

## Research Article

# Implications for Firms with Limited Information to Take Advantage of Reference Price Effect in Competitive Settings

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Received 4 November 2016; Revised 15 January 2017; Accepted 7 February 2017; Published 13 June 2017

Academic Editor: Mehrbakhsh Nilashi

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This paper studies internal reference price effects when competitive firms face reference price effects and make decisions based on partial information, where their decision-making mechanism is modeled by a dynamic adjustment process. It is shown that the evolution of this dynamic adjustment goes to stabilization if both adjustment speeds are small and the complexity of this evolution increases in adjustment speeds. It is proved that the necessary condition for flip bifurcation or Neimark-Sacker bifurcation will occur with the increase of adjustment speed in two special cases. What is more, numerical simulations show that these bifurcations do occur. Then, the impacts of parameters on stability and profits are investigated and some management insights for firms with limited information to take advantage of reference price effects are provided.

## 1. Introduction

Reference price plays an important role in consumer purchase decisions. In fact, consumers' past experiences contribute to the building of reference price. Consumers will compare this reference price with the current price and decide the current demand. The theoretical basis for reference price effect is the adaptation-level theory [1]. Adaptation-level theory holds that the expectation-based reference price is the adaptation level, which the current price is judged against [2]. In particular, if reference price is higher than the current price, the current price will be perceived as "low" and then consumers will be prone to buy more and demand increases. On the contrary, if reference price is lower than the current price, the current price will be perceived as "high" and then demand decreases. Nasiry and Popescu [3] provided an example to illustrate this effect. Apple used to sell digital songs at a low price (99 cents) for a long time. When the company raised these prices to \$1.29, the drop in sales was higher than expected. These lost sales were explained by the fact that the new price was perceived as very high when compared to the benchmark (99 cents), so demand dropped significantly.

Due to the significant effect of reference price, researchers have provided many construction models for reference price.

Kalyanaram and Winer [4] divided reference price into two types: internal reference price and external reference price. Internal reference price of one type of goods is related to its historic prices [5]; customers' memories of historic prices form the benchmark (internal reference price). External reference price refers to an external standard, which refers to contextual factors, such as prices of competing products and price of the same product in a different location [6]. In addition to purchase frequency, brand loyalty, and involvement levels, Chen [7] demonstrated that self-construal influences the impact of reference cues. Besides the above factors, advertising is often used to influence reference price as an operational management tool [8, 9].

Based on this comprehension, scholars have provided various implications for firms to take advantage of reference price effect [10–17]. These studies provided critical implications for firms to maximize their revenues via making use of reference price effect. However, in these models, firms are assumed to be fully rational; that is, firms not only master all necessary information but also choose the best solution determinately. In fact, there are many decision-makers, who face complex environments, and full information is very costly, even impossible, to master for them. Then, some bounded-rationality based decision-making mechanisms are adopted

by these decision-makers [18, 19]. However, implications for these firms to take advantage of reference price effect are still rare.

In this paper, we try to explore some useful implications for firms with limited information to take advantage of reference price effect in competitive settings. We consider a market with two competitive firms; they sell similar products to customers and compete in price. As the full market information is very difficult and costly to master, they make decisions based on partial information. We assume firms make decisions based on marginal profit, which can be estimated by market experiments. When marginal profit is positive, the firm should increase its product's price. Conversely, negative marginal profit leads to the decrease of price. We assume that firms estimate the marginal profits by a weighted mean value of past few prices rather than instant price to smooth possible deviations. Although we omit estimation deviation in the model, we also use this mechanism for the following reasons. First, weighted moving average is a widely used estimation method [20, 21]. Second, as will be shown in the following text, adopting an appropriate weight parameter, which belongs to  $(0.56, 0.75)$ , always benefits the stability of the system. Based on the nonlinear dynamic system theory, we study the evolution characteristics of this market and investigate the impacts of relevant factors to explore some managerial implications.

This study provides the following implications for firms with limited information to take advantage of reference price effect in competitive settings. First, there is a stability condition which must be satisfied to keep the dynamic system asymptotically stable. And firms get their maximal profits when the system is stable; the period of pricing cycle when the system loses its stability depends on the weight parameter and customer perception coefficient. Second, when customers use the most recent price as reference price, there is a threshold, which depends on customer perception coefficient and belongs to  $(0.56, 0.75)$ . The stable region of adjustment speeds increases in weight parameter when the weight parameter is less than this threshold and decreases in weight parameter when the weight parameter is greater than this threshold. What is more, if firms use instant prices to estimate marginal profits, the stable region of adjustment speeds increases in memory parameter. Third, a higher initial reference price relative to the equilibrium price generally benefits both firms when the adjustment process is asymptotically stable. However, this does not hold when the system is unstable, and firms should not improve initial reference prices blindly in this situation.

This paper is organized as follows. Section 2 introduces the related literature. Section 3 presents the dynamic game model, where firms face reference price effects and make decisions based on partial information. Management insights as well as main results from theoretical analysis and numerical observation are provided in Section 4. Finally, conclusions are given in Section 5.

## 2. Related Literature

This paper is related to the literature on dynamic pricing with reference price effect and implications on taking advantage of

reference price effect. The earliest study analyzing the impact of reference prices on pricing strategies was [10]. The author showed that a firm can increase its profit by considering reference price effects. After that, the impacts of reference price effects in various settings have been investigated. Fibich et al. [11] studied a continuous time and asymmetric reference effects model; they showed that the optimal price stabilizes at a steady-state price, which is below the optimal price without reference price effects. Popescu and Wu [12] studied the dynamic pricing problem of a monopolist firm under a discrete time model; they showed that if consumers are loss averse, optimal prices will converge to a constant steady-state price, otherwise, optimal policy cycles. Martín-Herrán and Taboubi [13] investigated whether the results on the efficiency of price coordination in bilateral monopolies still hold when the reference price effect is taken into account. They showed that, for some values of the initial reference price, there is a time interval where channel decentralization performs better than coordination. Zhang et al. [14] studied the dynamic pricing policy for supply chain firms under integrated channels and decentralized channels; they showed that the initial reference price of consumers plays an important role in determining whether price skimming policy or price penetration policy is more profitable. What is more, key parameters effects are also provided. Lin [16] studied the impact of price promotion in a supply chain with reference price effect. It is shown that the reference price effect could mitigate double marginalization effect and improve channel efficiency. This helps us to understand the fact that firms encourage consumers to recall reference price. Hu et al. [22] considered a gain-seeking reference price effect model and identified conditions on parameters such that a high-low pricing strategy is optimal. Besides these studies, some valuable insights are also reached by integrating pricing with other decision variables, such as advertising strategy [8, 23], replenishment policies [24, 25], and preservation technology investment [26]. Our study is significantly different from aforementioned papers in that we consider firms that only master partial information. We model the decision-making mechanism of these firms by a dynamic adjustment based on partial information, and then we explore managerial implications for these firms to take advantage of the reference price effect.

Our work is also closely related to the research on the decision-making mechanism of bounded rational players [27–29]. Among this stream of literature, these studies that investigate the dynamic adjustment of players with limited information in competitive settings are most close to our study. Wu et al. [30], Ma and Xie [31], and Elsadany [32] studied the evolution processes when the bounded rational players use naive expectation. Considering that players are heterogeneous, some scholars studied the influence of players' different adjustment mechanisms on the dynamics of game model (see [33–37]). Some papers extended the decision mechanisms by considering that firms may refer to more than one period when they make decisions, and then delay decision was utilized (see [38–41]). The impacts of players' exact estimation on the evolutions of the dynamic system were considered in [42, 43]. Besides the above researches,

Ma and Guo [44] studied the dynamic competition when one player adopts “one-period look-ahead” behavior. Ahmed et al. [45] and Agliari et al. [46] investigated the dynamic game process when bounded rational firms apply the gradient adjustment mechanism. Although the reference price effect does exist and play an important role in consumer purchase decisions, as far as we know, implication on how to take advantage of reference price effect in competitive settings has never been investigated for decision-makers with partial information, so we build this model to explore some useful implications.

### 3. Model

In this section, we set up a dynamic system model to represent the evolution process of prices and reference prices. We assume there are two firms (two players) in the market, labeled by  $i = 1, 2$ , respectively (we use  $i$  to denote both 1 and 2 unless otherwise specified). They offer similar products (firm  $i$  offers product  $i$ ) and form a price game. Firms make decisions in discrete time period  $t$ , ( $t = 1, 2, 3, \dots$ ). Their demand functions in the absence of reference price effect are

$$\widehat{D}_i(t) = a_i - b_i p_i(t) + d_i p_{3-i}(t), \quad (1)$$

where  $a_i, b_i, d_i > 0$ , and  $b_i > d_i$ . Traditionally,  $a_i$  is viewed as the size of the market base of product  $i$ ,  $b_i$  denotes the influence of its own price, and  $d_i$  denotes the influence of the substitute's price. We consider fixed unit-production cost  $c_i$  and assume that  $a_i - b_i c_i + d_i c_{3-i} > 0$ .

Reference price affects customer demand via the magnitude of perceived “gain” or “loss” relative to the reference point [25]. In this paper, we adopt the linear symmetric reference price effect model:  $R(r - p, r) = k(r - p)$ . Then, the demand functions in the presence of reference price effects are modeled as

$$D_i(t) = a_i - b_i p_i + d_i p_{3-i} + k_i (r_i - p_i), \quad (2)$$

where  $k_i$  is used to represent the effect of the perceived “gain” or “loss.” We follow the notion given in [47] that the effect of unit perceived “gain” or “loss” on demand is lower than that of unit real “gain” or “loss,” so we define  $k_i = \theta b_i$  for simplicity, where  $\theta \in [0, 1)$ . Profit functions are

$$\pi_i(t) = (p_i(t) - c_i) (a_i - b_i p_i + d_i p_{3-i} + \theta b_i (r_i - p_i)). \quad (3)$$

To model the evolution of reference prices, we adopt an exponential smoothing model. The exponential smoothing model, stemming from the adaptive expectation model, is the most commonly used updated model for reference price (see [12, 13, 25, 48]):

$$r_i(t+1) = \alpha r_i(t) + (1 - \alpha) p_i(t), \quad (4)$$

where  $r_i(t)$  is the reference price at time  $t$ ,  $p_i(t)$  is the actual price at time  $t$ , and  $\alpha \in [0, 1)$  denotes the memory parameter. Memory parameter  $\alpha$  captures the strength of past prices which the reference depends on.  $\alpha$  is high when customers have a long memory.  $\alpha$  is small when customers pay less

attention to past prices. If  $\alpha = 0$ , the reference price becomes the last period's price; this means customers only have one-period memory.

