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Research Article

Asymptotic Behavior of a Stochastic Two-Species Competition Model under the Effect of Disease

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This paper is concerned with a stochastic two-species competition model under the effect of disease. It is assumed that one of the competing populations is vulnerable to an infections disease. By the comparison theorem of stochastic differential equations, we prove the existence and uniqueness of global positive solution of the model. Then, the asymptotic pathwise behavior of the model is given via the exponential martingale inequality and Borel-Cantelli lemma. Next, we find a new method to prove the boundedness of the pth moment of the global positive solution. Then, sufficient conditions for extinction and persistence in mean are obtained. Furthermore, by constructing a suitable Lyapunov function, we investigate the asymptotic behavior of the stochastic model around the interior equilibrium of the deterministic model. At last, some numerical simulations are introduced to justify the analytical results. The results in this paper extend the previous related results.

1. Introduction

Populations that compete for common resources are well known among ecologists. They are classically modeled by observing their interactions that hinder the growth of both populations and are thus described by negative bilinear terms in all the relevant differential equations. The classic two-species Lotka-Volterra competition model takes the form

$$\begin{aligned} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= x_1(t)\left[r_1 - a_{11}x(t) - a_{12}x_2(t)\right], \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= x_2(t)\left[r_2 - a_{21}x(t) - a_{22}x_2(t)\right]. \end{aligned} \tag{1}$$

There is an extensive literature concerned with model (1) and related deterministic models (see [1–3] and the references therein) and we here do not mention them in detail.

As mentioned in [4], another major problem in today's modern society is the spread of infectious diseases. A detailed account of modeling and the study of epidemic diseases can be found in the literature [5, 6]. The population biology of infectious diseases has also been presented in [7]. In [8], the authors studied the dynamics of two competing species when one of them is subject to a disease. In [4], the authors assumed that $x(t)$ and $y(t)$ are competing for the same resource and assumed that the disease spreads only in one of the competing species, denoted by $y(t)$. They specified the healthy individuals $x(t)$, the healthy individuals $y_1(t)$, and the infected individuals of the latter species denoted by $y_2(t)$. Moreover, they studied the following two-competing-species model under the effect of infectious disease

$$\begin{aligned} \mathrm{d}x(t) &= x(t)\left[a - bx(t) - cy_1(t) - \eta y_2(t)\right]\mathrm{d}t, \\ \mathrm{d}y_1(t) &= y_1(t)\left[d - \hat{e}x(t) - f\left(y_1(t) + y_2(t)\right) - \delta y_2(t)\right]\mathrm{d}t, \\ \mathrm{d}y_2(t) &= y_2(t)\left[\delta y_1(t) - gx(t) - f\left(y_1(t) + y_2(t)\right)\right]\mathrm{d}t \end{aligned} \tag{2}$$

with initial value $x(0) = x_0$, $y_1(0) = y_{10}$, and $y_2(0) = y_{20}$. The parameters in (2) are defined as follows: a and d are the intrinsic growth rates of the populations $x(t)$ and $y_1(t)$, respectively. c is the loss rate in population $x(t)$ due to the competitor $y_1(t)$ and η is the loss rate in population $x(t)$ due to the competitor $y_2(t)$. $\hat{e}$ is the loss rate in population

$y_1(t)$ due to the competitor $x(t)$ and g is the loss rate in population $y_2(t)$ due to the competitor $x(t)$. b, f are intraspecific coefficients of $x(t)$, $y_1(t)$, and $y_2(t)$. δ is the transmission rate of the infection. Denote $\mathbb{R}_+^3 = \{(x, y.z) \in \mathbb{R}^3 : x > 0,\ y > 0,\ z > 0\}$. From [4], we know that all solutions of model (2) will lie in the region

$$B_1 = \left\{(x, y_1, y_2) \in \mathbb{R}_+^3 : 0 \le x \le \frac{a}{b},\ 0 \le y_1 \le \frac{d}{f},\ 0 \le y_2 \le \frac{\delta d}{f^2}\right\} \tag{3}$$

as $t \longrightarrow \infty$, for all positive initial values $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$. Moreover, the interior equilibrium $\widehat{E} = (\widehat{x}, \widehat{y}_1, \widehat{y}_2)$ of model (2) is feasible when $\delta > f$, $g(f+\delta) > \widehat{e}f$, $a > (ca_1 + \eta a_3)$, and $(\delta - f)(gf + g\delta - \widehat{e}f) > g\delta^2$. Here

$$\begin{aligned} \widehat{x} &= \frac{a - ca_1 - \eta a_3}{b + ca_2 + \eta a_4}, \\ \widehat{y}_1 &= a_1 + a_2\widehat{x}, \\ \widehat{y}_2 &= a_3 + a_4\widehat{x}, \end{aligned} \tag{4}$$

where $a_1 = df/\delta^2$, $a_2 = (gf + g\delta - \widehat{e}f)/\delta^2$, $a_3 = (\delta - f)/\delta^2$, $a_4 = (\delta - f)(gf + g\delta - \widehat{e}f)/f\delta^2 - g/f$.

