nu} + a'_{2\nu}) + (a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_3(A_I) &= J_3(B_I) \\
 &= \frac{(a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu}) - (a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu})}{8}, \\
 J_4(A_I) &= J_4(B_I) \\
 &= \frac{(a_{1\mu} + a_{2\mu} + a_{3\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{2\nu} + a'_{3\nu} + a'_{4\nu})}{8}, \\
 J_5(A_I) &= J_5(B_I) \\
 &= \frac{(a_{1\mu} + a_{3\mu}) - (a_{2\mu} + a_{4\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}, \\
 J_6(A_I) &= J_6(B_I) \\
 &= \frac{(a_{2\mu} + a_{4\mu}) - (a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) - (a'_{2\nu} + a'_{4\nu})}{8}, \\
 J_7(A_I) &= J_7(B_I) \\
 &= \frac{(a'_{3\nu} - a'_{1\nu}) + (a'_{2\nu} - a'_{4\nu}) - (a_{3\mu} - a_{1\mu}) - (a_{2\mu} - a_{4\mu})}{8}.
 \end{aligned} \tag{64}$$

But

$$\begin{aligned}
 J_8(A_I) &= \frac{(a_{1\mu} - a_{3\mu}) + (a_{4\mu} - a_{2\mu}) + (a'_{1\nu} - a'_{3\nu}) + (a'_{4\nu} - a'_{2\nu})}{8}, \\
 J_8(B_I) &= \frac{(a_{1\mu} - a_{3\mu}) + (a_{4\mu} - a_{2\mu}) + (a'_{1\nu} - a'_{3\nu}) + (a'_{4\nu} - a'_{2\nu})}{8} \\
 &\quad - \epsilon \implies
 \end{aligned} \tag{65}$$

$$J_8(A_I) > J_8(B_I).$$

Hence $A_I > B_I$.

4. A New Ranking Principle for Ordering Trapezoidal Intuitionistic Fuzzy Numbers

In this section a new ranking principle on trapezoidal intuitionistic fuzzy numbers is defined by using finite number of score functions defined in Sections 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, and 3.8 and we proved that the proposed ranking is a total order on the class of TrIFNs.

Definition 100. Let $A_I, B_I \in \text{TrIFN}$. An order relation $<$ on the set of TrIFNs is defined by the following:

If $J_1(A_I) < J_1(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I)$ then if $J_2(A_I) < J_2(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I)$ and $J_2(A_I) = J_2(B_I)$ then if $J_3(A_I) < J_3(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I), J_2(A_I) = J_2(B_I)$, and $J_3(A_I) = J_3(B_I)$ then if $J_4(A_I) < J_4(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I), J_2(A_I) = J_2(B_I), J_3(A_I) = J_3(B_I)$, and $J_4(A_I) = J_4(B_I)$ then if $J_5(A_I) < J_5(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I), J_2(A_I) = J_2(B_I), J_3(A_I) = J_3(B_I), J_4(A_I) = J_4(B_I)$, and $J_5(A_I) = J_5(B_I)$ then if $J_6(A_I) < J_6(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I), J_2(A_I) = J_2(B_I), J_3(A_I) = J_3(B_I), J_4(A_I) = J_4(B_I), J_5(A_I) = J_5(B_I)$, and $J_6(A_I) = J_6(B_I)$ then if $J_7(A_I) < J_7(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I), J_2(A_I) = J_2(B_I), J_3(A_I) = J_3(B_I), J_4(A_I) = J_4(B_I), J_5(A_I) = J_5(B_I), J_6(A_I) = J_6(B_I)$, and $J_7(A_I) = J_7(B_I)$ then if $J_8(A_I) < J_8(B_I)$ then A_I is smaller than B_I , denoted by $A_I < B_I$.

If $J_1(A_I) = J_1(B_I), J_2(A_I) = J_2(B_I), J_3(A_I) = J_3(B_I), J_4(A_I) = J_4(B_I), J_5(A_I) = J_5(B_I), J_6(A_I) = J_6(B_I), J_7(A_I) = J_7(B_I)$, and $J_8(A_I) = J_8(B_I)$ then $A_I = B_I$.

