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Research Article

Dynamic Analysis for a Kaldor–Kalecki Model of Business Cycle with Time Delay and Diffusion Effect

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The dynamics behaviors of Kaldor–Kalecki business cycle model with diffusion effect and time delay under the Neumann boundary conditions are investigated. First the conditions of time-independent and time-dependent stability are investigated. Then, we find that the time delay can give rise to the Hopf bifurcation when the time delay passes a critical value. Moreover, the normal form of Hopf bifurcations is obtained by using the center manifold theorem and normal form theory of the partial differential equation, which can determine the bifurcation direction and the stability of the periodic solutions. Finally, numerical results not only validate the obtained theorems, but also show that the diffusion coefficients play a key role in the spatial pattern. With the diffusion coefficients increasing, different patterns appear.

1. Introduction

Recently, business cycle, as one of the important economic phenomena, has received attractive attentions due its widely application in many fields such as economic decisions, macroeconomic regulation, and market regulation [1–8]. In order to understand the mechanisms of business cycle, many models are proposed. One of the most famous business cycle models is the Kaldor–Kalecki business cycle [9, 10], which is described as

$$\frac{dY(t)}{\partial t} = \alpha \left[I(Y(t), K(t)) - \gamma Y(t) \right]$$

$$\frac{dK(t)}{\partial t} = I(Y((t)), K(t)) - qK(t),$$

(1)

where $Y(t)$ is the gross product, $K(t)$ is the capital stock at time, α is the adjustment coefficient in the goods market, q is the depreciation rate of the capital stock, γ represents the propensity to save, and $I(Y(t), K(t))$ is the investment. Under this model, the dynamic behaviors are widely studied such as

stability, Hopf bifurcation, codimension-two bifurcation, and chaos [9–15].

It is well known that diffusion effects of economic activities are widespread phenomenon that existed all over the world. As a result of the impact of the growth pole, the diffusion effects are the main interactions in economic activities. So, the diffusion effect should be considered in the business cycle model. However, to the best of our knowledge, there are very few works on this field. Inspired by the observation, in this paper, based on the Kaldor–Kalecki model, we propose a novel business cycle with diffusion effect and time delay under the Neumann boundary conditions, which is as follows:

$$\frac{\partial Y(t,x)}{\partial t} = d_1 \Delta Y(t,x) + \alpha \left[I(Y(t,x)) - \beta K(t,x) - \gamma Y(t,x) \right]$$

$$\frac{\partial K(t,x)}{\partial t} = d_2 \Delta K(t,x) + I(Y((t-\tau),x)) - (q+\beta) K(t,x),$$

(2)

with its initial and boundary conditions given as follows:

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial x} = \frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial x} = 0, \quad t > 0, \ x \in \partial\Omega$$

$$u_i(t, x) = \phi_i(t, x), \quad i = 1, 2, \ t \in [-\tau, 0], \ x \in \overline{\Omega} \qquad (3)$$

$$v_i(t, x) = \phi_i(t, x), \quad i = 1, 2, \ t \in [-\tau, 0], \ x \in \overline{\Omega},$$

where $\alpha > 0$, $\beta > 0$, $q \in (0, 1)$, $\gamma \in (0, 1)$, and Ω is the market capacity. There are three contributions of this paper:

(1) Based on the Kaldor–Kalecki model, we propose a novel business cycle under the Neumann boundary conditions. Our model is a spatial-temporal model, which is more general than the existing models.

(2) The time-independent and time-dependent stability are investigated. Moreover, the conditions of the Hopf bifurcation are obtained.

(3) It is found that the diffusion coefficients play a key role in the spatial pattern. With the diffusion coefficients increasing, different patterns appear.

The rest paper is organized as follows. In Section 2, the time-independent stability, time-dependent stability, and the existence of Hopf bifurcation are obtained. In Section 3, the normal form of Hopf bifurcation is obtained. In Section 4, numerical results are given to validate the obtained theorems.

2. Local Stability and Hopf Bifurcation Analysis

Let (Y^*, K^*) be an equilibrium of (2) and $u_1(t, x) = Y(t, x) - Y^*$, $u_2(t, x) = K(t, x) - K^*$, $i(u(t, x)) = I(u(t, x)) - I^*$, and $s = S - S^*$; (2) can be rewrite as

$$\frac{\partial u_1(t, x)}{\partial t} = d_1 \Delta u_1(t, x)$$
$$+ \alpha \left[i(u_1(t, x)) - \beta u_2(t, x) - \gamma u_1(t, x) \right]$$
$$\frac{\partial u_2(t, x)}{\partial t} = d_2 \Delta u_2(t, x) + i(u_1((t - \tau), x)) \qquad (4)$$
$$- (q + \beta) u_2(t, x).$$

Taking the Taylor expansion of i at 0 yields

$$i(u) = c_1 u + c_2 u^2 + c_3 u^3 + \cdots, \qquad (5)$$

where $c_1 = i'(0)$, $c_2 = (1/2)i''(0)$, and $c_3 = (1/3!)i'''(0)$.

The linear parts of (4) are as follows:

$$\frac{\partial u_1(t, x)}{\partial t} = d_1 \Delta u_1(t, x) + \alpha(c_1 - \gamma) u_1(t, x)$$
$$- \alpha\beta u_2(t, x)$$
$$\frac{\partial u_2(t, x)}{\partial t} = d_2 \Delta u_2(t, x) + c_1 u_1((t - \tau), x) \qquad (6)$$
$$- (q + \beta) u_2(t, x).$$

Then, the characteristic of (6) is

$$\lambda y - D\Delta y - L(e^\lambda y) = 0, \quad y \in \text{dom}(\Delta) \setminus \{0\}. \qquad (7)$$

Following the method of [16], we define $-k^2$ ($k \in \mathfrak{R}_0 = \{0, 1, 2, \ldots\}$) as the eigenvalue of Δ under the Neumann boundary conditions on the X. Let $\beta_k^1 = (\gamma_k, 0,)^T$, $\beta_k^2 = (0, \gamma_k)^T$ be the corresponding eigenvectors, where $\gamma_k = \cos(kx)$, $k = 0, 1, 2, \ldots$. We use $\{\beta_k^1, \beta_k^2\}_{k=0}^\infty$ to construct a basis of the phase space X. Then, y can be expanded in the following form of Fourier on X:

$$y = \sum_{k=0}^\infty Y_k^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \\ \beta_k^4 \end{pmatrix},$$

$$Y_k = \begin{pmatrix} \langle y, \beta_k^1 \rangle \\ \langle y, \beta_k^2 \rangle \\ \langle y, \beta_k^3 \rangle \\ \langle y, \beta_k^4 \rangle \end{pmatrix}^T. \qquad (8)$$

Then, we have

$$L \left(\phi^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \\ \beta_k^4 \end{pmatrix} \right) = L(\phi)^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \\ \beta_k^4 \end{pmatrix}, \quad k \in N_0. \qquad (9)$$

By (7)–(9), we can obtain

$$\det \left(\begin{bmatrix} \lambda + d_1 k^2 & 0 \\ 0 & \lambda + d_2 k^2 \end{bmatrix} - \begin{bmatrix} \alpha(c_1 - \gamma) & -\alpha\beta \\ c_1 e^{-\lambda\tau} & -(q + \beta) \end{bmatrix} \right) = 0. \qquad (10)$$

