Research Article

Analytical Solution for the Time Fractional BBM-Burger Equation by Using Modified Residual Power Series Method

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In this study, a generalized Taylor series formula together with residual error function, which is named the residual power series method (RPSM), is used for finding the series solution of the time fractional Benjamin-Bona-Mahony-Burger (BBM-Burger) equation. The BBM-Burger equation is useful in describing approximately the unidirectional propagation of long waves in certain nonlinear dispersive systems. The numerical solution of the BBM-Burger equation is calculated by Maple. The numerical results show that the RPSM is reliable and powerful in solving the numerical solutions of the BBM-Burger equation compared with the exact solutions as well as the solutions obtained by homotopy analysis transform method through different graphical representations and tables.

1. Introduction

Today, fractional differential equations are more and more important in many fields, such as mathematics and dynamic systems [1, 2]. The persons who firstly proposed fractional differential equations were Leibniz and L'Hopital in 1695. Lakshmikantham and Vatsala [3] discussed the basic theory for the initial value problem involving Riemann-Liouville differential operators by fractional differential equations. Diethelm and Ford [4] proposed the analytical questions of existence and uniqueness of solutions by fractional differential equations. And many other academics studied different theories in fractional differential equations. However, most of the problems do not possess analytical solution, and thus, a lot of numerical methods have been developed to solve these fractional differential equations.

Many different methods are introduced to develop an approximate analytical solution for the fractional differential equations and systems, such as the variational iteration method [5], homotopy analysis transform [6], homotopy

asymptotic method [7], (G'/G) expansion method [8], polynomial least squares method [9], and finite difference method [10]. Recently, an analytical method based on power series expansion without linearization, discretization, or perturbation has been introduced and successfully applied to many kinds of fractional differential equations arising in strongly nonlinear and dynamic problems. The method was named residual power series method (RPSM) [11–27], which was used to find the analytical solution for several classes of time fractional differential equations. The residual power series method has been widely used in different fields. In [11], some important theorems that are related to the classical power series have been generalized to the fractional power series by El-Ajou et al. These theorems are constructed by using Caputo fractional derivatives. They also presented and discussed the explicit and approximate solutions of the nonlinear fractional KdV-Burgers equation with time-space-fractional derivatives in [12]. Moaddy et al. [13] proposed that the residual power series method can be applied to differential algebraic equation systems. Jaradat et al. [14]

solved the time fractional Drinfeld-Sokolov-Wilson system by residual power series method. In [15–21], residual power series method, as a powerful method, was used to solve the other time fractional differential equations. Residual power series method was also used for the time fractional Gardner [23] and Kawahara equations in [22], the time fractional Phi-4 equation in [24], the fractional population diffusion model [25], the generalized Burger-Huxley equation [26], and the time fractional two-component evolutionary system of order 2 [27].

In this paper, an analytical solution of the time fractional Benjamin-Bona-Mahony-Burger equation (called BBM-Burger equation) is proposed by residual power series method. The BBM-Burger equation describes the mathematical model of propagation of small-amplitude long waves in nonlinear dispersive media. It is well known that the BBM equation is a refinement of the KdV equation. The BBM-Burger equation and the KdV equation are relevant to the wave breaking models [28]. The KdV equation came from water waves, and the KdV equation as a model was used for long waves in many other physical systems. But in some physical system of long waves, the KdV equation was not applicable. So, the BBM-Burger was proposed; it described unidirectional propagation of long waves in a certain nonlinear dispersive system [28–30]. The integer order of the BBM-Burger equation can be written as

$$u_t - u_{xxt} - \alpha u_{xx} + uu_x + \beta u_x = 0, \quad x \in [x_L, x_R], \quad (1)$$

where α and β are positive constants and $x \in [x_L, x_R]$ is a domain partition.

In order to discuss the dynamic physical system, the time fractional BBM-Burger equation was proposed. The BBM-Burger equation can be written in time fractional operator form as [31]

$$D_t^\alpha u - u_{xxt} + u_x + \left(\frac{u^2}{2}\right)_x = 0, \quad t > 0, x \in I \subseteq \mathbb{R}, \alpha \in (0, 1], \quad (2)$$

where α is a parameter, which is the order of the time fractional derivative and is located in the range of (0,1]. The initial condition is

$$u(x, 0) = \text{sech}^2\left(\frac{x}{4}\right). \quad (3)$$

If $\alpha = 1$, the exact solution [32] is

$$u(x, t) = \text{sech}^2\left(\frac{x}{4} - \frac{t}{4}\right). \quad (4)$$

The rest of the paper is as follows. In Section 2, some basic definitions about the Caputo and modified residual power series method are introduced. In Section 3, we use residual power series method to solve the time fractional BBM-Burger equation specifically. Numerical results and

discussions are presented by graphics and charts in Section 4. At last, the conclusion was drawn in Section 5.