However, the population dynamics in the real world are often disturbed by some uncertain factors while the stochastic population model is more in line with actual situation. During the past decades, a great deal of attention has been paid to the study of stochastic biological model (see [9–20]). In [12], the authors discussed a two-species stochastic nonautonomous Lotka-Volterra competition model. Some sufficient conditions on the boundedness, extinction, nonpersistence in the mean, and weak persistence of solutions are established. In [13], the authors investigated the optimal harvesting problem for a stochastic delay competitive Lotka-Volterra model with Lévy jumps. In [14], the authors studied the permanence and asymptotical behavior of stochastic prey-predator model with Markovian switching. In [15], the authors investigated the stochastic competitive models in a polluted environment. In [16], the authors explored an impulsive stochastic infected predator-prey system with Lévy jumps and delays. In [20], the authors considered a stochastic susceptible-infective epidemic model in a polluted environment, which incorporates both environmental fluctuations and pollution.

Parameter perturbation induced by white noise is an important and common form to describe the effect of stochasticity. In this paper, we perturb the intrinsic growth rates a and d in model (2) with white noise; that is,

$$\begin{aligned} a &\longrightarrow a + \sigma_1 \dot{w}_1(t), \\ d &\longrightarrow d + \sigma_2 \dot{w}_2(t), \end{aligned} \tag{5}$$

where $w_1(t)$, $w_2(t)$ are mutually independent Brownian motions and σ_1^2, σ_2^2 denote the intensities of the white noise. Then corresponding to the deterministic model (2), the stochastic model takes the following form

$$\begin{aligned} dx(t) &= x(t)\left[a - bx(t) - cy_1(t) - \eta y_2(t)\right] dt \\ &\quad + \sigma_1 x(t)\, dw_1(t), \\ dy_1(t) &= y_1(t)\left[d - \widehat{e}x(t) - f\left(y_1(t) + y_2(t)\right) - \delta y_2(t)\right] dt \\ &\quad + \sigma_2 y_1(t)\, dw_2(t), \\ dy_2(t) &= y_2(t)\left[\delta y_1(t) - gx(t) - f\left(y_1(t) + y_2(t)\right)\right] dt \end{aligned} \tag{6}$$

with initial value $x(0) = x_0$, $y_1(0) = y_{10}$, and $y_2(0) = y_{20}$. Here $w = \{w_1(t), w_2(t), t \ge 0\}$ represents the two-dimensional standard Brownian motion defined on a compete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). All meanings of the parameters are exact to or similar to those for (2).

The remaining part of this paper is organized as follows. The proof of the existence and the uniqueness for the global positive solution of model (6) for any positive initial value is given in Section 2. An important asymptotic property of the model is obtained by using the exponential martingale inequality and Borel-Cantelli lemma in Section 3. In Section 4, the stochastically ultimate boundedness of the positive solution is examined. In Section 5, sufficient conditions for extinction and persistence in mean of model (6) are established. In Section 6, by constructing a suitable Lyapunov function, we investigate the asymptotic behaviors of the stochastic model (6) around the interior equilibrium of the deterministic model. Numerical simulations under certain parameters are presented to illustrate our main results in Section 7. Finally, a few comments will conclude the paper.

2. Existence and Uniqueness of Positive Solution

Since $x(t)$, $y_1(t)$, and $y_2(t)$ in model (6) are the size of the populations at time t, we are interested only in the positive solutions of model (6). However, the coefficients of (6) do not satisfy the linear growth condition; the classical theory of stochastic differential equations is not applicable directly. Next, by using comparison theorem of stochastic differential equations, we show that model (6) has unique positive global solution with positive initial value.

Theorem 1. *For any initial value $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$, model (6) has unique global solution $(x(t), y_1(t), y_2(t))$ defined on $t \ge 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one.*

Proof. For $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$, we first consider the following stochastic differential system

$$\begin{aligned}
&\mathrm{d}u(t) \\
&\quad = \left[a - \frac{\sigma_1^2}{2} - be^{u(t)} - ce^{v_1(t)} - \eta e^{v_2(t)}\right]\mathrm{d}t \\
&\qquad + \sigma_1 \mathrm{d}w_1(t), \\
&\mathrm{d}v_1(t) \\
&\quad = \left[d - \frac{\sigma_2^2}{2} - \widehat{e}e^{u(t)} - f\left(e^{v_1(t)} + e^{v_2(t)}\right) - \delta e^{v_2(t)}\right]\mathrm{d}t \\
&\qquad + \sigma_2 \mathrm{d}w_2(t), \\
&\mathrm{d}v_2(t) = \left[\delta e^{v_1(t)} - ge^{u(t)} - fe^{v_1(t)} - fe^{v_2(t)}\right]\mathrm{d}t
\end{aligned} \tag{7}$$

with the initial value $u(0) = \ln x_0$, $v_1(0) = \ln y_{10}$, $v_2(0) = \ln y_{20}$. Since the coefficients of (7) obey the local Lipschitz condition, and, hence, for given initial value $(u(0), v_1(0), v_2(0))$, (7) has a unique maximal local solution $(u(t), v_1(t), v_2(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. Therefore, by using Itô formula, it follows that $(x(t), y_1(t), y_2(t)) = (e^{u(t)}, e^{v_1(t)}, e^{v_2(t)})$ is the unique positive local solution of (6) with the initial value (x_0, y_{10}, y_{20}).

