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Research Article

Error Estimates for the Heterogeneous Multiscale Finite Volume Method of Convection-Diffusion-Reaction Problem

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Based on the heterogeneous multiscale method, this paper presents a finite volume method to solve multiscale convection-diffusion-reaction problem. The paper constructs an algorithm of the optimal order convergence rate in H^1-norm under periodic medias.

1. Introduction

This paper considers the multiscale method for the following convection-diffusion-reaction problem

$$\begin{cases} -\nabla\left(a^{\varepsilon}(x)\nabla u^{\varepsilon}(x)\right) + b^{\varepsilon}(x)\nabla u^{\varepsilon}(x) + c^{\varepsilon}(x)u^{\varepsilon}(x) = f(x), & x \in \Omega, \\ u^{\varepsilon}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) is a bounded convex polygonal domain with Lipschitz boundary $\partial\Omega$, $\varepsilon \ll 1$ is a positive parameter which signifies the multiscale nature of (1). This problem is related to groundwater and solute transport in porous media [1].

Optimal order convergence rate of classical finite element method based on piecewise linear polynomials relies on the H^2-norm of u^{ε}. As coefficients vary on a scale of ε, the solution u^{ε} may also oscillate at the same scale. A direct numerical solution of this multiscale problem is very difficult to derive unless the mesh size is sufficiently smaller. However, it is not feasible in practice since the amount of computation will increase sharply as the amount of calculation increases. On the other hand, from an engineering point of view, the macroscopic features of the solution are often of main interest and importance. According to homogenization theory [2, 3], there is a homogenized equation which can capture the macroscopic properties. In other words, there exist homogenized coefficients a^0, b^0, and c^0 so that

$$\begin{cases} -\nabla\left(a^{0}(x)\nabla u^{0}(x)\right) + b^{0}(x)\nabla u^{0}(x) + c^{0}(x)u^{0}(x) = f(x), & x \in \Omega, \\ u^{0}(x) = 0, & x \in \Omega. \end{cases} \tag{2}$$

Though there exist many classical methods for solving (2), unfortunately, in general, there are no explicit formulae for the homogenized coefficients, except that there are many restrictive assumptions on the media. Thus, developing numerical methods that can capture the effect of small scale on the large scale is an attractive subject. To overcome the above difficulties, many strategies have been established to solve the problems on grids which are coarser than the scale of oscillation; see [4–9] and references therein.

The multiscale finite volume (MSFV) method was first introduced by Jenny et al. for the elliptic problem of highly oscillatory coefficients [9]. Based on a special interpolation from the coarse mesh to the fine mesh, this method captured the effect of small scales on a coarse grid efficiently. It not only fitted into finite volume framework nicely but also kept both the conservation of coarse and fine scales. This method was widely applied in many situations, such as discrete

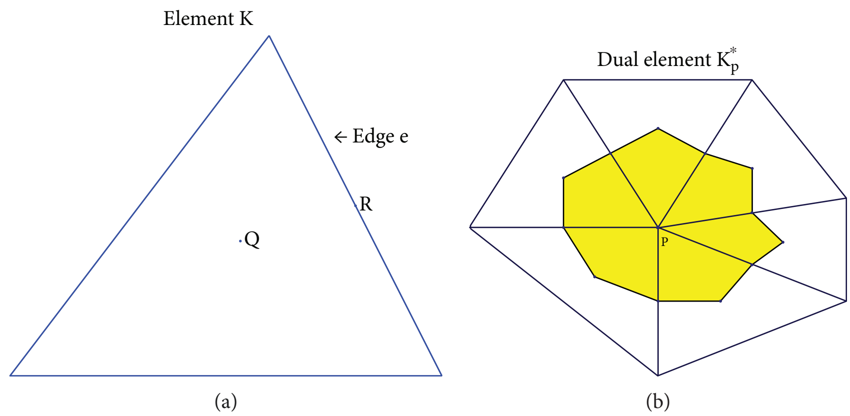

FIGURE 1: An element K (a) and its dual element (b) with respect to a node P.

