Some Generalized Pythagorean Fuzzy Bonferroni Mean Aggregation Operators with Their Application to Multiattribute Group Decision-Making

Runrong Zhang, 1 Jun Wang, 1 Xiaomin Zhu, 2 Meimei Xia, 1 and Ming Yu 3

1 School of Economics and Management, Beijing Jiaotong University, Beijing 100044, China
2 School of Mechanical, Electronic and Control Engineering, Beijing Jiaotong University, Beijing 100044, China
3 Department of Industrial Engineering, Tsinghua University, Beijing 100084, China

Correspondence should be addressed to Xiaomin Zhu; xmzhu@bjtu.edu.cn

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1. Introduction

Decision-making is a common and significant activity in daily life. In the past decades, decision-making problems in real life have become increasingly complicated because of the increasing complexity in economic and social management. Given the fuzziness and vagueness in decision-making, crisp numbers are inadequate and insufficient for managing real decision-making problems. In 1965, Zadeh introduced the concept of fuzzy set (FS) [1], which is an effective tool in handling fuzziness and uncertainty. However, FS only has a membership degree, which is unsuitable in managing several real decision-making problems. Atanassov [2] introduced the intuitionistic fuzzy set (IFS), which simultaneously has membership and nonmembership degrees, due to the shortcomings of FS. A few achievements on IFS have been reported [3, 4]. Mao et al. [5] introduced a few new cross-entropy and entropy measures for IFSs and applied them to decision-making. Liu and Teng [6] introduced the normal intuitionistic fuzzy numbers and several new normal intuitionistic fuzzy aggregation operators and applied them to multiattribute group decision-making (MAGDM). Lakshmana et al. [7] proposed a total order on the entire class of intuitionistic fuzzy numbers using upper, lower dense sequence in the interval [0, 1]. Lakshmana et al. [8] introduced a new principle for ordering trapezoidal intuitionistic fuzzy numbers. P. Liu and X. Liu [9] introduced the linguistic intuitionistic fuzzy set and a few linguistic intuitionistic fuzzy power Bonferroni mean (BM) aggregation operators by combining IFS and the linguistic terms set and applied them to MAGDM. Liu et al. [10] introduced the interval-valued...
intuitionistic fuzzy ordered weighted cosine similarity measure by combining the interval-valued intuitionistic fuzzy cosine similarity measure with the generalized ordered weighted averaging operator. Liu [11] used the Hamacher operations as basis to develop several new aggregation operators to fuse the interval-valued intuitionistic fuzzy information.

IFS is a powerful tool in decision-making. An extension of IFS, which is called the neutrosophic set [12], was introduced in 1999 to effectively address several real decision-making problems. In recent years, a few neutrosophic aggregation operators have been introduced [13–18]. A new extension of IFS, namely, the Pythagorean fuzzy set (PFS) [19], has been developed. The difference between PFS and IFS is that the square sum of the membership and nonmembership degrees is a maximum of one in PFS, whereas the sum of the membership and nonmembership degrees is a maximum of one in IFS. Several studies have been conducted on PFSs. Gou et al. [20] developed a few Pythagorean fuzzy functions and studied their fundamental properties. Zhang and Xu [21] introduced several operations for the Pythagorean fuzzy numbers (PFNs) and extended the technique for order preference by similarity to ideal solution (TOPSIS) method to solve MAGDM problems with Pythagorean fuzzy information. Several Pythagorean fuzzy aggregation operators have been introduced because aggregation operators are vital in decision-making [22].

2. Basic Concepts

This section reviews a few notions, such as IFS, PFS, and GBM.

2.1. IFS and PFS. In 1986, Atanassov [2] introduced IFS, which simultaneously has membership and nonmembership degrees.

Definition 1 (see [2]). Let X be an ordinary fixed set. An IFS A defined on X is expressed as follows:

\[ A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}, \] (1)

where \( \mu_A(x) \) and \( \nu_A(x) \) represent the membership and nonmembership degrees, respectively, thereby satisfying \( 0 \leq \mu_A(x) \leq 1, 0 \leq \nu_A(x) \leq 1, \) and \( \mu_A(x) + \nu_A(x) \leq 1 \). For convenience, the pair \((\mu, \nu)\) is called an intuitionistic fuzzy number (IFN) [39], in which \( \mu \in [0,1], \nu \in [0,1], \) and \( \mu + \nu \leq 1 \). The hesitancy degree is denoted by \( \pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \).

In 2014, Yager [19] introduced PFS, which is a generalization of IFS.

Definition 2 (see [19]). Let X be an ordinary fixed set; a PFS P defined on X is expressed as follows:

\[ P = \{(x, \mu_p(x), \nu_p(x)) \mid x \in X\}, \] (2)

where \( \mu_p(x) \) and \( \nu_p(x) \) are the membership and nonmembership degrees, respectively, thereby satisfying \( \mu_p(x) \in [0,1] \) and \( (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1 \). Thereafter, the indeterminacy degree is expressed by \( \pi_p(x) = \sqrt{1 - (\mu_p(x))^2 - (\nu_p(x))^2} \).

Peng and Yuan [27] introduced the comparison law for PFNs to compare two PFNs.

Definition 3 (see [27]). For any PFN \( p = (\mu, \nu) \), the score function of p is defined as \( s(p) = \mu^2 - \nu^2 \). For any two PFNs, such as \( p_1 = (\mu_1, \nu_1) \) and \( p_2 = (\mu_2, \nu_2) \), if \( s(p_1) > s(p_2) \), then \( p_1 > p_2 \); if \( s(p_1) = s(p_2) \), then \( p_1 = p_2 \).
Zhang and Xu [21] introduced a few operations for PFNs.

**Definition 4** (see [21]). Let \( p = (\mu, v) \), \( p_1 = (\mu_1, v_1) \), and \( p_2 = (\mu_2, v_2) \) be any three PFNs and \( \lambda \) be a positive real number. Thereafter,

1. \( p_1 \oplus p_2 = (\sqrt{\mu_1^2 + \mu_2^2 - \mu_1 \mu_2}, v_1 + v_2) \),
2. \( p_1 \odot p_2 = (\mu_1 \mu_2, \sqrt{v_1^2 + v_2^2 - v_1 v_2}) \),
3. \( \lambda p = (\sqrt{1 - (1 - \mu^2)^{\lambda}}, v^\lambda) \), and
4. \( p^\lambda = (\mu^\lambda, \sqrt{1 - (1 - v^2)^\lambda}) \).

2.2. GBM. Beliakov et al. [37] introduced GBM, which can consider the correlations of any three aggregated arguments because the traditional BM can only determine the interrelationship between any two arguments. Nevertheless, Xia et al. [38] highlighted that the GBM introduced by Beliakov et al. [37] has a drawback. Therefore, Xia et al. [38] introduced a new form of GBM. In the new GBM, the weights of the arguments are also considered.

