Research Article

The Domination Complexity and Related Extremal Values of Large 3D Torus

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Abstract

In this work, we consider certain graphs without loops or multiple edges. A dominating set in a graph \( G = (V, E) \) is a subset \( S \subseteq V \) such that each vertex in \( V \setminus S \) is adjacent to at least one vertex in \( S \). The domination number \( \gamma(G) \) of a graph \( G \) is the minimum cardinality of a dominating set in \( G \). A dominating set of a graph \( G \) is a \( \gamma(G) \) set if it has cardinality \( \gamma(G) \). A dominating set \( S \) is an independent dominating set, if no two vertices in \( S \) are adjacent. The independent domination number \( i(G) \) of a graph \( G \) is the minimum cardinality of an independent dominating set in \( G \). For \( S \subseteq V(G) \), the cardinality of a minimum dominating set of \( G \) containing \( S \) is denoted by \( \gamma(G | S) \), that is,

\[
\gamma(G | S) = \min \{|T|: T \text{ is a dominating set of } G \text{ and } S \subseteq T\},
\]

For two graphs \( G \) and \( H \), the Cartesian product \( G \square H \) is a graph with vertex set \( V(G) \times V(H) \) and two vertices \( (u_1, v_1) \) and \( (u_2, v_2) \) are adjacent if and only if either \( u_1 = u_2 \) and \( v_1, v_2 \in E(H) \) or \( v_1 = v_2 \) and \( u_1, u_2 \in E(G) \). The grid \( G_{j,k} \) is \( P_j \square P_k \), the cylinder \( C_{j,k} \) for \( j \geq 3 \) is \( C_j \square P_k \), the torus \( T_{j,k} \) for \( j \geq 3 \) and \( k \geq 3 \) is \( C_j \square C_k \), and the 3D torus \( T(m, n, k) \) for given \( m, n, k \) \( \geq 3 \) is \( C_m \square C_n \square C_k \). The subgraph of \( G \square H \) induced by \( V(G) \times \{v\} \) is isomorphic to \( G \). It is called a \( G \)-fiber and is denoted by \( G' \).

The domination number has attracted considerable attention in the general case [1, 2]. Due to a variety of applications, it has been studied on various types of graphs such as generalized Petersen graphs [8–11], hypercubes [12, 13], Fibonacci cubes [14], Kneser graphs [15–18], torus graphs [19–21], and grid graphs [22, 23]. Others are referred to [24–26].

In this work, we consider certain graphs without loops or multiple edges. A dominating set in a graph \( G = (V, E) \) is a subset \( S \subseteq V \) such that each vertex in \( V \setminus S \) is adjacent to at least one vertex in \( S \). The domination number \( \gamma(G) \) of a graph \( G \) is the minimum cardinality of a dominating set in \( G \). A dominating set of a graph \( G \) is a \( \gamma(G) \) set if it has cardinality \( \gamma(G) \). A dominating set \( S \) is an independent dominating set, if no two vertices in \( S \) are adjacent. The independent domination number \( i(G) \) of a graph \( G \) is the minimum cardinality of an independent dominating set in \( G \). For \( S \subseteq V(G) \), the cardinality of a minimum dominating set of \( G \) containing \( S \) is denoted by \( \gamma(G | S) \), that is,
sets in a type of Kneser graphs are also related to Steiner system [18]. The following result is a special case of Theorem 3.3 in [19]:

**Theorem 1.** \( \gamma(C_{5} \square C_{5}) = 5\ell_1 \ell_2 \) for any \( \ell_1, \ell_2 \geq 1 \).

Let \( \text{Aut}(G) \) be the automorphism group of graph \( G \). We say two subsets \( S_1 \subseteq V(G) \) and \( S_2 \subseteq V(G) \) are equivalent if there exists a mapping \( g \in \text{Aut}(G) \) such that \( S_1 = g(S_2) \). For a subset \( S \subseteq V(G) \), the orbit of \( S \) is the family \( S^{\text{Aut}(G)} = \{ g(S) \mid g \in \text{Aut}(G) \} \).