2. Modified Residual Power Series Method

In this section, the definition of the Caputo fractional is introduced systematically. And this section also presents the most details of the modified residual power series method. Fractional residual power series method is used to solve many kinds of differential equations, and this method is effective in calculating these equations.

Definition 1 (see [33]). Let $f(t): [0, +\infty) \rightarrow \mathbb{R}$ be a function and n be the upper positive integer of $\alpha (\alpha > 0)$. The Caputo fractional derivative is defined by

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, & n-1 < \alpha < n, \\ \frac{d^n f(x)}{dx^n}, & \alpha = n \in \mathbb{N}. \end{cases} \quad (5)$$

Theorem 1 (see [33]). *The Caputo fractional derivative of the power function satisfies*

$$D^\alpha x^q = \begin{cases} \frac{\Gamma(q+1)}{\Gamma(q+1-\alpha)} x^{q-\alpha}, & \alpha \leq q, \\ 0, & \alpha > q. \end{cases} \quad (6)$$

Below, we introduce some definitions and theorems related to the fractional power series used in this paper. These important theorems that are related to the fractional power series were presented by El-Ajou et al. [11, 12]. These theorems are constructed by using Caputo fractional derivatives.

Definition 2 (see [11, 12]). A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1 (t-t_0)^\alpha + c_2 (t-t_0)^{2\alpha} + \dots, \quad (7)$$

for $0 \leq n-1 < \alpha \leq n$ and $t \geq t_0$, is called fractional power series about $t = t_0$, where t is a variable and c_m are constants called the coefficients of the series.

Theorem 2 (see [11]). *Suppose that f has a fractional power series representation at $t = t_0$ of the form*

$$f(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha}, \quad 0 \leq n-1 < \alpha \leq n, t_0 \leq t < t_0 + R. \quad (8)$$

If $D^{m\alpha} f(t) \in (t_0, t_0 + R)$, $m = 0, 1, 2, \dots$, then coefficients c_m of (8) are given by the formula

$$c_m = \frac{D^{m\alpha} f(t_0)}{\Gamma(m\alpha + 1)}, \quad m = 0, 1, 2, \dots, \quad (9)$$

where $D^{m\alpha} = D^\alpha \cdot D^\alpha \dots D^\alpha$ (m – times) and R is the radius of convergence.

Definition 3 (see [12]). A power series of the form

$$\sum_{m=0}^{\infty} f_m(x)(t-t_0)^{k\alpha} = f_0(x) + f_1(x)(t-t_0)^\alpha + f_2(x)(t-t_0)^{2\alpha} + \dots, \quad (10)$$

for $0 \leq n-1 < \alpha \leq n$ and $t \geq t_0$, is called multiple fractional power series about $t = t_0$, where t is a variable and f_m are functions of x called the coefficients of the series.

Theorem 3 (see [11, 12]). Suppose that $u(x, t)$ has a multiple power series representation at $t = t_0$ of the form

$$u(x, t) = \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{k\alpha}, \quad (11)$$

$$0 \leq n-1 < \alpha \leq n, x \in I, t_0 \leq t < t_0 + R.$$

If $D_t^{m\alpha} u(x, t)$ are continuous on $I \times (t_0, t_0 + R)$, $m = 0, 1, 2, \dots$, then coefficients $f_m(x)$ of (11) are given as

$$f_m(x) = \frac{D_t^{m\alpha} u(x, t_0)}{\Gamma(m\alpha + 1)}, \quad m = 0, 1, 2, \dots, \quad (12)$$

where $D_t^{m\alpha} = \partial^{m\alpha} / \partial t^{m\alpha} = \partial^\alpha / \partial t^\alpha \cdot \partial^\alpha / \partial t^\alpha \dots \partial^\alpha / \partial t^\alpha$ (m – times) and $R = \min_{C \in I} R_C$, in which R_C is the radius of convergence of the fractional power series $\sum_{m=0}^{\infty} f_m(c)(t-t_0)^{m\alpha}$.