The characteristic equation of (10) is as follows:

$$\begin{pmatrix} \lambda + d_1 k^2 - \alpha(c_1 - \gamma) & \alpha\beta \\ -c_1 e^{-\lambda\tau} & \lambda + d_2 k^2 + (q + \beta) \end{pmatrix} = 0. \qquad (11)$$

The characteristic equation of (11) is

$$F(\lambda) = \lambda^2 + \varphi_1(k)\lambda + \varphi_2(k) + \alpha\beta c_1 e^{-\lambda\tau}, \qquad (12)$$

where

$$\varphi_1(k) = (d_1 + d_2)k^2 + q + \beta - \alpha(c_1 - \gamma)$$

$$\varphi_2(k) = d_1 d_2 k^4 + (d_1 q + d_1 \beta - d_2(c_1 - \gamma)\alpha)k^2 \qquad (13)$$
$$- (c_1 - \gamma)(q\alpha + \alpha\beta).$$

Lemma 1. *If $c_1 < \gamma$ holds, the positive equilibrium of system (2) is locally stable at $\tau = 0$.*

Proof. For $\tau = 0$, (12) can be rewrite as $F(\lambda) = \lambda^2 + \varphi_1(k)\lambda + \varphi_2(k) + \alpha\beta c_1$. As $c_1 < \gamma, \alpha > 0, \beta > 0, q \in (0,1)$, and $\gamma \in (0,1)$, one can obtain $\forall k \in N, \varphi_1(k) > 0$ and $\varphi_2(k) + \alpha\beta c_1 > \alpha(\gamma - c_1)q + \gamma q > 0$ which means the roots of (12) have negative part. Therefore, the positive equilibrium of system (2) is locally stable at $\tau = 0$. The proof is completed.

Substituting $\lambda = i\omega$ into (12) and separating real parts and imaginary parts yield

$$\alpha\beta c_1 \cos\omega\tau = \omega^2 - \varphi_2(k)$$
$$\alpha\beta c_1 \sin\omega\tau = \varphi_1(k)\omega. \tag{14}$$

According to $\sin^2\omega\tau + \cos^2\omega\tau = 1$, from (14), we have

$$F(\omega) = \omega^4 + \left(\varphi_1^2(k) - 2\varphi_2(k)\right)\omega^2 + \varphi_2^2(k) - (\alpha\beta c_1)^2$$
$$= 0. \tag{15}$$

By simple calculation, we have

$$\varphi_1^2(k) - 2\varphi_2(k)$$
$$= \left(\alpha(\gamma - c_1) + d_1 k^2\right)^2 + \left(q + \beta + d_2 k^2\right)^2 > 0 \tag{16}$$
$$\varphi_2^2(k) - (\alpha\beta c_1)^2$$
$$= \left(\alpha(\gamma - c_1) + d_1 k^2\right)^2 \left(q + \beta + d_2 k^2\right)^2 - (\alpha\beta c_1)^2.$$

As $d_i > 0, \ i = 1, 2$, one can obtain

$$\varphi_2^2(k) - (\alpha\beta c_1)^2$$
$$= \left(\alpha(\gamma - c_1) + d_1 k^2\right)^2 \left(q + \beta + d_2 k^2\right)^2 - (\alpha\beta c_1)^2 \tag{17}$$
$$> \alpha^2 (\gamma - c_1)^2 (q + \beta)^2 - (\alpha\beta c_1)^2.$$

It is easy to see if $(\gamma - c_1)(q + \beta) - (\beta c_1) > 0$, (15) has no positive roots. Combining with the Lemma 1, we have the following theorem. □

Theorem 2. *If $c_1 < \gamma$ and $(\gamma - c_1)(q + \beta) - (\beta c_1) > 0$, the positive equilibrium of system (2) is locally stable for any $\tau > 0$.*

In the following, one investigates the conditions of Hopf bifurcation of (2). By (14), one can obtain

$$\cos\omega\tau = \frac{\omega^2 - \varphi_2(k)}{\alpha\beta c_1}$$
$$\sin\omega\tau = \frac{\varphi_1(k)\omega}{\alpha\beta c_1}. \tag{18}$$

By simple calculation, one has

$$\frac{d\omega}{dk}$$
$$= -\frac{\left(4b_1 k^3 + 2b_2 k + b_3\right)\omega^2 + 8b_4 k^7 + 6b_5 k^5 + 4b_6 k^3 + 2b_7 k}{4\omega^3 + 2\left(b_1 k^4 + b_2 k^2 + b_3\right)\omega} \tag{19}$$
$$< 0,$$

where

$$b_1 = a_5^2 - 2a_1,$$
$$b_2 = 2(a_5 a_6 - a_2),$$
$$b_3 = a_6^2 + 2a_3,$$
$$b_4 = a_1^2,$$
$$b_5 = 2a_1 a_2,$$
$$b_6 = a_2^2 - 2a_1 a_3,$$
$$b_7 = -2a_2 a_3,$$
$$b_8 = a_3^2 - a_4^2, \tag{20}$$
$$a_1 = d_1 d_2,$$
$$a_2 = d_1 q + d_1\beta - d_2\alpha(c_1 - \gamma),$$
$$a_3 = \alpha(q + \beta)(c_1 - \gamma),$$
$$a_4 = \alpha\beta c_1,$$
$$a_5 = d_1 + d_2,$$
$$a_6 = q + \beta - \alpha(c_1 - \gamma).$$

If $c_1 < \gamma$, by simple calculation, one can obtain $b_i > 0, \ i = 1, 2, ..., 7$, and then one has $d\omega/dk < 0$. From (18), one has

$$\tan\omega\tau = \frac{\varphi_1(k)\omega}{\omega^2 - \varphi_2(k)}. \tag{21}$$

Taking the derivative of k in (21), one can obtain

$$\frac{d\tau}{dk} = -\left(\frac{1}{1 + e_0}\left(e_1\omega + \frac{e_2 k^6 + e_3 k^4 + e_4 k^2 + e_5}{\omega}\right)\right.$$
$$\left. + \frac{\tau}{\omega}\right)\frac{d\omega}{dk} + \frac{1}{1 + e_0}\left(e_6\omega^2 + e_7\right), \tag{22}$$

where

$$
e_0 = \left(\frac{\left(a_5 k^2 + a_6 \right) \omega}{\omega^2 - a_1 k^4 - a_2 k^2 + a_3} \right)^2 > 0,
$$

$$
e_1 = a_5 k^2 + a_6 > 0,
$$

$$
e_2 = a_1 a_5 > 0,
$$

$$
e_3 = a_2 a_5 + a_1 a_6, \tag{23}
$$

$$
e_4 = a_2 a_6 - a_3 a_5,
$$

$$
e_5 = -a_3 a_6,
$$

$$
e_6 = 2 a_5 k,
$$

$$
e_7 = 2 a_1 a_5 k^5 + 4 a_1 a_6 k^3 + 2 \left(a_2 a_6 + a_3 a_5 \right) k.
$$

If $c_1 < \gamma$, one can obtain $a_3 < 0$; by simple calculation, one can get $e_i > 0$, $i = 0, 1, \ldots, 7$. Combining with $d\omega/dk < 0$, one has

$$
\frac{d\tau}{dk} > 0, \tag{24}
$$

which means τ increase with the increasing of k. So, τ_0 must exist in $k = 0$. Now, one considers the case $k = 0$. Let $z = \omega^2$; (18) with $k = 0$ is as follows:

$$
F(z) = z^2 + \left(\alpha^2 \left(\gamma - c_1 \right)^2 + \left(q + \beta \right)^2 \right) z \\
+ \alpha^2 \left(\gamma - c_1 \right)^2 \left(q + \beta \right)^2 - \left(\alpha \beta c_1 \right)^2. \tag{25}
$$