**2. The Approaches**

2.1. ILP-Based Search. Let \( G \) be a graph and \( S \) be a dominating set of \( G \). We use a Boolean variable \( x_v \) to denote if a vertex \( v \) is in \( S \), that is, \( x_v = 1 \) if \( v \in S \).

The following is the well-known integer linear programming model for finding a minimum dominating set of \( G \):

\[
\begin{align*}
\text{ILPDominatingSet} &:= \text{Min} \quad \sum_{v \in V(G)} x_v \\
\text{subject to} &\quad \sum_{u \in N(v)} x_u \geq 1 \quad \text{for any } v \in V(G), \\
&\quad x_v \in \{0, 1\} \quad \text{for any } v \in V(G).
\end{align*}
\]

**2.2. Extending Dominating Sets from Inequivalent Seeds**. Sometimes, we can succeed to obtain the independent domination number \( i(G) \) of a graph \( G \) in a reasonable time but it fails to exhaustively search the domination number \( \gamma(G) \). In such cases, based on Observation 1, we can obtain \( i(G) \) first to assume there are two adjacent vertices in a minimum dominating set.

**Observation 1.** Let \( G \) be a graph. If there is a minimum dominating set \( S \) such that \( |S| < i(G) \), then \( S \) contains two adjacent vertices.

**Observation 2.** Let \( G \) be a graph and \( \mathcal{S} \) be a family of inequivalent subsets of \( V(G) \). If there is a minimum dominating set \( S \) of \( G \) containing a subset isomorphic to one element in \( \mathcal{S} \), then \( \gamma(G) = \min \{ \gamma(G \mid S) \mid S \in \mathcal{S} \} \).

Based on the above observations, we first select a family \( \mathcal{S} \) of inequivalent subsets of the vertex set of a considered graph \( G \) then extend each subset \( S \in \mathcal{S} \) to a minimum dominating set, that is, we compute \( \gamma(G \mid S) \). We apply the following procedure to compute the domination number of \( G \) (using a 2.66 GHz Intel Core (TM) i7-5600U CPU).

**Theorem 2.** \( \gamma(C_{21} \square C_{21}) = 95 \).

**Proof.** The upper bound follows from the dominating set depicted in Figure 1.
Next, we show that $\gamma_{C_{21}^2} \geq 95$. Consider a family of inequivalent subsets $\mathcal{S} = \{s_1, s_2, \ldots, s_8\}$ where

\[
s_1 = \{(1, 1), (1, 4), (4, 1), (4, 4)\}, \\
s_2 = \{(1, 1), (1, 4), (4, 1), (3, 4)\}, \\
s_3 = \{(1, 1), (1, 4), (4, 1), (5, 4)\}, \\
s_4 = \{(1, 1), (1, 4), (3, 1), (3, 4)\}, \\
s_5 = \{(1, 1), (1, 4), (3, 1), (5, 4)\}, \\
s_6 = \{(1, 1), (1, 4), (3, 1), (4, 3)\}, \\
s_7 = \{(1, 1), (1, 4), (3, 1), (4, 5)\}, \\
s_8 = \{(1, 1), (1, 4), (5, 1), (5, 4)\}.
\]

We run Procedure MinDomination to extend each $S \in \mathcal{S}$ for a minimum dominating set and determine $\gamma(G \mid S)$ for each $S \in \mathcal{S}$. It can be seen that $\gamma(G \mid S) \geq 95$ for each $S \in \mathcal{S}$. The results are presented in Table 1.

Moreover, we obtain that $\gamma' \geq 95$ in only 2774 seconds by running MinDomination. From the above computational results, we have $\gamma(C_{21}^2) \geq 95$.

**Theorem 3.** $\gamma(C_{22}^2) = 104$.

**Proof.** The upper bound follows from the dominating set depicted in Figure 2.

![Figure 2: A dominating set with 104 vertices in $C_{22}^2$.](image)

From the computational results above, we have $\gamma(C_{22}^2) \geq 104$.