Given the initial reference prices, the long-term profit maximization problem of each firm is

$$V_i = \max_{p_i(t)} \sum_{t=1}^{\infty} \rho^{t-1} \pi_i(r_i(t), p_i(t), p_j(t)), \quad (5)$$

where there is a discount rate of future profit when  $\rho \in [0, 1)$ ; discount rate is not considered when  $\rho = 1$ . The updated law of reference prices is (4).  $p_i(t)$  and  $p_j(t)$  are mutually dependent.

Solving the optimal strategy for each firm not only is very complicated but also needs complete information about the whole market. This condition is very tough in the real market. Usually, firms only master limited information and have to make decisions based on limited information. To model the decision-making processes of firms with limited information, in this paper, we adopt the widely used gradient mechanism, which provides a good approximation to the practical adjustment when only marginal profit is available. The gradient mechanism assumes that each firm gets its marginal profit with respect to its price via market experiments in each period:

$$\begin{aligned} \frac{\partial \pi_i(p_1(t), p_2(t))}{\partial p_i(t)} &= a_i - 2b_i p_i(t) (1 + \theta) + d_i p_{3-i}(t) \\ &\quad + \theta b_i r_i(t) + (1 + \theta) b_i c_i. \end{aligned} \quad (6)$$

Based on marginal profit, each makes a myopic decision to maximize its profit in the next period. If this marginal profit is positive (negative), firm  $i$  will increase (decrease) price  $i$  in the next period (see [31, 32, 49]):

$$\begin{aligned} p_i(t+1) &= p_i(t) + \alpha_i p_i(t) \frac{\partial \pi_i(p_1(t), p_2(t))}{\partial p_i(t)} = p_i(t) \\ &\quad + \alpha_i p_i(t) [a_i - 2b_i p_i(t) (1 + \theta) + d_i p_{3-i}(t) \\ &\quad + \theta b_i r_i(t) + (1 + \theta) b_i c_i], \end{aligned} \quad (7)$$

where  $\alpha_i > 0$  is the speed of adjustment.

We notice that firms may estimate the marginal profit by a weighted mean value of past few prices rather than instant price to smooth possible deviations (also cited as delay decision; see [38–41]). We include this mechanism here and consider one-period delay decision for simplification:

$$\begin{aligned} p_i(t+1) &= p_i(t) + \alpha_i p_i(t) \frac{\partial \pi_i(p_1^D(t), p_2^D(t))}{\partial p_i(t)} \\ &= p_i(t) + \alpha_i p_i(t) \{a_i \\ &\quad - 2b_i (1 + \theta) [w_i p_i(t) + (1 - w_i) p_i(t-1)] \\ &\quad + d_i [w_{3-i} p_{3-i}(t) + (1 - w_{3-i}) p_{3-i}(t-1)] \\ &\quad + \theta b_i r_i(t) + (1 + \theta) b_i c_i\}, \end{aligned} \quad (8)$$

where  $p_i^D(t) = w_i p_i(t) + (1-w_i)p_i(t-1)$  and  $w_i$  ( $w_i \in (0, 1]$ ) is the weight parameter or delay parameter.  $w_i \neq 1$  means firm  $i$  adopts one-period delay decision; otherwise, firm  $i$  does not adopt delay decision. Then, the dynamic game process of two myopic firms can be modeled as the following dynamic system:

$$\begin{aligned}
p_1(t+1) &= p_1(t) + \alpha_1 p_1(t) \{a_1 \\
&\quad - 2b_1(1+\theta)[w_1 p_1(t) + (1-w_1)x_1(t)] \\
&\quad + d_1[w_2 p_2(t) + (1-w_2)x_2(t)] + \theta b_1 r_1(t) \\
&\quad + (1+\theta)b_1 c_1\}, \\
p_2(t+1) &= p_2(t) + \alpha_2 p_2(t) \{a_2 \\
&\quad - 2b_2(1+\theta)[w_2 p_2(t) + (1-w_2)x_2(t)] \\
&\quad + d_2[w_1 p_1(t) + (1-w_1)x_1(t)] + \theta b_2 r_2(t) \\
&\quad + (1+\theta)b_2 c_2\}, \\
r_1(t+1) &= \alpha r_1(t) + (1-\alpha)p_1(t), \\
r_2(t+1) &= \alpha r_2(t) + (1-\alpha)p_2(t),
\end{aligned}$$

$$\begin{aligned}
x_1(t+1) &= p_1(t), \\
x_2(t+1) &= p_2(t).
\end{aligned} \tag{9}$$

#### 4. Main Results

The main goal of this section is to study the evolution characteristics of system (9) under different settings and provide some management insights for firms to make use of the reference price effect.

*4.1. Equilibrium Point and Stability Condition.* By setting  $p_i(t+1) = p_i(t)$ ,  $r_i(t+1) = r_i(t)$ , and  $x_i(t+1) = x_i(t)$ , we can get the fixed points of dynamic system (9). As the boundary fixed points ( $p_i = 0$ ) are meaningless, we only consider the Nash equilibrium point:  $E^* = (p_1^*, p_2^*, r_1^*, r_2^*, x_1^*, x_2^*)$ , where  $r_1^* = x_1^* = p_1^*$ ,  $r_2^* = x_2^* = p_2^*$ ,  $p_1^* = (2a_1 b_2 + 2b_1 b_2 c_1 + a_2 d_1 + b_2 c_2 d_1 + a_1 b_2 \theta + 3b_1 b_2 c_1 \theta + b_2 c_2 d_1 \theta + b_1 b_2 c_1 \theta^2) / (4b_1 b_2 - d_1 d_2 + 4b_1 b_2 \theta + b_1 b_2 \theta^2)$ , and  $p_2^* = (2a_2 b_1 + 2b_1 b_2 c_2 + a_1 d_2 + b_1 c_1 d_2 + a_2 b_1 \theta + 3b_1 b_2 c_2 \theta + b_1 c_1 d_2 \theta + b_1 b_2 c_2 \theta^2) / (4b_1 b_2 - d_1 d_2 + 4b_1 b_2 \theta + b_1 b_2 \theta^2)$ .

To investigate the local stability of  $E^*$ , we calculate its linear approximation. The Jacobian matrix of system (9) is

$$J = \begin{bmatrix} J_{11} & \alpha_2 d_2 p_2 w_1 & 1-\alpha & 0 & 1 & 0 \\ \alpha_1 d_1 p_1 w_2 & J_{22} & 0 & 1-\alpha & 0 & 1 \\ \alpha_1 d_1 p_1 \theta & 0 & \alpha & 0 & 0 & 0 \\ 0 & \alpha_2 b_2 p_2 \theta & 0 & \alpha & 0 & 0 \\ -2\alpha_1 b_1 p_1 (1-w_1)(1+\theta) & \alpha_2 d_2 p_2 (1-w_1) & 0 & 0 & 0 & 0 \\ \alpha_1 d_1 p_1 (1-w_2) & -2\alpha_2 b_2 p_2 (1-w_2)(1+\theta) & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tag{10}$$

where  $J_{11} = 1 - 2\alpha_1 b_1 p_1 w_1 (1+\theta) + \alpha_1 (a_1 + d_1 (p_2 w_2 + x_2 - w_2 x_2) + b_1 r_1 \theta + b_1 c_1 (1+\theta) - 2b_1 (p_1 w_1 + x_1 - w_1 x_1) (1+\theta))$ , and  $J_{22} = 1 - 2\alpha_2 b_2 p_2 w_2 (1+\theta) + \alpha_2 (a_2 + d_2 (p_1 w_1 + (1-w_1)x_1) + b_2 r_2 \theta + b_2 c_2 (1+\theta) - 2b_2 (p_2 w_2 + (1-w_2)x_2) (1+\theta))$ .

Then, the stability of  $E^*$  can be judged by the eigenvalues of  $J$  at point  $E^*$ .

The characteristic polynomial of (10) takes this form:  $f(\lambda) = \lambda^6 + \mu_1 \lambda^5 + \mu_2 \lambda^4 + \mu_3 \lambda^3 + \mu_4 \lambda^2 + \mu_5 \lambda + \mu_6 = 0$ . According to Jury stability criterion, system (9) is asymptotically stable at  $E^*$  if the following condition is satisfied (see [36]):

$$\begin{aligned}
1 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 &> 0 \\
1 - \mu_1 + \mu_2 - \mu_3 + \mu_4 - \mu_5 + \mu_6 &> 0 \\
1 - |\mu_6| &> 0, \\
|\varphi_0| - |\varphi_5| &> 0, \\
|\gamma_0| - |\gamma_4| &> 0 \\
|\nu_0| - |\nu_3| &> 0, \\
|\varepsilon_0| - |\varepsilon_2| &> 0,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\varphi_0 &= \mu_6^2 - 1, \\
\varphi_1 &= \mu_5 \mu_6 - \mu_1, \\
\varphi_2 &= \mu_4 \mu_6 - \mu_2, \\
\varphi_3 &= \mu_3 \mu_6 - \mu_3, \\
\varphi_4 &= \mu_2 \mu_6 - \mu_4, \\
\varphi_5 &= \mu_1 \mu_6 - \mu_5, \\
\gamma_0 &= \varphi_0^2 - \varphi_5^2, \\
\gamma_1 &= \varphi_0 \varphi_1 - \varphi_4 \varphi_5, \\
\gamma_2 &= \varphi_0 \varphi_2 - \varphi_3 \varphi_5, \\
\gamma_3 &= \varphi_0 \varphi_3 - \varphi_2 \varphi_5, \\
\gamma_4 &= \varphi_0 \varphi_4 - \varphi_1 \varphi_5, \\
\nu_0 &= \gamma_0^2 - \gamma_4^2,
\end{aligned}$$

$$\begin{aligned}
\nu_1 &= \gamma_0\gamma_1 - \gamma_3\gamma_4, \\
\nu_2 &= \gamma_0\gamma_2 - \gamma_2\gamma_4, \\
\nu_3 &= \gamma_0\gamma_3 - \gamma_1\gamma_4, \\
\varepsilon_0 &= \nu_0^2 - \nu_3^2, \\
\varepsilon_1 &= \nu_0\nu_1 - \nu_2\nu_3, \\
\varepsilon_2 &= \nu_0\nu_2 - \nu_1\nu_3.
\end{aligned} \tag{12}$$

**4.2. The Dynamic Features with respect to Adjustment Speed  $\alpha_i$ .** Adjustment speed is a parameter reflecting the character of the decision-maker. A radical firm prefers a big adjustment speed with expecting that its profit can increase quickly. A conservative player is more likely to adopt a small adjustment speed to reduce risk. By analyzing the influence of adjustment speeds, we can get the following proposition.

