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Research Article

The Domination Complexity and Related Extremal Values of Large 3D Torus

Zehui Shao,[1] **Jin Xu**,[1] **S. M. Sheikholeslami**,[2] **and Shaohui Wang**[3]

[1]*Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China*
[2]*Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran*
[3]*Department of Mathematics, Savannah State University, Savannah, GA 31404, USA*

Correspondence should be addressed to Shaohui Wang; shaohuiwang@yahoo.com

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Domination is a structural complexity of chemical molecular graphs. A dominating set in a (molecular) graph $G = (V, E)$ is a subset $S \subseteq V$ such that each vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of a graph G is the minimum size of a dominating set in G. In this paper, computer-aided approaches for obtaining bounds for domination number on torus graphs are here considered, and many new exact values and bounds are obtained.

1. Introduction

The structure of molecular graphs is a broad area among chemistry, biology, and mathematics. In particular, domination number or dominating complexity has attracted considerable attention in the general case [1–7], like coding theory and combinatorial design. Due to a variety of applications, it has been studied on various types of graphs such as generalized Petersen graphs [8–11], hypercubes [12, 13], Fibonacci cubes [14], Kneser graphs [15–18], torus graphs [19–21], and grid graphs [22, 23]. Others are referred to [24–26].

In this work, we consider certain graphs without loops or multiple edges. A dominating set in a graph $G = (V, E)$ is a subset $S \subseteq V$ such that each vertex in $V \setminus S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A dominating set of a graph G is a $\gamma(G)$ set if it has cardinality $\gamma(G)$. A dominating set S is an independent dominating set, if no two vertices in S are adjacent. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set in G. For $S \subseteq V(G)$, the cardinality of a minimum dominating set of G containing S is denoted by $\gamma(G \mid S)$, that is,

$$\gamma(G|S) = \min\{|T|: T \text{ is a dominating set of } G \text{ and } S \subseteq T\}. \tag{1}$$

For two graphs G and H, the *Cartesian product* $G\square H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. The gird $G_{j,k}$ is $P_j \square P_k$, the cylinder $C_{j,k}$ for $j \geq 3$ is $C_j \square P_k$, the torus $T_{j,k}$ for $j \geq 3$ and $k \geq 3$ is $C_j \square C_k$, and the 3D torus $T(m, n, k)$ for given m, n, $k \geq 3$ is $C_m \square C_n \square C_k$. The subgraph of $G\square H$ induced by $V(G) \times \{v\}$ is isomorphic to G. It is called a G-fiber and is denoted by G^v.

The domination number has attracted considerable attention in the general case [1, 2]. Due to a variety of applications, it has been studied on various types of graphs such as generalized Petersen graphs [8–11], hypercubes [12, 13], Fibonacci cubes [14], Kneser graphs [15, 16, 18], torus graphs [19–21], and grid graphs [22, 23]. Others are referred to [24–26].

Dominating sets in some special graphs have applied in coding theory and combinatorial design. For instance, there is a natural relation between dominating sets of torus graphs and covering codes in the Lee metric [21], and dominating

Procedure MinDomination(G, $\mathcal{S}$);
Input: a graph G and a family $\mathcal{S}$ of inequivalent subsets of $V(G)$.
begin
  minDom $\leftarrow +\infty$;
  for all $S \in \mathcal{S}$ **do**
    compute $\gamma(G \mid S)$ by solving the model *ILPDominatingSet* with additional constraints that $x_v = 1$ for each $v \in S$;
    if ($\gamma(G \mid S) <$ *minDom*) **then** *minDom* $\leftarrow \gamma(G \mid S)$;
  endfor
  Solve the model *ILPDominatingSet* with additional constraints that $\sum_{v \in T} x_v \leq |T| - 1$ for each T in the orbit of a set $S \in \mathcal{S}$ and let γ' be the objective value;
  return min{*minDOM*, γ'};
end.

Procedure 1

sets in a type of Kneser graphs are also related to Steiner system [18].

The following result is a special case of Theorem 3.3 in [19]:

Theorem 1. $\gamma(C_{5\ell_1} \square C_{5\ell_2}) = 5\ell_1\ell_2$ for *any* $\ell_1, \ell_2 \geq 1$.

Let $\mathrm{Aut}(G)$ be the automorphism group of graph G. We say two subsets $S_1 \subseteq V(G)$ and $S_2 \subseteq V(G)$ are equivalent if there exists a mapping $g \in \mathrm{Aut}(G)$ such that $S_1 = g(S2)$. For a subset $S \subseteq V(G)$, the orbit of S is the family $S^{\mathrm{Aut}(G)} = \{g(S) \mid g \in \mathrm{Aut}(G)\}$.

2. The Approaches

2.1. ILP-Based Search. Let G be a graph and S be a dominating set of G. We use a Boolean variable x_v to denote if a vertex v is in S, that is, $x_v = 1$ if $v \in S$.

The following is the well-known integer linear programming model for finding a minimum dominating set of G:

ILPDominatingSet

$$\min \sum_{v \in V(G)} x_v, \tag{2}$$

$$\text{subject to} \sum_{u \in N[v]} x_u \geq 1 \text{ for any } v \in V(G), \tag{3}$$

$$x_v \in \{0, 1\} \text{ for any } v \in V(G). \tag{4}$$

2.2. Extending Dominating Sets from Inequivalent Seeds. Sometimes, we can succeed to obtain the independent domination number $i(G)$ of a graph G in a reasonable time but it fails to exhaustively search the domination number $\gamma(G)$. In such cases, based on Observation 1, we can obtain $i(G)$ first to assume there are two adjacent vertices in a minimum dominating set.