Since $\alpha^2 (\gamma - c_1)^2 + (q+\beta)^2 > 0$, (25) has only one positive root of (25) if $(\gamma - c_1)(q + \beta) - (\beta c_1) < 0$. Let z be the positive root of (25); then one can obtain $\omega = \sqrt{z}$

$$
\tau_j^0 = \frac{1}{\omega} \arctan \frac{\varphi_1(0) \omega}{\omega^2 - \varphi_2(0)} + \frac{1}{\omega} j\pi, \quad j = 1, 2, \ldots. \tag{26}
$$

By (26), one can obtain $\tau_0 = \tau_0^0$.

Lemma 3 (see [16, 17]). *Consider the exponential polynomial*

$$
P\left(\lambda, e^{-\lambda \tau}, \ldots, e^{-\lambda \tau m} \right)
$$

$$
= \lambda^n + P_1^{(0)} \lambda^{n-1} + \cdots + P_{n-1}^{(0)} \lambda + P_n^{(0)}
$$

$$
+ \left[P_1^{(1)} \lambda^{n-1} + \cdots + P_{n-1}^{(1)} \lambda + P_n^{(1)} \right] e^{-\lambda \tau} + \cdots \tag{27}
$$

$$
+ \left[P_1^{(m)} \lambda^{n-1} + \cdots + P_{n-1}^{(m)} \lambda + P_n^{(m)} \right] e^{-\lambda \tau m},
$$

where $\tau_i \geq 0$ $(i = 1, 2, \ldots, m)$ *and* $p_j^{(i)}$ $(j = 1, 2, \ldots, m)$ *are constants. As* $(\tau_1, \tau_2, \ldots, \tau_m)$ *vary, the sum of the order of the zeroes of* $P(\lambda, e^{-\lambda \tau}, \ldots, e^{-\lambda \tau_m})$ *in the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

Theorem 4. *According to Lemmas 1 and 3, one has the following.*

If $c_1 < \gamma$ *and* $(\gamma - c_1)(q + \beta) - (\beta c_1) < 0$ *holds, system (2) is asymptotically stable for* $\tau \in [0, \tau_0)$. *System (2) undergoes a*

Hopf bifurcation at the origin when $\tau = \tau_0$; *that is, system (2) has a branch of periodic solutions bifurcating from the trivial solution near* $\tau = \tau_0$.

Remark 5. By incorporate diffusion effect into the Kaldor–Kalecki model of business cycle, a novel Kaldor–Kalecki model of business cycle with diffusion effect and time delay is proposed. Our model is a spatial-temporal model, which is more general than the existing business cycle [8, 9].

3. Direction and Stability of the Hopf Bifurcation

In this section, we give the normal form of Hopf bifurcation of (2) by using the method of [16, 18]. Let $\vartheta = \tau - \tau_0$ and normalize τ by $t \to t/\tau$; (2) can be rewritten as

$$
\dot{R}(t) = \tau_0 D U(t) + L(\tau_0)(U(t)) + F(U(t), \vartheta), \tag{28}
$$

where

$$
D = \operatorname{diag}(D_1, D_2),
$$

$$
L(\tau_0)(\varphi) = \tau_0 \left(B_0 \varphi(0) + B_1 \varphi(-\tau) \right),
$$

$$
B_0 = \begin{bmatrix} \alpha h & -\alpha \beta \\ 0 & -(q + \beta) \end{bmatrix}
$$

$$
B_1 = \begin{bmatrix} 0 & 0 \\ c_1 & 0 \end{bmatrix}.
$$

$$
F(\varphi, \vartheta) = \vartheta D \Delta \varphi(0) + L(\vartheta)(\varphi) + f^*(\varphi, \vartheta) \tag{29}
$$

$$
f^*(\varphi, \vartheta) = (\tau_0 + \vartheta) \begin{pmatrix} \alpha c_2 \varphi_1^2(0) + \alpha c_3 \varphi_1^3(0) \\ c_2 \varphi_1^2(-1) + c_3 \varphi_1^3(-1) \end{pmatrix},
$$

$$
c_1 = i'(0),
$$

$$
c_2 = \frac{1}{2} i''(0),
$$

$$
c_3 = \frac{1}{3} i'''(0).
$$

For $\varphi = (\varphi_1, \varphi_2)^T \in \zeta$, from Theorem 2, we know $\Lambda_0(-i\omega_0 \tau_0, i\omega_0 \tau_0)$ are the eigenvalues of the linear part of (28):

$$
\dot{U}(t) = \tau_0 D u(t) + L(\tau_0)(u(t)), \tag{30}
$$

$$
\dot{z}(t) = L(\tau_0)(z_t), \tag{31}
$$

where $L(\tau_0)$ is one parameter family of bounded linear operator in $C := C([-1, 0] \mathfrak{R}^2)$ into $\mathfrak{R}^2$.

Following the method of [16], we define $\eta(\theta, \tau)$ for $\theta \in [-1, 0]$, such that

$$
L^*(\tau_0)(\varphi) = \int_{-1}^0 d\eta(\theta, \tau_0) \varphi(\theta), \quad \varphi \in C, \tag{32}
$$

where

$$
\eta(\theta, \tau_0) = (\tau_0 + \vartheta) \left[B_0 \delta(\theta) + B_1 \delta(\theta + 1) \right]. \tag{33}
$$

$\delta(\theta)$ is Drac-delta function.

Let $A(\tau_0)$ be the infinitesimal generator of the semigroup induce by the solution of (28) and A^* denote the formal adjoint of $A(\tau_0)$ under the bilinear pairing; we have

$$\langle \psi, \varphi \rangle = \psi(0) \cdot \varphi(0)$$
$$- \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \psi(\xi - \theta) \, d\eta(\theta) \, \phi(\xi) \, d\xi, \tag{34}$$

for $\phi \in C$, $\psi \in C^* = C([0,1], \mathbb{R}^2)$. Then $A(\tau_0)$ and A^* are a pair of adjoint operators.