Now, by using the comparison theorem of stochastic differential equations, we show that $(u(t), v_1(t), v_2(t))$ is a global solution to system (7); that is, $\tau_e = \infty$ a.s. Let us consider the following two stochastic differential systems

$$\begin{aligned}
\mathrm{d}\Phi(t) &= \Phi(t)\left[a - b\Phi(t)\right]\mathrm{d}t + \sigma_1\Phi(t)\,\mathrm{d}w_1(t), \\
\mathrm{d}\Psi_1(t) &= \Psi_1(t)\left[d - f\Psi_1(t)\right]\mathrm{d}t + \sigma_2\Psi_1(t)\,\mathrm{d}w_2(t), \\
\mathrm{d}\Psi_2(t) &= \Psi_2(t)\left[\delta\Psi_1(t) - f\Psi_2(t)\right]\mathrm{d}t
\end{aligned} \tag{8}$$

with initial value $(\Phi(0), \Psi_1(0), \Psi_2(0)) = (x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$ and

$$\begin{aligned}
&\mathrm{d}\phi(t) \\
&\quad = \phi(t)\left[a - b\phi(t) - c\Psi_1(t) - \eta\Psi_2(t)\right]\mathrm{d}t \\
&\qquad + \sigma_1\phi(t)\,\mathrm{d}w_1(t), \\
&\mathrm{d}\psi_1(t) \\
&\quad = \psi_1(t)\left[d - \widehat{e}\Phi(t) - (f+\delta)\Psi_2(t) - f\psi_1(t)\right]\mathrm{d}t \\
&\qquad + \sigma_2\psi_1(t)\,\mathrm{d}w_2(t), \\
&\mathrm{d}\psi_2(t) = \psi_2(t)\left[-g\Phi(t) - f\Psi_1(t) - f\psi_2(t)\right]\mathrm{d}t
\end{aligned} \tag{9}$$

with initial value $(\phi(0), \psi_1(0), \psi_2(0)) = (x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$.

Thanks to [21, 22], systems (8) and (9) can be explicitly solved as follows:

$$\begin{aligned}
&\Phi(t) = \frac{\exp\left\{\left(a - \sigma_1^2/2\right)t + \sigma_1 w_1(t)\right\}}{1/x_0 + b\int_0^t \exp\left\{\left(a - \sigma_1^2/2\right)s + \sigma_1 w_1(s)\right\}\mathrm{d}s}, \\
&\Psi_1(t) \\
&\quad = \frac{\exp\left\{\left(d - \sigma_2^2/2\right)t + \sigma_2 w_2(t)\right\}}{1/y_{10} + f\int_0^t \exp\left\{\left(d - \sigma_2^2/2\right)s + \sigma_2 w_2(s)\right\}\mathrm{d}s}, \\
&\Psi_2(t) = \frac{\exp\left\{\delta\int_0^t \Psi_1(s)\,\mathrm{d}s\right\}}{1/y_{20} + f\int_0^t \exp\left\{\delta\int_0^s \Psi_1(r)\,\mathrm{d}r\right\}\mathrm{d}s}
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\phi(t) &= \frac{\exp\left\{\left(a - \sigma_1^2/2\right)t - \int_0^t \left[c\Psi_1(r) + \eta\Psi_2(r)\right]\mathrm{d}r + \sigma_1 w_1(t)\right\}}{1/x_0 + b\int_0^t \exp\left\{\left(a - \sigma_1^2/2\right)s - \int_0^s \left[c\Psi_1(r) + \eta\Psi_2(r)\right]\mathrm{d}r + \sigma_1 w_1(s)\right\}\mathrm{d}s}, \\
\psi_1(t) &= \frac{\exp\left\{\left(d - \sigma_2^2/2\right)t - \int_0^t \left[\widehat{e}\Phi(r) + (f+\delta)\Psi_2(r)\right]\mathrm{d}r + \sigma_2 w_2(t)\right\}}{1/y_{10} + f\int_0^t \exp\left\{\left(d - \sigma_2^2/2\right)s - \int_0^s \left[\widehat{e}\Phi(r) + (f+\delta)\Psi_2(r)\right]\mathrm{d}r + \sigma_2 w_2(s)\right\}\mathrm{d}s}, \\
\psi_2(t) &= \frac{\exp\left\{-\int_0^t \left[g\Phi(r) + f\Psi_1(r)\right]\mathrm{d}r\right\}}{1/y_{20} + f\int_0^t \exp\left\{-\int_0^s \left[g\Phi(r) + f\Psi_1(r)\right]\mathrm{d}r\right\}\mathrm{d}s}.
\end{aligned} \tag{11}$$

Note that the local solution $(x(t), y_1(t), y_2(t))$ is positive for $t \in [0, \tau_e)$. Then, by the comparison theorem of stochastic differential equations (see [23]) we have

$$\begin{aligned}
0 < \phi(t) \le x(t) \le \Phi(t) \quad \text{a.s.}, \\
0 < \psi_i(t) \le y_i(t) \le \Psi_i(t) \quad \text{a.s.}
\end{aligned} \tag{12}$$

for $t \in [0, \tau_e)$, $i = 1, 2$. Thus,

$$\begin{aligned}
\ln\phi(t) \le u(t) \le \ln\Phi(t) \quad \text{a.s.}, \\
\ln\psi_i(t) \le v_i(t) \le \ln\Psi_i(t) \quad \text{a.s.}
\end{aligned} \tag{13}$$

for $t \in [0, \tau_e)$, $i = 1, 2$.

