Hindawi
Complexity
Volume 2017, Article ID 5876371, 6 pages
https://doi.org/10.1155/2017/5876371

Research Article

Reachable Set Estimation for a Class of Nonlinear Time-Varying Systems

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Received 11 April 2017; Accepted 12 June 2017; Published 12 July 2017

Academic Editor: Pietro De Lellis

This paper studies the problem of reachable set estimation for a class of nonlinear time-varying systems with disturbances. New necessary and (or) sufficient conditions are derived for the existence of a ball such that all the solutions of the system converge asymptotically within it. Explicit estimation on the decay rate is also presented. The method used in this paper is motivated by that for positive systems, which is different from the conventional Lyapunov-Krasovskii functional method. A numerical example is given to illustrate the effectiveness of the obtained result.

1. Introduction

The problem of reachable set estimation (bounding) for linear systems has been addressed in [1–10], to name a few. In all of the aforementioned works, the involved system was mainly aimed at linear time invariant systems. Moreover, the Lyapunov-Krasovskii functional method was most commonly used, which is usually invalid to time-varying systems because they lead to either unsolvable matrix Riccati differential equations or indefinite linear matrix inequalities.

Recently, Hien and Trinh considered the problem of reachable set bounding for linear time-varying systems with delay and bounded disturbances for the first time in [11]. By using a new approach which does not involve the Lyapunov-Krasovskii functional method, an explicit delay-independent condition for state bounding of the system was given in terms of Metzler matrix. The reachable set bounding for a class of nonlinear perturbed time-delay systems was investigated in [12], where the involved nonlinear term satisfies linear growth condition. Recently, state bounding for homogeneous positive systems of degree one with time-varying delay and exogenous input was studied in [13]. In this paper, we will further study the reachable set estimation for a class of nonlinear time-varying systems without satisfying the linear growth condition or the homogeneous condition of degree one. For the particular case when the system is positive, we first establish a necessary and sufficient condition for reachable set bounding. Then, we extend the result to the more general case by using the comparison principle.

Positive systems, whose state trajectories remain nonnegative for all time provided that initial states are nonnegative, have received much attention in recent years (see the books [14, 15] and the references therein). When dealing with stability problems of positive systems, an approach independent of Lyapunov-Krasovskii functional was commonly used in [16–21]. Inspired by this, the paper will apply such a method to the reachable set estimation for nonlinear time-varying systems with disturbances. Necessary and (or) sufficient conditions have been established such that all the solutions of the system converge asymptotically within a ball determined by the upper bound of disturbances. Moreover, the explicit decay rate is also presented.

The paper is organized as follows. In Section 2, we present the notation used through this paper as well as preliminaries for our results. Section 3 then focuses on deriving explicit conditions under which all the solutions of the system converge asymptotically within a ball. Section 4 provides an illustrative example to show the effectiveness of the obtained result. The paper is concluded in Section 5.

2. Preliminaries

Throughout this paper, the following notation will be used. Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the set of n-dimensional real vectors and the n-dimensional real Euclidean space, respectively. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Metzler* if all its off-diagonal entries are nonnegative. For $x \in \mathbb{R}^n$, we denote by x_i the ith coordinate of x. For two vectors $x, y \in \mathbb{R}^n$, we write $x \succ y$ if $x_i > y_i$, $x \succeq y$ if $x_i \geq y_i$, $x \prec y$ if $x_i < y_i$, and $x \preceq y$ if $x_i \leq y_i$, $i = 1, 2, \dots, n$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$. For $x = (x_i) \in \mathbb{R}^n$, let $|x| = (|x_i|) \in \mathbb{R}^n_+$ and $\|x\|_\infty = \max_{i=1,2,\dots,n}|x_i|$. Given an n-dimensional vector $v = (v_i) \succ 0$, the weighted ∞-norm of the vector $x \in \mathbb{R}^n$ is defined by $\|x\|^v_\infty = \max_{i=1,2,\dots,n}(|x_i|/v_i)$. For $\varepsilon > 0$, denote a ball by $\mathscr{B}(\varepsilon) = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq \varepsilon\}$.

Consider the following continuous-time nonlinear time-varying system described by

$$\dot{x}(t) = \mathbf{F}(t, x(t)) + w(t), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) : [0, \infty) \to \mathbb{R}^n$ is the unknown disturbance input, and the vector field $\mathbf{F}(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz with respect to x, which ensures the existence and uniqueness of solutions of system (1) [22]. We now begin with the following definitions.