**Proposition 1.** *Nash equilibrium point of system (9) is asymptotically stable when the adjustment speeds of firms are very small.*

*Proof.* First, let us look at criterion (11) when  $\alpha_1 = 0$  and  $\alpha_2 = 0$ ; then, the subconditions of criterion (11) are

$$\begin{aligned}
1 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 &= 0, \\
1 - \mu_1 + \mu_2 - \mu_3 + \mu_4 - \mu_5 + \mu_6 &= 4(1 + \alpha)^2 > 0, \\
1 - |\mu_6| &= 1 > 0, \\
|\varphi_0| - |\varphi_5| &= 1 > 0, \\
|\gamma_0| - |\gamma_4| &= 1 - \alpha^2 > 0, \\
|\nu_0| - |\nu_3| &= (1 - \alpha)^3(1 + \alpha) > 0, \\
|\varepsilon_0| - |\varepsilon_2| &= 0.
\end{aligned} \tag{13}$$

As all subconditions are continuous in  $\alpha_1$  and  $\alpha_2$ , all subconditions, except the first and the last one, are still positive when both  $\alpha_1$  and  $\alpha_2$  are very small. Then, we consider whether the first one and last one hold for small adjustment speeds. Define  $g_1 = 1 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6$  and  $g_2 = |\varepsilon_0| - |\varepsilon_2| = \varepsilon_0 - \varepsilon_2$ , where  $\varepsilon_0 = \varepsilon_2 = (\alpha^2 - 1)^4 > 0$  when  $\alpha_1 = 0$  and  $\alpha_2 = 0$ . We can get their linear approximations:

$$\begin{aligned}
g_1(\alpha_1, \alpha_2) &= g_1(0, 0) + \frac{\partial g_1(0, 0)}{\partial \alpha_1} \alpha_1 + \frac{\partial g_1(0, 0)}{\partial \alpha_2} \alpha_2 + \frac{1}{2} \frac{\partial^2 g_1(0, 0)}{\partial \alpha_1^2} \alpha_1^2 + \frac{1}{2} \frac{\partial^2 g_1(0, 0)}{\partial \alpha_2^2} \alpha_2^2 + \frac{\partial^2 g_1(0, 0)}{\partial \alpha_1 \partial \alpha_2} \alpha_1 \alpha_2 \\
&= \frac{1}{b_1 b_2 (2 + \theta)^2 - d_1 d_2} (1 - \alpha)^2 (a_1 b_2 (2 + \theta) + b_1 b_2 c_1 (2 + 3\theta + \theta^2) + d_1 (a_2 + b_2 c_2 (1 + \theta))) \\
&\quad * (a_1 d_2 + a_2 b_1 (2 + \theta) + b_1 (1 + \theta) (c_1 d_2 + b_2 c_2 (2 + \theta))) \alpha_1 \alpha_2 + o(\alpha_1^2 + \alpha_2^2), \\
g_2(\alpha_1, \alpha_2) &= g_2(0, 0) + \frac{\partial g_2(0, 0)}{\partial \alpha_1} \alpha_1 + \frac{\partial g_2(0, 0)}{\partial \alpha_2} \alpha_2 + o(\sqrt{\alpha_1^2 + \alpha_2^2}) \\
&= \frac{b_1 (2 + \theta) (1 - \alpha)^5 (1 + \alpha + \alpha^2 + \alpha^3) (a_1 b_2 (2 + \theta) + a_2 d_1 + b_2 (1 + \theta) (c_2 d_1 + b_1 c_1 (2 + \theta)))}{b_1 b_2 (2 + \theta)^2 - d_1 d_2} \alpha_1 \\
&\quad + \frac{b_2 (2 + \theta) (1 - \alpha)^5 (1 + \alpha + \alpha^2 + \alpha^3) (a_1 d_2 + a_2 b_1 (2 + \theta) + b_1 (1 + \theta) (c_1 d_2 + b_2 c_2 (2 + \theta)))}{b_1 b_2 (2 + \theta)^2 - d_1 d_2} \alpha_2 \\
&\quad + o(\sqrt{\alpha_1^2 + \alpha_2^2}).
\end{aligned} \tag{14}$$

It is easy to see that  $g_1 > 0$  and  $g_2 > 0$  for very small adjustment speeds. So, the Nash equilibrium point is asymptotically stable.  $\square$

Now, we consider two special cases for system (9).

**Assumption 2.** Consumers only remember the most recent price in the reference price model ( $\alpha = 0$ ).

**Assumption 3.** Firms only consider the most recent price in the decision-making mechanism; that is, delay decision is not adopted ( $w_i = 1$ ).

Although these two assumptions seem restrictive, they can provide good approximation to some practical scenarios [22]. We use  $F(\lambda) = \lambda^4 + \xi_1 \lambda^3 + \xi_2 \lambda^2 + \xi_3 \lambda + \xi_4$  to denote the characteristic polynomial of four-dimensional systems. Then, the Jury condition is

$$\begin{aligned}
f_1 &= 1 + \xi_1 + \xi_2 + \xi_3 + \xi_4 > 0, \\
f_2 &= 1 - \xi_1 + \xi_2 - \xi_3 + \xi_4 > 0, \\
f_3 &= 1 - |\xi_4| > 0,
\end{aligned}$$

$$\begin{aligned} f_4 &= \psi_1 - |\psi_4| > 0, \\ f_5 &= \chi_1 - |\chi_3| > 0, \end{aligned} \quad (15)$$

where  $\psi_1 = 1 - \xi_4^2$ ,  $\psi_2 = \xi_1 - \xi_3 \xi_4$ ,  $\psi_3 = \xi_2 - \xi_2 \xi_4$ ,  $\psi_4 = \xi_3 - \xi_1 \xi_4$ ,  $\chi_1 = \psi_1^2 - \psi_4^2$ ,  $\chi_2 = \psi_1 \psi_2 - \psi_3 \psi_4$ ,  $\chi_3 = \psi_1 \psi_3 - \psi_2 \psi_4$ .

Under Assumption 2, we can get the following proposition.

**Proposition 4.** *When one adjustment speed is very small, Jury stability condition (15) will be violated with the increase of the other adjustment speed. If  $8w_1(1+\theta) - 3(2+\theta) > 0$ , the violation happens with one eigenvalue being minus one. If  $8w_1(1+\theta) - 3(2+\theta) < 0$ , the violation occurs with the modulus of a pair of conjugate complex eigenvalues being one.*

*Proof.* We assume one adjustment speed is very small (e.g.,  $\alpha_2$ ). Then, the Jury stability condition is

$$\begin{aligned} f_1 &= \frac{1}{b_1 b_2 (2+\theta)^2 - d_1 d_2} \alpha_1 \alpha_2 \left( a_1 b_2 (2+\theta) + b_1 b_2 c_1 (2+3\theta+\theta^2) + d_1 (a_2 + b_2 c_2 (1+\theta)) \right) (a_1 d_2 + a_2 b_1 (2+\theta) + b_1 (1+\theta) (c_1 d_2 + b_2 c_2 (2+\theta))) > 0, \\ f_2 &= \frac{-2}{b_1 b_2 (2+\theta)^2 - d_1 d_2} \left( 2d_1 d_2 + \alpha_1 b_1^2 b_2 c_1 (2+3\theta+\theta^2) (4w_1(1+\theta) - 2 - \theta) + b_1 (a_2 \alpha_1 d_1 (4w_1(1+\theta) - 2 - \theta) \right. \\ &\quad \left. + b_2 (\alpha_1 c_2 d_1 (1+\theta) (-2 - \theta + 4w_1(1+\theta)) - 2(2+\theta)^2 + a_1 \alpha_1 (2+\theta) (4w_1(1+\theta) - 2 - \theta))) \right) + o(1), \\ \frac{\partial f_2}{\partial \alpha_1} &= -\frac{2b_1 (4w_1(1+\theta) - 2 - \theta) (a_1 b_2 (2+\theta) + b_1 b_2 c_1 (2+3\theta+\theta^2) + d_1 (a_2 + b_2 c_2 (1+\theta)))}{b_1 b_2 (2+\theta)^2 - d_1 d_2} + o(1), \end{aligned} \quad (16)$$

$$\begin{aligned} \xi_4 &= \frac{1}{(d_1 d_2 - b_1 b_2 (2+\theta)^2)^2} \alpha_1 \alpha_2 \left( a_1 b_2 (2+\theta) + b_1 b_2 c_1 (2+3\theta+\theta^2) + d_1 (a_2 + b_2 c_2 (1+\theta)) \right) (d_1 d_2 (w_1 - 1 + w_2 \\ &\quad - w_1 w_2) + b_1 b_2 (2w_1(1+\theta) - 2 - \theta) (2w_2(1+\theta) - 2 - \theta)) (a_1 d_2 + a_2 b_1 (2+\theta) + b_1 (1+\theta) (c_1 d_2 + b_2 c_2 (2+\theta))), \\ f_3 &= 1 - |\xi_4| > 0 \text{ always holds,} \end{aligned}$$

$$\psi_1 = 1 + o(1),$$

$$\psi_4 = \frac{\alpha_1 b_1 (2w_1(1+\theta) - 2 - \theta) (a_1 b_2 (2+\theta) + b_1 b_2 c_1 (2+3\theta+\theta^2) + d_1 (a_2 + b_2 c_2 (1+\theta)))}{b_1 b_2 (2+\theta)^2 - d_1 d_2} + o(1).$$

If  $(2w_1(1+\theta) - 2 - \theta) > 0$ ,  $\psi_4$  is positive and increases in  $\alpha_1$ , and if  $(2w_1(1+\theta) - 2 - \theta) < 0$ ,  $\psi_4$  is negative and decreases in  $\alpha_1$ . So,  $f_4$  always decreases in  $\alpha_1$ .