Note that $\ln\phi(t)$, $\ln\Phi(t)$, $\ln\psi_i(t)$, and $\ln\Psi_i(t)$ exist on $[0,\infty)$. Hence $\tau_e = \infty$. Thus, for any initial value $(u(0), v_1(0), v_2(0)) = (\ln x_0, \ln y_{10}, \ln y_{20}) \in \mathbb{R}^3$, system (7) has a unique global solution $(u(t), v_1(t), v_2(t))$ on $[0,\infty)$ a.s. Therefore, for any initial value $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$, model (6) has a unique global positive solution $(x(t), y_1(t), y_2(t)) = (e^{u(t)}, e^{v_1(t)}, e^{v_2(t)})$ on $[0,\infty)$ a.s. The proof is therefore complete. □

3. Asymptotic Property

In this section, by using the exponential martingale inequality and Borel-Cantelli lemma, we investigate an important asymptotic property of positive solutions of model (6).

Theorem 2. *Let $(x(t), y_1(t), y_2(t))$ be the solution of model (6) with initial value $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$. Then*

$$\limsup_{t\longrightarrow\infty}\frac{\ln x(t)}{\ln t} \le 1,$$
$$\limsup_{t\longrightarrow\infty}\frac{\ln y_i(t)}{\ln t} \le 1 \text{ a.s.} \quad i = 1, 2. \tag{14}$$

Proof. For population x, applying Itô's formula to $e^t \ln x$ leads to

$$\begin{aligned} e^t \ln x(t) &= \ln x_0 + \int_0^t e^s \ln x(s)\,\mathrm{d}s \\ &\quad + \int_0^t e^s\left[a - bx(s) - cy_1(s) - \eta y_2(s) - \frac{\sigma_1^2}{2}\right]\mathrm{d}s \\ &\quad + M_1(t), \end{aligned} \tag{15}$$

where $M_1(t) = \int_0^t \sigma_1 e^s \mathrm{d}w_1(s)$ is a continuous local martingale vanishing at time 0 and the quadratic variation of $M_1(t)$ is

$$\langle M_1, M_1\rangle_t = \int_0^t \sigma_1^2 e^{2s}\mathrm{d}s. \tag{16}$$

Let $n = 1, 2, \cdots$, $\gamma > 0$, $\theta > 1$, and $0 < \varepsilon < 1$. Choose $T = n\gamma$, $\alpha = \varepsilon e^{-n\gamma}$, and $\beta = (\theta e^{n\gamma}\ln n)/\varepsilon$. By the exponential martingale inequality (see Theorem 1.7.4 in [24]), we deduce that

$$\mathbb{P}\left\{\sup_{0\le t\le T}\left[M_1(t) - \frac{\alpha}{2}\langle M_1, M_1\rangle_t\right] > \beta\right\} \le e^{-\alpha\beta} = \frac{1}{n^\theta}. \tag{17}$$

Since $\sum_{n=0}^{\infty}(1/n^\theta) < \infty$ for $\theta > 1$, the Borel-Cantelli lemma (see Lemma 1.2.4 in [24]) implies that there exists a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ and an integer-valued random variable $n_0 = n_0(\omega)$ such that for every $\omega \in \Omega_0$

$$M_1(t) \le \frac{\theta e^{n\gamma}\ln n}{\varepsilon} + \frac{\varepsilon e^{-n\gamma}}{2}\langle M_1, M_1\rangle_t \tag{18}$$

holds for all $0 \le t \le n\gamma$, $n \ge n_0$. Substituting the above inequality into (15), we see that

$$\begin{aligned} e^t \ln x(t) &\le \ln x_0 + \int_0^t e^s[\ln x(s) + a - bx(s)]\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_0^t \sigma_1^2 e^s\mathrm{d}s + \frac{\varepsilon e^{-n\gamma}}{2}\int_0^t \sigma_1^2 e^{2s}\mathrm{d}s \\ &\quad + \frac{\theta e^{n\gamma}\ln n}{\varepsilon} \end{aligned} \tag{19}$$

holds for all $0 \le t \le n\gamma$, $n \ge n_0$. Note that, for $0 \le s \le t \le n\gamma$,

$$\begin{aligned} \frac{1}{2}\varepsilon e^{-n\gamma}\sigma_1^2 e^{2s} - \frac{1}{2}\sigma_1^2 e^s &= \frac{1}{2}\sigma_1^2 e^s\left(\varepsilon e^{s-n\gamma} - 1\right) \\ &\le \frac{1}{2}\sigma_1^2 e^s(\varepsilon - 1) < 0. \end{aligned} \tag{20}$$

Therefore, for all $0 \le t \le n\gamma$, $n \ge n_0$, it follows from (19) that

$$\begin{aligned} e^t \ln x(t) &\le \ln x_0 + \int_0^t e^s[\ln x(s) + a - bx(s)]\,\mathrm{d}s \\ &\quad + \frac{\theta e^{n\gamma}\ln n}{\varepsilon}. \end{aligned} \tag{21}$$