**Theorem 4.** $\gamma(C_{23}^2) = 114$.

**Proof.** The upper bound follows from the dominating set depicted in Figure 3.

Next, we show that $\gamma(C_{23}^2) \geq 114$. First, it takes 12520 seconds to obtain $\gamma(C_{23}^2) = 115$ using Gurobi optimizer. Since $\gamma(C_{23}^2) \leq 114 \leq \gamma(C_{23}^2) = 115$, we deduce that any minimum dominating set has two adjacent vertices. Now, we may assume w.l.o.g. that $\{(0, 0), (0, 1)\}$ are in a minimum dominating set. We consider the following inequivalent family $\mathcal{S} = \{W_1, W_2, \ldots, W_8\}$, where

\[
W_1 = \{(1, 3), (2, 0), (3, 2), (0, 0), (0, 1)\}, \\
W_2 = \{(1, 3), (2, 0), (2, 2), (0, 0), (0, 1)\}, \\
W_3 = \{(1, 3), (2, 0), (4, 2), (0, 0), (0, 1)\}, \\
W_4 = \{(1, 3), (2, 0), (3, 1), (0, 0), (0, 1)\}, \\
W_5 = \{(1, 3), (2, 0), (3, 3), (0, 0), (0, 1)\}, \\
W_6 = \{(1, 3), (1, 0), (3, 2), (0, 0), (0, 1)\}, \\
W_7 = \{(1, 3), (1, 0), (2, 2), (0, 0), (0, 1)\}, \\
W_8 = \{(1, 3), (1, 0), (4, 2), (0, 0), (0, 1)\}.
\]

We run Procedure MinDomination to extend each $S \in \mathcal{S}$ for a minimum dominating set and determine $\gamma(G \mid S)$ for each $S \in \mathcal{S}$. It can be seen that $\gamma(G \mid S) \geq 114$ for each $S \in \mathcal{S}$, and the results are presented in Table 2.

Moreover, we obtain that $\gamma' \geq 114$ in 16753 seconds by running MinDomination. From the above computational results, we have $\gamma(C_{23}^2) \geq 114$.

**Theorem 5.** $\gamma(C_{24}^2) = 120$. 

From the above computational results, we have $\gamma(C_{24}^2) \geq 120$.

**Table 1: The results for extending seeds from $\delta$.**

<table>
<thead>
<tr>
<th>Seed (S)</th>
<th>$\gamma(G \mid S)$</th>
<th>Time (s)</th>
<th>Seed (S)</th>
<th>$\gamma(G \mid S)$</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>96</td>
<td>1805</td>
<td>$s_2$</td>
<td>95</td>
<td>2542</td>
</tr>
<tr>
<td>$s_3$</td>
<td>95</td>
<td>1918</td>
<td>$s_4$</td>
<td>95</td>
<td>1365</td>
</tr>
<tr>
<td>$s_5$</td>
<td>95</td>
<td>1032</td>
<td>$s_6$</td>
<td>95</td>
<td>1273</td>
</tr>
<tr>
<td>$s_7$</td>
<td>95</td>
<td>3892</td>
<td>$s_8$</td>
<td>96</td>
<td>1640</td>
</tr>
</tbody>
</table>

**Table 2: The results for extending seeds from $\delta$.**

<table>
<thead>
<tr>
<th>Seed (S)</th>
<th>$\gamma(G \mid S)$</th>
<th>Time (s)</th>
<th>Seed (S)</th>
<th>$\gamma(G \mid S)$</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>104</td>
<td>16753</td>
<td>$W_2$</td>
<td>114</td>
<td>16753</td>
</tr>
<tr>
<td>$W_3$</td>
<td>114</td>
<td>16753</td>
<td>$W_4$</td>
<td>114</td>
<td>16753</td>
</tr>
<tr>
<td>$W_5$</td>
<td>114</td>
<td>16753</td>
<td>$W_6$</td>
<td>114</td>
<td>16753</td>
</tr>
<tr>
<td>$W_7$</td>
<td>114</td>
<td>16753</td>
<td>$W_8$</td>
<td>114</td>
<td>16753</td>
</tr>
</tbody>
</table>

**Figure 2: A dominating set with 104 vertices in $C_{22}^2$.**

**Figure 3: A dominating set with 114 vertices in $C_{23}^2$.**
For each within 200000 seconds, we are able to \( \gamma(C_{24} \Box C_{24}) \geq 120 \). However, we failed to determine the exact values of \( \gamma(G | S) \) within 200000 seconds. Although we failed to determine the exact values of \( \gamma(G | S) \) within 200000 seconds, we are able to confirm that \( \gamma(G | S) \geq 144 \) for each \( S \in \mathcal{S} \). This suffices to confirm the lower bound 144, and the results are presented in Table 4.