The following theorem is proved to show the validity of Definition 100.

Theorem 101. Let $A_I, B_I \in \text{TrIFN}$. If $J_1(A_I) = J_1(B_I)$, $J_2(A_I) = J_2(B_I)$, $J_3(A_I) = J_3(B_I)$, $J_4(A_I) = J_4(B_I)$, $J_5(A_I) = J_5(B_I)$, $J_6(A_I) = J_6(B_I)$, $J_7(A_I) = J_7(B_I)$, and $J_8(A_I) = J_8(B_I)$ then $A_I = B_I$. That is, $a_{1\mu} = b_{1\mu}$, $a_{2\mu} = b_{2\mu}$, $a_{3\mu} = b_{3\mu}$, $a_{4\mu} = b_{4\mu}$, and $a'_{1\nu} = b'_{1\nu}$, $a'_{2\nu} = b'_{2\nu}$, $a'_{3\nu} = b'_{3\nu}$, $a'_{4\nu} = b'_{4\nu}$.

Proof. Let $A_I, B_I \in \text{TrIFN}$. Let us assume $J_1(A_I) = J_1(B_I)$, $J_2(A_I) = J_2(B_I)$, $J_3(A_I) = J_3(B_I)$, $J_4(A_I) = J_4(B_I)$, $J_5(A_I) = J_5(B_I)$, $J_6(A_I) = J_6(B_I)$, $J_7(A_I) = J_7(B_I)$, and $J_8(A_I) = J_8(B_I)$.

Claim 1 ($A_I = B_I$).

$$-J_1(A_I) + J_4(A_I) = -J_1(B_I) + J_4(B_I) \implies \quad (66)$$

$$(a_{3\mu} + a_{4\mu}) + (a'_{3\nu} + a'_{4\nu}) = (b_{3\mu} + b_{4\mu}) + (b'_{3\nu} + b'_{4\nu}),$$

$$J_4(A_I) + J_1(A_I) = J_4(B_I) + J_1(B_I) \implies \quad (67)$$

$$(a_{1\mu} + a_{2\mu}) + (a'_{1\nu} + a'_{2\nu}) = (b_{1\mu} + b_{2\mu}) + (b'_{1\nu} + b'_{2\nu}),$$

$$J_2(A_I) + J_4(A_I) = J_2(B_I) + J_4(B_I) \implies \quad (68)$$

$$(a_{1\mu} + a_{2\mu}) + (a'_{3\nu} + a'_{4\nu}) = (b_{1\mu} + b_{2\mu}) + (b'_{3\nu} + b'_{4\nu}),$$

$$-J_3(A_I) + J_4(A_I) = -J_3(B_I) + J_4(B_I) \implies \quad (69)$$

$$(a_{1\mu} + a_{2\mu}) + (a_{3\nu} + a_{4\nu}) = (b_{1\mu} + b_{2\mu}) + (b_{3\nu} + b_{4\nu}),$$

$$(68) - (66) \implies (a_{1\mu} + a_{2\mu}) - (a_{3\nu} + a_{4\nu}) \quad (70)$$

$$= (b_{1\mu} + b_{2\mu}) - (b_{3\nu} + b_{4\nu}),$$

$$(69) + (70) \implies (a_{1\mu} + a_{2\mu}) = (b_{1\mu} + b_{2\mu}), \quad (71)$$

$$(69) - (70) \implies (a_{3\mu} + a_{4\mu}) = (b_{3\mu} + b_{4\mu}). \quad (72)$$