fracture modeling [10], parabolic problem [11], and Maxwell's equations [12]. Recently, Weinan E. and Engquist established a general efficient methodology—heterogeneous multiscale method (HMM) [13], for the multiscale problems, which consists of two components: (a) selecting a macroscopic solver on a coarse mesh and (b) estimating the missing macroscale data by solving local fine-scale problems. The careful selection of the macro solver and local fine problems is a key issue for the method. A different choice of macroscopic solver will lead to a different heterogeneous multiscale method; see some examples for finite element method [8, 14, 15] and discontinuous finite element method [4, 16]. It is well known that the finite volume method has many advantages, such as keeping conservation and applying to complex regions. Thus, in this paper, we choose the finite volume method (FVM) introduced in [17] as the macroscopic solver. For convenience, the heterogeneous multiscale method taking finite volume method as the macroscopic solver is denoted as HMM-FVM. We will show that our method has optimal order convergence rate in H^1-norm for periodic medias.

The rest of this paper is organized as follows. To solve (1), Section 2 first constructs a HMM-FVM. Then, Section 3 will study the approximate solution and its error estimates for periodic media associated with (1) in detail. We finally conclude the paper in Section 4.

2. HMM-FVM for Convection-Diffusion-Reaction Problem

In this section, we first concisely describe the finite volume method for convection-diffusion-reaction problem in [17]. Then, using the method of [17] as a macroscopic solver of HMM, we derive the HMM-FVM in detail.

Let $\mathcal{T}_H$ be a quasi-uniform triangulation of the polygonal domain Ω. The barycenter dual decomposition $\mathcal{T}_H^*$ is constructed by connecting the barycenter to the midpoints of edges of each triangle element. Suppose K is a triangle element, denote e and Q, respectively, as an edge and barycenter of K and R the midpoint of e. Assume P is a nodal point and K_P^* is the dual element with respect to P (referring to Figure 1). Let H be the maximum length of the edges, N_H the set of all nodal points, $\dot{K}$ the vertices of K.

Let $V = H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$ and $V_H \subset V$ be the piecewise linear finite element space on $\mathcal{T}_H$, Π_H^* be the interpolation operator from V_H to the piecewise constant space on $\mathcal{T}_H^*$:

$$(\Pi_H^* v_H)(x) = v_H(P), x \in K_P^*, \forall P \in N_H, v_H \in V_H. \tag{3}$$

Then, the finite volume method [17] reads as finding $u_H \in V_H$ such that

$$A(u_H, \Pi_H^* v_H) = (f, \Pi_H^* v_H), \forall v_H \in V_H, \tag{4}$$

where

$$\begin{aligned} A(u_H, \Pi_H^* v_H) &= \sum_{P\in N_H} \left[-\int_{\partial K_P^*} \mathbf{n} \cdot \left(a^H \nabla u_H\right) \Pi_H^* v_H ds \right. \\ &\quad \left. + \int_{K_P^*} \left(b^H \nabla u_H + c^H u_H\right) \Pi_H^* v_H dx \right] \\ &= \sum_{K\in\mathcal{T}_H} \left[-\sum_{P\in\dot{K}} \int_{K\cap\partial K_P^*} \mathbf{n} \cdot \left(a^H \nabla u_H\right) \Pi_H^* v_H ds \right. \\ &\quad \left. + \int_K \left(b^H \nabla u_H + c^H u_H\right) \Pi_H^* v_H dx \right], \end{aligned} \tag{5}$$

$$(f, \Pi_H^* v_H) = \sum_{P\in N_H} \int_{K_P^*} f \Pi_H^* v_H dx = \sum_{K\in\mathcal{T}_H} \int_K f \Pi_H^* v_H dx. \tag{6}$$

We next construct our multiscale method with respect to (1).