Moreover, we obtain that \( \gamma' \geq 144 \) in 465326 seconds by running \textit{MinDomination}. From the above computational results, we have \( \gamma(C_{26} \Box C_{26}) \geq 144 \), contradicting with the initial assumption that \( \gamma(C_{24} \Box C_{24}) \leq 119 \). Thus, \( \gamma(C_{24} \Box C_{24}) = 120 \).

\textbf{Theorem 6.} \( \gamma(C_{26} \Box C_{26}) = 144 \).

\textit{Proof.} The upper bound follows from the dominating set depicted in Figure 5.

Next, we show that \( \gamma(C_{26} \Box C_{26}) \geq 144 \). Suppose, to the contrary, that \( \gamma(C_{26} \Box C_{26}) \leq 143 \).

First, it takes 938743 seconds to obtain \( i(C_{26} \Box C_{26}) = 144 \) using Gurobi optimizer. By assumption, \( \gamma(C_{26} \Box C_{26}) \leq 143 < i(C_{26} \Box C_{26}) = 144 \). By Observation 1, we conclude that any minimum dominating set has two adjacent vertices. Now, we may assume w.l.o.g. that \( \{0, 0\}, \{0, 1\} \) are in a minimum dominating set. We consider the inequivalent family \( \mathcal{S} = \{W_1, W_2, \ldots, W_6\} \) defined in the proof of Theorem 4. We run Procedure \textit{MinDomination} to extend each \( S \in \mathcal{S} \) for a minimum dominating set and determine \( \gamma(G | S) \) for each \( S \in \mathcal{S} \). It can be seen that \( \gamma(G | S) \geq 120 \) for each \( S \in \mathcal{S} \), and the results are presented in Table 3.

Moreover, we obtain that \( \gamma \geq 120 \) in 23764 seconds by running \textit{MinDomination}. From the above computational results, we have \( \gamma(C_{24} \Box C_{24}) \geq 120 \), contradicting with the

\begin{table}[h]
\centering
\caption{The results for extending seeds from \( \delta \).}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Seed (S) & \( \gamma(G | S) \) & Time (s) & Seed (S) & \( \gamma(G | S) \) & Time (s) \\
\hline
W_1 & 115 & 10535 & W_2 & 115 & 12360 \\
W_3 & 115 & 15515 & W_4 & 115 & 11740 \\
W_5 & 115 & 16170 & W_6 & 115 & 23005 \\
W_7 & 115 & 18650 & W_8 & 114 & 1325 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{The results for extending seeds from \( \delta \).}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Seed (S) & \( \gamma(G | S) \) & Time (s) & Seed (S) & \( \gamma(G | S) \) & Time (s) \\
\hline
W_1 & 144 & 120245 & W_2 & 144 & 171506 \\
W_3 & 144 & 187364 & W_4 & 144 & 189346 \\
W_5 & 144 & 196393 & W_6 & 144 & 193245 \\
W_7 & 144 & 179873 & W_8 & 144 & 162785 \\
\hline
\end{tabular}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{A dominating set with 120 vertices in \( C_{24} \Box C_{24} \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{A dominating set with 144 vertices in \( C_{26} \Box C_{26} \).}
\end{figure}
By Theorem 1, we have that if \( n \equiv 0 \pmod{5} \), then \( \gamma(C_n \square C_n) = n^2/5 \). Inspired by the result in [21] and the above computational results, the following conjecture is proposed.