From (71), (66), and (67), we get

$$(a'_{1\nu} + a'_{2\nu}) = (b'_{1\nu} + b'_{2\nu}), \quad (73)$$

$$(a'_{3\nu} + a'_{4\nu}) = (b'_{3\nu} + b'_{4\nu}), \quad (74)$$

$$J_4(A_I) - J_8(A_I) = J_4(B_I) - J_8(B_I) \implies \quad (75)$$

$$(a_{2\mu} + a_{3\mu}) + (a'_{2\nu} + a'_{3\nu}) = (b_{2\mu} + b_{3\mu}) + (b'_{2\nu} + b'_{3\nu}),$$

$$J_4(A_I) - J_7(A_I) = J_4(B_I) - J_7(B_I) \implies \quad (76)$$

$$(a_{2\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{4\nu}) = (b_{2\mu} + b_{3\mu}) + (b'_{1\nu} + b'_{4\nu}),$$

$$J_4(A_I) - J_6(A_I) = J_4(B_I) - J_6(B_I) \implies \quad (77)$$

$$(a_{1\mu} + a_{3\mu}) + (a'_{2\nu} + a'_{4\nu}) = (b_{1\mu} + b_{3\mu}) + (b'_{2\nu} + b'_{4\nu}),$$

$$J_4(A_I) - J_5(A_I) = J_4(B_I) - J_5(B_I) \implies \quad (78)$$

$$(a_{1\mu} + a_{3\mu}) + (a'_{1\nu} + a'_{3\nu}) = (b_{1\mu} + b_{3\mu}) + (b'_{1\nu} + b'_{3\nu}),$$

$$(75) - (76) \implies (a'_{2\nu} - a'_{1\nu}) + (a'_{3\nu} - a'_{4\nu}) \quad (79)$$

$$= (b'_{2\nu} - b'_{1\nu}) + (b'_{3\nu} - b'_{4\nu}),$$

$$(77) - (78) \implies (a'_{2\nu} - a'_{1\nu}) + (a'_{4\nu} - a'_{3\nu}) \quad (80)$$

$$= (b'_{2\nu} - b'_{1\nu}) + (b'_{4\nu} - b'_{3\nu}),$$

$$(79) + (80) \implies (a'_{2\nu} - a'_{1\nu}) = (b'_{2\nu} - b'_{1\nu}), \quad (81)$$

$$(80) - (79) \implies (a'_{4\nu} - a'_{3\nu}) = (b'_{4\nu} - b'_{3\nu}), \quad (82)$$

$$J_2(A_I) + J_8(A_I) = J_2(B_I) + J_8(B_I) \implies \quad (83)$$

$$(a_{1\mu} - a_{3\mu}) + (a'_{4\nu} - a'_{2\nu}) = (b_{1\mu} - b_{3\mu}) + (b'_{4\nu} - b'_{2\nu}),$$

$$-(J_6(A_I) + J_7(A_I)) = -(J_6(B_I) + J_7(B_I)) \implies \quad (84)$$

$$(a_{3\mu} - a_{4\mu}) + (a'_{4\nu} - a'_{3\nu}) = (b_{3\mu} - b_{4\mu}) + (b'_{4\nu} - b'_{3\nu}).$$

From (73) and (74) and (81) and (82), we get

$$\begin{aligned} a'_{1\nu} &= b'_{1\nu}, \\ a'_{2\nu} &= b'_{2\nu}, \\ a'_{3\nu} &= b'_{3\nu}, \\ a'_{4\nu} &= b'_{4\nu}. \end{aligned} \quad (85)$$

From (83), (84), and (85), we get

$$(a_{1\mu} - a_{3\mu}) = (b_{1\mu} - b_{3\mu}), \quad (86)$$

$$(a_{3\mu} - a_{4\mu}) = (b_{3\mu} - b_{4\mu}). \quad (87)$$

(72) + (87) $\implies a_{3\mu} = b_{3\mu}$ and (72) - (87) $\implies a_4 = b_4$. From (86) $a_{3\mu} = b_{3\mu}$ implies $a_{1\mu} = b_{1\mu}$. Put $a_{1\mu} = b_{1\mu}$ in (71) and we get $a_{2\mu} = b_{2\mu}$. Therefore $a_{1\mu} = b_{1\mu}$, $a_{2\mu} = b_{2\mu}$, $a_{3\mu} = b_{3\mu}$, $a_{4\mu} = b_{4\mu}$ and $a'_{1\nu} = b'_{1\nu}$, $a'_{2\nu} = b'_{2\nu}$, $a'_{3\nu} = b'_{3\nu}$, $a'_{4\nu} = b'_{4\nu}$, hence the proof. $\square$

Theorem 101 shows that the proposed ordering on TrIFNs satisfies Law of trichotomy and therefore the proposed ranking principle on TrIFNs gives total order on the class of TrIFNs.