Definition 1. System (1) is said to be positive if, for any initial condition $x(0) \in \mathbb{R}^n_+$, the corresponding state trajectory $x(t)$ remains nonnegative for all $t \geq 0$.

The following proposition gives a sufficient condition guaranteeing the positivity of system (1).

Proposition 2. *System (1) is positive if*

$$\begin{gathered} w(t) \succeq 0, \\ F_i(t, x) \geq 0, \\ t \geq 0,\ i = 1, 2, \dots, n,\ x \in \mathbb{R}^n_+,\ x_i = 0, \end{gathered} \tag{2}$$

where F_i is the ith coordinate of $\mathbf{F}$.

Proof. Let $x(t)$ be the solution of system (1) with the initial condition $x(0) \succeq 0$. In order to prove that $x(t) \succeq 0$ for all $t \geq 0$, it is sufficient to check that the vector $\dot{x}(t)$ does not point toward the outside of $\mathbb{R}^n_+$. This is equivalent to verifying that the components of the vector $\dot{x}(t) = \mathbf{F}(t, x(t)) + w(t)$ corresponding to the zero components of $x(t)$ are nonnegative, which can be derived from condition (2) immediately. The proof of Proposition 2 is complete. □

Definition 3 (see [21]). A vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be homogeneous of degree $p > 0$ if $\mathbf{f}(\lambda x) = \lambda^p \mathbf{f}(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$.

Definition 4 (see [23]). A continuous vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, which is continuously differentiable on $\mathbb{R}^n/\{0\}$, is said to be cooperative if the Jacobian matrix $(\partial \mathbf{f}/\partial x)(a)$ is Metzler for all $a \in \mathbb{R}^n/\{0\}$.

Denote by f_i the ith coordinate of $\mathbf{f}$. It is well known that a cooperative vector field satisfies the following Kamke condition (see [22, Remark 1.1, pp. 33]).

Proposition 5. *Let $\mathbf{f}$ be a cooperative vector field. For any two vectors $x, y \in \mathbb{R}^n/\{0\}$ satisfying $x \succeq y$ and $x_i = y_i$, one has $f_i(x) \geq f_i(y)$.*

3. Main Results

We first consider the following particular case of system (1), where

$$\begin{gathered} \mathbf{F}(t, x) \equiv \mathbf{f}(x), \\ w(t) \succeq 0, \\ \|w(t)\|_\infty \leq r, \\ t \geq 0, \end{gathered} \tag{3}$$

where the vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is cooperative and $r > 0$ is a constant. By Propositions 2 and 5, we see that system (1) is positive. For this case, we have the following necessary and sufficient condition for the reachable set bounding of system (1).

Theorem 6. *Assume that (3) holds, and the vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is cooperative and homogeneous of degree $p > 1$. Then the following two statements are equivalent.*

(i) For any initial condition $x(0) \in \mathbb{R}^n_+$ and any disturbance $w(t)$ satisfying (3), there exist nonnegative constants α, β, and γ such that the solution $x(t)$ of system (1) satisfies

$$\|x(t)\|_\infty \leq \alpha + (\beta + \gamma t)^{-1/(p-1)}, \quad t \geq 0, \tag{4}$$

where α depends on r, $\alpha = 0$ if $r = 0$, β is related to r and $x(0)$, and γ is independent of r and $x(0)$.

(ii) There exists an n-dimensional vector $v \succ 0$ such that $\mathbf{f}(v) \prec 0$.

In addition, if condition (ii) holds, we can choose $\alpha = \theta\rho$, $\beta = (K\rho)^{1-p}$, and $\gamma = (p-1)\eta\rho^{1-p}$, where $\rho = \max_{i=1,2,\dots,n} v_i$ and θ, K, and η are defined as follows:

$$\begin{gathered} \theta = \left(\frac{r}{-\max_{i=1,2,\dots,n} f_i(v)}\right)^{1/p}, \\ K = \begin{cases} 0, & \|x(0)\|^v_\infty \leq \theta, \\ \|x(0)\|^v_\infty - \theta, & \|x(0)\|^v_\infty > \theta, \end{cases} \\ \eta = -\max_{i=1,2,\dots,n} \frac{f_i(v)}{v_i}. \end{gathered} \tag{5}$$

Proof. (ii) ⇒ (i) Since system (1) is positive, each solution $x(t)$ of system (1) with the initial condition $x(0) \in \mathbb{R}^n_+$ satisfies