It is not hard to see that $i\tau_0\omega_0$ is an eigenvalue of $A(\tau_0)$ and $-i\tau_0\omega_0$ is an eigenvalue of A^*. Define $q_1(\theta) = (1, \alpha_1)^T e^{i\omega_0\tau_0\theta}$ and $q_2(\theta) = \overline{q_1}(\theta)$; then, one can obtain

$$Aq_1(0) = i\omega_0\tau_0 q_1(0). \tag{35}$$

Then, we have

$$\alpha_1 = \frac{\alpha h - i\omega_0\tau_0}{\alpha\beta} = \alpha_{11} + i\alpha_{12}, \tag{36}$$

where

$$\alpha_{11} = \frac{\alpha h}{\alpha\beta},$$
$$\alpha_{12} = -\frac{\omega_0\tau_0}{\alpha\beta}. \tag{37}$$

Define $q_1^*(\theta^*) = (1, \beta_1)^T e^{-i\omega_0\tau_0\theta^*}$ and $q_2^*(\theta^*) = \overline{q_1^*}(\theta^*)$; then, one can obtain

$$A^* q_1^*(0) = -i\omega_0\tau_0 q_1^*(0). \tag{38}$$

Then, we have

$$\beta_1 = \beta_{11} + i\beta_{12}, \tag{39}$$

where

$$\beta_{11} = -\frac{\alpha h \cos \omega_0\tau_0 - \sin \omega_0\tau_0 \omega_0\tau_0}{c_1},$$
$$\beta_{12} = -\frac{\omega_0\tau_0 \cos \omega_0\tau_0 + \sin \omega_0\tau_0 \alpha h}{c_1}. \tag{40}$$

Let $u(\theta) = (u_1(\theta), u_2(\theta))$ and $v(\theta) = (v_1(\theta), v_2(\theta))^T$ with

$$u_1(\theta) = \frac{q_1(\theta) + q_2(\theta)}{2}$$
$$= \left(\mathrm{Re}\left\{ e^{i\omega_0\tau_0\theta} \right\}, \mathrm{Re}\left\{ \alpha_1 e^{i\omega_0\tau_0\theta} \right\} \right)$$
$$= \begin{pmatrix} \cos \omega_0\tau_0\theta \\ \alpha_{11} \cos \omega_0\tau_0\theta - \alpha_{12} \sin \omega_0\tau_0\theta \end{pmatrix}$$

$$u_2(\theta) = \frac{q_1(\theta) - q_2(\theta)}{2i}$$
$$= \left(\lim\left\{ e^{i\omega_0\tau_0\theta} \right\}, \lim\left\{ \alpha_1 e^{i\omega_0\tau_0\theta} \right\} \right)$$
$$= \begin{pmatrix} \sin \omega_0\tau_0\theta \\ \alpha_{11} \sin \omega_0\tau_0\theta + \alpha_{12} \cos \omega_0\tau_0\theta \end{pmatrix} \tag{41}$$

for $\theta \in [-1, 0]$ and

$$v_1(\theta^*) = \frac{q_1^*(\theta^*) + q_2^*(\theta^*)}{2}$$
$$= \left(\mathrm{Re}\left\{ e^{-i\omega_0\tau_0\theta^*} \right\}, \mathrm{Re}\left\{ \beta_1 e^{-i\omega_0\tau_0\theta^*} \right\} \right)$$
$$= \begin{pmatrix} \cos \omega_0\tau_0\theta^* \\ \beta_{11} \cos \omega_0\tau_0\theta^* + \beta_{12} \sin \omega_0\tau_0\theta^* \end{pmatrix}$$

$$v_2(\theta^*) = \frac{q_1^*(\theta^*) - q_2^*(\theta^*)}{2i}$$
$$= \left(\lim\left\{ e^{-i\omega_0\tau_0\theta^*} \right\}, \lim\left\{ \beta_1 e^{-i\omega_0\tau_0\theta^*} \right\} \right)$$
$$= \begin{pmatrix} \sin \omega_0\tau_0\theta^* \\ -\beta_{11} \sin \omega_0\tau_0\theta^* + \beta_{12} \cos \omega_0\tau_0\theta^* \end{pmatrix} \tag{42}$$

for $\theta^* \in [-1, 0]$.

In the following, we define $(u, v^*) = (u_j, v^*_k)$ $j, k = 1, 2$ and construct a new basis v

$$v = (v_1, v_2) = (v^*, u)^{-1} v^*. \tag{43}$$

Let $f_k = (\beta_k^1 \ \beta_k^2)$ and $c \cdot f_0 = c_1\beta_0^1 + c_2\beta_0^2$ for $c = (c_1, c_2)^T \in \zeta$; the center space of linear equation (30) is given by

$$P_{CN}\zeta^* = u(v, \langle \varphi, f_k \rangle) \cdot f_k \quad \varphi \in \zeta^*. \tag{44}$$

And $\zeta^* = P_{CN}\zeta^* \oplus Q$, where Q denotes the complement subspace of $P_{CN}\zeta^*$ in ζ^*.

As $A(\tau_0)$ is the infinitesimal generator induced by the solution of (32), then (28) can be rewritten as

$$\dot{U}(t) = A_{\tau_0} U(t) + X_0 F(U_t, \vartheta), \tag{45}$$

where

$$X_0(\theta) = \begin{cases} 0, & -1 \leq \theta < 0 \\ I & \theta = 0. \end{cases} \tag{46}$$

By using the decomposition $\zeta^* = P_{CN}\zeta^* \oplus Q$ and (45), (30) can be written as

$$U(t) = U\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, \vartheta), \tag{47}$$

where $(x_1(t), x_2(t))^T = (v, \langle U(t), f_k \rangle)$, $h(x_1, x_2, \vartheta) \in Q$, and $h(0,0,0) = Dh(0,0,0) = 0$. By using center manifold, we can obtain

$$U(t) = u\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, 0). \tag{48}$$

Let $z = x_1 - ix_2$ and $q_1 = u_1 + iu_2$, from (48), we can obtain

$$R(t) = \frac{1}{2}(q_1 z + \overline{q}_1 \overline{z}) \cdot f_k + W(z, \overline{z}), \tag{49}$$

where

$$W(z, \overline{z}) = h\left(\frac{(z+\overline{z})}{2}, \frac{i(z-\overline{z})}{2}, 0\right)$$

$$\triangleq W_{20}\left(\frac{z^2}{2}\right) + W_{11} z\overline{z} + W_{02}\left(\frac{\overline{z}^2}{2}\right) + \cdots . \tag{50}$$

Following [18], we have

$$\dot{z} = i\omega_0\tau_0 z + g(z, \overline{z}), \tag{51}$$

where

$$g(z, \overline{z}) = (v_1 - iv_2)\left\langle f(U(t), 0), f_k \right\rangle$$

$$\triangleq g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{20}\frac{\overline{z}^2}{2} + \cdots \tag{52}$$

for $v(0) = (v_1(0), v_2(0))^T$.

By calculation, from (52), we have

$$\left\langle f(R(t), 0), f_k \right\rangle = \frac{\tau_0}{4}\frac{1}{\pi}\int_0^\pi \cos^3 kx \, dx \binom{d_{11}}{d_{21}}\frac{z^2}{2}$$

$$+ \frac{\tau_0}{4}\frac{1}{\pi}\int_0^\pi \cos^3 kx \, dx \binom{d_{12}}{d_{22}} z\overline{z}$$

$$+ \frac{\tau_0}{4}\frac{1}{\pi}\int_0^\pi \cos^3 kx \, dx \binom{d_{13}}{d_{23}}\frac{\overline{z}^2}{2}$$

$$+ \frac{\tau_0}{2}\binom{d_{14}}{d_{24}}\frac{z^2\overline{z}}{2}, \tag{53}$$

where

$$d_{11} = ac_2,$$

$$d_{12} = c_2 e^{-i\omega_0\tau_0},$$

$$d_{21} = ac_2,$$

$$d_{22} = c_2 e^{-i\omega_0\tau_0}$$

$$d_{31} = ac_2,$$

$$d_{32} = c_2 e^{-i\omega_0\tau_0}$$

$$d_{41} = ac_2 \left\langle \left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right)\cos kx, \cos kx \right\rangle \tag{54}$$