Remark 102. The validation of the proposed ranking principle on the subclasses of TrIFNs is listed as follows:

- (1) Accuracy score (J_4) function is sufficient to rank the set of real numbers $\mathfrak{R}$ totally, since J_1, J_2 , and J_3 scores become zero for any real number in $[0, 1]$.
- (2) J_1, J_4, J_5 , and J_8 give the total order on the class of trapezoidal fuzzy number.
- (3) J_1, J_2, J_3 , and J_4 together give the total order on the class of interval valued intuitionistic fuzzy number.
- (4) J_1, J_2, J_3, J_4, J_7 , and J_8 provide a complete ranking on the class of triangular intuitionistic fuzzy numbers.

TABLE 1: Significance of proposed method.

Other existing methods	Shortcomings of existing methods	Numerical example	Proposed method
Li [2] $R(A, \lambda) = \frac{a_1 + 4a_2 + a_3}{2(3 + a_3 - a_1)}$	$A_1 = \langle (a_1, a_2, a_3), 1, 0 \rangle,$ $A_2 = \left\langle \left(a_1 - \epsilon, a_2 + \frac{\epsilon}{2}, a_3 - \epsilon \right), 1, 0 \right\rangle$ $R(A_1) = R(A_2) \implies A_1 = A_2$	$A_1 = \langle (0, 0.25, 0.3), 1, 0 \rangle,$ $A_2 = \langle (0.1, 0.2, 0.4), 1, 0 \rangle$ $R(A_1) = R(A_2) = 0.19697 \implies A_1 = A_2$	$J_1(A_1) = J_1(A_2) = -0.075,$ $J_2(A_1) = J_2(A_2) = 0$ $J_3(A_1) = J_3(A_2) = 0$ $J_4(A_1) = 0.2 < J_4(A_2) = 0.225$ $J_4(A_2) > J_4(A_1) \implies A_2 > A_1$
Zeng et al., [12] $V_\lambda(A) = \lambda V_\mu(A) + (1 - \lambda)V_\gamma(A)$ where $V_\mu(A) = \frac{6}{(a_1 + 4a_2 + a_3)w_A^2},$ $V_\gamma(A) = \frac{6}{(a_1 + 4a_2 + a_3)(1 - u_A)^2}$ $A_\lambda(A) = \lambda A_\mu(A) + (1 - \lambda)A_\gamma(A)$ where $A_\mu(A) = \frac{(a_3 - a_1)w_A^2}{6},$ $A_\gamma(A) = \frac{(a_3 - a_1)(1 - u_A)^2}{6}$	$A_1 = \langle (a_1, a_2, a_3), 1, 0 \rangle$ $A_2 = \left\langle \left(a_1 - \epsilon, a_2 + \frac{\epsilon}{2}, a_3 - \epsilon \right), 1, 0 \right\rangle$ $V_\lambda(A_1) = V_\lambda(A_2),$ $A_\lambda(A_1) = A_\lambda(A_2) \implies A_1 = A_2$	$A_1 = \langle (0, 0.25, 0.3), 1, 0 \rangle,$ $A_2 = \langle (0.1, 0.2, 0.4), 1, 0 \rangle$ $V_\lambda(A_1) = V_\lambda(A_2) = 0.2167,$ $A_\lambda(A_1) = A_\lambda(A_2) = 0.05 \implies A_1 = A_2$	$J_1(A_1) = J_1(A_2) = -0.075,$ $J_2(A_1) = J_2(A_2) = 0$ $J_3(A_1) = J_3(A_2) = 0$ $J_4(A_1) = 0.2 < J_4(A_2) = 0.225$ $J_4(A_2) > J_4(A_1) \implies A_2 > A_1$
Dubey and Mehara, [9] $V(A, \lambda) = V_\mu(A) + \lambda(V_\gamma(A) - V_\mu(A))$ where $V_\mu(A) = \frac{6}{(a_1 + 4a_2 + a_3)w_A^2},$ $V_\gamma(A) = \frac{6}{(a_1 + 4a_2 + a_3)(1 - u_A)^2}$ $A(A, \lambda) = A_\gamma(A) - \lambda(A_\gamma(A) - A_\mu(A))$ where $A_\mu(A) = \frac{(a_3 - a_1)w_A^2}{6},$ $A_\gamma(A) = \frac{(a_3 - a_1)(1 - u_A)^2}{6}$ $F(A, \lambda) = V(A, \lambda) - A(A, \lambda)$	$A_1 = \langle (a_1, a_2, a_3), 1, 0 \rangle$ $A_2 = \left\langle \left(a_1 - \epsilon, a_2 + \frac{\epsilon}{2}, a_3 - \epsilon \right), 1, 0 \right\rangle$ $V(A_1, \lambda) = V(A_2, \lambda) \ \& \ A(A_1, \lambda) = A(A_2, \lambda)$ $\implies F(A_1, \lambda) = F(A_2, \lambda) \implies A_1 = A_2$	$A_1 = \langle (0, 0.25, 0.3), 1, 0 \rangle,$ $A_2 = \langle (0.1, 0.2, 0.4), 1, 0 \rangle$ $V(A_1, \lambda) = V(A_2, \lambda) = 0.2167,$ $A(A_1, \lambda) = A(A_2, \lambda) = 0.05$ $\implies F(A_1, \lambda) = F(A_2, \lambda) = 0.1667 \implies A_1 = A_2$	$J_1(A_1) = J_1(A_2) = -0.075$ $J_2(A_1) = J_2(A_2) = 0$ $J_3(A_1) = J_3(A_2) = 0$ $J_4(A_1) = 0.2 < J_4(A_2) = 0.225$ $J_4(A_2) > J_4(A_1) \implies A_2 > A_1$