**Conjecture 1.** Let \( n \geq 4 \). Then, we have

1. If \( n \equiv 1 \pmod{5} \), then \( \gamma(C_n \square C_n) = (n^2 + 2n - 8)/5 \).
2. If \( n \equiv 4 \pmod{5} \), then \( \gamma(C_n \square C_n) = (n^2 + n)/5 \).

3. Results and Discussions: Domination Number of 3D Torus

In this section, we determine the domination number of some 3D torus \( T_{m,n,k} \) where \( m, n, k \geq 3 \). Denote the vertices in \( S^i_{m,n} \) by \( v^i_{j,k} \) for \( 0 \leq i \leq m - 1 \) and \( 0 \leq j \leq n - 1 \). For any \( \gamma(T_{m,n,k}) \) set \( S \), let \( S^i_{m,n} = S \cap (C_m \square C_n)^i \) for \( i = 1, \ldots, k \).

**Theorem 7.** For \( k \geq 3 \), \( \gamma(T_{3,3,k}) \geq \lfloor 3k/2 \rfloor \).

**Proof.** It is easy to verify that \( \gamma(T_{3,3,k}) \geq 3k/2 \) for \( k = 3, 4, 5 \). Let \( k = 6 \) and \( S \) be a \( \gamma(T_{3,3,k}) \) set such that \( |S| = |S^1_{3,3}| + 5 \geq 1 \leq k \). We show that \( S^1_{3,3} \cap \emptyset \neq 0 \) for each \( i \). Suppose, to the contrary, that \( S^1_{3,3} \neq \emptyset \) for some \( i \), say \( i = 3 \). To dominate the vertices of \( S^i_{3,3} \), we must have \( |S^i_{3,3}| = 1 \). Clearly, the set

\[
S' = (S - \left(S^1_{3,3} \cup S^2_{3,3}\right)) \cup \left\{v^1_{0,0}, v^1_{2,2}, v^1_{1,3}, v^1_{3,1}, v^1_{3,0}, v^1_{0,2}, v^1_{2,0}\right\}
\]

\((7)\)

is a dominating set of \( T_{3,3,k} \) of size less than \( |S| \) which is a contradiction. Thus, \( |S^i_{3,3}| \geq 1 \) for each \( i \).

Next, we show that if \( |S \cap S^i_{3,3}| = |S \cap S^i_{3,3}| = 1 \) for some \( i \), then \(|S \cap S^i_{3,3}| \geq 1 \) and \(|S \cap S^i_{3,3}| \geq 1 \). Let \( S \cap S^i_{3,3} = S \cap S^1_{3,3} \). Assume without loss of generality that \( v^i_{1,1} \in S \). If \( v^i_{1,1} \in S \), then to dominate the vertices \( v^i_{0,0}, v^i_{2,2}, v^i_{0,2}, v^i_{2,0} \) for \( j = 2, 3 \), we must have \( \left\{v^i_{0,0}, v^i_{2,2}, v^i_{0,2}, v^i_{2,0}\right\} \subseteq S \) and so \( |S \cap S^i_{3,3}| \geq 4 \) for \( j = 1, 4 \).

If \( v^1_{1,1} \in S \) (the cases \( v^1_{1,0}, v^1_{0,1} \in S \), and \( v^1_{2,1} \in S \) are similar), then to dominate the vertices \( v^1_{2,2}, v^1_{0,2}, v^1_{2,0}, v^1_{0,0} \), we must have \( |S \cap S^i_{3,3}| \geq 4 \) and \( |S \cap S^i_{3,3}| \geq 5 \). If \( v^1_{0,0} \in S \) (the cases \( v^1_{2,2}, v^1_{0,2}, v^1_{0,0} \), and \( v^1_{2,0} \in S \) are similar), then to dominate the vertices \( v^1_{0,0}, v^1_{2,2}, v^1_{0,2}, v^1_{2,0} \), we must have \( |S \cap S^i_{3,3}| \geq 4 \) for \( i = 1, 4 \).