$$+ \frac{1}{4}ac_3\frac{1}{\pi}\int_0^\pi \cos^4 kx \, dx$$

$$d_{42} = c_2 \left\langle \left(2W_{11}^{(1)}(0) e^{-i\omega_0\tau_0} + W_{20}^{(1)}(0) e^{i\omega_0\tau_0}\right)\right.$$

$$\left. \cdot \cos kx, \cos kx \right\rangle + \frac{1}{4}c_3\frac{1}{\pi}\int_0^\pi \cos^4 kx \, dx$$

$$\left\langle W_{ij}^n(\theta), \cos kx \right\rangle = \frac{1}{\pi}\int_0^\pi W_{ij}^n(\theta)(x)\, dx,$$

$$i + j = 2, \ n = 1, 2, 3, 4.$$

It is easy to see $(1/\pi)\int_0^\pi \cos^3 kx \, dx = 0$ when $k \neq 0$. Let $(\rho_1, \rho_2) = v_1(0) - iv_2(0)$; compare the coefficients with (52); we can obtain

$$g_{20} = \begin{cases} 0 & k = \mathbb{N} \\ \dfrac{\tau}{4}(d_{11}\rho_1 + d_{21}\rho_2) & k = 0 \end{cases}$$

$$g_{11} = \begin{cases} 0 & k = \mathbb{N} \\ \dfrac{\tau}{4}(d_{12}\rho_1 + d_{22}\rho_2) & k = 0 \end{cases} \tag{55}$$

$$g_{02} = \begin{cases} 0 & k = \mathbb{N} \\ \dfrac{\tau}{4}(d_{13}\rho_1 + d_{23}\rho_2) & k = 0. \end{cases}$$

$$g_{21} = \frac{\tau}{2}(d_{14}\rho_1 + d_{24}\rho_2)$$

Because of $W_{20}(\theta)$ and $W_{11}(\theta)$ in the g_{21}, we need to determine them. It follows from (52) that

$$\dot{W}(z, \overline{z}) = W_{20} z\dot{z} + W_{11}\dot{z}\overline{z} + W_{11} z\dot{\overline{z}} + W_{02}\overline{z}\dot{\overline{z}}$$

$$+ \cdots , \tag{56}$$

$$A_{\tau_0}\dot{W}(z, \overline{z}) = A_{\tau_0}W_{20}\frac{z^2}{2} + A_{\tau_0}W_{11} z\overline{z} + A_{\tau_0}W_{02}\frac{\overline{z}^2}{2}$$

$$+ \cdots . \tag{57}$$

According to [18], we have

$$\dot{W} = A_{\tau_0}W + H(z, \overline{z}), \tag{58}$$

where

$$H(z, \overline{z}) = H_{20}\frac{z^2}{2} + H_{11} z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots$$

$$= X_0 f(U(t), 0)$$

$$- u\left(v, \left\langle X_0 f(U(t), 0), f_k \right\rangle\right) \cdot f_k. \tag{59}$$

By comparing (56), (58), and (59), we have

$$\left(A_{\tau_0} - 2i\omega_0\tau_0\right)W_{20}(\theta) = -H_{20}(\theta),$$

$$A_{\tau_0}W_{11}(\theta) = -H_{11}(\theta). \tag{60}$$

Expanding (59), we can obtain

$$H(z, \overline{z}) = -\frac{1}{2}\left[q_1(\theta) g_{20} + q_2(\theta)\overline{g}_{02}\right] \cdot f_k\frac{z^2}{2}$$

$$- \frac{1}{2}\left[q_1(\theta) g_{11} + q_2(\theta)\overline{g}_{11}\right] \cdot f_k z\overline{z} + \cdots . \tag{61}$$

According to (60), the following can be obtained:

$$H_{20}(\theta) = \begin{cases} 0 & k \in \mathbb{N} \\ -\dfrac{1}{2}\left[q_1(\theta) g_{20} + q_2 g_{11}(\theta)\right] \cdot f_0 & k = 0 \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0 & k \in \mathbb{N} \\ -\dfrac{1}{2}\left[q_1(\theta) g_{11} + q_2\overline{g}_{11}(\theta)\right] \cdot f_0 & k = 0. \end{cases} \tag{62}$$

According to (60), (62), and $q_1(\theta) = q_1(0)e^{i\omega_0\tau_0}$, we can obtain

$$W_{20}(\theta) = \frac{1}{2}\left[\frac{ig_{20}}{\omega_0\tau_0}q_1(\theta) + \frac{i\overline{g}_{20}}{3\omega_0\tau_0}q_2(\theta)\right] \cdot f_k$$

$$+ E_1e^{2i\omega_0\tau_0\theta} \tag{63}$$

$$W_{11}(\theta) = \frac{1}{2}\left[\frac{-ig_{11}}{\omega_0\tau_0}q_1(\theta) + \frac{i\overline{g}_{11}}{\omega_0\tau_0}q_2(\theta)\right] \cdot f_k + E_2,$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in \mathfrak{R}^2$, $E_2 = (E_2^{(1)}, E_2^{(2)}) \in \mathfrak{R}^2$, and E_1, E_2 are constant vector.

In order to seek appropriate E_1, E_2, we can obtain the following by the definition of A_{τ_0} and (60)

$$H_{20}(0) = 2i\omega_0\tau_0W_{20}(0) - \tau_0D\Delta W_{20}(0)$$

$$- L(\tau_0)W_{20}(\theta) \tag{64}$$

$$H_{11}(0) = -\tau_0D\Delta W_{11}(0) - L(\tau_0)W_{11}(\theta),$$

where

$$H_{20}(0)$$

$$= \begin{cases} \frac{\tau_0}{4}(d_{11}, d_{21})^T \cos^2 kx & k \in \mathbb{N} \\ \frac{\tau_0}{4}(d_{11}, d_{21})^T - \frac{1}{2}[q_1(0)g_{20} + q_2(0)\overline{g}_{02}] \cdot f_k & k = 0 \end{cases}$$

$$H_{11}(0) \tag{65}$$

$$= \begin{cases} \frac{\tau_0}{4}(d_{12}, d_{22})^T \cos^2 kx & k \in \mathbb{N} \\ \frac{\tau_0}{4}(d_{12}, d_{22})^T - \frac{1}{2}[q_1(0)g_{11} + q_2(0)\overline{g}_{11}] \cdot f_k & k = 0. \end{cases}$$

Then, we can obtain

$$E_1$$

$$= \frac{1}{4}\begin{pmatrix} 2i\omega_0 + d_1k^2 - \alpha(c_1 - \gamma) & \alpha\beta \\ -c_1e^{-2i\omega_0\tau} & 2i\omega_0 + d_2k^2 + (q + \beta) \end{pmatrix}^{-1}$$

$$\cdot \begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix} \cos^2 kx \tag{66}$$

$$E_2 = \frac{1}{4}\begin{pmatrix} d_1k^2 - \alpha(c_1 - \gamma) & \alpha\beta \\ -c_1 & d_2k^2 + (q + \beta) \end{pmatrix}^{-1}$$

$$\cdot \begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix} \cos^2 kx.$$