Observation 1. Let G be a graph. If there is a minimum dominating set S such that $|S| < i(G)$, then S contains two adjacent vertices.

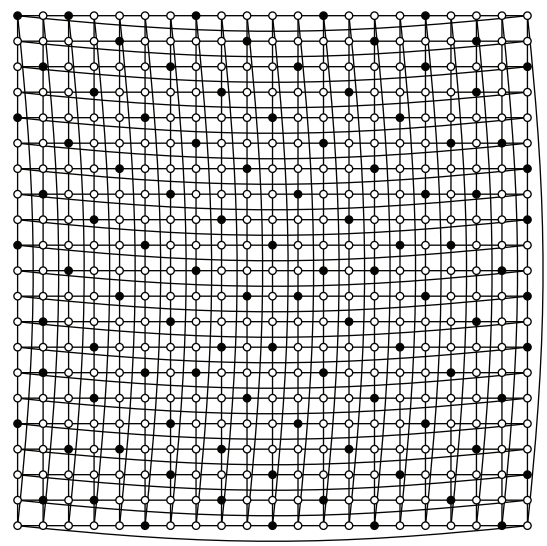

Figure 1: A dominating set with 95 vertices in $C_{21} \square C_{21}$.

Observation 2. Let G be a graph and $\mathcal{S}$ be a family of inequivalent subsets of $V(G)$. If there is a minimum dominating set S of G containing a subset isomorphic to one element in $\mathcal{S}$, then $\gamma(G) = \min\{\gamma(G \mid S) \mid S \in \mathcal{S}\}$.

Based on the above observations, we first select a family $\mathcal{S}$ of inequivalent subsets of the vertex set of a considered graph G then extend each subset $S \in \mathcal{S}$ to a minimum dominating set, that is, we compute $\gamma(G \mid S)$. We apply the following procedure to compute the domination number of G (using a 2.66 GHz Intel Core (TM) i7-5600U CPU).

2.3. Numerical Results. In this section, we determine the domination number of five torus.

Theorem 2. $\gamma(C_{21} \square C_{21}) = 95$.

Proof. The upper bound follows from the dominating set depicted in Figure 1.

Table 1: The results for extending seeds from $\mathcal{S}$.

Seed (S)	$\gamma(G \mid S)$	Time (s)	Seed (S)	$\gamma(G \mid S)$	Time (s)
s_1	96	1805	s_2	95	2542
s_3	95	1918	s_4	95	1365
s_5	95	1032	s_6	95	1273
s_7	95	3892	s_8	96	1640

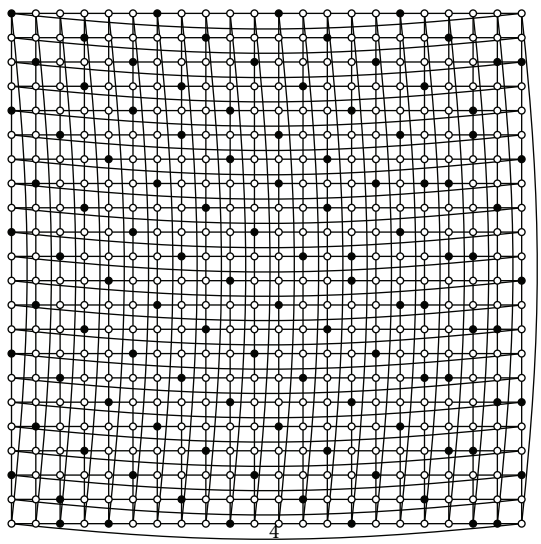

Figure 2: A dominating set with 104 vertices in $C_{22}\square C_{22}$.

Next, we show that $\gamma(C_{21}\square C_{21}) \geq 95$. Consider a family of inequivalent subset $\mathcal{S}\{s_1, s_2, \ldots, s_8\}$ where

$$\begin{aligned}
s_1 &= \{(1,1),(1,4),(4,1),(4,4)\},\\
s_2 &= \{(1,1),(1,4),(4,1),(3,4)\},\\
s_3 &= \{(1,1),(1,4),(4,1),(5,4)\},\\
s_4 &= \{(1,1),(1,4),(3,1),(3,4)\},\\
s_5 &= \{(1,1),(1,4),(3,1),(5,4)\},\\
s_6 &= \{(1,1),(1,4),(3,1),(4,3)\},\\
s_7 &= \{(1,1),(1,4),(3,1),(4,5)\},\\
s_8 &= \{(1,1),(1,4),(5,1),(5,4)\}.
\end{aligned} \tag{5}$$

We run Procedure *MinDomination* to extend each $S \in \mathcal{S}$ for a minimum dominating set and determine $\gamma(G \mid S)$ for each $S \in \mathcal{S}$. It can be seen that $\gamma(G \mid S) \geq 95$ for each $S \in \mathcal{S}$. The results are presented in Table 1.

Moreover, we obtain that $\gamma' \geq 95$ in only 2774 seconds by running *MinDomination*. From the above computational results, we have $\gamma(C_{21}\square C_{21}) \geq 95$.

Theorem 3. $\gamma(C_{22}\square C_{22}) = 104$.