Now, the RPSM can be proposed by

$$u(x, t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}. \quad (13)$$

In order to obtain the approximate value of (13), the form of the i th series of $u(x, t)$ is proposed. Then the truncated series $u_i(x, t)$ is defined by

$$u_i(x, t) = \sum_{n=0}^i f_n(x) \frac{t^{n\alpha}}{\Gamma(1 + n\alpha)}. \quad (14)$$

If $t = 0$, $u(x, 0) = f_0(x)$. We define the i th residual function as follows:

$$\text{Res}_i(x, t) = D_t^\alpha u_i - u_{i, \text{xx}t} + u_{i, x} + \left(\frac{u_i^2}{2} \right)_x. \quad (15)$$

In order to get $f_n(x)$, $n \in N^*$, we look for the solution of

$$D_t^{(n-1)\alpha} \text{Res}_n(x, 0) = 0, \quad n \in N^*, \quad (16)$$

where $N^* = \{1, 2, 3, \dots\}$.

3. Solution of the Time Fractional BBM-Burger Equation by Residual Power Series Method

The purpose of this paper is to use modified residual power series method to solve the time fractional BBM-Burger equation. The initial condition of the time fractional BBM-Burger equation is (3), and the exact solution of the time fractional BBM-Burger equation is (4). In this section, we use residual power series method to solve the time fractional BBM-Burger equation specifically.

$\text{Res}_i(x, t)$ is the i th residual function of (2), which is defined as

$$\text{Res}_i(x, t) = D_t^\alpha u_i(x, t) - u_{i, \text{xx}t}(x, t) + u_{i, x}(x, t) + \left(\frac{u_i^2(x, t)}{2} \right)_x. \quad (17)$$

Step 1. For $i = 1$, the residual function of the time fractional BBM-Burger equation can be written as

$$\text{Res}_1(x, t) = D_t^\alpha u_1(x, t) - u_{1, \text{xx}t}(x, t) + u_{1, x}(x, t) + \left(\frac{u_1^2(x, t)}{2} \right)_x, \quad (18)$$

where $u_1(x, t)$ can be written by (13) as

$$u_1(x, t) = f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1 + \alpha)}. \quad (19)$$

Then we get

$$\begin{aligned} \text{Res}_1(x, t) &= D_t^\alpha u_1(x, t) - u_{1, \text{xx}t}(x, t) + u_{1, x}(x, t) + \left(\frac{u_1^2(x, t)}{2} \right)_x \\ &= f_1(x) - f_1''(x) \frac{\alpha t^{\alpha-1}}{\Gamma(1 + \alpha)} + f_0'(x) + f_1'(x) \frac{t^\alpha}{\Gamma(1 + \alpha)} \\ &\quad + \left[f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1 + \alpha)} \right] \\ &\quad * \left[f_0'(x) + f_1'(x) \frac{t^\alpha}{\Gamma(1 + \alpha)} \right] \\ &= f_1(x) - \frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \\ &\quad - \frac{1}{2} \left[\text{sech}^2\left(\frac{x}{4}\right) + \frac{f_1 x t^\alpha}{\Gamma(1 + \alpha)} \right] \\ &\quad * \text{sech}^2\left(\frac{x}{4}\right) * \tanh\left(\frac{x}{4}\right). \end{aligned} \quad (20)$$

For $t = 0$, we have

$$\text{Res}_1(x, t)|_{t=0} = f_1(x) + f_0'(x) + f_0(x) * f_0'(x). \quad (21)$$

In addition, since

$$f_0(x) = u(x, 0) = \text{sech}^2\left(\frac{x}{4}\right), \quad (22)$$

and

$$f'_0(x) = -\frac{1}{2} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right), \quad (23)$$

then according to $\operatorname{Res}_1(x, 0) = 0$, we have

$$f_1(x) = \frac{1}{2} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right). \quad (24)$$