**Definition 5** (see [38]). Let \( p, q, r \geq 0 \) and \( a_i \) \( (i = 1, 2, \ldots, n) \) be a collection of nonnegative crisp numbers with the weight vector being \( \omega = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_i \in [0, 1] \) \( (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^{n} w_i = 1 \). If

\[
\text{GWBM}^{p,q,r}(a_1, a_2, \ldots, a_n)
\]

\[
= \left( \sum_{i,j,k=1}^{n} w_i w_j w_k a_i^p a_j^q a_k^r \right)^{1/(p+q+r)},
\]

then GWBM\(^{p,q,r}\) is called GWBM.

### 3. The Generalized Pythagorean Fuzzy Weighted Bonferroni Mean

This section extends GWBM and GWBGM to fuse the Pythagorean fuzzy information and proposes several new Pythagorean fuzzy aggregation operators.

**Definition 7.** Let \( s, t, r > 0 \) and \( p_i = (\mu_i, v_i) \) \( (i = 1, 2, \ldots, n) \) be a collection of PFNs with their weight vector being \( \omega = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_i \in [0, 1] \) \( (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^{n} w_i = 1 \). If

\[
\text{GPFWM}^{s,t,r}(p_1, p_2, \ldots, p_n)
\]

\[
= \left( \sum_{i,j,k=1}^{n} w_i w_j w_k (p_i^s \odot p_j^t \odot p_k^r) \right)^{1/(s+t+r)},
\]

then GPFWM\(^{s,t,r}\) is called the generalized Pythagorean fuzzy weighted Bonferroni mean (GPFWM).

We can obtain the following theorem according to Definition 4.

**Theorem 8.** Let \( s, t, r > 0 \) and \( p_i = (\mu_i, v_i) \) \( (i = 1, 2, \ldots, n) \) be a collection of PFNs. The aggregated value by GPFWM is also a PFN and

\[
\text{GPFWM}^{s,t,r}(p_1, p_2, \ldots, p_n) = \left( \sqrt{1 - \prod_{i,j,k=1}^{n} (1 - \mu_i^2 \mu_j^2 \mu_k^2)} \right)^{1/(s+t+r)},
\]

where

\[
p_i^s = (\mu_i, \sqrt{1 - (1 - v_i^2)^s}),
\]

\[
p_j^t = (\mu_j, \sqrt{1 - (1 - v_j^2)^t}),
\]

\[
p_k^r = (\mu_k, \sqrt{1 - (1 - v_k^2)^r}).
\]

Thus,

\[
p_i^s \odot p_j^t \odot p_k^r.
\]
\[ = \left( \mu_i \mu_j \mu_k, \sqrt{1 - (1 - v_i^2)^r (1 - v_j^2)^r (1 - v_k^2)^r} \right). \tag{8}\]

Moreover,
\[
\bigoplus_{i,j,k=1}^{n} w_i w_j w_k \left( p_i^r \otimes p_j^r \otimes p_k^r \right)
\]
\[
= \left( \sqrt{1 - \prod_{i,j,k=1}^{n} \left( 1 - \mu_i^2 \mu_j^2 \mu_k^2 \right)^{w_i w_j w_k}} \right)^{1/(s+t+r)}.
\tag{10}\]

Therefore,
\[
\text{GPFWBM}^{s+t+r} (p_1, p_2, \ldots, p_n) = \left( \sqrt{1 - \prod_{i,j,k=1}^{n} \left( 1 - \mu_i^2 \mu_j^2 \mu_k^2 \right)^{w_i w_j w_k}} \right)^{1/(s+t+r)}.
\tag{11}\]

Hence, (6) is maintained.

Therefore,
\[
0 \leq \left( \sqrt{1 - \prod_{i,j,k=1}^{n} \left( 1 - \mu_i^2 \mu_j^2 \mu_k^2 \right)^{w_i w_j w_k}} \right)^{1/(s+t+r)} \leq 1,
\tag{12}\]
\[
0 \leq \sqrt{1 - \prod_{i,j,k=1}^{n} \left( 1 - (1 - v_i^2)^r (1 - v_j^2)^r (1 - v_k^2)^r \right)^{w_i w_j w_k}}^{1/(s+t+r)} \leq 1.
\]

Therefore,
\[
\left( \sqrt{1 - \prod_{i,j,k=1}^{n} \left( 1 - \mu_i^2 \mu_j^2 \mu_k^2 \right)^{w_i w_j w_k}} \right)^{1/(s+t+r)}
\]
Theorem 9 (idempotency). If \( p_i (i = 1, 2, \ldots, n) \) are equal, that is, \( p_i = p = (\mu, \nu) \), then

**GPFWBM**_{\mu,\nu} (p_1, p_2, \ldots, p_n) = p. 

Proof.

\[
\begin{align*}
\text{GPFWBM}_{\mu,\nu} (p_1, p_2, \ldots, p_n) & = \left( \bigoplus_{i,j,k=1}^n w_i w_j w_k (p_i \otimes p_j \otimes p_k) \right)^{1/(s+t+r)} \\
& = \left( \sum_{i,j,k=1}^n w_i w_j w_k p \right)^{1/(s+t+r)} \\
& = p.
\end{align*}
\]

Moreover, GPFWBM has the following properties.

Theorem 10 (monotonicity). Let \( p_i (i = 1, 2, \ldots, n) \) and \( q_i = (\mu_i, \nu_i) (i = 1, 2, \ldots, n) \) be two collections of PFNs. If \( \mu_p \leq \mu_q \) and \( \nu_p \geq \nu_q \), holds for all \( i \), then

**GPFWBM**_{\mu,\nu} (p_1, p_2, \ldots, p_n) \leq **GPFWBM**_{\mu,\nu} (q_1, q_2, \ldots, q_n). 

Proof. Let \( \text{GPFWBM}_{\mu,\nu} (p_1, p_2, \ldots, p_n) = (\mu_p, \nu_p) \) and \( \text{GPFWBM}_{\mu,\nu} (q_1, q_2, \ldots, q_n) = (\mu_q, \nu_q) \). Given that \( \mu_p \leq \mu_q \), we can obtain

\[
\begin{align*}
\mu_p^{2s} \mu_p^{2t} \mu_p^{2r} & \leq \mu_q^{2s} \mu_q^{2t} \mu_q^{2r}, \\
\left( 1 - \mu_p^{2s} \mu_p^{2t} \mu_p^{2r} \right) w_i w_j w_k & \geq \left( 1 - \mu_q^{2s} \mu_q^{2t} \mu_q^{2r} \right) w_i w_j w_k, \\
1 - \prod_{i,j,k=1}^n \left( 1 - \mu_p^{2s} \mu_p^{2t} \mu_p^{2r} \right) w_i w_j w_k & \leq 1 - \prod_{i,j,k=1}^n \left( 1 - \mu_q^{2s} \mu_q^{2t} \mu_q^{2r} \right) w_i w_j w_k.
\end{align*}
\]

Therefore,

\[
\left( \left( \prod_{i,j,k=1}^n \left( 1 - \mu_p^{2s} \mu_p^{2t} \mu_p^{2r} \right) w_i w_j w_k \right)^{1/(s+t+r)} \right)^2 \leq \left( \left( \prod_{i,j,k=1}^n \left( 1 - \mu_q^{2s} \mu_q^{2t} \mu_q^{2r} \right) w_i w_j w_k \right)^{1/(s+t+r)} \right)^2,
\]

which means \( \mu_p^2 \leq \mu_q^2 \). Similarly, we can obtain \( \nu_p^2 \geq \nu_q^2 \).