We now investigate  $f_5$ .  $\chi_3$  is a quadratic function of  $\alpha_1$  and  $\chi_3 = 1$  when  $\alpha_1 = 0$ . Solving  $\chi_3 = 0$ , we can get two solutions:

$$\begin{aligned} \alpha_1^* &= \frac{4b_1 b_2 + 4b_1 b_2 \theta + b_1 b_2 \theta^2 - d_1 d_2}{b_1 (2+\theta - 2w_1(1+\theta)) (2a_1 b_2 + 2b_1 b_2 c_1 + a_2 d_1 + b_2 c_2 d_1 + a_1 b_2 \theta + 3b_1 b_2 c_1 \theta + b_2 c_2 d_1 \theta + b_1 b_2 c_1 \theta^2)} + o(1), \\ \alpha_1^{**} &= \frac{4b_1 b_2 + 4b_1 b_2 \theta + b_1 b_2 \theta^2 - d_1 d_2}{2b_1 w_1 (1+\theta) (2a_1 b_2 + 2b_1 b_2 c_1 + a_2 d_1 + b_2 c_2 d_1 + a_1 b_2 \theta + 3b_1 b_2 c_1 \theta + b_2 c_2 d_1 \theta + b_1 b_2 c_1 \theta^2)} + o(1). \end{aligned} \quad (17)$$

*Case 1* ( $2+\theta - 2w_1(1+\theta) < 0$ ).  $\chi_3 = 0$  has only one positive solution  $\alpha_1^{**}$  and  $\chi_3 > 0$  ( $\chi_3 < 0$ ) when  $0 < \alpha_1 < \alpha_1^{**}$  ( $\alpha_1 > \alpha_1^{**}$ ). Then, solving  $f_5 = \chi_1 - |\chi_3| = 0$ , we can get  $\bar{\alpha}_1 = 2(4b_1 b_2 + 4b_1 b_2 \theta + b_1 b_2 \theta^2 - d_1 d_2) / (b_1 (4w_1(1+\theta) - 2 -$

$\theta)(2a_1 b_2 + 2b_1 b_2 c_1 + a_2 d_1 + b_2 c_2 d_1 + a_1 b_2 \theta + 3b_1 b_2 c_1 \theta + b_2 c_2 d_1 \theta + b_1 b_2 c_1 \theta^2)) + o(1) > \alpha_1^{**}$ .  $f_2$  decreases in  $\alpha_1$ , and  $f_2 = o(1)$  when  $\alpha_1 = \bar{\alpha}_1$ . What is more,  $f_4 > 0$  ( $0 < \psi_4 = (4w_1(1+\theta) - 2(2+\theta)) / (4w_1(1+\theta) - (2+\theta)) + o(1) < 1$ ) when

$\alpha_1 = \bar{\alpha}_1$ . We can get that all subconditions are satisfied when  $0 < \alpha_1 < \bar{\alpha}_1 - \varepsilon$ , where  $\varepsilon$  is an appropriately small positive constant. When  $\alpha_1 = \bar{\alpha}_1 - \varepsilon$ , the characteristic polynomial has one eigenvalue close to but less than +1, one eigenvalue close to but bigger than -1, and two eigenvalues with the product of their moduli less than 1. If the characteristic polynomial has a pair of conjugate complex numbers, the moduli of conjugate complex numbers are less than one. When  $\alpha_1$  increased from  $\alpha_1^* - \varepsilon$  to  $\alpha_1^* + \varepsilon$ ,  $f_2$  changed from positive to negative; then, a single eigenvalue passes through -1 with other eigenvalues still inside the unit circle.

*Case 2* ( $2 + \theta - 2w_1(1 + \theta) > 0$  and  $8w_1(1 + \theta) - 3(2 + \theta) > 0$ ).  $\chi_3 = 0$  has two positive solutions  $\alpha_1^*$  and  $\alpha_1^{**}$  and  $\alpha_1^* > \alpha_1^{**}$ . Then, we can get that  $f_5 = \chi_1 - |\chi_3| = 0$  has two solutions  $\alpha_1^*$  and  $\bar{\alpha}_1$  and  $\alpha_1^* > \bar{\alpha}_1 > \alpha_1^{**}$ . Then, Case 2 is similar to Case 1.

*Case 3* ( $4w_1(1 + \theta) - 2 - \theta > 0$  and  $8w_1(1 + \theta) - 3(2 + \theta) < 0$ ).  $\chi_3 = 0$  has two positive solutions  $\alpha_1^*$  and  $\alpha_1^{**}$ , and  $\alpha_1^* > \alpha_1^{**}$ . Then, we can get that  $f_5 = \chi_1 - |\chi_3| = 0$  has only one positive solution  $\alpha_1^*$ ,  $(\partial f_5 / \partial \alpha_1)|_{\alpha_1 = \bar{\alpha}_1 + 0} = -b_1(2 + \theta)(a_1 b_2(2 + \theta) + b_1 b_2 c_1(2 + 3\theta + \theta^2) + d_1(a_2 + b_2 c_2(1 + \theta)))/(b_1 b_2(2 + \theta)^2 - d_1 d_2) + o(1) < 0$ .  $\psi_4$  decreases in  $\alpha_1$  and  $\psi_4 = -1 + o(1)$  ( $f_4 = o(1)$ ) when  $\alpha_1 = \alpha_1^*$ . What is more,  $f_2$  decreases in  $\alpha_1$ ,  $f_2 =$

$-2(8w_1(1 + \theta) - 3(2 + \theta))/(2 + \theta - 2w_1(1 + \theta)) + o(1) > 0$  when  $\alpha_1 = \alpha_1^*$ . Above all, all subconditions are satisfied when  $\alpha_1 \in (0, \alpha_1^* - \varepsilon)$ . When  $\alpha_1$  increased from  $\alpha_1^* - \varepsilon$  to  $\alpha_1^* + \varepsilon$ , the Jury stability condition was violated with  $f_1 > 0$  and  $f_2 > 0$ , so a pair of conjugate complex eigenvalues occur with their moduli equal to one.

*Case 4* ( $4w_1(1 + \theta) - 2 - \theta < 0$ ).  $\chi_3 = 0$  has two positive solutions  $\alpha_1^*$  and  $\alpha_1^{**}$  and  $\alpha_1^* < \alpha_1^{**}$ . Then, Case 4 is similar to Case 3.

Above all, if  $8w_1(1 + \theta) - 3(2 + \theta) > 0$ , the violation happens with one eigenvalue being minus one. If  $8w_1(1 + \theta) - 3(2 + \theta) < 0$ , the violation occurs with the modulus of a pair of conjugate complex eigenvalues being one.  $\square$

Under Assumption 3, the following proposition holds.

**Proposition 5.** For system (9) without delay decision ( $w_i = 1$ ), if the adjustment speed of one firm is very small, Jury stability condition (15) can be violated by one eigenvalue passing through minus one.

*Proof.* When delay decision is not considered, system (9) turns to a four-dimensional system. We investigate the case when the adjustment speed of one firm (e.g., firm 2) is a very small positive constant:

$$f_1 = \frac{1}{b_1 b_2 (2 + \theta)^2 - d_1 d_2} (\alpha - 1)^2 \alpha_1 \alpha_2 (a_1 b_2 (2 + \theta) + b_1 b_2 c_1 (2 + 3\theta + \theta^2) + d_1 (a_2 + b_2 c_2 (1 + \theta))) (a_1 d_2 + a_2 b_1 (2 + \theta) + b_1 (1 + \theta) (c_1 d_2 + b_2 c_2 (2 + \theta))) > 0,$$

$$f_2 = \frac{2(1 + \alpha) (8b_1 b_2 \theta + 2b_1 b_2 \theta^2 + 2\alpha b_1 b_2 (2 + \theta)^2 + 8b_1 b_2 - 2(1 + \alpha) d_1 d_2)}{b_1 b_2 (2 + \theta)^2 - d_1 d_2} - \frac{2(1 + \alpha) \alpha_1}{b_1 b_2 (2 + \theta)^2 - d_1 d_2} (2b_1 b_2 c_2 d_1 + 5b_1 b_2 c_2 d_1 \theta + 3b_1 b_2 c_2 d_1 \theta^2 + \alpha b_1 b_2 c_2 d_1 (1 + \theta) (2 + \theta) + a_1 \alpha b_1 b_2 (2 + \theta)^2 + a_1 b_1 b_2 (4 + 8\theta + 3\theta^2) + a_2 b_1 d_1 (2 + 3\theta + \alpha (2 + \theta)) + b_1^2 b_2 c_1 (2 + 3\theta + \theta^2) (2 + 3\theta + \alpha (2 + \theta))) + o(1).$$

We can see that  $f_2$  decreases in  $\alpha_1$ . Solving  $f_2 = 0$  gives

$$\alpha_1 = \frac{1}{b_1 (2 + 3\theta + \alpha (2 + \theta)) (a_1 b_2 (2 + \theta) + b_1 b_2 c_1 (2 + 3\theta + \theta^2) + d_1 (a_2 + b_2 c_2 (1 + \theta)))} * 2(1 + \alpha) (b_1 b_2 (2 + \theta)^2 - d_1 d_2) + o(1) \triangleq \bar{\alpha}_1.$$

What is more, when  $\alpha_1 = \bar{\alpha}_1$ ,  $\xi_4 = -\alpha(2\theta + \alpha(2 + \theta) + \alpha^2(2 + \theta))/(2 + 3\theta + \alpha(2 + \theta)) + o(1)$ ,

$$f_3 = 1 - |\xi_4| \approx \frac{(1 - \alpha) (2 + 3\theta + 2\alpha (2 + \theta) + \alpha^2 (2 + \theta))}{2 + 3\theta + \alpha (2 + \theta)} > 0,$$

$$f_4 \approx \frac{(1 - \alpha)^3 (1 + \alpha) (2 + \theta) (2 + 3\theta + 2\alpha (2 + \theta) + \alpha^2 (2 + \theta))}{(2 + 3\theta + \alpha (2 + \theta))^2} > 0.$$

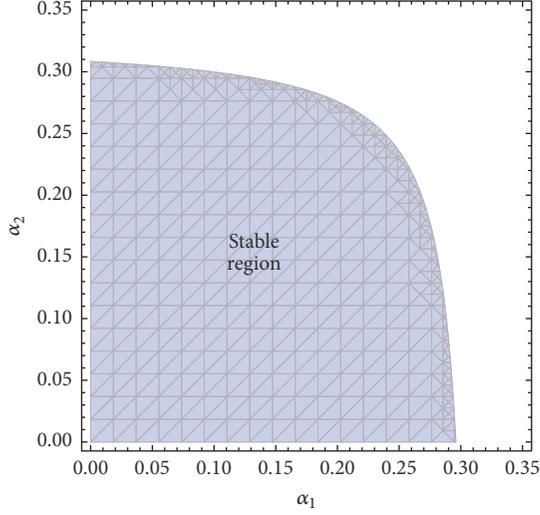


FIGURE 1: The stable region of adjustment speeds when  $\theta = 0.5$ ,  $\alpha = 0.5$ , and  $w_i = 0.5$ .

$$f_5 = \chi_1 - |\chi_3| = \chi_1 + \chi_3 = 0,$$

$$\left. \frac{\partial f_5}{\partial \alpha_1} \right|_{\alpha_1 = \tilde{\alpha}_1} < 0.$$