Let us consider function $q(x) = \ln x + a - bx$ on $(0,\infty)$. It is easy to show that q has maximum value for $x = 1/b > 0$ and maximum value of function q is $q_{\max} = \ln(1/b) + a - 1$. Denote $K_1 \doteq (\ln(1/b) + a - 1) \vee 1$. Then

$$e^t \ln x(t) \le \ln x_0 + K_1 e^t + \frac{\theta e^{n\gamma}\ln n}{\varepsilon} \tag{22}$$

holds for all $0 \le t \le n\gamma$, $n \ge n_0$. Therefore, for all $0 \le (n-1)\gamma \le t \le n\gamma$, $n \ge n_0$, we have

$$\frac{\ln x(t)}{\ln t} \le \frac{\ln x_0}{e^t\ln t} + \frac{K_1}{\ln t} + \frac{\theta e^\gamma \ln n}{\varepsilon\ln[(n-1)\gamma]}. \tag{23}$$

Letting $n \longrightarrow \infty$ (and so $t \longrightarrow \infty$), we obtain

$$\limsup_{t\longrightarrow\infty}\frac{\ln x(t)}{\ln t} \le \frac{\theta e^\gamma}{\varepsilon}. \tag{24}$$

Moreover, letting $\theta \downarrow 1$, $\gamma \downarrow 0$, and $\varepsilon \uparrow 1$, we can get

$$\limsup_{t\longrightarrow\infty}\frac{\ln x(t)}{\ln t} \le 1 \text{ a.s.} \tag{25}$$

For population y, denote $y(t) = y_1(t) + y_2(t)$ and $y_0 = y_{10} + y_{20}$. It follows from (6) that

$$\begin{aligned} \mathrm{d}y(t) &= \left[dy_1(t) - \hat{e}x(t)y_1(t) - gx(t)y_2(t) - fy^2(t)\right]\mathrm{d}t \\ &\quad + \sigma_2 y_1(t)\,\mathrm{d}w_2(t). \end{aligned} \tag{26}$$

Applying Itô's formula to $e^t \ln y$ leads to

$$\begin{aligned} e^t \ln y(t) &\le \ln y_0 + \int_0^t e^s[\ln y(s) + d - fy(s)]\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_0^t \sigma_2^2 e^s\frac{y_1^2(s)}{y^2(s)}\mathrm{d}s + M_2(t), \end{aligned} \tag{27}$$

where $M_2(t) = \int_0^t \sigma_2 e^s (y_1(s)/y(s)) \mathrm{d}w_2(s)$ is a continuous local martingale with initial value $M_2(0) = 0$ and the quadratic variation of $M_2(t)$ is

$$\langle M_2, M_2 \rangle_t = \int_0^t \sigma_2^2 e^{2s} \frac{y_1^2(s)}{y^2(s)} \mathrm{d}s. \quad (28)$$

Similarly, we can derive that there exists a set $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$ and an integer-valued random variable $n_1 = n_1(\omega)$ such that, for every $\omega \in \Omega_1$ and $n \geq n_1$,

$$M_2(t) \leq \frac{\theta e^{n\gamma} \ln n}{\varepsilon} + \frac{\varepsilon e^{-n\gamma}}{2} \langle M_2, M_2 \rangle_t \quad (29)$$

holds for all $0 \leq t \leq n\gamma$. Moreover, for $0 \leq s \leq t \leq n\gamma$, we have

$$\frac{1}{2}\varepsilon e^{-n\gamma} \sigma_2^2 e^{2s} - \frac{1}{2}\sigma_2^2 e^s < 0. \quad (30)$$

Substituting these inequalities into (27), we see that

$$\begin{aligned} e^t \ln y(t) &\leq \ln y_0 + \int_0^t e^s [\ln y(s) + d - fy(s)] \mathrm{d}s \\ &\quad + \frac{\theta e^{n\gamma} \ln n}{\varepsilon} \end{aligned} \quad (31)$$

holds for all $0 \leq t \leq n\gamma$, $n \geq n_1$. Similarly, for almost all $0 \leq s \leq n\gamma$, there exists a positive constant $K_2 \doteq (\ln(1/f)+d-1) \vee 1$ such that $\ln y(s) + d - fy(s) \leq K_2$. This, together with (31), yields

$$e^t \ln y(t) \leq \ln y_0 + K_2 e^t + \frac{\theta e^{n\gamma} \ln n}{\varepsilon} \quad (32)$$

for all $0 \leq t \leq n\gamma$, $n \geq n_0$. Thus,

$$\limsup_{t \to \infty} \frac{\ln y(t)}{\ln t} \leq 1 \text{ a.s.} \quad (33)$$

Since $y_1(t)$ and $y_2(t)$ are positive for all $t \geq 0$, it follows that

$$\limsup_{t \to \infty} \frac{\ln y_i(t)}{\ln t} \leq \limsup_{t \to \infty} \frac{\ln y(t)}{\ln t} \leq 1 \text{ a.s.}, \quad i = 1, 2. \quad (34)$$

The proof is therefore complete. □

Remark 3. It follows from $\lim_{t \to \infty} (\ln t / t) = 0$ that

$$\begin{aligned} &\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leq 0, \\ &\limsup_{t \to \infty} \frac{\ln y_i(t)}{t} \leq 0 \text{ a.s.}, \quad i = 1, 2. \end{aligned} \quad (35)$$

Hence, the sample Lyapunov exponents of the solutions of model (6) are less than or equal to zero.