By the numerical integration for the barycenter quadrature rule, (5) and (6) can be written as

$$A_{FVM}(u_H, \Pi_H^* v_H) = \sum_{K\in\mathcal{T}_H}\Bigg[-\sum_{P\in\dot{K}}|K\cap\partial K_P^*|\mathbf{n} \cdot \left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H + |K|\Big(b^H(Q)\nabla u_H + c^H(Q)u_H(Q)\Big)\Pi_H^* v_H\Bigg], \tag{7}$$

$$(f, \Pi_H^* v_H)_H = \sum_{K\in\mathcal{T}_H}|K|f(Q)\Pi_H^* v_H. \tag{8}$$

Thus, the barycenter quadrature approximation of the FVM can be read as finding $u_H \in V_H$ such that

$$A_{FVM}(u_H, \Pi_H^* v_H) = (f, \Pi_H^* v_H)_H, \forall v_H \in V_H. \tag{9}$$

To obtain error estimate, this paper considers the coefficients of scale separation. Assume that the multiscale coefficients $a^\varepsilon(x)$, $b^\varepsilon(x)$, and $c^\varepsilon(x)$ have the forms $a(x, x/\varepsilon)$, $b(x, x/\varepsilon)$, and $c(x, x/\varepsilon)$. Furthermore, assume that $a(x, y)$, $b(x, y)$, and $c(x, y)$ are smooth in x and periodic in y with respect to the unit cube Y.

In the absence of explicit expressions of a^H, b^H, and c^H, $a^H(Q)\nabla u_H$, $b^H(Q)\nabla u_H$, and $c^H(Q)$ can be approximated by

$$\begin{aligned} a^H(Q)\nabla u_H &\simeq \frac{1}{|K_\delta(Q)|}\int_{K_\delta(Q)} a\left(Q, \frac{x}{\varepsilon}\right)\nabla R(u_H)dx, \\ b^H(Q)\nabla u_H &\simeq \frac{1}{|K_\delta(Q)|}\int_{K_\delta(Q)} b\left(Q, \frac{x}{\varepsilon}\right)\nabla R(u_H)dx, \end{aligned} \tag{10}$$

$$c^H(Q) \simeq \frac{1}{|K_\delta(Q)|}\int_{K_\delta(Q)} c\left(Q, \frac{x}{\varepsilon}\right)dx, \tag{11}$$

where $R(u_H)$ is the solution of the following microcell problem

$$\begin{cases} -\nabla\cdot\left(a\left(Q, \dfrac{x}{\varepsilon}\right)\nabla R(u_H)\right) = 0 \text{ in } K_\delta(Q), \\ R(u_H) = u_H, \text{ on } \partial K_\delta(Q), \end{cases} \tag{12}$$

and $K_\delta(Q)$ is a cube of size δ centered at Q.

Thus, the HMM-FVM can be summarized as the following variational problem: $\forall v_H \in V_H$, find $u_H \in V_H$ satisfying (9)–(11).

3. A Priori Error Estimate

This section deduces the main result of the paper—optimal H^1 error estimate (Theorem 2). To this end, we first introduce Lemma 1 and Lemma 2 to derive its modeling error. Then, by regarding the HMM-FVM as a perturbation of the linear finite element method (Theorem 1), we obtain the optimal error estimate of HMM-FVM.