Proof. The upper bound follows from the dominating set depicted in Figure 2.

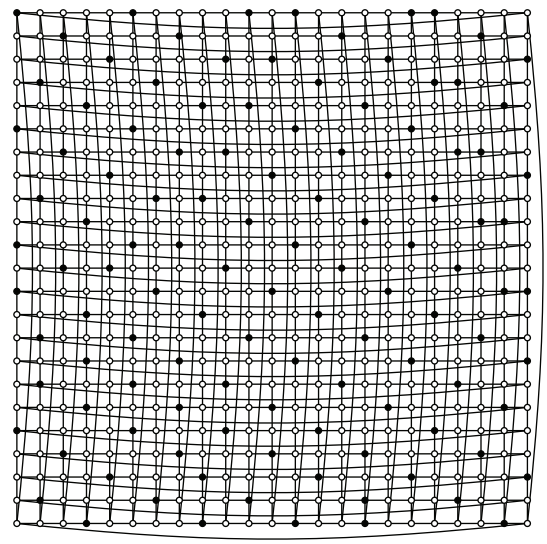

Figure 3: A dominating set with 114 vertices in $C_{23}\square C_{23}$.

From the computational results above, we have $\gamma(C_{22}\square C_{22}) \geq 104$.

Theorem 4. $\gamma(C_{23}\square C_{23}) = 114$.

Proof. The upper bound follows from the dominating set depicted in Figure 3.

Next, we show that $\gamma(C_{23}\square C_{23}) \geq 114$. First, it takes 12520 seconds to obtain $i(C_{23}\square C_{23}) = 115$ using Gurobi optimizer. Since $\gamma(C_{23}\square C_{23}) \leq 114 < i(C_{23}\square C_{23}) = 115$, we deduce that any minimum dominating set has two adjacent vertices. Now, we may assume w.l.o.g. that $\{(0,0),(0,1)\}$ are in a minimum dominating set. We consider the following inequivalent family $\mathcal{S} = \{W_1, W_2, \ldots, W_8\}$, where

$$\begin{aligned}
W_1 &= \{(1,3),(2,0),(3,2),(0,0),(0,1)\},\\
W_2 &= \{(1,3),(2,0),(2,2),(0,0),(0,1)\},\\
W_3 &= \{(1,3),(2,0),(4,2),(0,0),(0,1)\},\\
W_4 &= \{(1,3),(2,0),(3,1),(0,0),(0,1)\},\\
W_5 &= \{(1,3),(2,0),(3,3),(0,0),(0,1)\},\\
W_6 &= \{(1,3),(1,0),(3,2),(0,0),(0,1)\},\\
W_7 &= \{(1,3),(1,0),(2,2),(0,0),(0,1)\},\\
W_8 &= \{(1,3),(1,0),(4,2),(0,0),(0,1)\}.
\end{aligned} \tag{6}$$

We run Procedure *MinDomination* to extend each $S \in \mathcal{S}$ for a minimum dominating set and determine $\gamma(G \mid S)$ for each $S \in \mathcal{S}$. It can be seen that $\gamma(G \mid S) \geq 114$ for each $S \in \mathcal{S}$, and the results are presented in Table 2.

Moreover, we obtain that $\gamma' \geq 114$ in 16753 seconds by running *MinDomination*. From the above computational results, we have $\gamma(C_{23}\square C_{23}) \geq 114$.

Theorem 5. $\gamma(C_{24}\square C_{24}) = 120$.

Table 2: The results for extending seeds from $\mathcal{S}$.

Seed (S)	$\gamma(G \mid S)$	Time (s)	Seed (S)	$\gamma(G \mid S)$	Time (s)
W_1	115	10535	W_2	115	12360
W_3	115	15515	W_4	115	11740
W_5	115	16170	W_6	115	23005
W_7	115	18650	W_8	114	1325

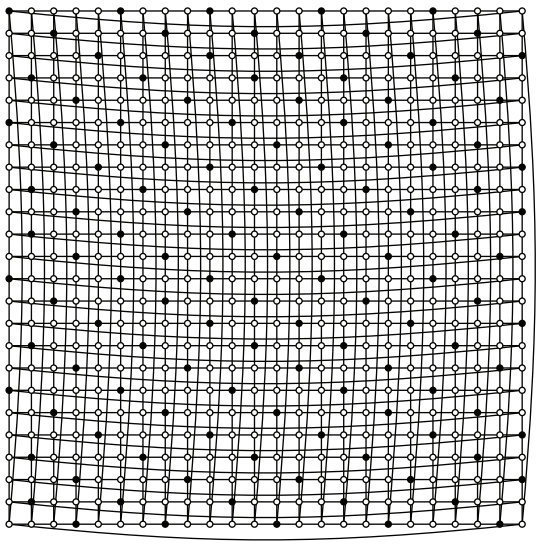

Figure 4: A dominating set with 120 vertices in $C_{24}\square C_{24}$.

Table 3: The results for extending seeds from $\mathcal{S}$.

Seed (S)	$\gamma(G \mid S)$	Time (s)	Seed (S)	$\gamma(G \mid S)$	Time (s)
W_1	121	40	W_2	122	135
W_3	122	206	W_4	122	75
W_5	122	17	W_6	122	110
W_7	122	685	W_8	122	90

Proof. The upper bound follows from the dominating set depicted in Figure 4.