4. Stochastically Ultimate Boundedness

In this section, we continue to examine the stochastically ultimate boundedness which means that the solution is ultimately bounded with the large probability. Firstly, its definition will be given.

Definition 4 (see [25]). The solutions of model (6) are called stochastically ultimate bounded, if, for any $\varepsilon \in (0, 1)$, there exist three positive constants $H_1 = H_1(\varepsilon)$, $H_2 = H_2(\varepsilon)$, and $H_3 = H_3(\varepsilon)$ such that the solution $(x(t), y_1(t), y_2(t))$ of model (6) with any initial value in $\mathbb{R}_+^3$ satisfies

$$\begin{aligned} &\limsup_{t \to \infty} \mathbb{P}\{x(t) > H_1\} < \varepsilon, \\ &\limsup_{t \to \infty} \mathbb{P}\{y_1(t) > H_2\} < \varepsilon, \\ &\limsup_{t \to \infty} \mathbb{P}\{y_2(t) > H_3\} < \varepsilon. \end{aligned} \quad (36)$$

Now, we prove that the solutions of model (6) are uniformly bounded in the pth moment by using the Bernoulli equation. Then, the stochastically ultimate boundedness follows directly by Chebyshev's inequality.

Lemma 5. *For any positive constants p, α, and β, the Bernoulli equation*

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = p\beta\varphi(t) - p\alpha\varphi^{1+1/p}(t), \quad (37)$$

with the initial value $\varphi(0) = x_0 > 0$, has the solution

$$\varphi(t) = \left[\frac{\beta}{\alpha\left(1 - e^{-\beta t} + (\beta/\alpha) x_0^{-1/p} e^{-\beta t}\right)}\right]^p. \quad (38)$$

Proof. Obviously, (37) is equivalent to

$$\frac{1}{p}\varphi^{-(1+1/p)}(t) \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} - \beta\varphi^{-1/p}(t) = -\alpha. \quad (39)$$

Multiplying the above equation by integral factor $-e^{\beta t}$, we obtain

$$-\frac{1}{p}e^{\beta t}\varphi^{-(1+1/p)}(t) \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} + \beta e^{\beta t}\varphi^{-1/p}(t) = \alpha e^{\beta t}, \quad (40)$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\beta t}\varphi^{-1/p}(t)\right] = \alpha e^{\beta t}. \quad (41)$$

Integrating both sides of the above equation from 0 to t yields

$$\varphi^{-1/p}(t) = x_0^{-1/p} e^{-\beta t} + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right), \quad (42)$$

which implies

$$\varphi(t) = \left[\frac{\beta}{\alpha\left(1 - e^{-\beta t} + (\beta/\alpha) x_0^{-1/p} e^{-\beta t}\right)}\right]^p. \quad (43)$$

The proof is therefore complete. □

Lemma 6. *For any* $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$, *let* $(x(t), y_1(t), y_2(t))$ *be the solution of model (6) with initial value* (x_0, y_{10}, y_{20}). *Then, for any* $p \geq 1$,

$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[x^p(t)\right] \leq \left[\frac{a + ((p-1)/2)\sigma_1^2}{b}\right]^p,$$
$$\limsup_{t\longrightarrow\infty} \mathbb{E}\left[y_i^p(t)\right] \leq \left[\frac{d + ((p-1)/2)\sigma_2^2}{f}\right]^p, \quad i = 1, 2. \tag{44}$$

That is, the solutions of model (6) are uniformly bounded in the pth moment.

Proof. For population x, applying Itô's formula to x^p leads to

$$\begin{aligned} x^p(t) = x_0^p &+ \int_0^t px^p(s) \\ &\cdot \left[a + \frac{p-1}{2}\sigma_1^2 - bx(s) - cy_1(s) - \eta y_2(s)\right] \mathrm{d}s \\ &+ \int_0^t p\sigma_1 x^p(s)\,\mathrm{d}w_1(s). \end{aligned} \tag{45}$$

Taking expectation on both sides of the above inequality, we can derive

$$\begin{aligned} \mathbb{E}\left[x^p(t)\right] = x_0^p &+ \mathbb{E}\int_0^t px^p(s) \\ &\cdot \left[a + \frac{p-1}{2}\sigma_1^2 - bx(s) - cy_1(s) - \eta y_2(s)\right] \mathrm{d}s, \end{aligned} \tag{46}$$

which implies the differentiability of $\mathbb{E}[x^p(t)]$. Denote $\beta_1 \doteq a + ((p-1)/2)\sigma_1^2$. Then, using Hölder inequality $(\mathbb{E}x^p)^{1/p} \leq (\mathbb{E}x^{p+1})^{1/(p+1)}$, we obtain that

$$\begin{aligned} \frac{\mathrm{d}\mathbb{E}\left[x^p(t)\right]}{\mathrm{d}t} &\leq p\beta_1\mathbb{E}\left[x^p(t)\right] - pb\mathbb{E}\left[x^{p+1}(t)\right] \\ &\leq p\beta_1\mathbb{E}\left[x^p(t)\right] - pb\left(\mathbb{E}\left[x^p(t)\right]\right)^{1+1/p}. \end{aligned} \tag{47}$$