(20)

Assume that  $\varepsilon$  is an appropriately small positive constant. When  $\alpha_1 = \tilde{\alpha}_1 - \varepsilon$ , the Jury condition is satisfied. Then, the characteristic polynomial has one eigenvalue close to but less than +1, one eigenvalue close to but bigger than -1, and two eigenvalues with the product of their moduli less than 1. If the characteristic polynomial has conjugate complex numbers, the moduli of conjugate complex numbers are less than one. As the moduli of eigenvalues are continuous in  $\alpha_1$ , when  $\alpha_1$  increases from  $\tilde{\alpha}_1 - \varepsilon$  to  $\tilde{\alpha}_1 + \varepsilon$ ,  $f_2$  changes from positive to negative, and then a single eigenvalue passes through minus one with other eigenvalues still inside the unit circle.  $\square$

Although a single eigenvalue becoming minus one and the modulus of a pair of conjugate complex eigenvalues being equal to one are necessary conditions for the existence of flip bifurcation and Neimark-Sacker bifurcation, respectively, they constitute strong evidence combined with numerical simulations which show that such bifurcations do occur [50]. In the following text, we will resort to some numerical simulations to demonstrate these propositions given above and explore more useful results. To do the simulations, we take the basic parameter values as follows:  $a_1 = 4.8$ ,  $a_2 = 4.5$ ,  $b_1 = 3.5$ ,  $b_2 = 3.3$ ,  $c_1 = 0.01$ ,  $c_2 = 0.01$ ,  $d_1 = 2.1$ , and  $d_2 = 2.2$ , and then we can get  $p_1^* = 0.734$  and  $p_2^* = 0.747$ .

Figure 1 shows the stable region of adjustment speeds in system (9). Figure 2 gives the bifurcation diagram of prices with respect to  $\alpha_1$  when other parameters are fixed, where the blue orbit represents the evolution of  $p_1$ , the red orbit represents the evolution of  $p_2$ , and the black orbit shows the

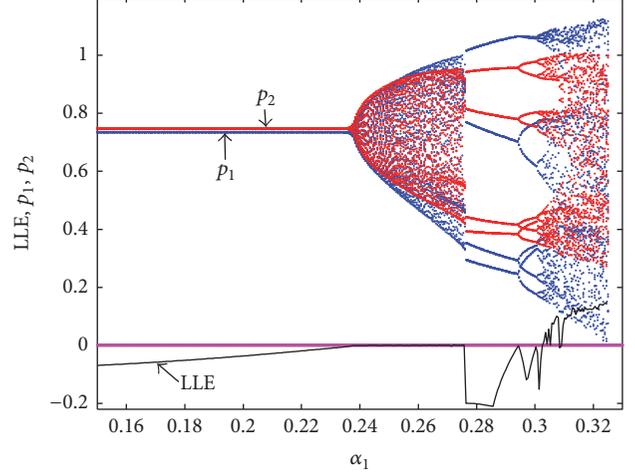


FIGURE 2: The bifurcation diagram of prices and LLE with respect to  $\alpha_1$  when  $\theta = 0.5$ ,  $\alpha = 0.5$ ,  $w_i = 0.5$ , and  $\alpha_2 = 0.25$ .

largest Lyapunov exponent (LLE) of this system. LLE is widely used to mark chaos; it is positive when chaos occurs [51]. From Figures 1 and 2, we can see that the Nash equilibrium point is locally stable when  $\alpha_1$  and  $\alpha_2$  are small; this is consistent with Proposition 1. At the same time, if one firm increases its adjustment speed to a certain value, system (9) will become unstable and sink into closed invariant curve via Neimark-Sacker bifurcation. A further increase of its adjustment speed will lead to a series of period-doubling bifurcations. System (9) sinks into chaos eventually via period-doubling bifurcation.

We simulate the attractors of system (9) in different states and show them in Figure 3, where Figure 3(a) corresponds to the fixed point attractor, Figure 3(b) corresponds to the closed invariant curve attractor, and Figure 3(c) corresponds to the chaos attractor. We can also see that the complexity of this system increases when adjustment speeds increase.

Figures 4 and 5 show the evolution characteristics of the system given in Propositions 4 and 5, respectively, where the first figure in Figure 4 corresponds to  $8w_1(1 + \theta) - 3(2 + \theta) < 0$  and the second one corresponds to  $8w_1(1 + \theta) - 3(2 + \theta) > 0$ . We can see clearly from them that the corresponding bifurcations do occur in this system. What is more important is that although Propositions 4 and 5 give local bifurcation characteristics (bifurcation characteristics when one adjustment speed is very small), they also can shed light on the global bifurcation characteristics (bifurcation characteristics with any adjustment speeds). As is shown in Figures 6–8, the bifurcation type when the system loses its stability is the same for any adjustment speed in each figure, where different colors represent different periods (grey means divergence). What is more, from Figure 9, we can see that firms get their maximal profits when the system is stable.

Above all, we can get that adjustment speeds have obvious impacts on the dynamics of system (9). The system is stable when both adjustment speeds are small. System (9) will lose

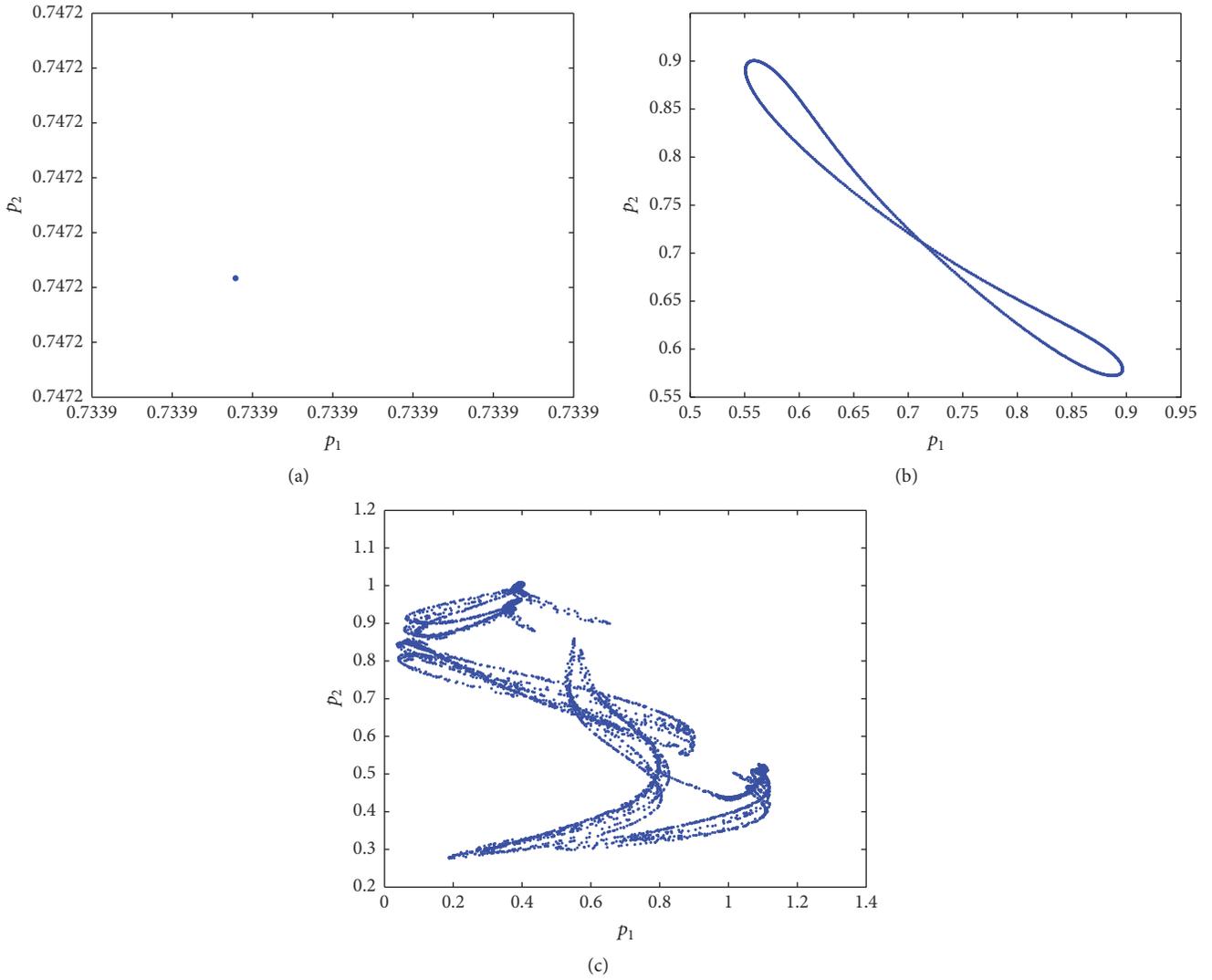


FIGURE 3: Attractor when  $\theta = 0.5$ ,  $\alpha = 0.5$ ,  $w_i = 0.5$ , and  $\alpha_2 = 0.25$ : (a)  $\alpha_1 = 0.2$ , (b)  $\alpha_1 = 0.25$ , and (c)  $\alpha_1 = 0.32$ .

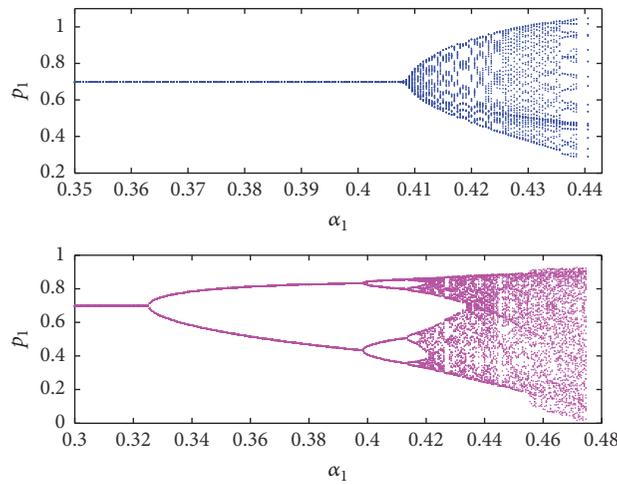


FIGURE 4: The bifurcation diagram of  $p_1$  with respect to  $\alpha_1$  when  $\theta = 0.6$ ,  $\alpha_2 = 0.01$ , and  $\alpha = 0$ , where  $w_i = 0.5$  in the first figure and  $w_i = 0.8$  in the second figure.