If \( \mu_p^2 < \mu_q^2 \) because \( \nu_p^2 \geq \nu_q^2 \), then GPFWBM_{\mu,\nu} (p_1, p_2, \ldots, p_n) < GPFWBM_{\mu,\nu} (q_1, q_2, \ldots, q_n);

If \( \mu_p^2 = \mu_q^2 \) and \( \nu_p^2 > \nu_q^2 \), then GPFWBM_{\mu,\nu} (p_1, p_2, \ldots, p_n) < GPFWBM_{\mu,\nu} (q_1, q_2, \ldots, q_n);

If \( \mu_p^2 = \mu_q^2 \) and \( \nu_p^2 = \nu_q^2 \), then GPFWBM_{\mu,\nu} (p_1, p_2, \ldots, p_n) = GPFWBM_{\mu,\nu} (q_1, q_2, \ldots, q_n).

Therefore, the proof of Theorem 10 is completed.

Theorem 11 (boundedness). Let \( p_i = (\mu_i, \nu_i) (i = 1, 2, \ldots, n) \) be a collection of PFNs. If \( p^+ = (\max(\mu_i), \min(\nu_i)) \) and \( p^- = (\min(\mu_i), \max(\nu_i)) \), then

\[
p^- \leq \text{GPFWBM}_{\mu,\nu} (p_1, p_2, \ldots, p_n) \leq p^+.
\]

Proof. From Theorem 9, we can obtain

\[
\text{GPFWBM}_{\mu,\nu} (p^+, p^+, \ldots, p^+) = p^+,
\]

\[
\text{GPFWBM}_{\mu,\nu} (p^+, p^+, \ldots, p^-) = p^-.
\]

From Theorem 10, we can obtain

\[
\text{GPFWBM}_{\mu,\nu} (p^-, p^-, \ldots, p^-) \leq \text{GPFWBM}_{\mu,\nu} (p_1, p_2, \ldots, p_n) \leq \text{GPFWBM}_{\mu,\nu} (p^+, p^+, \ldots, p^+).
\]
Therefore, \( p^- \leq \text{GPFWBM}^{s,t,r}(p_1, p_2, \ldots, p_n) \leq p^+ \).

Thereafter, we extend GWBGM to PFSs and introduce the generalized Pythagorean fuzzy weighted Bonferroni geometric mean (GPFWBGM).

Definition 12. Let \( s, t, r > 0 \) and \( p_i = (\mu_i, v_i) \) (\( i = 1, 2, \ldots, n \)) be a collection of PFNs with their weight vector being \( \omega = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \). If

\[
\text{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) = \left( 1 - \prod_{i,j,k=1}^{n} (1 - (1 - \mu_i^s)(1 - \mu_j^t)(1 - \mu_k^r)) \right)^{1/(s+t+r)}
\]

then \( \text{GPFWBGM}^{s,t,r} \) is called GPFWBGM.

We can obtain the following theorem based on Definition 4.

Theorem 13. Let \( s, t, r > 0 \) and \( p_i = (\mu_i, v_i) \) (\( i = 1, 2, \ldots, n \)) be a collection of PFNs. The aggregated value by GPFWBGM is also a PFN and

\[
\text{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) = \left( \prod_{i,j,k=1}^{n} (1 - (1 - \mu_i^2)(1 - \mu_j^2)(1 - \mu_k^2)) \right)^{1/(s+t+r)}
\]

Proof. Through Definition 4, we can obtain

\[
s p_i = \left( \sqrt{1 - (1 - \mu_i^2)}, v_i \right),
\]

\[
t p_j = \left( \sqrt{1 - (1 - \mu_j^2)}, v_j \right),
\]

\[
r p_k = \left( \sqrt{1 - (1 - \mu_k^2)}, v_k \right),
\]

\[
sp \oplus tp \oplus rp = \left( \sqrt{1 - (1 - \mu_i^2)(1 - \mu_j^2)(1 - \mu_k^2)}, v_i v_j v_k \right).
\]

Thereafter,

\[
\text{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) = \left( 1 - \prod_{i,j,k=1}^{n} (1 - (1 - \mu_i^2)(1 - \mu_j^2)(1 - \mu_k^2)) \right)^{1/(s+t+r)}
\]
Hence, (24) is maintained. Thereafter,

\[
0 \leq \sqrt{1 - \prod_{i,j,k=1}^{n} \left( 1 - (1 - \mu_i^2)^{s} (1 - \mu_j^2)^{t} (1 - \mu_k^2)^{r} \right)}^{1/(s+t+r)} w_i w_j w_k \leq 1, \tag{30}
\]

\[
0 \leq \left( 1 - \prod_{i,j,k=1}^{n} \left( 1 - V_i^{2s} V_j^{2t} V_k^{2r} \right)^{w_i w_j w_k} \right)^{1/(s+t+r)} \leq 1. \tag{31}
\]

Therefore,

\[
\left( \sqrt{1 - \prod_{i,j,k=1}^{n} \left( 1 - (1 - \mu_i^2)^{s} (1 - \mu_j^2)^{t} (1 - \mu_k^2)^{r} \right)}^{1/(s+t+r)} \right)^2 + \left( \prod_{i,j,k=1}^{n} \left( 1 - V_i^{2s} V_j^{2t} V_k^{2r} \right)^{w_i w_j w_k} \right)^{1/(s+t+r)} = 1
\]

\[
- \left( 1 - \prod_{i,j,k=1}^{n} \left( 1 - (1 - \mu_i^2)^{s} (1 - \mu_j^2)^{t} (1 - \mu_k^2)^{r} \right)^{w_i w_j w_k} \right)^{1/(s+t+r)} + \left( 1 - \prod_{i,j,k=1}^{n} \left( 1 - V_i^{2s} V_j^{2t} V_k^{2r} \right)^{w_i w_j w_k} \right)^{1/(s+t+r)} \leq 1
\]

\[
- \left( 1 - \prod_{i,j,k=1}^{n} \left( 1 - (1 - \mu_i^2)^{s} (1 - \mu_j^2)^{t} (1 - \mu_k^2)^{r} \right)^{w_i w_j w_k} \right)^{1/(s+t+r)} + \left( 1 - \prod_{i,j,k=1}^{n} \left( 1 - V_i^{2s} V_j^{2t} V_k^{2r} \right)^{w_i w_j w_k} \right)^{1/(s+t+r)} = 1, \tag{32}
\]

thereby completing the proof.