Next, we show that $\gamma(C_{24}\square C_{24}) \geq 120$. Suppose, to the contrary, that $\gamma(C_{24}\square C_{24}) \leq 119$.

First, it takes 18974 seconds to obtain $i(C_{24}\square C_{24}) = 120$ using Gurobi optimizer. By assumption, $\gamma(C_{24}\square C_{24}) \leq 119 < i(C_{24}\square C_{24}) = 120$. By Observation 1, we conclude that any minimum dominating set has two adjacent vertices. Now, we may assume w.l.o.g. that $\{(0,0),(0,1)\}$ are in a minimum dominating set. We consider the inequivalent family $\mathcal{S} = \{W_1, W_2, \ldots, W_8\}$ defined in the proof of Theorem 4. We run Procedure *MinDomination* to extend each $S \in \mathcal{S}$ for a minimum dominating set and determine $\gamma(G \mid S)$ for each $S \in \mathcal{S}$. It can be seen that $\gamma(G \mid S) \geq 120$ for each $S \in \mathcal{S}$, and the results are presented in Table 3.

Moreover, we obtain that $\gamma' \geq 120$ in 23764 seconds by running *MinDomination*. From the above computational results, we have $\gamma(C_{24}\square C_{24}) \geq 120$, contradicting with the initial assumption that $\gamma(C_{24}\square C_{24}) \leq 119$. Thus, $\gamma(C_{24}\square C_{24}) = 120$.

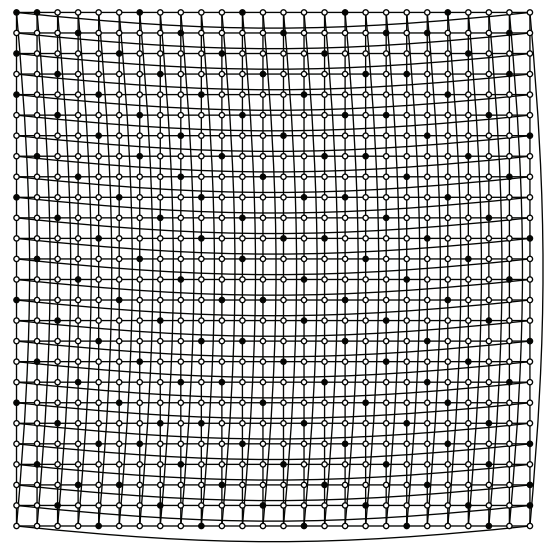

Figure 5: A dominating set with 144 vertices in $C_{26}\square C_{26}$.

Table 4: The results for extending seeds from $\mathcal{S}$.

Seed (S)	$\gamma(G \mid S)$	Time (s)	Seed (S)	$\gamma(G \mid S)$	Time (s)
W_1	144	120245	W_2	≥144	171506
W_3	≥144	187364	W_4	≥144	189346
W_5	≥144	196393	W_6	≥144	193245
W_7	≥144	179873	W_8	≥144	162785

Theorem 6. $\gamma(C_{26}\square C_{26}) = 144$.

Proof. The upper bound follows from the dominating set depicted in Figure 5.

Next, we show that $\gamma(C_{26}\square C_{26}) \geq 144$. Suppose, to the contrary, that $\gamma(C_{26}\square C_{26}) \leq 143$.

First, it takes 938743 seconds to obtain $i(C_{26}\square C_{26}) = 144$ using Gurobi optimizer. By assumption, $\gamma(C_{26}\square C_{26}) \leq 143 < i(C_{26}\square C_{26}) = 144$. By Observation 1, we conclude that any minimum dominating set has two adjacent vertices. Now, we may assume w.l.o.g. that $\{(0,0),(0,1)\}$ are in a minimum dominating set. We consider the inequivalent family $\mathcal{S} = \{W_1, W_2, \ldots, W_8\}$ defined in the proof of Theorem 4. We run Procedure *MinDomination* to extend each $S \in \mathcal{S}$ for a minimum dominating set with a time limit 200000 seconds. Although we failed to determine the exact values of $\gamma(G \mid S)$ within 200000 seconds, we are able to confirm that $\gamma(G \mid S) \geq 144$ for each $S \in \mathcal{S}$. This suffices to confirm the lower bound 144, and the results are presented in Table 4.

Moreover, we obtain that $\gamma' \geq 144$ in 465326 seconds by running *MinDomination*. From the above computational results, we have $\gamma(C_{26}\square C_{26}) \geq 144$, contradicting with the initial assumption that $\gamma(C_{26}\square C_{26}) \leq 143$. Thus, $\gamma(C_{26}\square C_{26}) = 144$.

By Theorem 1, we have that if $n \equiv 0 \pmod 5$, then $\gamma(C_n \square C_n) = n^2/5$. Inspired by the result in [21] and the above computational results, the following conjecture is proposed.

Conjecture 1. *Let $n \geq 4$. Then, we have*

(i) if $n \equiv 1$ (mod 5), then $\gamma(C_n \square C_n) = (n^2 + 2n - 8)/5$,

(ii) if $n \equiv 4$ (mod 5), then $\gamma(C_n \square C_n) = (n^2 + n)/5$.

3. Results and Discussions: Domination Number of 3D Torus

In this section, we determine the domination number of some 3D torus $T_{m,n,k}$, where $m, n, k \geq 3$. Denote the vertices in $S^i_{m,n}$ by $v^k_{i,j}$ for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. For any $\gamma(T_{m,n,k})$ set S, let $S^i_{m,n} = S \cap (C_m \square C_n)^i$ for $i = 1, \ldots, k$.