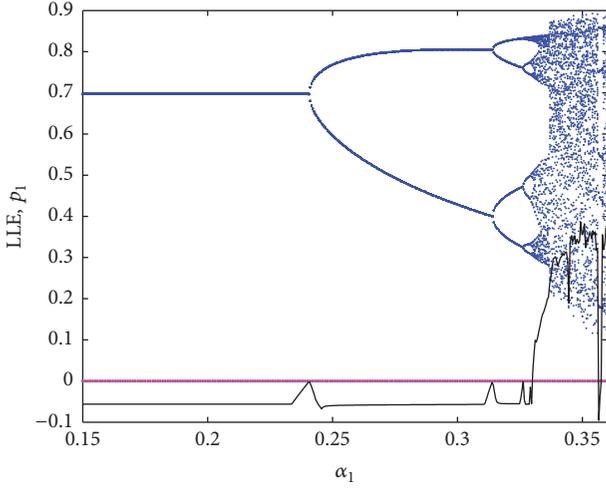


FIGURE 5: The bifurcation diagram of price and LLE with respect to  $\alpha_1$  when  $\theta = 0.6$ ,  $\alpha = 0.5$ ,  $w_i = 1$ , and  $\alpha_2 = 0.01$ .

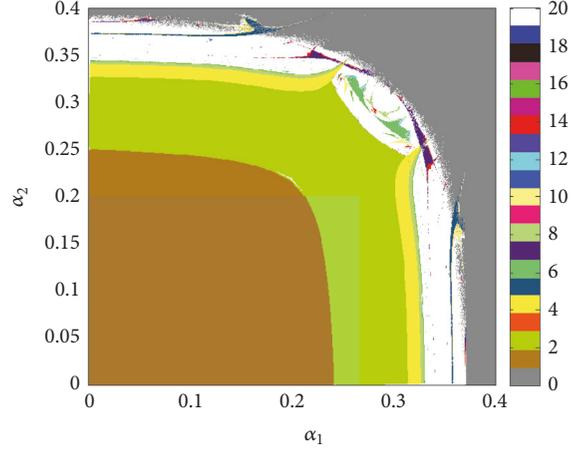


FIGURE 8: Parameter basin in  $(\alpha_1, \alpha_2)$  plane when  $\theta = 0.6$ ,  $\alpha = 0.5$ , and  $w_i = 1$ .

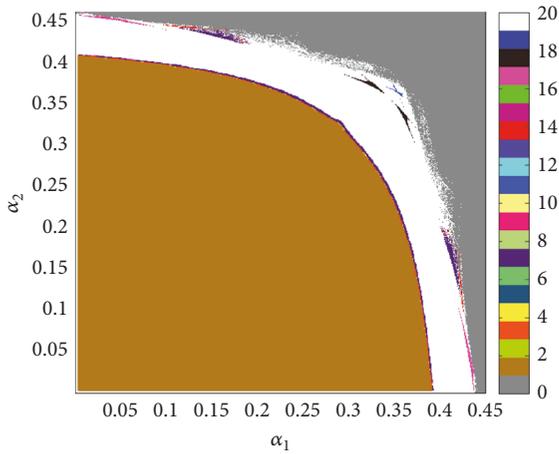


FIGURE 6: Parameter basin in  $(\alpha_1, \alpha_2)$  plane when  $\theta = 0.6$ ,  $\alpha = 0$ , and  $w_i = 0.5$ .

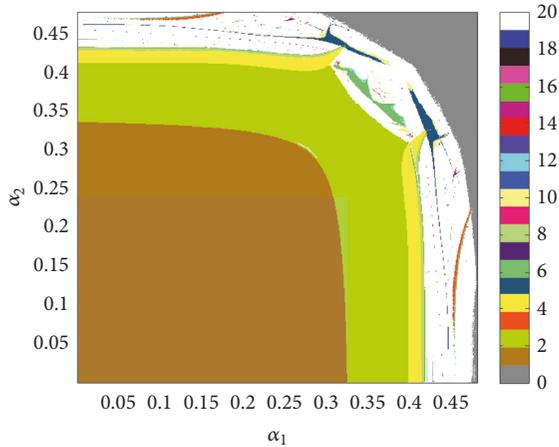


FIGURE 7: Parameter basin in  $(\alpha_1, \alpha_2)$  plane when  $\theta = 0.6$ ,  $\alpha = 0$ , and  $w_i = 0.8$ .

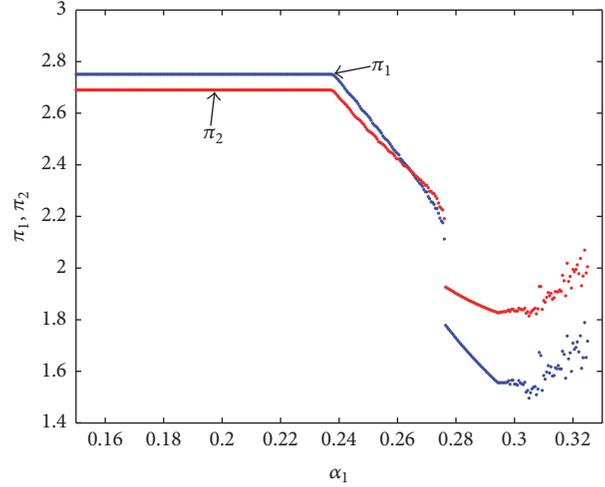


FIGURE 9: The average profits with respect to  $\alpha_1$  when  $\theta = 0.5$ ,  $w_i = 0.5$ , and  $\alpha_2 = 0.25$ .

its stability and bifurcation or even chaos may happen when adjustment speeds increase. Stabilization is a preferable state for firms both from the perspective of dynamic complexity and from the perspective of profit maximization.

4.3. *The Impacts of Memory Parameter  $\alpha$  and Delay Decision Coefficient  $w$ .* Memory parameter  $\alpha$  captures the strength of past prices which the reference price depends on.  $\alpha$  is high (low) when customers have a long (short) memory. At the same time, players may adopt delay decision. In this section, the influence of  $\alpha$  and  $w$  on the dynamics of system (9) under Assumptions 2 and 3 will be studied.

**Proposition 6.**  $\alpha_i^*$  increases in  $w_i$ ,  $\bar{\alpha}_i$  decreases in  $w_i$ , and  $\bar{\alpha}_i$  increases in  $\alpha$ .

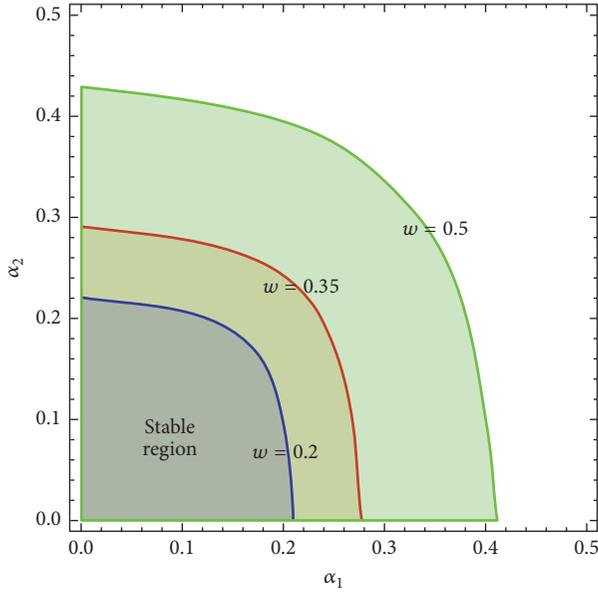


FIGURE 10: The stable region of adjustment speeds when  $\theta = 0.6$ ,  $\alpha = 0$ , and  $w_i = w$ .

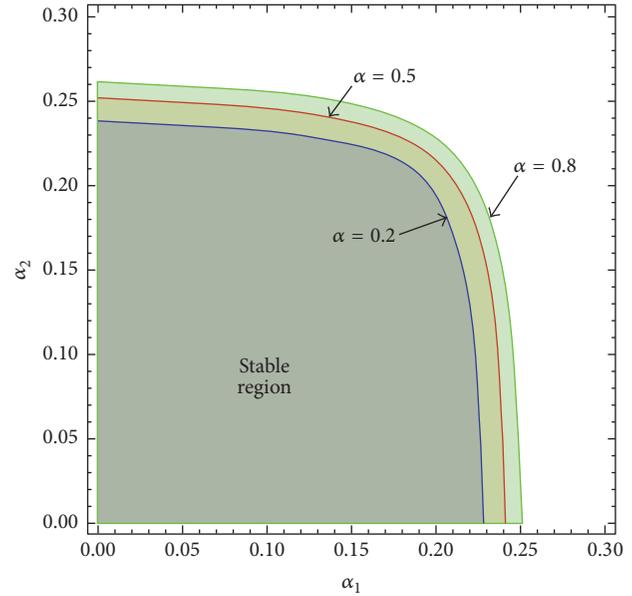


FIGURE 12: The stable region of adjustment speeds when  $\theta = 0.6$  and  $w_i = 1$ .

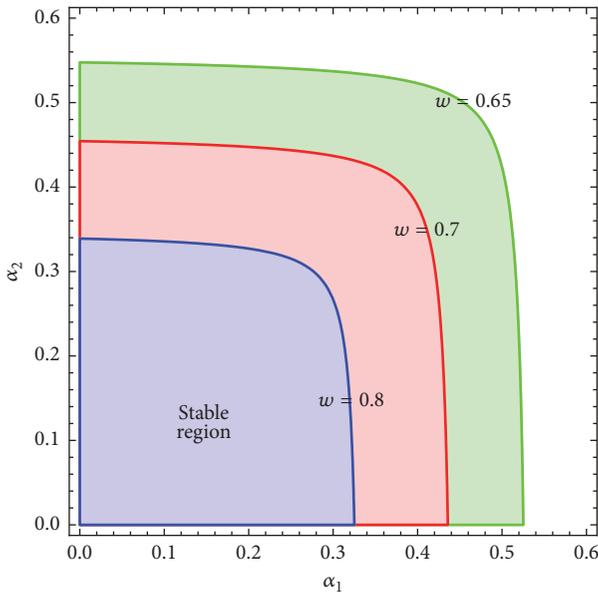


FIGURE 11: The stable region of adjustment speeds when  $\theta = 0.6$ ,  $\alpha = 0$ , and  $w_i = w$ .

As it is easy to prove Proposition 6 by calculating partial derivatives  $\partial \alpha_i^* / \partial w_i$ ,  $\partial \tilde{\alpha}_i / \partial w_i$ , and  $\partial \tilde{\alpha}_i / \partial \alpha$ , we omit this proof here.