Similar to GPFWM, the GPFWBGM has the same properties. The proofs of these properties are similar to that of the properties of GPFWM. Accordingly, the proofs are omitted to save space.

**Theorem 14.** Let \( s, t, r > 0 \) and \( p_i = (\mu_i, V_i) (i = 1, 2, \ldots, n) \) be a collection of PFNs.

1. **Idempotency.** If \( p_i (i = 1, 2, \ldots, n) \) are equal, that is, \( p_i = p = (\mu, v) \), then

\[
\text{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) = p. \tag{33}
\]

2. **Monotonicity.** Let \( q_i = (\mu_i, V_i) (i = 1, 2, \ldots, n) \) be two collections of PFNs. If \( \mu_{p_i} \leq \mu_{p_i} \) and \( V_{p_i} \geq V_{q_i} \) holds for all \( i \), then

\[
\text{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) \leq \text{GPFWBGM}^{s,t,r}(q_1, q_2, \ldots, q_n). \tag{34}
\]

(3) **Boundedness.** If \( p^* = (\max(\mu_i), \min(V_i)) \) and \( p^- = (\min(\mu_i), \max(V_i)) \), then

\[
p^- \leq \text{GPFWBGM}^{s,t,r}(p_1, p_2, \ldots, p_n) \leq p^*. \tag{35}
\]

4. **Dual Generalized Pythagorean Fuzzy Weighted BM**

The primary advantage of BM is that it can determine the interrelationship between arguments. However, the traditional BM can only consider the correlations of any two aggregated arguments. Thereafter, Beliakov et al. [37] extended the traditional BM and introduced GBM, which can determine the correlations between any three aggregated arguments. Xia et al. [38] introduced GBWM and GBWGM given that the GBM introduced by Beliakov et al. [37] still has a few drawbacks. However, GBWM and GBWGM can only consider the interrelationship between any three aggregated arguments. We introduce a new generalization of the traditional BM because the correlations are ubiquitous.
among all arguments. The new generalization of the traditional BM is called the dual GBM (DGBM) to distinguish the new aggregation operator from the GBM introduced by Beliakov et al. [37] and Xia et al. [38]. Furthermore, we develop the dual generalized weighted BM (DGWBM) and dual generalized weighted Bonferroni geometric mean (DGWBGM) to consider the weights of the arguments.

Definition 15. Let \( a_i (i = 1, 2, \ldots, n) \) be a collection of nonnegative crisp numbers with the weight vector being \( w = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_i \in [0, 1] (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^n w_i = 1 \). If

\[
\text{DGWBGM}^R (a_1, a_2, \ldots, a_n)
\]

\[
= \left( \sum_{i,j=1}^n \left( \prod_{j=1}^n w_j a_i^j \right) \right)^{1/\sum_{i=1}^n r_j},
\]

where \( R = (r_1, r_2, \ldots, r_n)^T \) is the parameter vector with \( r_j \geq 0 (i = 1, 2, \ldots, n) \), then DGWBM \(^R\) is called DGWBM.

Several special cases can be obtained given the change of the parameter vector.

(1) If \( R = (\lambda, 0, 0, \ldots, 0) \), then we obtain

\[
\text{DGWBGM}^{(\lambda,0,0,\ldots,0)} (a_1, a_2, \ldots, a_n) = \left( \sum_{i=1}^n w_i a_i^1 \right)^{1/\lambda},
\]

which is the generalized weighted averaging operator.

(2) If \( R = (s, t, 0, 0, \ldots, 0) \), then we obtain

\[
\text{DGWBGM}^{(s,t,0,0,\ldots,0)} (a_1, a_2, \ldots, a_n) = \left( \sum_{i,j=1}^n w_j a_i^j a_j^2 \right)^{1/(s+t)},
\]

which is the weighted BM.

(3) If \( R = (s, t, r, 0, 0, \ldots, 0) \), then we obtain

\[
\text{DGWBGM}^{(s,t,r,0,0,\ldots,0)} (a_1, a_2, \ldots, a_n) = \left( \sum_{i,j,k=1}^n w_j w_k a_i^j a_j^k \right)^{1/(s+t+k)},
\]

which is the GWBM.

Definition 16. Let \( a_i (i = 1, 2, \ldots, n) \) be a collection of nonnegative crisp numbers with the weight vector being \( w = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_i \in [0, 1] (i = 1, 2, \ldots, n) \) and \( \sum_{i=1}^n w_i = 1 \). If

\[
\text{DGWBGM}^R (a_1, a_2, \ldots, a_n)
\]

\[
= \frac{1}{\sum_{j=1}^n r_j} \left( \prod_{i,j=1}^n \left( \sum_{j=1}^n r_j a_i^j \right)^{w_j} \right),
\]

where \( R = (r_1, r_2, \ldots, r_n)^T \) is the parameter vector with \( r_j \geq 0 (i = 1, 2, \ldots, n) \), then DGWBM \(^R\) is called DGWBM.

Similar to the DGWBM, we can consider some special cases given the change of the parameter vector.

(1) If \( R = (\lambda, 0, 0, \ldots, 0) \), then we obtain

\[
\text{DGWBGM}^{(\lambda,0,0,\ldots,0)} (a_1, a_2, \ldots, a_n) = \frac{1}{\lambda} \left( \prod_{j=1}^n \left( \lambda a_j^j \right)^{w_j} \right),
\]

which is the generalized weighted geometric averaging operator.

(2) If \( R = (s, t, 0, 0, \ldots, 0) \), then we obtain

\[
\text{DGWBGM}^{(s,t,0,0,\ldots,0)} (a_1, a_2, \ldots, a_n) = \frac{1}{s+t} \prod_{i,j=1}^n (s a_i + t a_j)^{w_i w_j},
\]

which is the weighted Bonferroni geometric mean.

(3) If \( R = (s, t, r, 0, 0, \ldots, 0) \), then we obtain

\[
\text{DGWBGM}^{(s,t,r,0,0,\ldots,0)} (a_1, a_2, \ldots, a_n) = \frac{1}{s+t+r} \prod_{i,j,k=1}^n (s a_i + t a_j + r a_k)^{w_i w_j w_k},
\]

which is the GWBM.