$x(t) \geq 0$ for all $t \geq 0$. For any $\delta > 1$, we can conclude from the definition of K that

$$\frac{x_i(0)}{v_i} \leq \theta + K < \delta(\theta + K), \quad i = 1, 2, \ldots, n. \tag{6}$$

Set

$$z_i(t) = \frac{x_i(t)}{v_i} - \delta\theta - \left[(\delta K)^{1-p} + (p-1)\eta t\right]^{-1/(p-1)}, \quad t \geq 0, \; i = 1, 2, \ldots, n. \tag{7}$$

From (6), we have $z_i(0) < 0$. By the continuity of $z_i(t)$ at $t = 0$, there exists a real number $t_1 > 0$ such that $z_i(t) < 0$ for $i = 1, 2, \ldots, n$ and $t \in [0, t_1)$. We now prove that $z_i(t) < 0$ for $i = 1, 2, \ldots, n$ and $t \geq 0$. Otherwise, there exist a real number $t_2 > t_1$ and an index $k \in \{1, 2, \ldots, n\}$ such that $z_i(t) < 0$ for $i = 1, 2, \ldots, n$ and $t \in [0, t_2)$, and $z_k(t_2) = 0$. Therefore,

$$\dot{z}_k(t_2) \geq 0, \tag{8}$$

$$\frac{x_i(t_2)}{v_i} \leq \delta\theta + \left[(\delta K)^{1-p} + (p-1)\eta t_2\right]^{-1/(p-1)}, \quad i = 1, 2, \ldots, n, \tag{9}$$

$$\frac{x_k(t_2)}{v_k} = \delta\theta + \left[(\delta K)^{1-p} + (p-1)\eta t_2\right]^{-1/(p-1)}.$$

By Proposition 5 and the homogeneity of $\mathbf{f}$, we obtain

$$\begin{aligned} f_k(x(t_2)) &\leq f_k\left(\left[\delta\theta + \left((\delta K)^{1-p} + (p-1)\eta t_2\right)^{-1/(p-1)}\right] v\right) \\ &= \left[\delta\theta + \left((\delta K)^{1-p} + (p-1)\eta t_2\right)^{-1/(p-1)}\right]^p f_k(v). \end{aligned} \tag{10}$$

From (1) and the definition of $z_i(t)$, we have

$$\begin{aligned} \dot{z}_k(t_2) &= \frac{\dot{x}_k(t_2)}{v_k} + \eta\left[(\delta K)^{1-p} + (p-1)\eta t_2\right]^{-p/(p-1)} \\ &= \frac{f_k(x(t_2)) + w_k(t_2)}{v_k} \\ &\quad + \eta\left[(\delta K)^{1-p} + (p-1)\eta t_2\right]^{-p/(p-1)}. \end{aligned} \tag{11}$$

By using the basic inequality $(a+b)^q \geq a^q + b^q$ for $a, b \geq 0$ and $q > 1$, we get from (10) and (11) that

$$\begin{aligned} &\dot{z}_k(t_2) \\ &\quad \leq \frac{1}{v_k}\left[\delta^p\theta^p f_k(v) + r\right] \\ &\quad + \left[(\delta K)^{1-p} + (p-1)\eta t_2\right]^{-p/(p-1)}\left[\frac{f_k(v)}{v_k} + \eta\right]. \end{aligned} \tag{12}$$

Since $\delta > 1$, by the definitions of θ and η, we obtain

$$\begin{aligned} &\delta^p\theta^p f_k(v) + r < \theta^p f_k(v) + r \leq 0, \\ &\frac{f_k(v)}{v_k} + \eta \leq 0. \end{aligned} \tag{13}$$

Combining this with (12), we have $\dot{z}_k(t_2) < 0$, which contradicts (8). Consequently, $z_i(t) < 0$ for $i = 1, 2, \ldots, n$ and $t \geq 0$. That is, for any $\delta > 1$,

$$\frac{x_i(t)}{v_i} < \delta\theta + \left[(\delta K)^{1-p} + (p-1)\eta t\right]^{-1/(p-1)}, \quad t \geq 0, \; i = 1, 2, \ldots, n. \tag{14}$$

As δ tends to 1 from the right, we have

$$\frac{x_i(t)}{v_i} \leq \theta + \left[K^{1-p} + (p-1)\eta t\right]^{-1/(p-1)}, \quad t \geq 0, \; i = 1, 2, \ldots, n, \tag{15}$$

which implies (4) with α, β, and γ defined as in Theorem 6.