Step 2. For $i = 2$, the residual function of the time fractional BBM-Burger equation can be written as

$$\begin{aligned} \operatorname{Res}_2(x, t) = & D_t^\alpha u_2(x, t) - u_{2,xxx}(x, t) \\ & + u_{2,x}(x, t) + \left(\frac{u_2^2(x, t)}{2}\right)_x, \end{aligned} \quad (25)$$

with the condition

$$u_2(x, t) = f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (26)$$

Therefore, we can attain

$$\begin{aligned} \operatorname{Res}_2(x, t) = & D_t^\alpha u_2(x, t) - u_{2,xxx}(x, t) + u_{2,x}(x, t) + \left(\frac{u_2^2(x, t)}{2}\right)_x \\ = & f_1(x) + \frac{f_2(x)t^\alpha}{\Gamma(1+\alpha)} - f_1''(x) \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)} \\ & - f_2''(x) \frac{2\alpha t^{2\alpha-1}}{\Gamma(1+2\alpha)} + f_0'(x) \\ & + f_1'(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2'(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \left(f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ & * \left(f_0'(x) + f_1'(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2'(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right) \\ = & f_2(x) - \frac{1}{4} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\ & + \frac{1}{2} \operatorname{sech}^2\left(\frac{x}{4}\right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right)\right) \\ & - \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \\ & \cdot \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right)\right) + \left[\frac{1}{2} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right. \\ & \left. + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) + \frac{f_2(x)t^\alpha}{\Gamma(1+\alpha)}\right] \end{aligned}$$

$$\begin{aligned} & * \left[-\frac{1}{2} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) + \frac{t^\alpha}{\Gamma(1+\alpha)} \right. \\ & * \left(-\frac{1}{4} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \right. \\ & \cdot \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right) - \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\ & \left. \left. + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right) \right) \right] \\ & + \left[\operatorname{sech}^2\left(\frac{x}{4}\right) + \frac{f_2(x)t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)} \right. \\ & * \left(\frac{1}{2} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right. \\ & \left. + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right) \\ & * \left(-\frac{1}{4} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \right. \\ & \cdot \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right) - \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\ & \left. \left. + \frac{1}{2} \operatorname{sech}^4\left(\frac{x}{4}\right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right) \right) \right]. \end{aligned} \quad (27)$$

Then we solve $D_t^\alpha \operatorname{Res}_2(x, 0) = 0$; thus,

$$\begin{aligned} f_2(x) = & \frac{7}{8} \operatorname{sech}^6\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) - \frac{1}{8} \operatorname{sech}^6\left(\frac{x}{4}\right) \\ & + \frac{5}{4} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) - \frac{1}{4} \operatorname{sech}^4\left(\frac{x}{4}\right) \\ & + \frac{3}{8} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) - \frac{1}{8} \operatorname{sech}^2\left(\frac{x}{4}\right). \end{aligned} \quad (28)$$

Step 3. For $i = 3$, the residual function of the time fractional BBM-Burger equation can be written by

$$\begin{aligned} \operatorname{Res}_3(x, t) = & D_t^\alpha u_3(x, t) - u_{3,xxx}(x, t) \\ & + u_{3,x}(x, t) + \left(\frac{u_3^2(x, t)}{2}\right)_x, \end{aligned} \quad (29)$$

with the condition

$$\begin{aligned} u_3(x, t) = & f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \\ & + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \end{aligned} \quad (30)$$

Then we can get

$$\operatorname{Res}_3(x, t) = D_t^\alpha u_3(x, t) - u_{3,xxx}(x, t) + u_{3,x}(x, t) + \left(\frac{u_3^2(x, t)}{2}\right)_x$$

$$\begin{aligned}
&= f_1(x) + \frac{f_2(x)t^\alpha}{\Gamma(1+\alpha)} + \frac{f_3(x)t^{2\alpha}}{\Gamma(1+2\alpha)} - f_1''(x) \frac{\alpha t^{\alpha-1}}{\Gamma(1+\alpha)} \\
&\quad - f_2''(x) \frac{2\alpha t^{2\alpha-1}}{\Gamma(1+2\alpha)} - f_3''(x) \frac{3\alpha t^{3\alpha-1}}{\Gamma(1+3\alpha)} + f_0'(x) \\
&\quad + f_1'(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2'(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad + f_3'(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \left(f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} \right. \\
&\quad \left. + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + f_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) \\
&\quad * \left(f_0'(x) + f_1'(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2'(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \right. \\
&\quad \left. + f_3'(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right). \tag{31}
\end{aligned}$$