We extend DGWBM and DGWBGM to PFSs, as well as introduce several new aggregation operators for fusing the Pythagorean fuzzy information.

Definition 17. Let \( p_i = (\mu_i, v_i) (i = 1, 2, \ldots, n) \) be a collection of PFNs with their weight vector being \( w = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_i \in [0, 1] \) and \( \sum_{i=1}^n w_i = 1 \). Thereafter, the dual generalized Pythagorean fuzzy weighted Bonferroni mean (DGPFWBM) is defined as

\[
\text{DGPFWBM}^R (p_1, p_2, \ldots, p_n)
\]

\[
= \left( \bigoplus_{i=1, j=1, k=1}^n \left( \bigotimes_{j=1}^n w_j p^T_{i,j} \right)^{1/(s+t+k)} \right)^{1/\sum_{j=1}^n r_j},
\]

where \( R = (r_1, r_2, \ldots, r_n)^T \) is the parameter vector with \( r_i \geq 0 (i = 1, 2, \ldots, n) \).

We can derive the following theorem based on Definition 4.

Theorem 18. Let \( p_i = (\mu_i, v_i) (i = 1, 2, \ldots, n) \) be a collection of PFNs. Hence, the aggregated value by DGPFWBM is also PFN and

\[
\text{DGPFWBM}^R (p_1, p_2, \ldots, p_n)
\]

\[
= \left( \left[ 1 - \prod_{i,j=1, k=1}^n \left( 1 - \prod_{j=1}^n (1 - (1 - \mu_i^j) v_i^j) \right) \right]^{1/\sum_{j=1}^n r_j} \right) \frac{1}{1 - \prod_{i,j=1, k=1}^n \left( 1 - \prod_{j=1}^n (1 - (1 - v_i^j)^2) \right) \left[ 1 - \prod_{i,j=1, k=1}^n \left( 1 - \prod_{j=1}^n (1 - (1 - v_i^j)^2) \right) \right]^{1/\sum_{j=1}^n r_j}}.
\]
Proof. Through Definition 4, we obtain

\[
p_{i}^{r_{j}} = \left( \mu_{i}^{r_{j}} \sqrt{1 - (1 - \nu_{i}^{r_{j}})^{\gamma}} \right),
\]

\[
w_{i}^{r_{j}} p_{i}^{r_{j}} = \left( \sqrt{1 - (1 - \mu_{i}^{2r_{j}})^{w_{i}}}, \sqrt{1 - (1 - \nu_{i}^{r_{j}})^{w_{i}}} \right).
\]

Therefore,

\[
\bigotimes_{j=1}^{n} w_{i}^{r_{j}} p_{i}^{r_{j}} = \left( \prod_{j=1}^{n} \sqrt{1 - (1 - (1 - \mu_{i}^{2r_{j}})^{w_{i}})}, \prod_{j=1}^{n} \sqrt{1 - (1 - (1 - \nu_{i}^{r_{j}})^{w_{i}})} \right).
\]

Thus,

\[
\left( \prod_{j=1}^{n} \left( \sqrt{1 - \prod_{i=1}^{n} \left( (1 - (1 - \mu_{j}^{2r_{j}})^{w_{i}}) \right) w_{i}} \right) \right)^{1/\Sigma_{i=1}^{n} r_{j}}
\]

\[
\left( \prod_{j=1}^{n} \left( \sqrt{1 - \prod_{i=1}^{n} \left( (1 - (1 - \nu_{j}^{r_{j}})^{w_{i}}) \right) w_{i}} \right) \right)^{1/\Sigma_{i=1}^{n} r_{j}}
\]

Therefore,

\[
\bigoplus_{i,j,d_{i},d_{j}=1}^{n} \left( \bigotimes_{j=1}^{n} w_{i}^{r_{j}} p_{i}^{r_{j}} \right)
\]

\[
= \left( \prod_{j=1}^{n} \left( \sqrt{1 - \prod_{i=1}^{n} \left( (1 - (1 - \mu_{j}^{2r_{j}})^{w_{i}}) \right) w_{i}} \right) \right)^{1/\Sigma_{i=1}^{n} r_{j}}.
\]

Thus, (45) is maintained. Therefore,

\[
0 \leq \left( \prod_{j=1}^{n} \left( \sqrt{1 - \prod_{i=1}^{n} \left( (1 - (1 - \mu_{j}^{2r_{j}})^{w_{i}}) \right) w_{i}} \right) \right)^{1/\Sigma_{i=1}^{n} r_{j}} \leq 1.
\]

In addition,

\[
\left( \left( \prod_{j=1}^{n} \left( \sqrt{1 - \prod_{i=1}^{n} \left( (1 - (1 - \mu_{j}^{2r_{j}})^{w_{i}}) \right) w_{i}} \right) \right)^{1/\Sigma_{i=1}^{n} r_{j}} \right)^{2}
\]

\[
+ \left( \left( \prod_{j=1}^{n} \left( \sqrt{1 - \prod_{i=1}^{n} \left( (1 - (1 - \nu_{j}^{r_{j}})^{w_{i}}) \right) w_{i}} \right) \right)^{1/\Sigma_{i=1}^{n} r_{j}} \right)^{2}
\]

\[
\leq \left( \prod_{i,j,d_{i},d_{j}=1}^{n} \left( \sqrt{1 - \prod_{i=1}^{n} \left( (1 - (1 - \mu_{j}^{2r_{j}})^{w_{i}}) \right) w_{i}} \right) \right)^{1/\Sigma_{i=1}^{n} r_{j}}
\]

thereby completing the proof.

Moreover, DGPFWBM has the following properties.

Theorem 19 (monotonicity). Let \( p_{i} = (\mu_{p_{i}}, \nu_{p_{i}}) \) (\( i = 1, 2, \ldots, n \)) and \( q_{i} = (\mu_{q_{i}}, \nu_{q_{i}}) \) (\( i = 1, 2, \ldots, n \)) be two collections of PFNs. If \( \mu_{p_{i}} \leq \mu_{q_{i}} \) and \( \nu_{p_{i}} \geq \nu_{q_{i}} \) holds for all \( i \), then

\[
\text{DGPFWBM}^{\alpha} (p_{1}, p_{2}, \ldots, p_{n}) \leq \text{DGPFWBM}^{\alpha} (q_{1}, q_{2}, \ldots, q_{n}).
\]
Proof. Let $\text{DGPFWBM}^R(p_1, p_2, \ldots, p_n) = (\mu_p, v_p)$ and $\text{DGPFWBM}^R(q_1, q_2, \ldots, q_n) = (\mu_q, v_q)$. Given that $\mu_p \leq \mu_q$, we obtain

\[
1 - \left(1 - \mu_{p_i}\right)^{w_{ij}} \leq 1 - \left(1 - \mu_{q_i}\right)^{w_{ij}},
\]

\[
1 - \left(1 - \mu_{q_i}\right)^{w_{ij}} \leq \left(1 - \mu_{q_i}\right)^{w_{ij}}.
\]