Then, from Lemma 5 and the comparison theorem, it follows that

$$\mathbb{E}\left[x^p(t)\right] \leq \left[\frac{\beta_1}{b\left(1 - e^{-\beta_1 t} + (\beta_1/b)\,x_0^{-1}e^{-\beta_1 t}\right)}\right]^p. \tag{48}$$

Note that $p \geq 1$. Then $\beta_1 = a + ((p-1)/2)\sigma_1^2 > 0$. Thus,

$$\begin{aligned} \limsup_{t\longrightarrow\infty} \mathbb{E}\left[x^p(t)\right] &\leq \left[\frac{\beta_1}{b}\right]^p \\ &= \left[\frac{a + ((p-1)/2)\sigma_1^2}{b}\right]^p. \end{aligned} \tag{49}$$

For population y, denote $y(t) = y_1(t) + y_2(t)$ and $y_0 = y_{10} + y_{20}$. Applying Itô's formula to y^p leads to

$$\begin{aligned} y^p(t) = &\int_0^t \Big\{py^{p-1}(s) \\ &\cdot \left[dy_1(s) - \hat{e}x(s)y_1(s) - gx(s)y_2(s) - fy^2(s)\right] \\ &+ \frac{p(p-1)}{2}\sigma_2^2 y_1^2(s)\,y^{p-2}(s)\Big\}\,\mathrm{d}s \\ &+ \int_0^t p\sigma_2 y_1(s)\,y^{p-1}(s)\,\mathrm{d}w_2(s) + y_0^p. \end{aligned} \tag{50}$$

Taking expectation on both sides of the above inequality, we can derive

$$\begin{aligned} \mathbb{E}\left[y^p(t)\right] = y_0^p &+ \mathbb{E}\int_0^t \Big\{py^{p-1}(s) \\ &\cdot \left[dy_1(s) - \hat{e}x(s)y_1(s) - gx(s)y_2(s) - fy^2(s)\right] \\ &+ \frac{p(p-1)}{2}\sigma_2^2 y_1^2(s)\,y^{p-2}(s)\Big\}\,\mathrm{d}s, \end{aligned} \tag{51}$$

which implies the differentiability of $\mathbb{E}[y^p(t)]$. Denote $\beta_2 \doteq d + ((p-1)/2)\sigma_2^2$. Using the Hölder inequality, we have

$$\begin{aligned} \frac{\mathrm{d}\mathbb{E}\left[y^p(t)\right]}{\mathrm{d}t} &\leq p\beta_2\mathbb{E}\left[y^p(t)\right] - pf\mathbb{E}\left[y^{p+1}(t)\right] \\ &\leq p\beta_2\mathbb{E}\left[y^p(t)\right] - pf\left(\mathbb{E}\left[y^p(t)\right]\right)^{1+1/p}. \end{aligned} \tag{52}$$

Then, from Lemma 5 and the comparison theorem, it follows that

$$\mathbb{E}\left[y^p(t)\right] \leq \left[\frac{\beta_2}{f\left(1 - e^{-\beta_2 t} + (\beta_2/f)\,y_0^{-1}e^{-\beta_2 t}\right)}\right]^p. \tag{53}$$

Note that $p \geq 1$. Then $\beta_2 = d + ((p-1)/2)\sigma_2^2 > 0$. Thus,

$$\begin{aligned} \limsup_{t\longrightarrow\infty} \mathbb{E}\left[y^p(t)\right] &\leq \left[\frac{\beta_2}{f}\right]^p \\ &= \left[\frac{d + ((p-1)/2)\sigma_2^2}{f}\right]^p. \end{aligned} \tag{54}$$

Since $y_1(t)$ and $y_2(t)$ are positive for all $t \geq 0$, it follows that, for $i = 1, 2$,

$$\begin{aligned} \limsup_{t\longrightarrow\infty} \mathbb{E}\left[y_i^p(t)\right] &\leq \limsup_{t\longrightarrow\infty} \mathbb{E}\left[y^p(t)\right] \\ &\leq \left[\frac{d + ((p-1)/2)\sigma_2^2}{f}\right]^p. \end{aligned} \tag{55}$$

The proof is therefore complete. □

According to Chebyshev's inequality and the application of Lemma 6, we have the following result.

Theorem 7. *Solutions of model (6) are stochastically ultimate bounded.*

Proof. Let $(x(t), y_1(t), y_2(t))$ be the solution of model (6) with any initial values $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$. From Lemma 6, it follows that

$$\limsup_{t\longrightarrow\infty} \mathbb{E}[x(t)] \le \frac{a}{b},$$
$$\limsup_{t\longrightarrow\infty} \mathbb{E}[y_i(t)] \le \frac{d}{f}, \quad i = 1, 2. \tag{56}$$

Now, for any $\varepsilon \in (0, 1)$, let $H_1 = a/b\varepsilon + 1$ and $H_2 = H_3 = d/f\varepsilon + 1$. Then by Chebyshev's inequality

$$\mathbb{P}\{x(t) > H_1\} \le \frac{\mathbb{E}[x(t)]}{H_1},$$
$$\mathbb{P}\{y_1(t) > H_2\} \le \frac{\mathbb{E}[y_1(t)]}{H_2}, \tag{57}$$
$$\mathbb{P}\{y_2(t) > H_3\} \le \frac{\mathbb{E}[y_2(t)]}{H_3}.$$