So, from $D_t^{2\alpha} \text{Res}_3(x, 0) = 0$, we get

$$\begin{aligned}
f_3(x) &= \frac{35}{16} \text{sech}^8\left(\frac{x}{4}\right) \tanh^3\left(\frac{x}{4}\right) - \frac{11}{16} \text{sech}^8\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \\
&\quad + \frac{17}{4} \text{sech}^6\left(\frac{x}{4}\right) \tanh^3\left(\frac{x}{4}\right) - \frac{13}{8} \text{sech}^6\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \\
&\quad + \frac{39}{16} \text{sech}^4\left(\frac{x}{4}\right) \tanh^3\left(\frac{x}{4}\right) - \frac{19}{16} \text{sech}^4\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \\
&\quad + \frac{3}{8} \text{sech}^2\left(\frac{x}{4}\right) \tanh^3\left(\frac{x}{4}\right) - \frac{1}{4} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right). \tag{32}
\end{aligned}$$

Step 4. For $i = 4$, the residual function of the time fractional BBM-Burger equation can be written as

$$\begin{aligned}
\text{Res}_4(x, t) &= D_t^\alpha u_4(x, t) - u_{4,xx}(x, t) \\
&\quad + u_{4,x}(x, t) + \left(\frac{u_4^2(x, t)}{2} \right)_x, \tag{33}
\end{aligned}$$

with the condition

$$\begin{aligned}
u_4(x, t) &= f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&\quad + f_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + f_4(x) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}. \tag{34}
\end{aligned}$$

Using the same method, through the equation of $D_t^{3\alpha} \text{Res}_4(x, 0) = 0$, we can get $f_4(x)$ as follows:

$$\begin{aligned}
f_4(x) &= \frac{385}{64} \text{sech}^8\left(\frac{x}{4}\right) \tanh^4\left(\frac{x}{4}\right) - \frac{35}{16} \text{sech}^8\left(\frac{x}{4}\right) \\
&\quad \cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^8 \tanh\left(\frac{x}{4}\right) \\
&\quad * \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^3 - \frac{51}{16} \text{sech}^8\left(\frac{x}{4}\right) \tanh^2
\end{aligned}$$

$$\begin{aligned}
&\cdot \left(\frac{x}{4} \right) + \frac{153}{16} \text{sech}^6\left(\frac{x}{4}\right) \tanh^4\left(\frac{x}{4}\right) + \frac{11}{16} \text{sech}^8\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^8 \tanh\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right) - \frac{17}{4} \text{sech}^8\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^6 \tanh\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^3 + \frac{11}{64} \text{sech}^8\left(\frac{x}{4}\right) \\
&- \frac{193}{32} \text{sech}^6\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) + \frac{273}{64} \text{sech}^4\left(\frac{x}{4}\right) \tanh^4 \\
&\cdot \left(\frac{x}{4} \right) + \frac{13}{8} \text{sech}^8\left(\frac{x}{4}\right) \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \right)^6 \tanh\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right) - \frac{39}{16} \text{sech}^8\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^4 \tanh\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^3 + \frac{13}{32} \text{sech}^6\left(\frac{x}{4}\right) \\
&- \frac{53}{16} \text{sech}^4\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) + \frac{15}{32} \text{sech}^2\left(\frac{x}{4}\right) \tanh^4\left(\frac{x}{4}\right) \\
&+ \frac{19}{16} \text{sech}^8\left(\frac{x}{4}\right) \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^4 \tanh \\
&\cdot \left(\frac{x}{4} \right) \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right) - \frac{3}{8} \text{sech}^8\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^2 \tanh\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^3 + \frac{19}{64} \text{sech}^4\left(\frac{x}{4}\right) \\
&- \frac{15}{32} \text{sech}^2\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) + \frac{1}{4} \text{sech}^8\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right)^{2*} \tanh\left(\frac{x}{4}\right) \\
&\cdot \left(-\frac{1}{2} \text{sech}^2\left(\frac{x}{4}\right) \tanh\left(\frac{x}{4}\right) \right) - \text{sech}^8\left(\frac{x}{4}\right) \\
&- \frac{35}{8} \text{sech}^8\left(\frac{x}{4}\right) \tanh^4\left(\frac{x}{4}\right) + \frac{105}{16} \text{sech}^8\left(\frac{x}{4}\right) \tanh^2 \\
&\cdot \left(\frac{x}{4} \right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right) + \frac{11}{8} \text{sech}^8\left(\frac{x}{4}\right) \tanh^2 \\
&\cdot \left(\frac{x}{4} \right) - \frac{11}{16} \text{sech}^8\left(\frac{x}{4}\right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right) \\
&- \frac{51}{8} \text{sech}^6\left(\frac{x}{4}\right) \tanh^4\left(\frac{x}{4}\right) + \frac{51}{4} \text{sech}^6\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\
&* \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right) + \frac{39}{16} \text{sech}^6\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\
&- \frac{13}{8} \text{sech}^6\left(\frac{x}{4}\right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right) \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{39}{16} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh^4\left(\frac{x}{4}\right) + \frac{117}{16} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\
& \cdot \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right)\right) + \frac{19}{16} \operatorname{sech}^4\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\
& - \frac{19}{16} \operatorname{sech}^4\left(\frac{x}{4}\right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right)\right) \\
& - \frac{3}{16} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh^4\left(\frac{x}{4}\right) + \frac{9}{8} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\
& \cdot \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right)\right) + \frac{1}{8} \operatorname{sech}^2\left(\frac{x}{4}\right) \tanh^2\left(\frac{x}{4}\right) \\
& - \frac{1}{4} \operatorname{sech}^2\left(\frac{x}{4}\right) \left(\frac{1}{4} - \frac{1}{4} \tanh^2\left(\frac{x}{4}\right)\right)^2 + \frac{1}{16} \operatorname{sech}^2\left(\frac{x}{4}\right).
\end{aligned} \tag{35}$$