Therefore,

\[
\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \mu_{p_i}\right)^{w_{ij}}\right)\right)^{1/\sum_{j=1}^{r_i}} \leq \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \mu_{q_i}\right)^{w_{ij}}\right)\right)^{1/\sum_{j=1}^{r_i}},
\]

\[
\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \mu_{q_i}\right)^{w_{ij}}\right)\right)^{2} \leq \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \mu_{q_i}\right)^{w_{ij}}\right)\right)^{2},
\]

\[
\leq \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \mu_{q_i}\right)^{w_{ij}}\right)\right)^{1/\sum_{j=1}^{r_i}},
\]

thus, $\mu_p^2 \leq \mu_q^2$. Similarly, we can obtain $v_p^2 \geq v_q^2$.

If $\mu_p^2 < \mu_q^2$ because $v_p^2 \geq v_q^2$, then

\[
\text{DGPFWBM}^R(p_1, p_2, \ldots, p_n) < \text{DGPFWBM}^R(q_1, q_2, \ldots, q_n);
\]

If $\mu_p^2 = \mu_q^2$ and $v_p^2 > v_q^2$, then $\text{DGPFWBM}^R(p_1, p_2, \ldots, p_n) < \text{DGPFWBM}^R(q_1, q_2, \ldots, q_n)$;

If $\mu_p^2 = \mu_q^2$ and $v_p^2 = v_q^2$, then $\text{DGPFWBM}^R(p_1, p_2, \ldots, p_n) = \text{DGPFWBM}^R(q_1, q_2, \ldots, q_n)$.

Therefore, $\text{DGPFWBM}^R(p_1, p_2, \ldots, p_n) \leq \text{DGPFWBM}^R(q_1, q_2, \ldots, q_n)$ and the proof of Theorem 19 is completed. \qed

Theorem 20 (boundedness). Let $p_i = (\mu_{p_i}, v_{p_i})$ ($i = 1, 2, \ldots, n$) be a collection of PFNs. If $p^+ = (\max_i(\mu_{i}), \min_i(v_{i}))$ and $p^- = (\min_i(\mu_{i}), \max_i(v_{i}))$, then

\[
\text{DGPFWBM}^R(p^+, p^-, \ldots, p^-) \leq \text{DGPFWBM}^R(p_1, p_2, \ldots, p_n)
\]

\[
\leq \text{DGPFWBM}^R(p^+, p^+, \ldots, p^+).
\]

Proof. According to Theorem 18, we can obtain

\[
\text{DGPFWBM}^R(p^+, p^-, \ldots, p^-) = \left(\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \min \mu_{i}\right)^{2} w_{ij}\right)\right)^{1/\sum_{j=1}^{r_i}}\right),
\]

\[
\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \max v_{i}\right)^{2} w_{ij}\right)\right)^{1/\sum_{j=1}^{r_i}}.
\]

\[
\text{DGPFWBM}^R(p_1, p_2, \ldots, p_n) = \left(\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \mu_{p_i}\right)^{2} w_{ij}\right)\right)^{1/\sum_{j=1}^{r_i}}\right),
\]

\[
\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \mu_{q_i}\right)^{2} w_{ij}\right)\right)^{1/\sum_{j=1}^{r_i}}.
\]
According to Theorem 19, we can obtain
\[
\text{DGPFWBGM}^R(p^*, p^*, \ldots, p^*) \leq \text{DGPFWBGM}^R(p_1, p_2, \ldots, p_n) \leq \text{DGPFWBGM}^R(p^+, p^+, \ldots, p^+).
\]

We extend DGBWGM to PFSs and introduce the dual generalized Pythagorean fuzzy weighted Bonferroni geometric mean (DGPFWBGM) operator.

**Definition 21.** Let \( p_i = (\mu_i, \nu_i) (i = 1, 2, \ldots, n) \) be a collection of PFNs with their weight vector being \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \), thereby satisfying \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{n} \omega_i = 1 \). If
\[
\text{DGPFWBGM}^R(p_1, p_2, \ldots, p_n) = \frac{1}{\sum_{j=1}^{n} r_j} \left( \prod_{i=1, j_2, \ldots, j_n=1}^{n} \left( \sqrt[n]{\prod_{j=1}^{n} (r_j p_{i,j})} \right)^{\omega_i} \right),
\]

where \( R = (r_1, r_2, \ldots, r_n)^T \) is the parameter vector with \( r_i \geq 0 \) (\( i = 1, 2, \ldots, n \)); then \( \text{DGPFWBGM}^R \) is called the DGPFWBGM operator. We obtain the following theorem based on Definition 4.

**Theorem 22.** Let \( p_i = (\mu_i, \nu_i) (i = 1, 2, \ldots, n) \) be a collection of PFNs. The aggregated value by the DGPFWBGM operator is also PFN and

\[
\sqrt{1 - \left( 1 - \prod_{i,j_2,\ldots,j_n=1}^{n} \left( 1 - \left( 1 - \left( 1 - \max_{j=1}^{n} \mu_{i,j} \right)^{2r_j} \omega_j \right) \right) \right)^{\frac{1}{\sum_{j=1}^{n} r_j}}},
\]

The proof of Theorem 22 is similar to that of Theorem 18; thus, such proof is omitted to save space.

Similar to DGPFWBGM, we can obtain the following properties of DGPFWBGM. The proofs of these properties are likewise omitted to save space.

**Theorem 23.** Let \( p_i = (\mu_{p_i}, \nu_{p_i}) (i = 1, 2, \ldots, n) \) be a collection of PFNs.

1. **Monotonicity.** Let \( q_i = (\mu_{q_i}, \nu_{q_i}) (i = 1, 2, \ldots, n) \) be a collection of PFNs. If \( \mu_{p_i} \leq \mu_{q_i} \) and \( \nu_{p_i} \geq \nu_{q_i} \) holds for all \( i \), then

\[
\text{DGPFWBGM}^R(p_1, p_2, \ldots, p_n) \leq \text{DGPFWBGM}^R(q_1, q_2, \ldots, q_n).
\]

2. **Boundedness.** If \( p^+ = (\max_i(\mu_i), \min_i(\nu_i)) \) and \( p^- = (\min_i(\mu_i), \max_i(\nu_i)) \), then

\[
\text{DGPFWBGM}^R(p^+, p^+, \ldots, p^+) \leq \text{DGPFWBGM}^R(p_1, p_2, \ldots, p_n) \leq \text{DGPFWBGM}^R(p^+, p^-, \ldots, p^-).
\]
5. **Novel Approach to MAGDM with Pythagorean Fuzzy Information**

This section introduces a novel approach to MAGDM under the Pythagorean fuzzy environment. A typical MAGDM problem with the Pythagorean fuzzy information can be described as follows. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of alternatives and \( G = \{G_1, G_2, \ldots, G_n\} \) be a set of attributes with the weight vector being \( w = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_j \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \). Several decision makers are organized to decide over alternatives. For the attribute \( G_j \) \( (j = 1, 2, \ldots, n) \) of alternative \( x_i \) \( (i = 1, 2, \ldots, m) \), the decision makers are required to use PFNs to express their preference assessments. Moreover, a Pythagorean fuzzy decision matrix can be obtained by \( P = (p_{ij})_{m \times n} \). A novel approach based on the dual generalized Pythagorean fuzzy BM aggregation operators is introduced to solve this problem.