Theorem 7. *For $k \geq 3$, $\gamma(T_{3,3,k}) \geq \lceil 3k/2 \rceil$.*

Proof. It is easy to verify that $\gamma(T_{3,3,k}) \geq 3k/2$ for $k = 3, 4, 5$. Let $k \geq 6$ and S be a $\gamma(T_{3,3,k})$ set such that $|\{S^i_{3,3} : |S \cap S^i_{3,3}| \geq 5 \text{ and } 1 \leq i \leq k\}|$ is as small as possible. First, we show that $|S^i_{3,3}| \neq 0$ for each i. Suppose, to the contrary, that $|S^i_{3,3}| = 0$ for some i, say $i = 3$. To dominate the vertices of $S^3_{3,3}$, we must have $|S^2_{3,3}| + |S^4_{3,3}| \geq 9$. Clearly, the set

$$S' = \left(S - \left(S^2_{3,3} \cup S^4_{3,3}\right)\right) \cup \cdot \left\{v^1_{0,0}, v^1_{2,2}, v^1_{1,1}, v^3_{0,2}, v^3_{2,0}, v^4_{1,1}, v^5_{0,0}, v^5_{2,2}\right\} \tag{7}$$

is a dominating set of $T_{3,3,k}$ of size less than $|S|$ which is a contradiction. Thus, $|S^i_{3,3}| \geq 1$ for each i.

Next, we show that if $|S \cap S^i_{3,3}| = |S \cap S^{i+1}_{3,3}| = 1$ for some i, then $|S \cap S^{i-1}_{3,3}| \geq 3$ and $|S \cap S^{i+2}_{3,3}| \geq 3$. Let $|S \cap S^i_{3,3}| = |S \cap S^{i+1}_{3,3}| = 1$ for some i, say $i = 2$. Assume without loss of generality that $v^2_{1,1} \in S$. If $v^3_{1,1} \in S$, then to dominate the vertices $v^j_{0,0}, v^j_{2,2}, v^j_{0,2}, v^j_{2,0}$ for $j = 2, 3$, we must have $\{v^j_{0,0}, v^j_{2,2}, v^j_{0,2}, v^j_{2,0} \mid j = 14\} \subseteq S$ and so $|S \cap S^j_{3,3}| \geq 4$ for $j = 1, 4$.

If $v^3_{1,2} \in S$ (the cases $v^3_{1,0} \in S$, $v^3_{0,1} \in S$, and $v^3_{2,1} \in S$ are similar), then to dominate the vertices $v^2_{0,0}, v^2_{2,2}, v^2_{0,2}, v^2_{2,0}, v^3_{0,0}, v^3_{0,1}, v^3_{1,0}, v^3_{2,0}, v^3_{2.1}$, we must have $|S \cap S^1_{3,3}| \geq 4$ and $|S \cap S^4_{3,3}| \geq 5$. If $v^3_{0,0} \in S$ (the cases $v^3_{0,2} \in S$, $v^3_{2,0} \in S$, and $v^3_{2,2} \in S$ are similar), then to dominate the vertices $v^2_{0,2}, v^2_{2,0}, v^2_{2,2}, v^3_{1,2}, v^3_{2,1}, v^3_{2,2}$, we must have $|S \cap S^j_{3,3}| \geq 3$ for $j = 1, 4$.

Let $w_i = |S \cap S^i_{3,3}|$ for each i and define f on $\{1, \ldots, k\}$ by

$$f(i) \begin{cases} w_i, & \text{if } w_{i,1} \geq 2 \text{ or } w_{i-1} = 1 \text{ and } w_i = 2, \\ w_i - 1, & \text{if } w_{i,1} = 1 \text{ and } w_i \geq 3, \\ w_i + 1, & \text{if } w_i = w_{i-1} = 1. \end{cases} \tag{8}$$

Clearly, $\sum_{i=1}^k f(i) = |S|$ and $f(i) + f(i+1) \geq 3$ for each i. It follows that

$$2\gamma(T_{3,3,k}) = 2|S| = 2\sum_{i=1}^k f(i) = \sum_{i=1}^k (f(i) + f(i+1)) \geq 3k, \tag{9}$$

yielding $\gamma(T_{3,3,k}) \geq 3k/2$. Since $\gamma(T_{3,3,k})$ is integer, we obtain the desired bound.

Theorem 8. *Let $k \geq 3$ be an integer. Then*

(i) $\gamma(T_{3,3,k}) = \lceil 3k/2 \rceil$ if $k = 6$ or $k \equiv 0, 1, 3$ (mod 4),

(ii) $\gamma(T_{3,3,k}) \leq \lceil 3k/2 \rceil + 1$ if $k \equiv 2$ (mod 4) and $k \neq 6$.

Proof. Let $G = T_{3,3,k}$ and

$$A = \left\{v^{4\ell+1}_{1,1}, v^{4\ell+2}_{0,0}, v^{4\ell+2}_{2,2}, v^{4\ell+3}_{1,1}, v^{4\ell+4}_{0,2}, v^{4\ell+4}_{2,0} \middle| 0 \leq \ell \leq \left\lfloor \frac{k}{4} \right\rfloor - 1\right\}. \tag{10}$$

(i) If $k \equiv 0$ (mod 4), then A is a dominating set of G of size $3k/2$, if $k \equiv 1$ (mod 4), then $A \cup \{v^k_{0,2}, v^k_{2,0}\}$ is a dominating set of G with cardinality $2k$, and if $k \equiv 3$ (mod 4), then $A \cup \{v^{k-2}_{0,0}, v^{k-2}_{2,2}, v^{k-1}_{1,1}, v^k_{0,2}, v^k_{2,0}\}$ is a dominating set of G of size $(3k+1)/2$.