Thus, the approximate solution of the time fractional BBM-Burger equation is

$$\begin{aligned}
u_4(x, t) = & f_0(x) + f_1(x) \frac{t^\alpha}{\Gamma(1+\alpha)} + f_2(x) \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
& + f_3(x) \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + f_4(x) \frac{t^{4\alpha}}{\Gamma(1+4\alpha)},
\end{aligned} \tag{36}$$

where $f_0(x)$ is given in the initial condition at (3) and $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ are given at (24)–(35).

4. Results and Discussion

In this section, the approximate analytical solution of the time fractional BBM-Burger equation by using residual power series method is calculated. We can compare the exact solution of the BBM-Burger equation with the analytical approximate solution by graphics and charts.

In Figure 1, the approximate solutions and the exact solutions are presented by drawing three-dimensional graphics. Figure 1(a) presents the approximate solution, where $\alpha = 0.9$, and Figure 1(b) presents the exact solution at $\alpha = 1$.

In Figure 1, when α approaches 1, the approximate solution is close to the exact solution. So, we can conclude that when α approaches 1, the three-dimensional graphic is accurate and when α approaches 0, the three-dimensional graphic is inaccurate. In such phenomena, one can say that when alpha approaches 0, the solution bifurcates or admits chaotic behavior.

In Figure 2, the three-dimensional graphics show the influence of different α on analytical solutions. Figure 2(a) presents approximate solutions when $\alpha = 0.25$, and Figure 2(b) presents approximate solutions when $\alpha = 0.5$. In Figure 3, the three-dimensional graphics show approximate solutions when $\alpha = 0.8$ and $\alpha = 1$ (Figures 3(a) and 3(b), respectively).

In Figures 2 and 3, we find that the larger the value of α is, the smoother is the plane. As parameter α increases, the graphics get closer and closer to the exact solution of the graphic.

For any $\alpha \in (0, 1]$, the exact value of $\operatorname{Res}(x, t)$ is 0. The difference between the 4th approximate solutions and the exact solutions can be shown by the value of $\operatorname{Res}(x, t)$. In Figure 4,

the two-dimensional graphics show the influence of different α and x on the value of $\operatorname{Res}(x, t)$. We can find the impact of different α and x on $\operatorname{Res}_4(x, t)$ at $t = 0.01$ and $t = 0.001$. In Figure 4, the different colors present different curves. In Figure 4(a), the parameter t is 0.01; in Figure 4(b), the parameter t is 0.001.

As shown in Figure 4, if the values of α and t are fixed, when $|x| > 8$, the values of $\operatorname{Res}_4(x, t)$ are close to 0. When the values of x are not in this interval, the values of $\operatorname{Res}_4(x, t)$ are not close to 0. For $x \in (-8, 8)$, if the values of x and t are fixed, the value of $|\operatorname{Res}_4(x, t)|$ decreases with an increase in α , and if the values of x and α are fixed, the value of $|\operatorname{Res}_4(x, t)|$ increases with a decrease in t .