**Step 1.** The two types of attributes are benefit and cost attributes. Xu and Hu [40] introduced the normalization regulation for intuitionistic fuzzy decision matrix, which can be extended to the Pythagorean fuzzy decision matrix. Therefore, the decision matrix should be normalized by

\[
 p_{ij} = \begin{cases} 
 (\mu_{ij}, v_{ij}), & G_j \in I_1, \\
 (v_{ij}, \mu_{ij}), & G_j \in I_2, 
\end{cases} 
\]

where \( I_1 \) and \( I_2 \) represent the benefit and cost attributes, respectively. Thereafter, a normalized decision matrix can be obtained.

**Step 2.** For the alternative \( x_i \) \( (i = 1, 2, \ldots, m) \), we utilize the DGPFWBM operator

\[
 p_i = \text{DGPFWBM}^{R}(p_{i1}, p_{i2}, \ldots, p_{in}) = \left( \left( 1 - \prod_{i_1,j_2,\ldots,j_m=1}^{n} \left( 1 - (1 - (1 - \mu_{ij})^2)^{v_{ij}} \right) ^{w_{ij}} \right)^{1/\sum_{j=1}^{n} r_{ij}}, \right)
\]

or the DGPFWBGM operator

\[
 p_i = \text{DGPFWBGM}^{R}(p_{i1}, p_{i2}, \ldots, p_{in}) = \left( \left( 1 - \prod_{i_1,j_2,\ldots,j_m=1}^{n} \left( 1 - (1 - (1 - \mu_{ij})^2)^{v_{ij}} \right) ^{w_{ij}} \right)^{1/\sum_{j=1}^{n} r_{ij}}, \right)
\]

which can be extended to the Pythagorean fuzzy environment. A typical MAGDM problem with the Pythagorean fuzzy information can be described as follows. Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of alternatives and \( G = \{G_1, G_2, \ldots, G_n\} \) be a set of attributes with the weight vector being \( w = (w_1, w_2, \ldots, w_n)^T \), thereby satisfying \( w_j \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \). Several decision makers are organized to decide over alternatives. For the attribute \( G_j \) \( (j = 1, 2, \ldots, n) \) of alternative \( x_i \) \( (i = 1, 2, \ldots, m) \), the decision makers are required to use PFNs to express their preference assessments. Moreover, a Pythagorean fuzzy decision matrix can be obtained by \( P = (p_{ij})_{m \times n} \). A novel approach based on the dual generalized Pythagorean fuzzy BM aggregation operators is introduced to solve this problem.

CAAT organizes several experts to form a committee that will evaluate four major domestic airlines, namely, UNI Air (\( x_1 \)), Transasia (\( x_2 \)), Mandarin (\( x_3 \)), and Daily Air (\( x_4 \)). The experts are required to evaluate the four airlines using the following four aspects: (1) booking and ticketing service (\( G_1 \)), (2) check-in and boarding process (\( G_2 \)), (3) cabin service (\( G_3 \)), and (4) responsiveness (\( G_4 \)). The weight vector of the attributes is \( w = (0.15, 0.25, 0.35, 0.25)^T \). For the attribute \( G_j \) \( (j = 1, 2, 3, 4) \) of airline \( x_i \) \( (i = 1, 2, 3, 4) \), the experts are required to utilize the PFN \( p_{ij} = (\mu_{ij}, v_{ij}) \) to express their assessments. Moreover, a Pythagorean fuzzy decision matrix \( P = (p_{ij})_{4 \times 4} \) \( (i, j = 1, 2, 3, 4) \) can be obtained (see Table 1). We utilize the newly introduced decision-making approach to solve this problem.

6. **Numerical Example**

We provide a numerical example adopted from [24] and a few comparative analyses to illustrate the validity of the new approach. The Civil Aviation Administration of Taiwan (CAAT) aims to determine the best airline in Taiwan. Hence,
Step 3. Therefore, let \( R = (1, 1, 1, 1) \) aggregate the attribute values. Therefore, we can obtain a set of overall values. In this step, let \( R = (1, 1, 1, 1) \).

\[
\begin{align*}
    p_1 &= (0.7543, 0.2324), \\
    p_2 &= (0.8424, 0.0004), \\
    p_3 &= (0.7664, 0.0211), \\
    p_4 &= (0.8387, 0.0131).
\end{align*}
\]

Step 3. The scores of \( p_i \) \((i = 1, 2, 3, 4)\) are calculated based on Definition 3 to obtain \( s(p_1) = 0.5150, s(p_2) = 0.7103, \) and \( s(p_4) = 0.7032 \). Therefore, the rank of the overall values is \( p_2 > p_4 > p_3 > p_1 \).

Step 4. The alternative \( x_i \) \((i = 1, 2, 3, 4)\) is ranked based on the rank of \( p_i \) \((i = 1, 2, 3, 4)\) to obtain \( x_2 > x_4 > x_3 > x_1 \). Therefore, \( x_2 \) is the best alternative. That is, Daily Air is the best airline in Taiwan.

6.2. Influence of the Parameter Vector \( R \) on the Final Result.

The prominent characteristic of the DGPFWBM and the DGPFWBGM operators is that they can consider the interrelationship among all PFNs. The parameter vector \( R \) plays a crucial role in the final result. We may obtain a different ranking result by assigning different values to \( R \). We set a different weight vector \( R \) and discuss the ranking results. Tables 3 and 4 provide further details.

Tables 3 and 4 show that the different ranking results can be obtained by assigning different values in the parameter vector \( R \). Therefore, the DGPFWBM and the DGPFWBGM operators are considerably flexible by using a parameter vector. Table 3 shows that the best alternatives are consistently the same, although the ranking results are different by using different parameter vectors. That is, the final results become increasingly objective by considering the interrelationship among all the attribute values. The best alternative is consistently \( x_2 \) regardless of the parameter vector. Table 4 shows that the ranking results increase and become steady with the increase of values in parameter vector \( R \). These features of the DGPFWBM and DGPFWBGM operators are crucial in real decision-making problems. Accordingly, we can assign a weight vector with large values to the DGPFWBM and the DGPFWBGM operators for steady and reliable final results.