Let \( w_i = |S \cap S^i_{3,3}| \) for each \( i \) and define \( f \) on \( \{1, \ldots, k\} \) by

\[
f(i) = \begin{cases} 
2 & \text{if } w_{i+1} \geq 2 \text{ or } w_{i-1} = 1 \text{ and } w_i = 2, \\
1 & \text{if } w_{i+1} = 1 \text{ and } w_i \geq 3, \\
0 & \text{if } w_i = w_{i-1} = 1.
\end{cases}
\]

\((8)\)

Clearly, \( \sum_{i=1}^k f(i) = |S| \) and \( f(i) + f(i+1) \geq 3 \) for each \( i \). It follows that

\[
2\gamma(T_{3,3,k}) = 2|S| = 2 \sum_{i=1}^k f(i) = \sum_{i=1}^k (f(i) + f(i+1)) \geq 3k,
\]

yielding \( \gamma(T_{3,3,k}) \geq 3k/2 \). Since \( \gamma(T_{3,3,k}) \) is integer, we obtain the desired bound.

**Theorem 8.** Let \( k \geq 2 \) be an integer. Then

1. If \( k \equiv 6 \) or \( k \equiv 0, 1, 3 \pmod{4} \), then \( \gamma(T_{3,3,k}) \leq 3k/2 \).
2. If \( k \equiv 2 \pmod{4} \) and \( k \neq 6 \), then \( \gamma(T_{3,3,k}) \leq 3k/2 + 1 \).

**Proof.** Let \( G = T_{3,3,k} \) and

\[
A = \left\{v^i_{0,0}, v^i_{2,2}, v^i_{0,2}, v^i_{2,0}, v^i_{1,3}, v^i_{3,1}, v^i_{3,0}, v^i_{0,2} \mid 0 \leq i \leq \left\lfloor \frac{k}{4} \right\rfloor - 1 \right\}.
\]

\((9)\)

(i) If \( k \equiv 0 \pmod{4} \), then \( A \cup \{v^k_{0,0}, v^k_{2,2}\} \) is a dominating set of \( G \) with cardinality \( 2k \), and if \( k = 3 \pmod{4} \), then \( A \cup \{v^k_{0,0}, v^k_{2,2}, v^k_{0,2}, v^k_{2,0}\} \) is a dominating set of \( G \) with cardinality \( 3k \).

(ii) If \( k \equiv 2 \pmod{4} \), then \( A \cup \{v^k_{0,0}, v^k_{2,2}, v^k_{0,2}, v^k_{2,0}\} \) is a dominating set of \( G \) with cardinality \( 3k \).

Remark 1. We have succeed to compute all the exact values of \( \gamma(T_{3,3,k}) \) for \( k \leq 34 \) and \( \gamma(C_3 \square C_4) = \left\lfloor \frac{3k}{2} \right\rfloor + 1 \) only if \( k \in \{10, 14, 22, 26, 34\} \).

**Theorem 9.** For \( k \geq 3 \), \( \gamma(T_{3,3,k}) = 2k \).

**Proof.** Let \( G = T_{3,3,k} \). First, we show that \( \gamma(G) \leq 2k \). Let

\[
A = \left\{v^i_{0,0}, v^i_{2,2}, v^i_{0,2}, v^i_{2,0}, v^i_{1,3}, v^i_{3,1} \mid 0 \leq i \leq \left\lfloor \frac{k}{4} \right\rfloor - 1 \right\}.
\]

\((10)\)

(i) If \( k \equiv 0 \pmod{3} \), then \( A \cup \{v^k_{0,0}, v^k_{2,2}\} \) is a dominating set of \( G \) with cardinality \( 2k \). If \( k \equiv 1 \pmod{3} \), then \( A \cup \{v^k_{0,0}, v^k_{2,2}, v^k_{0,2}, v^k_{2,0}\} \) is a dominating set of \( G \) with cardinality \( 2k \). If \( k \equiv 2 \pmod{3} \), then \( A \cup \{v^k_{0,0}, v^k_{2,2}, v^k_{0,2}, v^k_{2,0}, v^k_{1,3}, v^k_{3,1}\} \) is a dominating set of \( G \) with cardinality \( 2k \). Therefore, we have \( \gamma(G) \leq 2k \).