If $k = 6$, then set $\{v^1_{0,0}, v^1_{1,0}, v^2_{2,1}, v^3_{0,2}, v^3_{1,2}, v^4_{2,0}, v^5_{0,0}, v^5_{1,1}, v^6_{2,2}\}$ is a dominating set of G with cardinality 9. Therefore, we have $\gamma(G) \leq \lceil 3k/2 \rceil$ when $k = 6$ or $k \equiv 0, 1, 3$ (mod 4).

(ii) If $k \equiv 2$ (mod 4), then the set $A \cup \{v^{k-1}_{0,0}, v^{k-1}_{1,1}, v^k_{1,1}, v^k_{2,2}\}$ is a dominating set of G with cardinality $(3k/2) + 1$. Therefore, we have $\gamma(G) \leq \lceil 3k/2 \rceil + 1$ if $k = 4p + 2$ for $p \geq 2$.

Remark 1. We have succeed to compute all the exact values of $\gamma(T_{3,3,k})$ for $k \leq 34$ and $\gamma(C_3 \square C_3 \square C_k) = \lceil 3k/2 \rceil + 1$ only if $k \in \{10, 14, 22, 26, 34\}$.

Theorem 9. *For $k \geq 3$, $\gamma(T_{3,4,k}) = 2k$.*

Proof. Let $G = T_{3,4,k}$. First, we show that $\gamma(G) \leq 2k$. Let

$$A = \left\{v^{3\ell+1}_{0,3}, v^{3\ell+1}_{1,1}, v^{3\ell+2}_{2,0}, v^{3\ell+2}_{2,2}, v^{3\ell+3}_{0,1}, v^{3\ell+3}_{1,3} \middle| 0 \leq \ell \leq \left\lfloor \frac{k}{3} \right\rfloor - 1\right\}. \tag{11}$$

If $k \equiv 0$ (mod 3), then A is a dominating set of G with cardinality $2k$. If $k \equiv 1$ (mod 3), then $A \cup \{v^k_{2,0}, v^k_{2,2}\}$ is a dominating set of G with cardinality $2k$. If $k \equiv 2$ (mod 3), then $A \cup \{v^{k-1}_{2,0}, v^{k-1}_{2,2}, v^k_{0,3}, v^k_{1,1}\}$ is a dominating set of G with cardinality $2k$. Therefore, we have $\gamma(G) \leq 2k$.

Now, we show that $\gamma(G) \geq 2k$. It is not hard to see that $\gamma(G) \geq 2k$ for $k = 3, 4$. Assume $k \geq 5$ and let S be a $\gamma(G)$ set. Denote the vertices in $S^i_{3,4}$ by $v^k_{i,j}$ for $0 \leq i \leq 2$ and $0 \leq j \leq 3$.

Table 5: Exact values of some 3D torus $T_{m,n,k}$.

$(m,n)\setminus k$	3	4	5	6	7	8	9	10
(3,3)	5	6	8	9	11	12	14	16
(3,4)		8	10	12	14	16	18	20
(3,5)			13	15	18	20	23	25
(3,6)				18	21	24	27	30
(3,7)					25	28	32	35
(3,8)						32	36	40
(3,9)							41	45
(4,4)		12	15	16	20	23	24	29
(4,5)			18	20	25	28	30	36
(4,6)				24	28	32	36	40
(4,7)					34	38	42	48
(5,5)			20	26	31	34	38	40
(5,6)				30	35	40	45	50

First, we show that $|S^i_{3,4}| \neq 0$ for each i. Suppose, to the contrary, that $|S^i_{3,4}| = 1$ for some i, say $i = 3$. To dominate the vertices of $S^3_{3,4}$, we have $|S^2_{3,4}| + |S^4_{3,4}| > 12$. Clearly, the set

$$S' = \left(S - \left(S^2_{3,4} \cup S^4_{3,4}\right)\right) \cup \cdot \left\{v^1_{0,3}, v^1_{1,1}, v^2_{2,0}, v^2_{2,2}, v^3_{0,1}, v^3_{1,3}, v^4_{2,0}, v^4_{2,2}, v^5_{0,3}, v^5_{1,1}\right\} \tag{12}$$

is a dominating set of G of size less than $|S|$ which is a contradiction. Thus, $|S^i_{3,4}| \geq 1$ for each i. Next, we show that there is no i such that $|S^i_{3,4}| + |S^{i+1}_{3,4}| \leq 3$. Suppose, to the contrary, that $|S^i_{3,4}| + |S^{i,1}_{3,4}| \leq 3$ for some i, say $i = 1$. We may assume without a loss of generality that $|S^1_{3,4}| \leq 2$ and $|S^2_{3,4}| = 1$ and that $v^2_{0,3} \in S$. To dominate the vertices of $S^2_{3,4}$, we must have $|S \cap S^3_{3,4}| \geq 5$. Now, the set

$$S' = \left(S - S^3_{3,4}\right) \cup \left\{v^2_{1,1}, v^3_{2,0}, v^3_{2,2}, v^3_{0,1}, v^3_{1,3}\right\} \tag{13}$$

is a $\gamma(G)$ set contradicting the choice of S. Hence, $|S^i_{3,4}| + |S^{i+1}_{3,4}| \geq 4$ for each i. Thus,

$$\begin{aligned}\gamma(G) = 2|S| &= 2\sum_{i=1}^{k}\left|S \cap S^i_{3,4}\right| \\ &= \sum_{i-1}^{k}\left(\left|S \cap S^i_{3,4}\right| + \left|S \cap S^{i+1}_{3,4}\right|\right) \geq 4k,\end{aligned} \tag{14}$$

and this completes the proof.