Lemma 1 ([14, 18, 19]). *There exists a constant C independent of H, ε, and δ such that for a^H, b^H,*

$$\left|a_{ij}^H - a_{ij}^0\right| \le C\frac{\varepsilon}{\delta}, \quad i, j = 1, 2, \tag{13}$$

$$\left|b_i^H - b_i^0\right| \le C\frac{\varepsilon}{\delta}, \quad i = 1, 2, \tag{14}$$

where $a_{ij}^0 = 1/|Y|\int_Y a_{ik}(x, y)(\delta_{kj} + (\partial x^j/\partial y_k)(x, y))dy$ $b_i^0 = 1/|Y|\int_Y (b_i(x, y) + a_{ik}(\partial\eta/\partial y_k)(x, y))dy$. *Here,* χ^j *and* η *are, respectively, the periodic solutions of* $-\nabla_y\cdot(a(x, y)\nabla_y\chi^j(x, y)) = \nabla_y\cdot(a(x, y)e_j)$, $-\nabla_y\cdot(a(x, y)\nabla_y\eta(x, y)) = \nabla_y\cdot b(x, y)$ *with zero mean (i.e.,* $\int_Y\chi^j dy = 0$, $\int_Y\eta dy = 0$*).*

Lemma 2 ([18, 20]). *Given domain K_δ with* $\operatorname{diam}(K) = \delta$, *let $\varphi(s, y)$ defined in Y be a Y − periodic function in y, where Y is a unit cube and $s \in R$ is fixed. Then*

$$\left|\frac{1}{|Y|}\int_Y \varphi(s, y)dy - <\varphi\left(s, \frac{x}{\varepsilon}\right)>_K\right| \le C\frac{\varepsilon}{\delta}, \tag{15}$$

where C is independent of ε, δ, and s.

Mathematically, the finite element method for the homogenization problem (2) is equivalent to finding $u_H \in V_H$ such that

$$B(u_H, v_H) = (f, v_H), \forall v_H \in V_H, \tag{16}$$

where

$$B(u_H, v_H) = \sum_{K\in\mathcal{T}_H}\int_K\left(a^0\nabla u_H\cdot\nabla v_H + b^0\nabla u_H v_H + c^0 u_H v_H\right)dx, \tag{17}$$

$$(f, v_H) = \sum_{K\in\mathcal{T}_H}\int_K f v_H dx. \tag{18}$$

Applying barycenter quadrature to (17), the bilinear form can be refined as

$$\tilde{B}(u_H, v_H) = \sum_{K\in\mathcal{T}_H}|K|\big(a^0(Q)\nabla u_H\bullet\nabla v_H + b^0(Q)\nabla u_H v_H(Q) + c^0(Q)u_H(Q)v_H(Q)\big). \tag{19}$$

Finally, to estimate the priori error estimate, we borrow inf-sup condition of bilinear form B given in [17].

Lemma 3 [17]. *For* $0 < H < h_0$,

$$\|u_H\|_{1,\Omega} \le C \sup_{0 \ne v_H \in V_H} \frac{B(u_H, v_H)}{\|v_H\|_{1,\Omega}}, \forall u_H \in V_H. \tag{20}$$

The following theorem characterizes the difference between the bilinear form of HMM-FVM and FEM, which will play the key role in the subsequent analysis.

Theorem 1. $\forall u_H, v_H \in V_H$, *we have*

$$|B(u_H, v_H) - A_{FVM}(u_H, v_H)| \le C\left(H + \frac{\varepsilon}{\delta}\right)\|u_H\|_{1,\Omega}\|v_H\|_{1,\Omega}, \tag{21}$$

where C is a positive constant independent of ε, δ, *and* H.

Proof. The bilinear form can be split into three terms

$$\begin{aligned} |B(u_H, v_H) - A_{FVM}(u_H, v_H)| &\le |B(u_H, v_H) - \tilde{B}(u_H, v_H)| \\ &+ \left|\tilde{B}(u_H, v_H) - \tilde{A}(u_H, v_H)\right| + \left|\tilde{A}(u_H, v_H)\right. \\ &\left. - A_{FVM}(u_H, v_H)\right| := \varepsilon_1(u_H, v_H) + \varepsilon_2(u_H, v_H) \\ &+ \varepsilon_3(u_H, v_H), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \tilde{A}(u_H, v_H) = \sum_{K \in \mathcal{T}_H} |K| \Big(& a^H(Q)\nabla u_H \cdot \nabla v_H + b^H(Q)\nabla u_H v_H(Q) \\ & + c^H(Q) u_H(Q) v_H(Q)\Big). \end{aligned} \tag{23}$$