7. Conclusions

PFS is a powerful tool for expressing the fuzziness, uncertainty, and hesitancy of decision makers. This research extends the GWBM and GWBGM operators to the Pythagorean fuzzy environment, as well as introduces the GPFWBM and the GPFWBGM operators. First, we extend the GWBM and GWBGM operators, as well as develop the DGPFWBM and DGPFWBGM operators, because the two operators can only consider the interrelationship between any two IFNs. The prominent advantage of the DGPFWBM and DGPFWBGM operators is that they can consider the interrelationship among all the arguments being fused. Moreover, we extend the GWBGM and DGWBM operators to the Pythagorean fuzzy environment, as well as develop the DGPFWBGM operators. Thereafter, the new operators are used as bases to propose a novel approach to MAGDM with Pythagorean fuzzy information. We apply the new approach to illustrate its validity to the problem of selecting the best airline. Moreover, we investigate the influence of the parameter vector \( R \) on the ranking results to show the advantages of the new approach. The limitation of the DGPFWFBM and DGPFWFBGM operators is that the calculation process may be more complicated than the existing Pythagorean fuzzy aggregation operators as they can consider the interrelationship between all PFNs. Therefore, the calculation process of the proposed method to MAGDM is little more complicated than existing methods. The focus of future research is to reduce complexity of the calculation of the proposed method.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
Complexity

Table 2: Comparison of the different operators.

<table>
<thead>
<tr>
<th>Aggregation operators</th>
<th>Whether the operator can capture the interrelationship between any two PFNs</th>
<th>Whether the operator can capture the interrelationship between any three PFNs</th>
<th>Whether the operator can capture the interrelationship among all PFNs</th>
<th>Whether a parameter vector exists to manipulate the ranking results</th>
</tr>
</thead>
<tbody>
<tr>
<td>PFWA [23]</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
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<td>No</td>
<td>No</td>
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<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Table 3: Ranking results by assigning different values to parameter vector $R$ in the DGPFWBM operator.

<table>
<thead>
<tr>
<th>Parameter $R$</th>
<th>Scores of overall values</th>
<th>Ranking results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1, 1, 1)$</td>
<td>$s(p_1) = 0.5150 \ s(p_2) = 0.7103 \ s(p_3) = 0.5869 \ s(p_4) = 0.7032$</td>
<td>$x_2 &gt; x_4 &gt; x_3 &gt; x_1$</td>
</tr>
<tr>
<td>$(2, 2, 2, 2)$</td>
<td>$s(p_1) = 0.3742 \ s(p_2) = 0.6487 \ s(p_3) = 0.4906 \ s(p_4) = 0.5944$</td>
<td>$x_2 &gt; x_4 &gt; x_3 &gt; x_1$</td>
</tr>
<tr>
<td>$(3, 3, 3, 3)$</td>
<td>$s(p_1) = 0.3657 \ s(p_2) = 0.6517 \ s(p_3) = 0.4572 \ s(p_4) = 0.5584$</td>
<td>$x_2 &gt; x_4 &gt; x_3 &gt; x_1$</td>
</tr>
<tr>
<td>$(4, 4, 4, 4)$</td>
<td>$s(p_1) = 0.3872 \ s(p_2) = 0.6613 \ s(p_3) = 0.4417 \ s(p_4) = 0.5411$</td>
<td>$x_2 &gt; x_4 &gt; x_3 &gt; x_1$</td>
</tr>
<tr>
<td>$(5, 5, 5, 5)$</td>
<td>$s(p_1) = 0.4148 \ s(p_2) = 0.6743 \ s(p_3) = 0.4361 \ s(p_4) = 0.5330$</td>
<td>$x_2 &gt; x_4 &gt; x_3 &gt; x_1$</td>
</tr>
<tr>
<td>$(8, 8, 8, 8)$</td>
<td>$s(p_1) = 0.4870 \ s(p_2) = 0.6904 \ s(p_3) = 0.4435 \ s(p_4) = 0.5303$</td>
<td>$x_2 &gt; x_4 &gt; x_3 &gt; x_1$</td>
</tr>
<tr>
<td>$(10, 10, 10, 10)$</td>
<td>$s(p_1) = 0.5215 \ s(p_2) = 0.7004 \ s(p_3) = 0.4550 \ s(p_4) = 0.5339$</td>
<td>$x_2 &gt; x_4 &gt; x_3 &gt; x_1$</td>
</tr>
</tbody>
</table>

Table 4: Ranking results by assigning different values to parameter vector $R$ in the DGPFWBGM operator.

<table>
<thead>
<tr>
<th>Parameter $R$</th>
<th>Scores of overall values</th>
<th>Ranking results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1, 1, 1)$</td>
<td>$s(p_1) = -0.9636 \ s(p_2) = -0.6197$</td>
<td>$x_4 &gt; x_3 &gt; x_2 &gt; x_1$</td>
</tr>
<tr>
<td>$(2, 2, 2, 2)$</td>
<td>$s(p_1) = -0.7322 \ s(p_2) = -0.4985$</td>
<td>$x_4 &gt; x_3 &gt; x_2 &gt; x_1$</td>
</tr>
<tr>
<td>$(3, 3, 3, 3)$</td>
<td>$s(p_1) = -0.5973 \ s(p_2) = -0.4662$</td>
<td>$x_4 &gt; x_3 &gt; x_2 &gt; x_1$</td>
</tr>
<tr>
<td>$(4, 4, 4, 4)$</td>
<td>$s(p_1) = -0.5155 \ s(p_2) = -0.4324$</td>
<td>$x_4 &gt; x_3 &gt; x_2 &gt; x_1$</td>
</tr>
<tr>
<td>$(5, 5, 5, 5)$</td>
<td>$s(p_1) = -0.4705 \ s(p_2) = -0.4068$</td>
<td>$x_4 &gt; x_3 &gt; x_2 &gt; x_1$</td>
</tr>
<tr>
<td>$(8, 8, 8, 8)$</td>
<td>$s(p_1) = -0.4178 \ s(p_2) = -0.3663$</td>
<td>$x_4 &gt; x_3 &gt; x_2 &gt; x_1$</td>
</tr>
<tr>
<td>$(10, 10, 10, 10)$</td>
<td>$s(p_1) = -0.4056 \ s(p_2) = -0.3538$</td>
<td>$x_4 &gt; x_3 &gt; x_2 &gt; x_1$</td>
</tr>
</tbody>
</table>
Acknowledgments

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References