Based on Figures 10–12, we can get the following observations. (1) The stable region of adjustment speeds is a convex set in  $(\alpha_1, \alpha_2)$  plane, so  $\tilde{\alpha}_i$  defined in Proposition 5 is also the boundary values of stable region of adjustment speeds. What is more valuable is that (2) the proposition given by Proposition 6 not only applies to boundary values (when

one adjustment speed is very small), but also applies to the whole stable region of adjustment speeds. Then, we can get the impacts of parameters on the whole stable region of adjustment speeds. Under Assumption 2, there is  $w^* = 3(2 + \theta)/8(1 + \theta) \in (0.56, 0.75)$ , and the stable region increases in  $w$  ( $w_i = w$ ) firstly ( $w < w^*$ ) and then decreases in it ( $w > w^*$ ). The stable region of adjustment speeds increases in memory parameter if delay decision is not adopted.

Above all, the stable region increases in memory parameter when delay decision is not adopted. Firms can expand the stable region by encouraging consumers to recall past prices in this case. When consumers only remember the most recent price, the impacts of delay decision depend on the delay parameter and customer perception coefficient. Firms should be cautious in choosing delay parameter in this situation.

**4.4. The Influence of Customer Perception Coefficient  $\theta$ .** Perception coefficient  $\theta$  ( $\theta \in [0, 1)$ ) captures the percentage of perceived “gain” or “loss” effect in response to the real price effect.  $\theta = 0$  corresponds to the demand in the absence of reference price effect; then demands only depend on current prices. The perceived “gain” or “loss” influences demand when  $\theta \neq 0$ , and a bigger value of  $\theta$  means a bigger influence of perceived “gain” or “loss” effect. We can get the following proposition on  $\theta$ .

**Proposition 7.** *The equilibrium price  $p_i^*$  and equilibrium profit  $\pi_i^*$  decrease in perception coefficient  $\theta$  as long as  $a_i - b_i c_i - c_{3-i} d_i > 0$ .*

*Proof.* As the proofs for  $p_2^*$  and  $\pi_2^*$  are similar to that for  $p_1^*$  and  $\pi_1^*$ , we take  $p_1^*$  and  $\pi_1^*$  as an example.

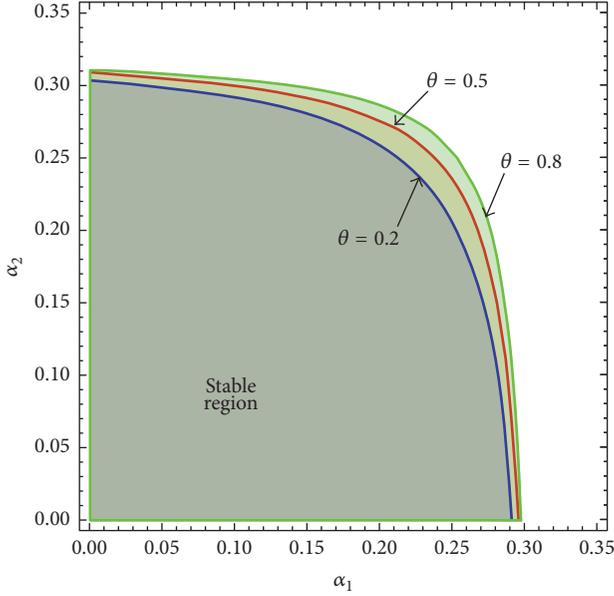


FIGURE 13: The stable region of adjustment speeds when  $\alpha = 0.5$  and  $w_i = 0.5$ .

$$\frac{\partial p_1^*}{\partial \theta} = \frac{-b_2 [c_2 d_1^2 d_2 + b_1 b_2 (2 + \theta)^2 (a_1 - b_1 c_1) + a_1 d_1 d_2 + b_1 d_1 (2a_2 (2 + \theta) + b_2 c_2 \theta (2 + \theta) + c_1 d_2 (3 + 2\theta))]}{[d_1 d_2 - b_1 b_2 (2 + \theta)^2]^2}. \quad (21)$$

It is easy to get  $\partial p_1^* / \partial \theta < 0$  as long as  $a_1 - b_1 c_1 > 0$ .

$$\frac{\partial \pi_1^*}{\partial \theta} = \frac{-b_1}{(-d_1 d_2 + b_1 b_2 (2 + \theta)^2)^3} * A * B, \quad (22)$$

where  $A = [-b_2(2 + \theta)(a_1 - b_1 c_1) - d_1(a_2 + c_1 d_2 + b_2 c_2(1 + \theta))] < 0$  and  $B = [-4b_1 b_2 d_1 (a_2 - b_2 c_2) - 4a_1 b_2 d_1 d_2 - a_2 d_1^2 d_2 - 3b_2 c_2 d_1^2 d_2 - c_1 d_1^2 d_2^2 - 4b_1 b_2^2 \theta (a_1 - b_1 c_1 - c_2 d_1) - 8a_2 b_1 b_2 d_1 \theta - 3a_1 b_2 d_1 d_2 \theta - 5b_1 b_2 c_1 d_1 d_2 \theta - 3b_2 c_2 d_1^2 d_2 \theta - 4b_1 b_2^2 \theta^2 (a_1 - b_1 c_1) - 3a_2 b_1 b_2 d_1 \theta^2 - b_1 b_2^2 c_2 d_1 \theta^2 - 3b_1 b_2 c_1 d_1 d_2 \theta^2 - b_1 b_2^2 \theta^3 (a_1 - b_1 c_1) - b_1 b_2^2 c_2 d_1 \theta^3] < 0$ , so we can get  $\partial \pi_1^* / \partial \theta < 0$ .  $\square$

We show the stable region boundary of  $(\alpha_1, \alpha_2)$  in Figure 13. From Figure 13, we can see that the stable region of  $(\alpha_1, \alpha_2)$  increases in  $\theta$ . That is to say, a large perception coefficient of customer is beneficial to the stability of system (9), and the system is more likely to be stable when customers pay more attention to the perceived “gain” or “loss.” Figure 14 shows the bifurcation diagram of  $p_1$  and  $r_1$  with respect to  $\theta$ . From Figure 14, we can get that the system turns to a stable state when the value of  $\theta$  is increased to a certain value. Figure 15 gives firms’ average profits corresponding to Figure 14. We can know that firms’ profits decrease in  $\theta$  under period 4 status and stabilization status, but profits are not monotonous under the closed invariant curve status. They firstly increase in  $\theta$  and then decrease in  $\theta$  after getting their maximal values. What is more, they get their global maximal values in closed invariant curve status with  $\theta \neq 0$ .

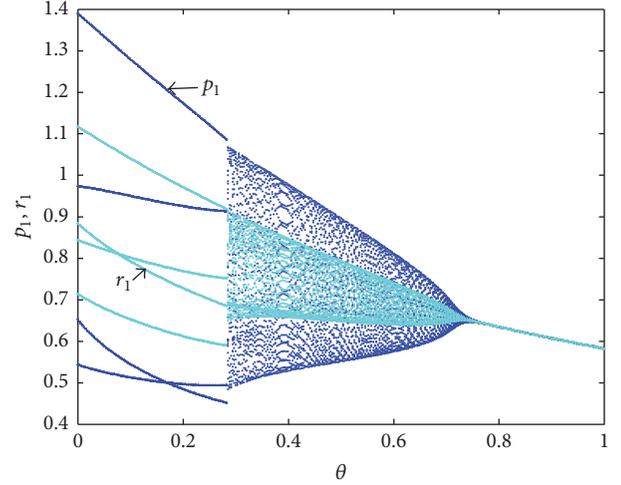


FIGURE 14: The bifurcation diagram of  $p_1$  and  $r_1$  with respect to  $\theta$  when  $\alpha = 0.5$ ,  $w_i = 0.5$ , and  $\alpha_1 = \alpha_2 = 0.25$ .

Above all, equilibrium price and equilibrium profit decrease in customers’ perception coefficient, but the stable region of adjustment speeds increases in it. Compared with the state without reference price effect ( $\theta = 0$ ), customer perception coefficient  $\theta \neq 0$  is disadvantageous to firms’ profits when Nash equilibrium is asymptotically stable. But it can be beneficial to profits when Nash equilibrium is unstable. So, when the evolution is stable, firms should try to attenuate customer perception coefficient. But when the system is unstable, firms should try to keep a suitable customer perception coefficient.

**4.5. The Influence of Initial Reference Price.** Reference price is determined by customers but also can be influenced by firms’ activities, such as advertising. Reference price in advertisements (advertised reference price) can draw consumers’ attention and influence consumers’ product evaluations [52]. In this section, we will study the influence of initial reference price on profits and provide some managerial insights about advertising. As the attractor is independent of initial states if initial states belong to the basin of attraction of the same attractor, we only need to consider this influence before system (9) gets its final state (attractor).

Figure 16 shows players’ average profits of the first 100 periods with respect to initial pricing  $p_1(1)$  and initial reference price  $r_1(1)$  when Nash equilibrium point is asymptotically stable, where  $x_1(1) = p_1(1)$ ,  $x_2(1) = p_2(1) = 0.7$ ,  $r_2(1) = 1$ ,  $\theta = 0.5$ ,  $\alpha = 0.5$ ,  $w_i = 0.5$ , and  $\alpha_i = 0.2$ . Figures 17 and 18 show the same content as Figure 16 when system

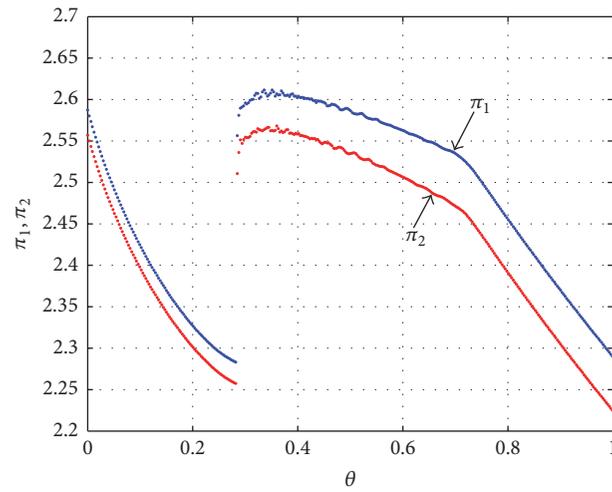


FIGURE 15: The average profits with respect to  $\theta$  when  $\alpha = 0.5$ ,  $w_i = 0.5$ , and  $\alpha_1 = \alpha_2 = 0.25$ .