Denote by

$$E_K(f) = \int_K f(x)dx - |K| f(Q). \tag{24}$$

By the standard estimate of [21], we get

$$|E_K(a(x)p(x)q(x))| \le CH\|a\|_{1,\infty}\|p\|_{0,K}\|q\|_{0,K}. \tag{25}$$

Accordingly, the numerical quadrature error $\varepsilon_1(u_H, v_H)$ is

$$\begin{aligned} \varepsilon_1(u_H, v_H) \le \sum_{K \in \mathcal{T}_H} & \left(\left|E_K\left(a^0 \nabla u_H \cdot \nabla v_H\right)\right| + \left|E_K\left(b^0 u_H \nabla v_H\right)\right|\right. \\ & \left. + \left|E_K\left(c^0 u_H v_H\right)\right|\right) \le CH\|u_H\|_{1,\Omega}\|v_H\|_{1,\Omega}. \end{aligned} \tag{26}$$

We next utilize Lemma 1 and Lemma 2 to estimate the modeling error $\varepsilon_2(u_H, v_H)$. By the definition of c^H in (11) and Lemma 2, we see that the error between c^H and c^0 can be treated as

$$\left|c^H - c^0\right| \le C\frac{\varepsilon}{\delta}, \tag{27}$$

where $c^0 = 1/|Y|\int_Y c(x, y)dy$ and C is a positive constant independent of ε, δ, and H. Let

$$e(HMM) = \max_{x \in \Omega}\left\{\left|a_{ij}^H - a_{ij}^0\right|, \left|b_i^H - b_i^0\right|, \left|c^H - c^0\right|\right\}. \tag{28}$$

By Lemma 1 and (27),

$$e(HMM) \le C\frac{\varepsilon}{\delta}. \tag{29}$$

Thus, the modeling error $\varepsilon_2(u_H, v_H)$ can be further estimated as

$$\begin{aligned} \varepsilon_2(u_H, v_H) &\le Ce(HMM)\|u_H\|_{1,\Omega}\|v_H\|_{1,\Omega} \\ &\le C\frac{\varepsilon}{\delta}\|u_H\|_{1,\Omega}\|v_H\|_{1,\Omega}. \end{aligned} \tag{30}$$

It remains to estimate the term $\varepsilon_3(u_H, v_H)$ which concerns three components. We next start this work.

By Green's formula, the first term of $\tilde{A}(u_H, v_H)$ is

$$\begin{aligned} |K| a^H(Q)\nabla u_H \nabla v_H &= \int_K a^H(Q)\nabla u_H \nabla v_H dx \\ &= -\int_K \nabla\left(a^H(Q)\nabla u_H\right) v_H dx \\ &\quad + \int_{\partial K} n \cdot \left(a^H(Q)\nabla u_H\right) v_H ds \\ &= \int_{\partial K} n \cdot \left(a^H(Q)\nabla u_H\right) v_H ds. \end{aligned} \tag{31}$$

The first term of $A_{FVM}(u_H, v_H)$ can be derived similarly

$$\begin{aligned} & -\sum_{P \in \dot{K}} |K \cap \partial K_P^*| n \cdot \left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H \\ &= -\sum_{P \in \dot{K}} \int_{K \cap \partial K_P^*} n \cdot \left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H ds \\ &= -\sum_{P \in \dot{K}} \int_{\partial(K \cap \partial K_P^*)} n \cdot \left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H ds \\ &\quad + \int_{\partial K} n \cdot \left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H ds \\ &= -\int_K \nabla\left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H dx \\ &\quad + \int_{\partial K} n \cdot \left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H ds \\ &= \int_{\partial K} n \cdot \left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H ds. \end{aligned} \tag{32}$$