TABLE 1: Continued.

Other existing methods	Shortcomings of existing methods	Numerical example	Proposed method
<p>Nehi, [7]</p> $C_\mu^k(A) = \frac{a_2 + a_3}{a_1 - a_2 + a_4 - a_3}, \quad k \in \mathfrak{R}^+$ $C_\nu^k(A) = \frac{b_1 + b_4}{2(k+2)}$ $C_\mu^k(A) = \frac{2}{b_2 - b_1 + b_3 - b_4}, \quad k \in \mathfrak{R}^+$	$A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$ $B = \langle (a_1 + \epsilon, a_2 + \epsilon, a_3 - \epsilon, a_4 - \epsilon), (b_1 + \epsilon, b_2 + \epsilon, b_3 - \epsilon, b_4 - \epsilon) \rangle$ $C_\mu^k(A) = C_\mu^k(B), C_\nu^k(A) = C_\nu^k(B)$ $\implies A = B$	$A = \langle (0.2, 0.3, 0.5, 0.6), (0, 0.1, 0.6, 0.7) \rangle$ $B = \langle (0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.5, 0.6) \rangle$ $C_\mu^k(A) = C_\mu^k(B) = 0.4, C_\nu^k(A) = C_\nu^k(B) = 0.35$ $\implies A = B$	$J_1(A) = -0.2225 < J_1(B) = -0.125$ $\implies A < B$
<p>Ye, [8]</p> $EV(A) = \frac{a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4}{8}$	$A_1 = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle,$ $A_2 = \langle (a_1 + \epsilon, a_2 + \epsilon, a_3 + \epsilon, a_4 + \epsilon), (b_1 - \epsilon, b_2 - \epsilon, b_3 - \epsilon, b_4 - \epsilon) \rangle$ $EV(A_1) = EV(A_2) \implies A_1 = A_2$	$A_1 = \langle (0.1, 0.2, 0.3, 0.4), (0.1, 0.15, 0.55, 0.6) \rangle,$ $A_2 = \langle (0.2, 0.3, 0.4, 0.5), (0, 0.05, 0.45, 0.5) \rangle$ $EV(A_1) = EV(A_2) = 0.3 \implies A_1 = A_2$	$J_1(A_1) = J_1(A_2) = -0.1625,$ $J_2(A_1) = J_2(A_2) = 0.0625,$ $J_3(A_1) = 0.05, J_3(A_2) = -0.05$ $J_3(A_2) < J_3(A_1) \implies A_2 < A_1$
<p>De and Das, [26]</p> $V(\bar{A}) = V_\mu(\bar{A}) + \lambda(V_\nu(\bar{A}) - V_\mu(\bar{A}))$ $A(\bar{A}_1) = A_\nu(\bar{A}) + \lambda(A_\nu(\bar{A}) - A_\mu(\bar{A}))$ <p>where $V_\mu(\bar{A}) = \frac{(a_1 + 2a_2 + 2a_3 + a_4)(w_{\bar{A}})}{(a_1 + 2a_2 + 2a_3 + a_4)(1 - u_{\bar{A}})}$</p> $V_\nu(\bar{A}) = \frac{6}{(a_4 - a_1 - 2a_2 + 2a_3)(w_{\bar{A}})}$ $A_\mu(\bar{A}) = \frac{(a_4 - a_1 - 2a_2 + 2a_3)(w_{\bar{A}})}{6}$ $A_\nu(\bar{A}) = \frac{(a_4 - a_1 - 2a_2 + 2a_3)(1 - u_{\bar{A}})}{6}$	$\bar{A}_1 = \langle (a_1, a_2, a_3, a_4), 1, 0 \rangle$ $\bar{A}_2 = \left\langle \left(a_1 - \epsilon, a_2 + \frac{\epsilon}{2}, a_3 + \frac{\epsilon}{2}, a_4 - \epsilon \right), 1, 0 \right\rangle$ $V(\bar{A}_1) = V(\bar{A}_2) \text{ and } A(\bar{A}_1) = A(\bar{A}_2)$ $\implies \bar{A}_1 = \bar{A}_2$	$\bar{A}_1 = \langle (0.20, 0.35, 0.40, 0.55), 1, 0 \rangle,$ $\bar{A}_2 = \langle (0.10, 0.40, 0.45, 0.45), 1, 0 \rangle$ $V(\bar{A}_1) = V(\bar{A}_2) = 0.2333,$ $A(\bar{A}_1) = A(\bar{A}_2) = 0.0833 \implies \bar{A}_1 = \bar{A}_2$	$J_1(\bar{A}_1) = J_1(\bar{A}_2) = -0.1$ $J_2(\bar{A}_1) = J_2(\bar{A}_2) = 0$ $J_3(\bar{A}_1) = J_3(\bar{A}_2) = 0$ $J_4(\bar{A}_1) = 0.375 > J_4(\bar{A}_2) = 0.35$ $\implies \bar{A}_1 > \bar{A}_2$

TABLE 1: Continued.