Now, we can calculate $W_{20}(\theta)$ and $W_{11}(\theta)$; g_{21} is also expressed, and then the following important parameter can be obtained

$$C_1(0) = \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\text{Re}\{C_1(0)\}}{\text{Re}\{\lambda'(\tau_0)\}},$$

$$\tag{67}$$

$$\beta_2 = 2\text{Re}\{C_1(0)\},$$

$$T_2 = -\frac{\text{Im}\{C_1(0)\} + \mu_2\text{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}.$$

Theorem 6. *By Theorem 4, one has the following results:*

(i) *The sign of μ_2 can determine the direction of Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau > \tau_0$ ($\tau < \tau_0$).*

(ii) *The sign of β_2 determines the stability of the bifurcating periodic solutions: if $\beta_2 < 0$, ($\beta_2 > 0$) the bifurcation periodic solutions are stable (unstable).*

(iii) *The sign of T_2 determines the period of the bifurcating periodic solutions: if $T_2 > 0$ ($T_2 < 0$) the period increases (decreases).*

4. Numerical Simulations

In this section, two simulations are given to validate the obtained theorems. Let $d_1 = d_2 = 0.1$, $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.25$, $q = 0.05$, $i(Y(t,x)) = e^{Y(t,x)}/(1 + e^{Y(t,x)})$, and system (2) can be rewrite as follows:

$$\frac{\partial Y(t,x)}{\partial t}$$

$$= 0.1 * \Delta Y(t,x)$$

$$+ 3\left(\frac{e^{Y(t,x)}}{1 + e^{Y(t,x)}} - 0.2Y(t,x) - 0.25K(t,x)\right) \tag{68}$$

$$\frac{\partial K(t,x)}{\partial t}$$

$$= 0.1 * \Delta K(t,x) + \frac{e^{Y(t,x)}}{1 + e^{Y(t,x)}} - 0.25K(t,x),$$

the Neumann boundary conditions of system (2) is as follows:

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial x} = \frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial x} = 0, \quad t > 0, \ x \in \partial\Omega, \tag{69}$$

and the initial conditions is as follows:

$$u_1(t,x) = v_1(t,x) = u_2(t,x) = v_2(t,x) = 0.1. \tag{70}$$

By simple calculations, we can obtain the equilibrium is $(0.498, 2.489)$, $c_1 < \gamma(\gamma - c_1)(q + \beta) - (\beta c_1) = -0.0198 < 0$, $\omega_0 = 0.366$, and $\tau_0 \approx 2.3805$. Consider $\tau = 2.2 < \tau_0$, by Theorem 4, system (2) is asymptotically stable, which is verified in Figures 1 and 2.

Consider $\tau = 2.6 > \tau_0$; by Theorem 4, system (2) is unstable and a Hopf bifurcation occurs. By simple calculation, we can obtain the parameter $C_1(0) = 7.0694 - 0.892020i$, $\mu_2 = 7.3686 > 0$, and $\beta_2 = 14.1388 > 0$ which means Hopf bifurcation is supercritical since $\mu_2 > 0$. The simulations are shown in Figures 3 and 4, which indicate that there is a stable limit cycle.

In the following, we investigate the effect of diffusion on the dynamics of system (2). Let $d_i = 0.1, 0.5$ and $\tau = 2.8 > \tau_0$; the diagrams are shown in Figures 4–7. From Figure 4, we can see that system (2) have a periodic solution when $d_i = 0.1$. With D increasing to 0.5, the periodic solution disappear,
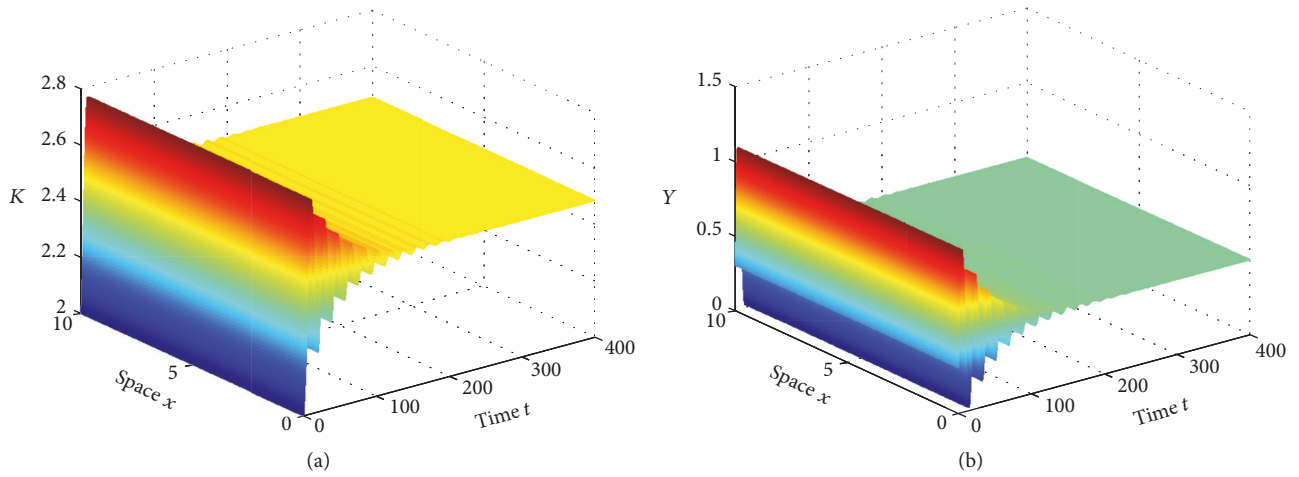
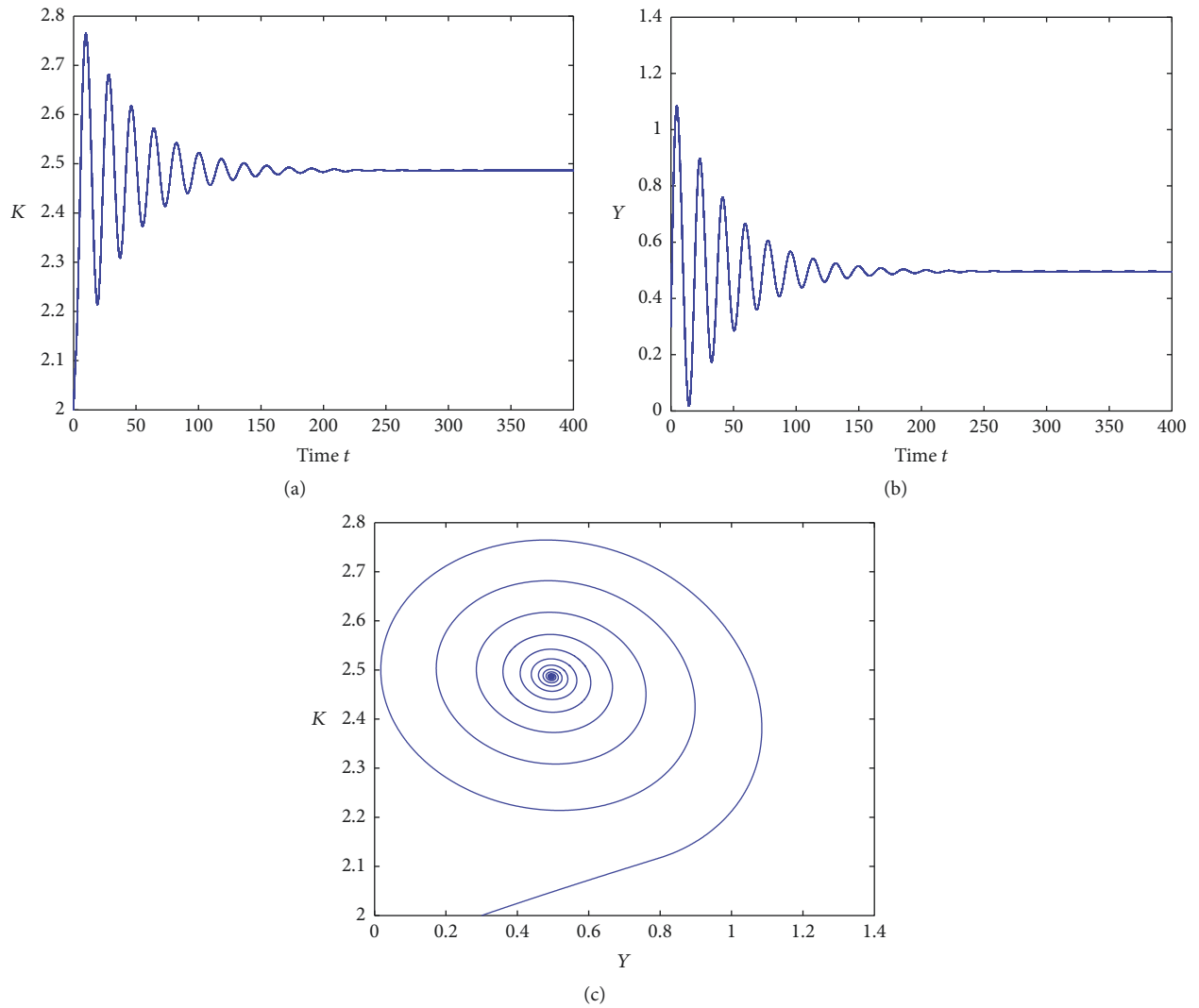FIGURE 1: The temporal solution of system (2) with $\tau = 2.2$.