The combination of the two gives

$$\sum_{K\in\mathcal{T}_H}|K|a^H(Q)\nabla u_H\nabla v_H - \sum_{K\in\mathcal{T}_H}\left(-\sum_{P\in\dot{K}}|K\cap\partial K_P^*|n\cdot\left(a^H(Q)\nabla u_H\right)\Pi_H^* v_H\right) = \sum_{K\in\mathcal{T}_H}\left(\sum_{e\in\partial K}\int_e n\cdot\left(a^H(Q)\nabla u_H\right)(v_H-\Pi_H^* v_H)ds\right). \tag{33}$$

Recall the interpolation operator Π_H^* (see Lemma 2.1 in [17]) satisfying

$$\|v_H-\Pi_H^* v_H\|_{0,K}\le CH|v_H|_{1,K},\ \forall v_H\in V_H. \tag{34}$$

From (29), (33), and (34), the last term $\varepsilon_3(u_H,v_H)$ can be estimated as

$$\begin{aligned}
\varepsilon_3(u_H,v_H) &= \sum_{K\in\mathcal{T}_H}\left[\sum_{e\in\partial K}\int_e n\cdot\left(a^H(Q)\nabla u_H\right)(v_H-\Pi_H^* v_H)ds \right.\\
&\quad\left. +|K|\left(b^H(Q)\nabla u_H+c^H(Q)u_H\right)(v_H-\Pi_H^* v_H)\right]\\
&= \sum_{K\in\mathcal{T}_H}|K|\left(b^H(Q)\nabla u_H+c^H(Q)u_H\right)(v_H-\Pi_H^* v_H)\\
&\le \sum_{K\in\mathcal{T}_H}|K|\left(\left\|b^H\right\|_{0,\infty,K}\|\nabla u_H\|_{0,\infty,K}\right.\\
&\quad \left.+\left\|c^H\right\|_{0,\infty,K}\|u_H\|_{0,\infty,K}\right)\|v_H-\Pi_H^* v_H\|_{0,\infty,K}\\
&\le C\sum_{K\in\mathcal{T}_H}\left(\left\|b^H\right\|_{0,\infty,K}\|\nabla u_H\|_{0,2,K}\right.\\
&\quad\left.+\left\|c^H\right\|_{0,\infty,K}\|u_H\|_{0,2,K}\right)\|v_H-\Pi_H^* v_H\|_{0,2,K}\\
&\le C\left(\left\|b^H\right\|_{0,\infty,\Omega}+\left\|c^H\right\|_{0,\infty,\Omega}\right)\|u_H\|_{1,\Omega}\|v_H\\
&\quad -\Pi_H^* v_H\|_{0,\Omega}\le CH\left(\left\|b^H\right\|_{0,\infty,\Omega}+c^H{}_{0,\infty,\Omega}\right)\\
&\quad\cdot\|u_H\|_{1,\Omega}|v_H|_{1,\Omega}\le CH\left(\left\|b^0\right\|_{0,\infty,\Omega}+\left\|c^0\right\|_{0,\infty,\Omega}\right.\\
&\quad\left.+e(HMM)\right)\|u_H\|_{1,\Omega}|v_H|_{1,\Omega}\\
&\le C(H+e(HMM))\|u_H\|_{1,\Omega}|v_H|_{1,\Omega}\\
&\le C\left(H+\frac{\varepsilon}{\delta}\right)\|u_H\|_{1,\Omega}|v_H|_{1,\Omega}.
\end{aligned} \tag{35}$$

By the estimates of $\varepsilon_1(u_H,v_H)$, $\varepsilon_2(u_H,v_H)$, and $\varepsilon_3(u_H,v_H)$, we finally get

$$|B(u_H,v_H)-A_{FVM}(u_H,v_H)|\le C\left(H+\frac{\varepsilon}{\delta}\right)\|u_H\|_{1,\Omega}\|v_H\|_{1,\Omega}. \tag{36}$$

With the preparation of Theorem 1, we summarize the main result in Theorem 2.