In Figure 5, the two-dimensional graphics show the impact of different x and α on the value of $|\operatorname{Res}_4(x, t)|$. If the value of x is fixed, the relationship of t and $\operatorname{Res}_4(x, t)$ is presented in the each panel of Figure 5. Different colors present different α .

In Figure 5, we can see that if the values of α and t are fixed, the value of $|\operatorname{Res}_4(x, t)|$ decreases with an increase in constant x . If the value of t and x are fixed, the value of $|\operatorname{Res}_4(x, t)|$ decreases with an increase in α . If the value of α and x are fixed, the value of $|\operatorname{Res}_4(x, t)|$ decreases with an increase in $t \in (0, 0.2)$. As shown in Figures 5(a)–5(d), the value of $|\operatorname{Res}_4(x, 0)|$ is a constant and $|\operatorname{Res}_4(x, 0)| \neq 0$. However, if $t \rightarrow 0$, the value of $|\operatorname{Res}_4(x, t)| \rightarrow 0$, which is because the power series approximate solution of $u_4(x, t)$ is a generalized Taylor expansion at $t_0 = 0$. If $t \rightarrow t_0$, the precision of the power series approximate solution of $u_4(x, t)$ is higher.

Then we compare the exact solution with the approximate solution for the time fractional BBM-Burger equation. The absolute error is

$$\operatorname{Error}(x, t) = |u(x, t)_{\text{exact}} - u(x, t)_{\text{RPSM}}|. \tag{37}$$

In Table 1, we present the exact solutions, the 4th-term approximate solutions by residual power series method, and the absolute errors. We can find the relationship between x and t and the solutions of $u(x, t)_{\text{exact}}$ and $u(x, t)_{\text{RPSM}}$.

In Table 1, we can conclude that the greater the absolute value of x is, the smaller the $u(x, t)_{\text{exact}}$ and $u_4(x, t)_{\text{RPSM}}$ values are. At the same condition of x , we can find that the smaller the value of t is, the smaller the absolute errors are. There are two conditions at the same value of t —one is that the absolute errors are smallest when the value of x is zero and the other is that the smaller the values of absolute x are, the larger are the values of absolute errors. In general, we can find that the absolute errors with different x and t between the 4th-term analytical approximate solutions and the exact solutions are within the acceptable range. The range of magnitude of absolute errors is from 10^{-3} to 10^{-7} .

As shown in Table 2, we compare the 4th-term approximate solutions by residual power series method (RPSM) with the 5th-term approximate solutions by fractional homotopy analysis transform method (FHATM) in [31].

By comparing 4th-term approximate solutions by residual power series method (RPSM) with the 5th-term approximate solutions by fractional homotopy analysis transform

FIGURE 1: 3D graphics of the exact and approximate solutions.

FIGURE 2: Approximate solution $u_4(x, t, \alpha = 0.25, 0.5)$. FIGURE 3: Approximate solution $u_4(x, t, \alpha = 0.8, 1)$.

FIGURE 4: The impact of different t and α on $\text{Res}_4(x, t)$. FIGURE 5: The impact of different t and α on $|\text{Res}_4(x, t)|$.

TABLE 1: Solutions for $\alpha = 1$.

x	t	$u(x, t)_{\text{exact}}$	$u_4(x, t)_{\text{RPSM}}$	$\text{Error}(x, t)_{\text{RPSM}}$
-15	0.001	0.0022087888	0.0022087864	2.44×10^{-9}
	0.01	0.0021988825	0.0021988584	2.41×10^{-8}
	0.1	0.0021022279	0.0021019994	2.29×10^{-7}
-10	0.001	0.0265791118	0.0265787633	3.49×10^{-7}
	0.01	0.0264613640	0.0264579103	3.45×10^{-6}
	0.1	0.0253118301	0.0252801959	3.16×10^{-5}
-5	0.001	0.2802959508	0.2802626306	3.33×10^{-5}
	0.01	0.2792275395	0.2788971245	3.30×10^{-4}
	0.1	0.2687235583	0.2656871698	3.30×10^{-3}
0	0.001	0.9999999376	0.9999997500	1.88×10^{-7}
	0.01	0.9999937500	0.9999750000	1.88×10^{-6}
	0.1	0.9993752604	0.9974997396	1.88×10^{-3}
5	0.001	0.2805338222	0.2805672046	3.34×10^{-5}
	0.01	0.2816062545	0.2819428891	3.37×10^{-4}
	0.1	0.2925122644	0.2961686088	3.66×10^{-3}
10	0.001	0.0266053480	0.0266056972	3.49×10^{-7}
	0.01	0.0267237277	0.0267272511	3.52×10^{-6}
	0.1	0.0279364632	0.0279748560	3.84×10^{-5}
15	0.001	0.0022109963	0.0022109987	2.44×10^{-9}
	0.01	0.0022209571	0.0022209817	2.46×10^{-8}
	0.1	0.0023230642	0.0023233232	2.59×10^{-7}