Applying analogous approaches described in Section 2, we can obtain substantial results on domination of multidimensional torus. The exact values of some 3D torus $T_{m,n,k}$ are presented in Table 5. Let

$$T^k_n = \underbrace{C_n \square C_n \square, \ldots, \square C_m}_{k \text{ times}}. \tag{15}$$

Note that when $n = 3$, the domination of T^k_n corresponds to the classic ternary covering codes of length k. By the results of [27], it is clear that $\gamma(T^3_3) = 5$, $\gamma(T^4_3) = 9$, and $\gamma(T^5_3) = 27$. Using the above approaches, we obtained some new bounds and exact values of $\gamma(T^k_n)$. The results are presented in Table 6.

Table 6: Bounds and exact values of $\gamma(T^k_n)$.

(n,k)	(21,2)	(22,2)	(23,2)	(24,2)	(25,2)	(26,2)	(27,2)
$\gamma(T^k_n)$	95	104	114	120	125	144	152–155
(n,k)	(3,3)	(4,3)	(5,3)	(6,3)	(7,3)		
$\gamma(T^k_n)$	5	12	20	36	49		
(n,k)	(3,4)	(4,4)	(5,4)				
$\gamma(T^k_n)$	9	32	73–85				
(n,k)	(3,5)	(4,5)					
$\gamma(T^k_n)$	27	100–141					

Moreover, some dominating sets corresponding to upper bounds are also listed below. Inspired by the computational results, the following conjecture is proposed.

Conjecture 2.

(i) For $3 < n \leq k$, $\gamma(C_3 \square C_n \square C_k) = \lceil nk/2 \rceil$,

(ii) For $n, k \geq 3$, $\gamma(C_6 \square C_n \square C_k) = nk$.

$\gamma(C_4 \square C_4 \square C_4) \leq 12$: Aγ − set : $\left\{v^1_{0,0}, v^3_{0,2}, v^4_{0,2}, v^1_{1,0}, v^2_{1,0}, v^3_{1,2}, v^4_{2,1}, v^1_{2,2}, v^4_{2,3}, v^3_{3,0}, v^2_{3,1}, v^2_{3,3}\right\}$.

$\gamma(C_5 \square C_5 \square C_5) \leq 20$: Aγ − set : $\left\{v^1_{0,0}, v^4_{0,0}, v^4_{0,2}, v^2_{0,3}, v^2_{1,1}, v^5_{1,1}, v^5_{1,3}, v^3_{1,4}, v^4_{2,0}, v^1_{2,2}, v^3_{2,2}, v^1_{2,4}, v^2_{3,0}, v^5_{3,1}, v^2_{3,3}, v^4_{3,3}, v^3_{4,1}, v^1_{4,2}, v^3_{4,4}, v^5_{4,4}\right\}$.

$\gamma(C_6 \square C_6 \square C_6) \leq 36$: Aγ − set : $\left\{v^1_{0,0}, v^4_{0,0}, v^1_{0,2}, v^4_{0,2}, v^1_{0,4}, v^4_{0,4}, v^3_{1,1}, v^6_{1,1}, v^2_{1,3}, v^5_{1,3}, v^2_{1,5}, v^5_{1,5}, v^1_{2,0}, v^4_{2,0}, v^2_{2,2}, v^5_{2,2}, v^3_{2,4}, v^6_{2,4}, v^2_{3,1}, v^5_{3,1}, v^1_{3,3}, v^4_{3,3}, v^3_{3,5}, v^6_{3,5}, v^1_{4,0}, v^4_{4,0}, v^3_{4,2}, v^6_{4,2}, v^2_{4,4}, v^5_{4,4}, v^2_{5,1}, v^5_{5,1}, v^3_{5,3}, v^6_{5,3}, v^3_{5,5}, v^6_{5,5}\right\}$.

$\gamma(C_7 \square C_7 \square C_7) \leq 49$: Aγ − set : $\left\{v^1_{0,0}, v^4_{0,1}, v^7_{0,2}, v^3_{0,3}, v^6_{0,4}, v^2_{0,5}, v^5_{0,6}, v^6_{1,0}, v^2_{1,1}, v^5_{1,2}, v^1_{1,3}, v^4_{1,4}, v^7_{1,5}, v^3_{1,6}, v^4_{2,0}, v^7_{2,1}, v^3_{2,2}, v^6_{2,3}, v^2_{2,4}, v^5_{2,5}, v^1_{2,6}, v^2_{3,0}, v^5_{3,1}, v^1_{3,2}, v^4_{3,3}, v^7_{3,4}, v^3_{3,5}, v^6_{3,6}, v^7_{4,0}, v^3_{4,1}, v^6_{4,2}, v^2_{4,3}, v^5_{4,4}, v^1_{4,5}, v^4_{4,6}, v^5_{5,0}, v^1_{5,1}, v^4_{5,2}, v^7_{5,3}, v^3_{5,4}, v^6_{5,5}, v^2_{5,6}, v^3_{6,0}, v^6_{6,1}, v^2_{6,2}, v^5_{6,3}, v^1_{6,4}, v^4_{6,5}, v^7_{6,6}$.