Theorem 2. *Denote by u^0 and u_H the solution of (2) and (9), respectively. Then for sufficient small H and ε/δ, we have*

$$\left\|u^0-u_H\right\|_{1,\Omega}\le C\left(H+\frac{\varepsilon}{\delta}\right), \tag{37}$$

where C is a positive constant independent of ε, δ, and H.

Proof. In order to estimate the error between u^0 and u^H, we separate it into two parts

$$\left\|u^0-u_H\right\|_{1,\Omega}\le\left\|u^0-u_H^0\right\|_{1,\Omega}+\left\|u_H^0-u_H\right\|_{1,\Omega}, \tag{38}$$

where u_H^0 is the numerical solution of (16), which is FEM formula of the homogenized problem (2).

From the standard error estimate [21], the first part of (38) can be estimated as

$$\left\|u^0-u_H^0\right\|_{1,\Omega}\le CH\left\|u^0\right\|_{1,\Omega}. \tag{39}$$

It remains to treat the second part of (38). By Lemma 3, we have

$$\begin{aligned}
\left\|u_H^0-u_H\right\|_{1,\Omega} &\le C\sup_{0\ne v_H\in V_H}\frac{B\left(u_H^0-u_H,v_H\right)}{\|v_H\|_{1,\Omega}}\\
&\le C\sup_{0\ne v_H\in V_H}\left(\frac{|B(u_H,v_H)-A_{FVM}(u_H,\Pi_H^* v_H)|}{\|v_H\|_{1,\Omega}}\right.\\
&\quad\left.+\frac{\left|(f,v_H)_H-(f,\Pi_H^* v_H)_H\right|}{\|v_H\|_{1,\Omega}}\right)\coloneqq T_1+T_2.
\end{aligned} \tag{40}$$

According to Theorem 1, T_1 can be estimated as

$$\begin{aligned}
T_1 &\le C\left(H+\frac{\epsilon}{\delta}\right)\|u_H\|_{1,\Omega}\\
&\le C\left(H+\frac{\epsilon}{\delta}\right)\left(\left\|u^0\right\|_{1,\Omega}+\left\|u_H-u^0\right\|_{1,\Omega}\right).
\end{aligned} \tag{41}$$

On the other hand, the numerator of T_2 in (40) is

$$\begin{aligned}
(f,v_H)_H-(f,\Pi_H^* v_H)_H &= \sum_K\int_K f v_H dx-\sum_K|K|f(Q)\Pi_H^* v_H\\
&= \sum_K\int_K f v_H dx-\sum_K\int_K f(Q)\Pi_H^* v_H dx\\
&= \sum_K\int_K f v_H dx-\sum_K\int_K f(Q) v_H dx\\
&= \sum_K\int_K (f(x)-f(Q)) v_H dx\\
&\le CH\sum_K\|\nabla f\|_{0,2,K}\|v_H\|_{0,2,K}\\
&\le CH|f|_{1,\Omega}\|v_H\|_{0,\Omega}\\
&\le CH|f|_{1,\Omega}\|v_H\|_{1,\Omega}.
\end{aligned} \tag{42}$$

Thus,

$$T_2 \leq CH|f|_{1,\Omega}. \quad (43)$$

We see (37) follows immediately by combining the results of (38)–(43).

4. Conclusion

This paper investigates the multiscale convection-diffusion-reaction problem. Based on the heterogeneous multiscale method, we first choose a finite volume method as macroscopic solver elaborately which can keep conservation and can be applied to complex regions. Then, we construct a suitable HMM-FVM strategy to solve the problem. We show that this method possesses the optimal order convergence rate in H^1-norm for periodic medias. As a matter of fact, this method can be applied in practice to multiscale problems without assumptions of periodic and scale separation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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