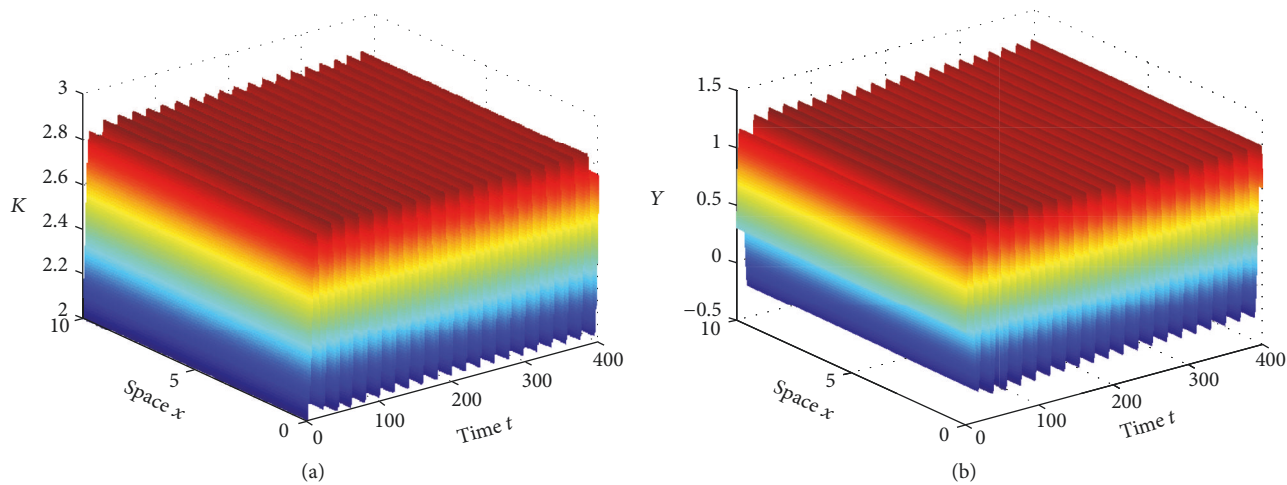FIGURE 2: The temporal solution of system (2) with $\tau = 2.2$, $x = 5$.

(a)

(b)

FIGURE 3: The periodic solutions diagram of system (2) with $\tau = 2.6$.
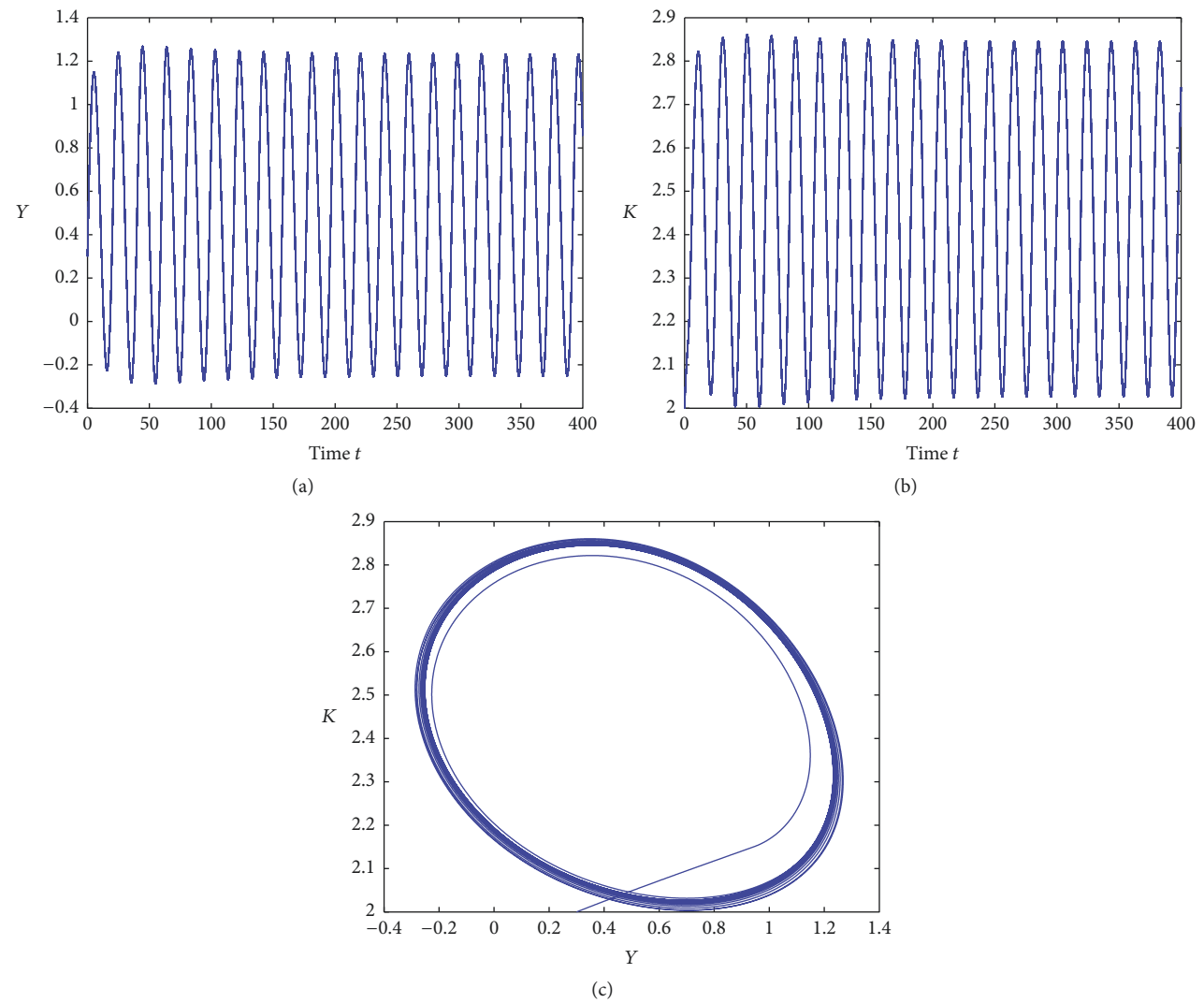