Now, we show that \( \gamma(G) \geq 2k \). It is not hard to see that \( \gamma(G) \geq 2k \) for \( k = 3, 4 \). Assume \( k \geq 5 \) and let \( S \) be a \( \gamma(G) \) set. Denote the vertices in \( S_{3,3} \) by \( v^i_{j,k} \) for \( 0 \leq i \leq 2 \) and \( 0 \leq j \leq 3 \).
First, we show that \(|S_{3,4}| \neq 0\) for each \(i\). Suppose, to the contrary, that \(|S_{3,4}| = 1\) for some \(i\), say \(i = 3\). To dominate the vertices of \(S_{3,4}\), we have \(|S_{2i}^2 + |S_{3,4}^1| > 12\). Clearly, the set
\[
S' = (S - (S_{3,4}^2 \cup S_{3,4}^1)) \cup \{v_{0,3,1}^1, v_{1,1,0}^1, v_{2,2,0}^1, v_{0,1,1}^3, v_{1,3,0}^3, v_{2,0,0}^3, v_{2,2,2}^3, v_{0,3,1}^1, v_{1,1,0}^1 \}
\]
(12)
is a dominating set of \(G\) of size less than \(|S|\) which is a contradiction. Thus, \(|S_{3,4}^1| \geq 1\) for each \(i\). Next, we show that there is no \(i\) such that \(|S_{3,4}^1| + |S_{3,4}^1| \leq 3\). Suppose, to the contrary, that \(|S_{3,4}^1| + |S_{3,4}^1| \leq 3\) for some \(i\), say \(i = 1\). We may assume without a loss of generality that \(|S_{3,4}^1| \leq 2\) and \(v_{0,2}^3 \in S\). To dominate the vertices of \(S_{3,4}^2\), we must have \(|S \cap S_{3,4}^1| \geq 5\). Now, the set
\[
S' = (S - S_{3,4}^1) \cup \{v_{1,2,1}^1, v_{2,2,0}^3, v_{0,1,1}^3, v_{1,3,0}^3 \}
\]
is a \(\gamma(G)\) set contradicting the choice of \(S\). Hence, \(|S_{3,4}^1| + |S_{3,4}^1| \geq 4\) for each \(i\). Thus,
\[
\gamma(G) = 2|S| = 2 \sum_{i=1}^{k} |S \cap S_{3,4}^i| = \sum_{i=1}^{k} (|S \cap S_{3,4}^i| + |S \cap S_{3,4}^1|) \geq 4k,
\]
(14)
and this completes the proof.

Applying analogous approaches described in Section 2, we can obtain substantial results on domination of multidimensional torus. The exact values of some 3D torus \(T_{m,n,k}\) are presented in Table 5.

Let
\[
T_n^k = C_m \square C_{m \square \ldots \square C_m},
\]
(15)
Note that when \(n = 3\), the domination of \(T_n^k\) corresponds to the classic ternary covering codes of length \(k\). By the results of [27], it is clear that \(\gamma(T_3^3) = 5\), \(\gamma(T_3^4) = 9\), and \(\gamma(T_3^5) = 27\). Using the above approaches, we obtained some new bounds and exact values of \(\gamma(T_n^k)\). The results are presented in Table 6.

Moreover, some dominating sets corresponding to upper bounds are also listed below. Inspired by the computational results, the following conjecture is proposed.

Conjecture 2.

(i) For \(3 < n \leq k\), \(\gamma(C_n \square C_n \square C_n) = \lceil nk/2 \rceil\).

(ii) For \(n, k \geq 3\), \(\gamma(C_n \square C_n \square C_k) = nk\).

of [27], it is clear that \(\gamma(T_3^3) = 5\), \(\gamma(T_3^4) = 9\), and \(\gamma(T_3^5) = 27\). Using the above approaches, we obtained some new bounds and exact values of \(\gamma(T_n^k)\). The results are presented in Table 6.

Moreover, some dominating sets corresponding to upper bounds are also listed below. Inspired by the computational results, the following conjecture is proposed.
Conflicts of Interest

The data used to support the findings of this study are available from the corresponding author upon request.

References