(i) $\Rightarrow$ (ii) Assume that condition (4) holds for any initial condition $x(0) \in \mathbb{R}_+^n$ and any w satisfying (3). In particular, each solution of the system $\dot{x}(t) = \mathbf{f}(x(t))$ with the nonnegative initial condition satisfies

$$\|x(t)\|_\infty \leq (\beta + \gamma t)^{-1/(p-1)}. \tag{16}$$

That is, system (1) without disturbances is globally asymptotically stable. Invoking [23, Proposition 4.1], we have that there exists a vector $v \succ 0$ such that $\mathbf{f}(v) \prec 0$. This completes the proof of Theorem 6. □

Next, we study the reachable set bounding for system (1) without condition (3). Based on a straightforward computation, we can get

$$D_+\left|x_i(t)\right| \leq F_i(t, x(t)) \operatorname{sign}(x_i(t)) + \left|w_i(t)\right|, \quad i = 1, 2, \ldots, n, \; t \geq 0, \tag{17}$$

where $D_+|x_i(t)|$ denotes the derivative of $|x_i(t)|$ from the right and $w(t) \in BF_r([0, \infty), \mathbb{R}^n)$. The right-hand side of the above inequality is interpreted as $|F_i(t, x(t))|$ when $x_i(t) = 0$.

Similar to Definitions 3 and 4, a vector field $\mathbf{f}(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be cooperative, if it is continuously differentiable on $\mathbb{R}^n/\{0\}$ with respect to x, and the Jacobian matrix $(\partial\mathbf{f}/\partial x)(t, a)$ is Metzler for $a \in \mathbb{R}^n/\{0\}$ and $t \geq 0$. The vector field $\mathbf{f}$ is said to be homogeneous of degree $p > 0$ if $\mathbf{f}(t, \lambda x) = \lambda^p \mathbf{f}(t, x)$ for $x \in \mathbb{R}^n$, $\lambda > 0$, and $t \geq 0$. In the following, assume that the vector field $\mathbf{F}$ satisfies the following.

(iii) There exists a vector field $\mathbf{f}(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ which is cooperative and homogeneous of degree $p > 1$, such that

$$F_i(t, x) \operatorname{sign}(x_i) \leq f_i(t, |x|), \quad i = 1, 2, \ldots, n, \; t \geq 0, \; x \in \mathbb{R}_n, \tag{18}$$

where $F_i(t,x)\operatorname{sign}(x_i)$ is also interpreted as $|F_i(t,x)|$ when $x_i = 0$.

Since $\mathbf{f}$ is cooperative and homogeneous, we can get from Propositions 2 and 5 that the following system is positive:

$$\dot{x}(t) = \mathbf{f}(t, x(t)) + |w(t)|, \quad t \geq 0. \tag{19}$$

By using the comparison principle, we have that solution $x(t)$ of system (1) with the initial condition $x(0) \in \mathbb{R}^n$ satisfies

$$|x(t)| \preceq \overline{x}(t), \quad t \geq 0, \tag{20}$$

where $\overline{x}(t)$ is the solution of system (19) with the initial condition $\overline{x}(0) = |x(0)|$. Consequently, by using Theorem 6, we have the following sufficient condition for reachable set estimation of system (1).

Theorem 7. *Assume that condition (iii) holds. If there exists a vector $v \succ 0$ such that*

$$\mathbf{f}(t, v) \prec 0, \quad t \geq 0, \tag{21}$$

then for any initial condition $x(0) \in \mathbb{R}^n$ and any disturbance $w(t)$, the solution $x(t)$ of system (1) satisfies

$$\|x(t)\|_\infty^v \leq \widetilde{\theta} + \left(\widetilde{K}^{1-p} + (p-1)\int_0^t \widetilde{\eta}(s)\,ds\right)^{-1/(p-1)}, \quad t \geq 0, \tag{22}$$

where

$$\begin{aligned}
\widetilde{\theta} &= \left(\max_{i=1,2,\dots,n} \max_{t\geq 0} \frac{|w_i(t)|}{-f_i(t,v)}\right)^{1/p} < +\infty, \\
\widetilde{K} &= \begin{cases} 0, & \|x(0)\|_\infty^v \leq \widetilde{\theta}, \\ \|x(0)\|_\infty^v - \widetilde{\theta}, & \|x(0)\|_\infty^v > \widetilde{\theta}, \end{cases} \\
\widetilde{\eta}(t) &= -\max_{i=1,2,\dots,n} \frac{f_i(t,v)}{v_i}.
\end{aligned} \tag{23}$$