(a)

(b)

(c)

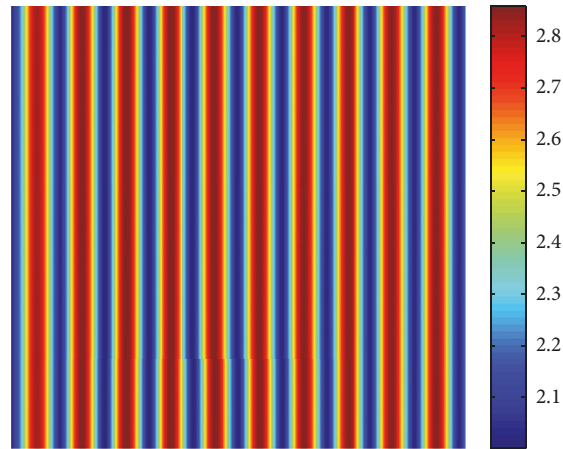FIGURE 4: The periodic solutions diagram of system (2) with $\tau = 2.6$, $x = 5$.

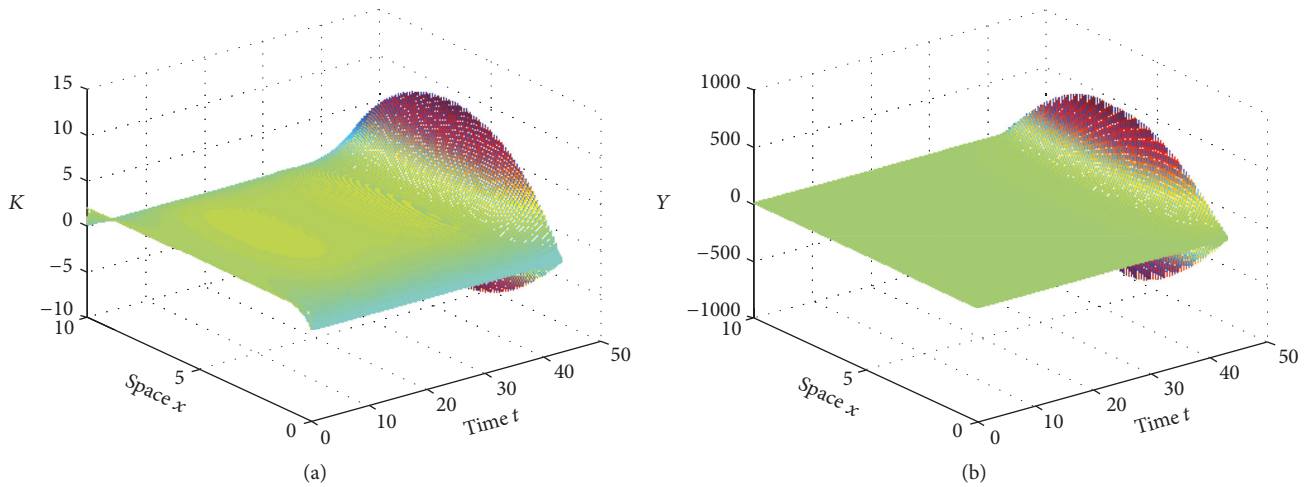FIGURE 5: The spatial pattern of system (2) with $\tau = 2.6$, $d_i = 0.1$.

(a)

(b)

FIGURE 6: The temporal solution of system (2) with $\tau = 2.6$, $d_i = 0.5$.
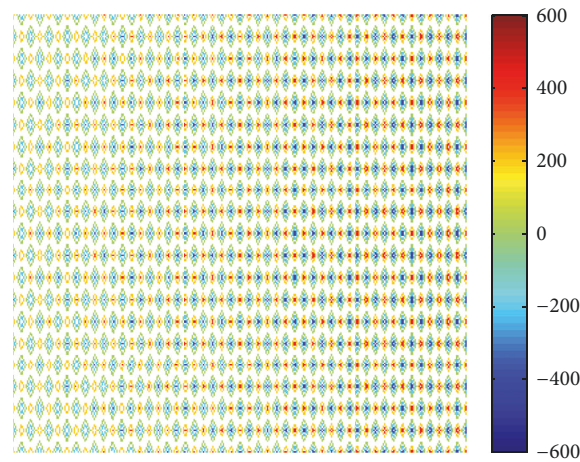
FIGURE 7: The spatial pattern of system (2) with $\tau = 2.6$, $d_i = 0.5$.

system (2) becomes divergent, which is shown in Figure 6. Moreover, from Figures 5–7, we can see that the spatial pattern with $d_i = 0.1$ is different from the spatial pattern with $d_i = 0.5$. To summarize, it can be seen the diffusion coefficient affects the pattern formation of system (2).

5. Conclusions

It is well known that diffusion effects of economic activities are widespread phenomenon that existed all over the world. As a result of the impact of the growth pole, the diffusion effects are the main interactions in economic activities. So, the diffusion effect should be considered in the business cycle model. In this paper, we consider a Kaldor–Kalecki business cycle model with diffusion effect and time delay under the Neumann boundary conditions. First the time-independent and time-dependent stability are investigated. Then, we find that the time delay can give rise to the Hopf bifurcation when the time delay passes a critical value. Moreover, the normal form of Hopf bifurcation is obtained. Finally, numerical results not only validate the obtained theorems, but also show that the diffusion coefficients play a key role in the spatial pattern. With the diffusion coefficients increasing, different patterns appear.
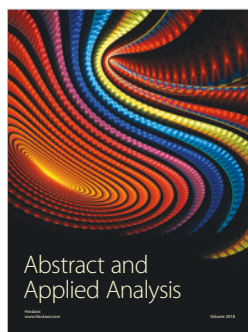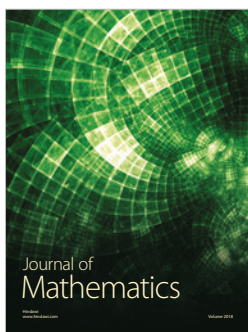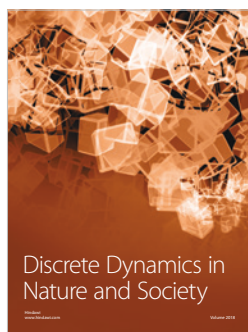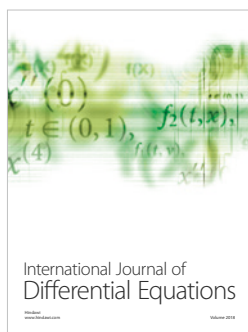
Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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