Other existing methods	Shortcomings of existing methods	Numerical example	Proposed method
Kumar and Kaur, [11] $M_{\mu}^{\beta,k}(A) = \frac{\beta(a_2 + a_3)}{2} + \frac{(1 - \beta)(a_1 + a_4)}{(1 - \beta)(a_2 - a_1 + a_3 - a_4)(k + 1)} + \frac{2}{2(k + 2)}$, $M_{\nu}^{\beta,k}(A) = \frac{\beta(a'_2 + a'_3)}{2} + \frac{(1 - \beta)(a'_1 + a'_4)}{(1 - \beta)(a'_2 - a'_1 + a'_3 - a'_4)(k + 1)} + \frac{2}{2(k + 2)}$, $k \in \mathfrak{R}^+ \text{ \& } \beta \in [0, 1]$	$A = \langle (a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4) \rangle$ $B = \langle (a_1 - \epsilon, a_2 - \epsilon, a_3 + \epsilon, a_4 + \epsilon), (a'_1 - \epsilon, a'_2 - \epsilon, a'_3 + \epsilon, a'_4 + \epsilon) \rangle$ $M_{\nu}^{\beta,k}(A) = M_{\nu}^{\beta,k}(B), M_{\nu}^{\beta,k}(A) = M_{\nu}^{\beta,k}(B)$ $A = B$	$A = \langle (0.2, 0.3, 0.5, 0.6), (0.0, 0.1, 0.6, 0.7) \rangle$ $B = \langle (0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.5, 0.6) \rangle$ $M_{\nu}^{\beta,k}(A) = M_{\nu}^{\beta,k}(B) = 0.4, M_{\nu}^{\beta,k}(A) = M_{\nu}^{\beta,k}(B) = 0.35$ $A = B$	$J_1(A) = -0.225, J_1(B) = -0.125$ $J_1(A) < J_1(B) \implies A < B$
Wan and Dong, [13] $m_{\mu}(A) = \frac{1}{2}[\underline{m}_{\mu}(A) + \overline{m}_{\mu}(A)]$ $m_{\nu}(A) = \frac{1}{2}[\underline{m}_{\nu}(A) + \overline{m}_{\nu}(A)]$ $m_{\mu}(A) = \frac{1}{6}(a + 2b + 2c + d)w_A$ $m_{\nu}(A) = \frac{1}{6}(a + 2b + 2c + d)(1 - u_A)$	$A_1 = \langle (a_1, a_2, a_3, a_4), 1, 0 \rangle$ $A_2 = \langle (a_1 - \epsilon, a_2 - \epsilon, a_3 + \epsilon, a_4 + \epsilon), 1, 0 \rangle$ $m_{\mu}(A_1) = m_{\nu}(A_1) = m_{\mu}(A_2) = m_{\nu}(A_2)$ $\implies A_1 = A_2$	$A_1 = \langle (0.40, 0.45, 0.50, 0.55), 1, 0 \rangle$, $A_2 = \langle (0.30, 0.35, 0.60, 0.65), 1, 0 \rangle$ $m_{\mu}(A_1) = m_{\nu}(A_1) = m_{\mu}(A_2) = m_{\nu}(A_2) = 0.475$ $\implies A_1 = A_2$	$J_1(A_1) = -0.05, J_1(A_2) = -0.15$ $J_1(A_2) < J_1(A_1) \implies A_2 < A_1$

5. Significance of the Proposed Method

In this section, significance of our proposed method is shown by comparing our proposed method with all other existing techniques. Table 1 will show the shortcoming of existing methods and the advantage of our method. In Table 1, any trapezoidal intuitionistic fuzzy number is considered as normal.

That is, $A = \langle (a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4), 1, 0 \rangle$. For example, let $A = \langle (0.2, 0.3, 0.5, 0.6), (0, 0.1, 0.6, 0.7) \rangle$ and $B = \langle (0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.5, 0.6) \rangle$ be any two TrIFNs. Then by applying Nehi's [7] approach, we get $C_\mu^k(A) = C_\mu^k(B) = 0.4$ and $C_\nu^k(A) = C_\nu^k(B) = 0.35$ and this implies that A and B are equal which is illogical. By applying the total order relation defined in Definition 100, we get $J_1(A) = 0.225 > J_1(B) = 0.125 \Rightarrow A > B$. Hence our proposed method ranks A as better.

6. Conclusion

In this paper, a complete ranking on the class of trapezoidal intuitionistic fuzzy numbers using eight different scores is proposed and compared with existing techniques using illustrative examples. A complete ranking on the class of TrIFNs yields the better results in MADM problems, fuzzy information systems, and artificial intelligence involving intuitionistic trapezoidal fuzzy information and hence our proposed method may be applied to control systems and other engineering fields. Our proposed method is conceptually easy to understand and operationally easy to use.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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