TABLE 2: Comparison between $\text{Error}(x, t)_{\text{RPSM}}$ and $\text{Error}(x, t)_{\text{FHATM}}^{[31]}$ at $\alpha = 1$.

x	t	$u(x, t)_{\text{exact}}$	$u_4(x, t)_{\text{RPSM}}$	$\text{Error}(x, t)_{\text{RPSM}}$	$\text{Error}(x, t)_{\text{FHATM}}^{[31]}$
10	0.01	2.672×10^{-2}	2.673×10^{-2}	3.523×10^{-6}	4.529×10^{-5}
	0.001	2.661×10^{-2}	2.661×10^{-2}	3.492×10^{-7}	4.501×10^{-6}
15	0.01	2.221×10^{-3}	2.221×10^{-3}	2.464×10^{-8}	3.717×10^{-6}
	0.001	2.211×10^{-3}	2.211×10^{-3}	2.441×10^{-9}	3.697×10^{-7}
20	0.01	1.825×10^{-4}	1.825×10^{-4}	1.663×10^{-10}	3.034×10^{-7}
	0.001	1.817×10^{-4}	1.817×10^{-4}	1.640×10^{-11}	3.018×10^{-8}

TABLE 3: Comparison the 4th residual function for different α and x at $t = 0.01$.

α	$\text{Res}_4(10, 0.01)$	$\text{Res}_4(15, 0.01)$	$\text{Res}_4(20, 0.01)$	$\text{Res}_4(25, 0.01)$
0.1	-0.3464731857	-0.0043621541	-0.0003541594	-0.0000290446
0.3	-0.1648036880	-0.0030670425	-0.0002505805	-0.0000205608
0.5	-0.0618059110	-0.0017109445	-0.0001402342	-0.0000115097
0.7	-0.0206388560	-0.0008737493	-0.0000716902	-0.0000058845
0.9	-0.0062622125	-0.0004139411	-0.0000339747	-0.0000027888

method (FHATM), we can find the absolute errors are smaller than the results in [31]. The absolute error by using RPSM is one order of magnitude smaller than that by using

FHATM. So, residual power series method is efficient and accurate for solving the time fractional BBM-Burger equation. In addition, in Table 2, we can conclude that when the

value of parameter t is smaller, the absolute error is smaller and when the value of parameter x is larger, the absolute error is smaller; in contrast, when the value of parameter α approaches 1, the absolute error is smaller.

We compare the value of the 4th residual function for different α at $t = 0.01$ in Table 3.

In Table 3, we get the solutions of the 4th residual function $\text{Res}_4(x, t)$. If the value of t and α are fixed, the value of $|\text{Res}_4(x, t)|$ decreases with an increase in x . If the value of t and x are fixed, the value of $|\text{Res}_4(x, t)|$ decreases with an increase in α . So, when the value of $\text{Res}_4(x, t)$ is close to 0, the approximate solutions are close to exact solutions and the approximate solutions are accurate. In Table 3, we can conclude that when the value of α is approaching 1, the solutions are more accurate.

In conclusion, the residual power series method is a powerful method to solve the analytical approximate solution of the time fractional BBM-Burger equation in $|x| > 10$ and $t \in (0, 0.2)$.

5. Conclusion

In this paper, we discuss the analytical solution of the time fractional BBM-Burger equation by using residual power series method (RPSM). The time fractional BBM-Burger equation is calculated by Maple in Windows 7 (64 bit). The analytical solution is presented by graphics and datum. Results show that the analytical solutions by residual power series method are close to the exact solution. In general, RPSM is an effective and convenient method in finding analytical solution for the time fractional BBM-Burger equation and other long waves in certain nonlinear dispersive systems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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