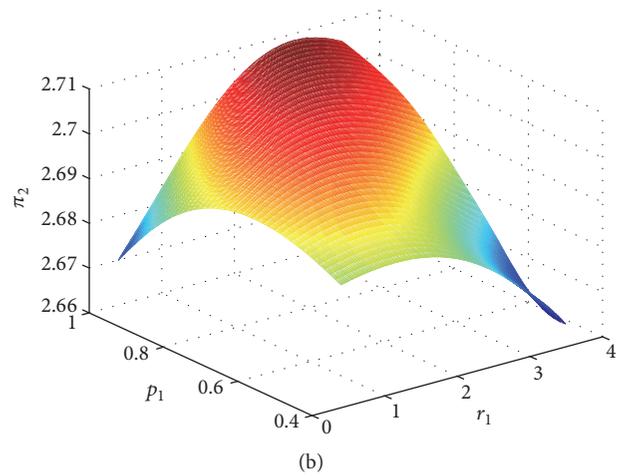
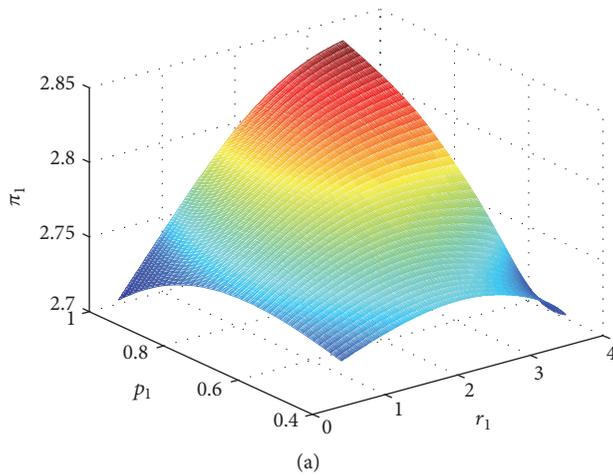


FIGURE 16: The average profits with respect to initial  $r_1$  and  $p_1$  when  $\theta = 0.5$ ,  $\alpha = 0.5$ ,  $w_1 = w_2 = 0.5$ , and  $\alpha_1 = \alpha_2 = 0.2$ .

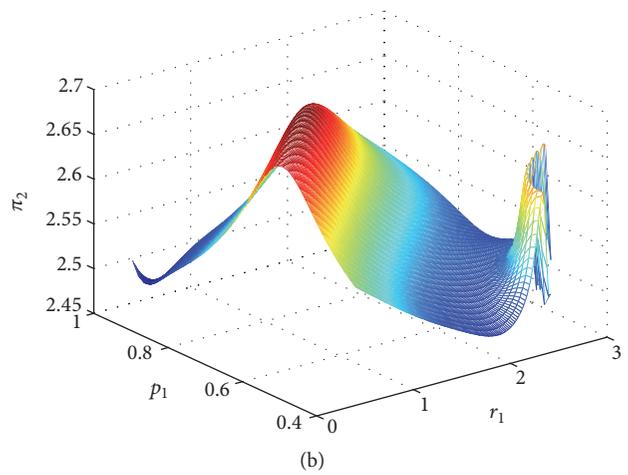
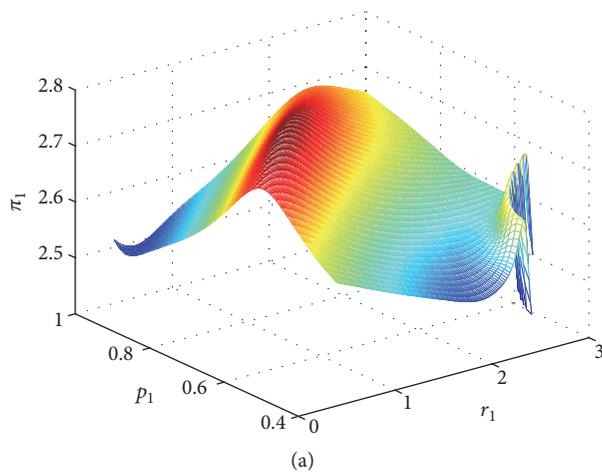


FIGURE 17: The average profits with respect to initial  $r_1$  and  $p_1$  when  $\theta = 0.5$ ,  $\alpha = 0.5$ ,  $w_1 = w_2 = 0.5$ , and  $\alpha_1 = \alpha_2 = 0.25$ .

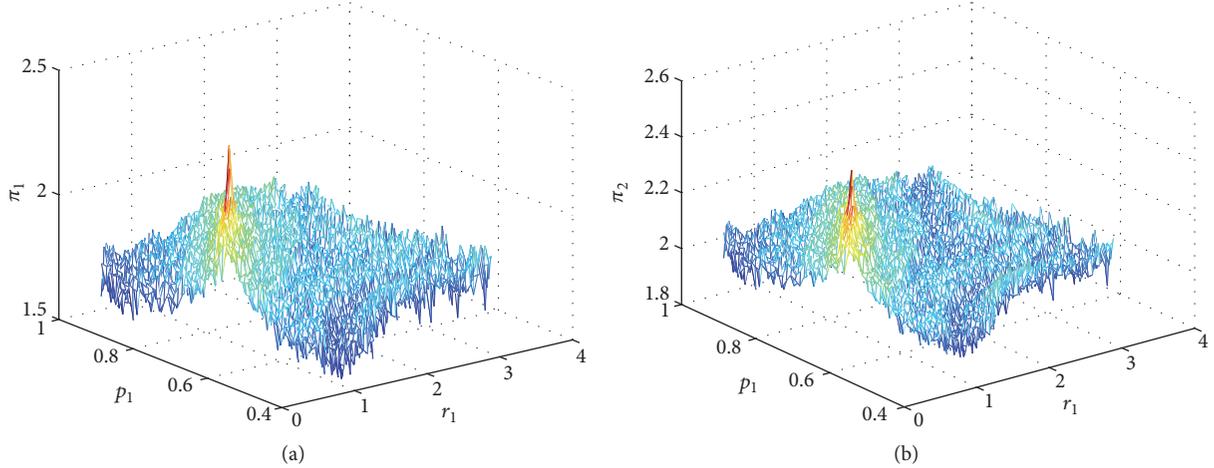


FIGURE 18: The average profits with respect to initial  $r_1$  and  $p_1$  when  $\theta = 0.5$ ,  $\alpha = 0.5$ ,  $w_1 = w_2 = 0.5$ ,  $\alpha_1 = 0.32$ , and  $\alpha_2 = 0.25$ .

TABLE 1: Maximal profits and their locations.

Figure	Figure 16(a)	Figure 16(b)	Figure 17(a)	Figure 17(b)	Figure 18(a)	Figure 18(b)
$r_1$	3.84	2.98	1.14	1.06	0.81	0.81
$p_1$	1	1	0.73	0.71	0.7	0.7
$\pi_i$	2.838	2.708	2.762	2.698	2.35	2.434

(9) is in closed invariant curve state (Figure 17) and chaos state (Figure 18), respectively, where parameters are the same as those in Figure 16 except for  $\alpha_i = 0.25$  (in Figure 17) and  $\alpha_1 = 0.32$  and  $\alpha_2 = 0.25$  (in Figure 18). Table 1 shows the maximal profits and their locations for Figures 16–18.

We can know that when the Nash equilibrium point is asymptotically stable, (1) profits get their maximal values when both initial pricing and initial reference price are obviously bigger than the equilibrium price. In this situation, advertising is generally beneficial to both firms' profits if the advertising cost is very small compared with the selling profit. But when the system is in closed invariant curve state or chaos state, (2) firms' profits get their peak values when initial pricing and initial reference price are close to equilibrium point. A higher initial reference price relative to equilibrium price, despite improving the current profit, also enlarges the vibration of the evolution process.

## 5. Conclusions

This paper investigates the effect of internal reference price in competitive settings for firms that only master partial information, where reference prices are assumed to evolve according to an exponential smoothing process, and firms' decision-making mechanism is modeled as a dynamic adjustment based on estimated marginal profits. By investigating the evolution characteristics of this dynamic adjustment and the impacts of key parameters, this study provides the following implications for firms with limited information to take advantage of reference price effect in competitive settings.

First, firms get their maximal profits when this evolution converges to the Nash equilibrium point, so firms should let the stability condition be satisfied. Fortunately, it is proved that the stability condition is satisfied when firms' adjustment speeds are small, so firms should adopt small adjustment speeds when adjusting their decisions. Second, if customers' reference prices are the last paid prices, there is a threshold belonging to  $(0.56, 0.75)$ . The stable region of adjustment speeds increases in delay parameter when the delay parameter is less than this threshold, and the stable region decreases in delay parameter when the delay parameter is greater than this threshold. So, firms can benefit from delay decision by adopting appropriate delay parameters, and this threshold increases when customer perception coefficient decreases. What is more, when delay decision is not adopted, the stable region of adjustment speeds increases in memory parameter, so the implication that firms should encourage consumers to recall reference price given in [16] also applies in limited information settings. A lower customer perception coefficient is beneficial when the system is stable. However, it can be harmful when the system is unstable, so firms should try to keep a suitable customer perception coefficient to maximize their profits when evolution is fluctuant. Third, a higher initial reference price (relative to equilibrium price) of one product generally benefits both firms when the Nash equilibrium point is asymptotically stable. But it also can be harmful when the Nash equilibrium point is unstable. This is an interesting result; if the adjustment process is unstable, too much advertising may be harmful even when the advertising cost is negligible. So, firms should take the potential evolution process into consideration before implementing their advertising strategies.

There are several extensions deserving further investigation. First, this paper only considers the internal reference price effect; the results can be more practical if both internal reference price effect and external reference price effect are taken into consideration at the same time. Second, it is assumed that firms are myopic in this paper; reference price effects are not directly reflected in their decision-making mechanism; researchers can investigate the case where firms try to maximize the sum profits of two or more periods. Then, researchers may get some more interesting results. Third, random factors are unavoidable and play important roles in the real market. Incorporating random factors into the study of reference price effect makes implications more profound and robust, so this is another interesting and challenging research direction.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

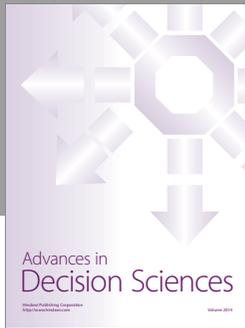
### Acknowledgments

This research was supported by the National Natural Science Foundation of China (nos. 61273231 and 71571131), Doctoral Fund of Ministry of Education of China (Grant no. 20130032110073), and Tianjin University Innovation Fund.

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