Proof. Based on the above analysis, it is sufficient to prove that (22) holds for each solution of system (19) with any nonnegative initial condition. Denote $x(t)$ by the solution of system (22) with $x(0) \in \mathbb{R}^n_+$. For any $\delta > 1$, let

$$\begin{aligned}
z_i(t) = {} & \frac{x_i(t)}{v_i} - \delta\widetilde{\theta} \\
& - \left[\left(\delta\widetilde{K}\right)^{1-p} + (p-1)\int_0^t \widetilde{\eta}(s)\,ds\right]^{-1/(p-1)}, \\
& t \geq 0,\ i = 1, 2, \dots, n.
\end{aligned} \tag{24}$$

Since $z_i(0) < 0$, $i = 1, 2, \dots, n$, we show that $z_i(t) < 0$ for $t \geq 0$. Otherwise, there exist a real number $t_2 > 0$ and an index $k \in \{1, 2, \dots, n\}$ such that $z_i(t) < 0$, $i = 1, 2, \dots, n$, $t \in [0, t_2)$, and $z_k(t_2) = 0$. It implies that $\dot{z}_k(t_2) \geq 0$,

$$\begin{aligned}
\frac{x_i(t_2)}{v_i} &\leq \delta\widetilde{\theta} + \left[\left(\delta\widetilde{K}\right)^{1-p} + (p-1)\int_0^{t_2} \widetilde{\eta}(s)\,ds\right]^{-1/(p-1)}, \\
& \qquad i = 1, 2, \dots, n, \\
\frac{x_k(t_2)}{v_k} &= \delta\widetilde{\theta} + \left[\left(\delta\widetilde{K}\right)^{1-p} + (p-1)\int_0^{t_2} \widetilde{\eta}(s)\,ds\right]^{-1/(p-1)}.
\end{aligned} \tag{25}$$

Note that $\mathbf{f}(t, x)$ is a cooperative and homogeneous vector field of degree p. Based on the analysis in Theorem 6, we can eventually get

$$\begin{aligned}
\dot{z}_k(t_2) \leq {} & \frac{1}{v_k}\left[\delta^p \widetilde{\theta}^p f_k(t_2, v) + r_k(t_2)\right] \\
& + \left[\left(\delta\widetilde{K}\right)^{1-p} + (p-1)\int_0^{t_2} \widetilde{\eta}(s)\,ds\right]^{-p/(p-1)} \\
& \cdot \left[\frac{f_k(t_2, v)}{v_k} + \widetilde{\eta}(t_2)\right].
\end{aligned} \tag{26}$$

By the definitions of $\widetilde{\theta}$ and $\widetilde{\eta}$, we conclude that $\dot{z}_k(t_2) < 0$, which is a contradiction. The remainder of the proof is similar to that of Theorem 6. This completes the proof of Theorem 7. □

Remark 8. Compared with Theorem 6, we do not require in Theorem 7 that the initial condition $x(0) \in \mathbb{R}^n_+$ and the disturbance input $w(t) \succeq 0$ for $t \geq 0$. Moreover, the disturbance $w(t)$ may be unbounded.

Remark 9. If $\int_0^\infty \widetilde{\eta}(s)ds = +\infty$, then Theorem 7 guarantees that all the solutions of system (1) converge asymptotically within an ellipsoid.

Remark 10. Assume that there exists a vector field $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{f}(t, x) \preceq \mathbf{g}(x)$ for $t \geq 0$ and $x \succ 0$. Suppose also that there exist positive constants r_i, $i = 1, 2, \dots, n$, such that $|w_i(t)| \leq r_i$ for $t \geq 0$. Then we get $\widetilde{\theta} = (\max_{i=1,2,\dots,n}(r_i/-g_i(v)))^{1/p}$. It can be seen from Theorem 7 that the bound of the reachable set is determined by the bound of disturbances, the choice of v, and the value of p. When the bound of disturbances and the value of p are given, an appropriate vector v can be chosen to guarantee a minimal bound of the reachable set by solving some nonlinear optimization problem (for details, please see analysis in the following numerical example).