$\gamma(C_4 \square C_4 \square C_4 \square C_4) \leq 32$: Aγ − set : $\{(0,0,0,0), (0,0,3,2), (0,1,1,2), (0,1,2,0), (0,2,0,1), (0,2,3,3), (0,3,1,3), (0,3,2,1), (1,0,1,1), (1,0,2,3), (1,1,0,3), (1,1,3,1), (1,2,1,0), (1,2,2,2), (1,3,0,2), (1,3,3,0), (2,0,0,3), (2,0,2,0), (2,1,1,1), (2,1,3,2), (2,2,1,3), (2,2,3,0), (2,3,0,1), (2,3,2,2), (3,0,1,2), (3,0,3,1), (3,1,0,0), (3,1,2,3), (3,2,0,2), (3,2,2,1), (3,3,1,0), (3,3,3,3)\}$.

$$\gamma(C_4\square C_4\square C_4\square C_4\square C_4) \le 141 : A\gamma - \text{set} : \{(0,0,0,0,0), (0,0,0,0,1), (0,0,0,3,1), (0,0,1,3,3), (0,0,2,2,1), (0,0,2,2,2), (0,0,3,0,3), (0,0,3,1,3), (0,1,0,2,0), (0,1,0,3,2), (0,1,1,1,0), (0,1,1,1,3), (0,1,2,0,2), (0,1,2,3,0), (0,1,3,1,1), (0,2,0,1,2), (0,2,0,2,0), (0,2,1,0,1), (0,2,1,0,3), (0,2,1,2,2), (0,2,2,1,2), (0,2,2,2,0), (0,2,3,0,0), (0,2,3,2,3), (0,2,3,3,1), (0,3,0,1,3), (0,3,0,3,3), (0,3,1,1,1), (0,3,1,2,0), (0,3,2,0,0), (0,3,2,3,2), (0,3,3,1,1), (0,3,3,2,3), (1,0,0,1,3), (1,0,0,2,2), (1,0,0,3,0), (1,0,1,1,3), (1,0,2,0,1), (1,0,2,1,0), (1,0,2,2,0), (1,0,2,3,2), (1,0,3,3,0), (1,1,0,1,1), (1,1,1,0,3), (1,1,1,2,1), (1,1,1,3,1), (1,1,2,0,0), (1,1,2,1,2), (1,1,3,0,2), (1,1,3,1,1), (1,1,3,2,3), (1,2,0,0,3), (1,2,0,3,1), (1,2,1,1,0), (1,2,1,2,3), (1,2,1,3,3), (1,2,2,1,2), (1,2,2,2,2), (1,2,2,3,3), (1,2,3,3,1), (1,3,0,2,1), (1,3,0,2,2), (1,3,1,0,0), (1,3,1,0,2), (1,3,2,0,0), (1,3,3,1,3), (1,3,3,2,1), (1,3,3,3,1), (1,3,3,3,2), (2,0,0,0,2), (2,0,0,2,1), (2,0,1,1,1), (2,0,1,2,0), (2,0,2,0,3), (2,0,2,2,1), (2,0,3,0,3), (2,0,3,1,2), (2,1,0,0,0), (2,1,0,3,3), (2,1,1,2,0), (2,1,1,2,2), (2,1,2,0,2), (2,1,2,3,0), (2,1,3,0,1), (2,1,3,1,3), (2,1,3,3,1), (2,2,0,1,2), (2,2,0,3,3), (2,2,1,0,1), (2,2,1,1,3), (2,2,2,0,1), (2,2,2,2,1), (2,2,3,1,0), (2,2,3,2,0), (2,2,3,2,2), (2,3,0,0,1), (2,3,0,1,0), (2,3,1,0,2), (2,3,1,3,1), (2,3,1,3,2), (2,3,2,1,1), (2,3,2,2,3), (2,3,3,0,3), (2,3,3,2,3), (3,0,0,1,2), (3,0,0,2,3), (3,0,1,0,0), (3,0,1,0,2), (3,0,1,1,2), (3,0,1,3,3), (3,0,2,0,1), (3,0,3,1,0), (3,0,3,2,0), (3,0,3,3,2), (3,1,0,0,0), (3,1,0,0,3), (3,1,1,3,1), (3,1,2,1,0), (3,1,2,1,1), (3,1,2,2,3), (3,1,3,2,2), (3,1,3,3,3), (3,2,0,0,2), (3,2,0,1,1), (3,2,0,2,1), (3,2,1,1,3), (3,2,1,3,0), (3,2,2,0,3), (3,2,2,3,2), (3,2,3,0,2), (3,2,3,2,3), (3,3,0,0,2), (3,3,0,3,0), (3,3,1,1,3), (3,3,1,2,1), (3,3,1,3,1), (3,3,2,1,3), (3,3,2,3,0), (3,3,3,0,2), (3,3,3,2,2), (3,3,3,3,0)\}. \tag{16}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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