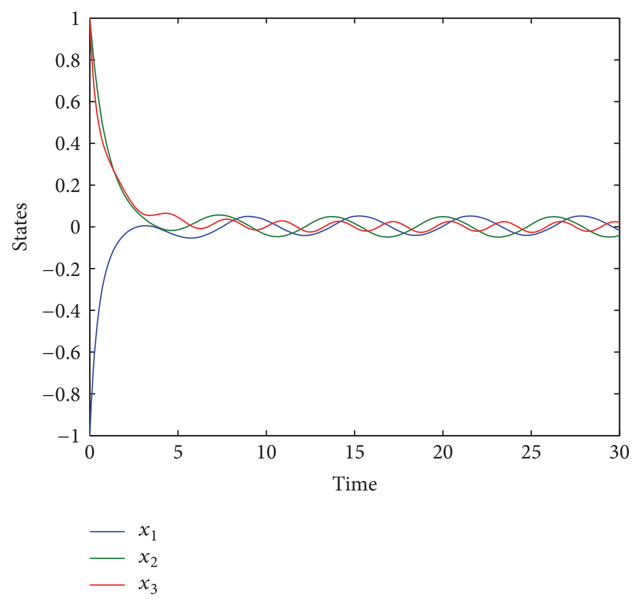

FIGURE 1: The state trajectories of system (1) with $x(0) = (-1, 1, 1)^T$.

4. Numerical Example

Consider system (1) on $\mathbb{R}^3$ with $w(t) = 0.05(\sin t, \cos t, \sin 2t)^T$ and

$$\mathbf{F}(t,x) = \begin{bmatrix} -2|x_1|^{3/2}\operatorname{sign}(x_1) + \dfrac{t}{1+t}\sqrt{|x_1x_2x_3|} \\ (\sin t)\sqrt{|x_1x_2x_3|} - 2|x_2|^{3/2}\operatorname{sign}(x_2) \\ (\cos t)\sqrt{|x_1x_2x_3|} - 2|x_3|^{3/2}\operatorname{sign}(x_3) \end{bmatrix}. \quad (27)$$

We see that condition (18) holds for the vector field

$$\mathbf{f}(t,x) \equiv \mathbf{f}(x) = \begin{bmatrix} -2|x_1|^{3/2} + \sqrt{|x_1x_2x_3|} \\ \sqrt{|x_1x_2x_3|} - 2|x_2|^{3/2} \\ \sqrt{|x_1x_2x_3|} - 2|x_3|^{3/2} \end{bmatrix}, \quad (28)$$

which is cooperative and homogeneous of degree $p = 3/2$. If we choose $v = (1,1,1)^T$, then (21) is valid. A straightforward computation yields that $\tilde{\theta} = 0.05^{2/3} = 0.1357$ and $\tilde{\eta}(t) \equiv 1$. Therefore, all the solutions of system (1) converge asymptotically within the ball $\mathscr{B}(0.1357)$ by Theorem 7. If we choose the initial condition $x(0) = (-1,1,1)^T$, then $\widetilde{K}^{1-p} = (1-0.1357)^{-1/2} = 1.0756$, and hence the solution $x(t)$ of the system satisfies $\|x(t)\|_\infty \le 0.1357 + (1.0756 + 0.5t)^{-2}$. The simulation result is presented in Figure 1.

In the above numerical example, a better estimation of the reachable set can be obtained by choosing appropriate vector v. In fact, the minimal bound of the reachable set can be chosen to be $\min_{v>0} g(v)$ subject to $\mathbf{f}(v) \prec 0$, which is a nonlinear optimization problem, where $g(v) = (0.05/-\max_{i=1,2,\ldots,n} f_i(v))^{2/3} \max_{i=1,2,\ldots,n} v_i$.

5. Conclusion

In this paper, we have studied the problem of reachable set estimation for a class of continuous-time nonlinear time-varying systems with disturbances. When the involved system is a particular positive system, we establish a necessary and sufficient condition such that all the solutions of the system converge asymptotically within a ball. By using the comparison principle, an explicit sufficient condition for reachable set bounding of the general system is also presented. In both cases, the decay rate is estimated precisely. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained result. The state bounding for both the continuous-time system with delay and the discrete-time system is an interesting problem, which will be studied in the next paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This paper is supported by the National Natural Science Foundation of China under Grant nos. 61473133, 11671227, and 61374074 and the Natural Science Foundation of Shandong Province under Grant no. JQ201119.

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