Hence,

$$\limsup_{t\longrightarrow\infty} \mathbb{P}\{x(t) > H_1\} \le \limsup_{t\longrightarrow\infty} \frac{\mathbb{E}[x(t)]}{H_1} < \varepsilon,$$
$$\limsup_{t\longrightarrow\infty} \mathbb{P}\{y_1(t) > H_2\} \le \limsup_{t\longrightarrow\infty} \frac{\mathbb{E}[y_1(t)]}{H_2} < \varepsilon, \tag{58}$$
$$\limsup_{t\longrightarrow\infty} \mathbb{P}\{y_2(t) > H_3\} \le \limsup_{t\longrightarrow\infty} \frac{\mathbb{E}[y_2(t)]}{H_3} < \varepsilon.$$

The proof is therefore complete. □

5. Extinction and Persistence

In this section, we will investigate the extinction and persistence of the population. In order to obtain our main results, several lemmas will be given. For the sake of convenience and simplicity, we first introduce the following notation:

$$\langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)\,ds. \tag{59}$$

Lemma 8 (see [26]). *Suppose* $x \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$ *and* $F \in C(\Omega \times [0, +\infty), (-\infty, +\infty))$.

(I) If there are two positive constants λ_0 *and* T *such that*

$$\ln x(t) \le \lambda t - \lambda_0 \int_0^t x(s)\,ds + F(t), \quad a.s., \qquad \text{for } t \ge T, \tag{60}$$

where $\lim_{t\longrightarrow\infty}(F(t)/t) = 0$, *then*

$$\limsup_{t\longrightarrow\infty} \langle x(t)\rangle \le \frac{\lambda}{\lambda_0}, \quad a.s. \quad \lambda \ge 0,$$
$$\lim_{t\longrightarrow\infty} x(t) = 0, \quad a.s. \quad \lambda < 0. \tag{61}$$

(II) If there are three positive constants λ_0, T, *and* λ *such that*

$$\ln x(t) \ge \lambda t - \lambda_0 \int_0^t x(s)\,ds + F(t), \quad a.s., \qquad \text{for } t \ge T, \tag{62}$$

where $\lim_{t\longrightarrow\infty}(F(t)/t) = 0$, *then*

$$\liminf_{t\longrightarrow\infty} \langle x(t)\rangle \ge \frac{\lambda}{\lambda_0} \quad a.s. \tag{63}$$

The following theorem investigates how the intensity of noise affects the competition population.

Theorem 9. *Let* $(x(t), y_1(t), y_2(t))$ *be the solution of model (6) with any initial value* $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$.

(i) *If* $a - 0.5\sigma_1^2 < 0$, *then* $\lim_{t\longrightarrow\infty} x(t) = 0$ *a.s.*
If $a - 0.5\sigma_1^2 = 0$, *then* $\lim_{t\longrightarrow\infty}\langle x(t)\rangle = 0$ *a.s.*
If $a - 0.5\sigma_1^2 > 0$, *then* $\limsup_{t\longrightarrow\infty}\langle x(t)\rangle \le (a - 0.5\sigma_1^2)/b$ *a.s.*

(ii) *If* $d - 0.5\sigma_2^2 < 0$, *then* $\lim_{t\longrightarrow\infty} y_1(t) = 0$ *a.s.*
If $d - 0.5\sigma_2^2 = 0$, *then* $\lim_{t\longrightarrow\infty}\langle y_1(t)\rangle = 0$ *a.s.*
If $d - 0.5\sigma_2^2 > 0$, *then* $\limsup_{t\longrightarrow\infty}\langle y_1(t)\rangle \le (d - 0.5\sigma_2^2)/f$ *a.s.*

(iii) *If* $d - 0.5\sigma_2^2 < 0$, *then* $\lim_{t\longrightarrow\infty}\langle y_2(t)\rangle = 0$ *a.s.*

Proof. (i) For population x, using Itô's formula results in

$$\begin{aligned}\ln x(t) &= \int_0^t \left[\left(a - 0.5\sigma_1^2\right) - bx(s) - cy_1(s) - \eta y_2(s)\right] ds + \sigma_1 w_1(t) + \ln x_0 \\ &\le \left(a - 0.5\sigma_1^2\right)t - b\int_0^t x(s)\,ds + \sigma_1 w_1(t) + \ln x_0.\end{aligned} \tag{64}$$

Clearly, Brownian motion $w_1(t)$ is a real-valued continuous local martingale vanishing at time 0. Then, from the strong law of large numbers [24], it follows that

$$\lim_{t\longrightarrow\infty} \frac{w_1(t)}{t} = 0. \tag{65}$$

Thus,

$$\lim_{t\longrightarrow\infty} \frac{\sigma_1 w_1(t) + \ln x_0}{t} = 0. \tag{66}$$

Note that $x(t) > 0$. Then $\liminf_{t\longrightarrow\infty}\langle x(t)\rangle \ge 0$. Thus, from Lemma 8, it follows that

$$\lim_{t\longrightarrow\infty} x(t) = 0, \quad a.s. \quad a - 0.5\sigma_1^2 < 0,$$
$$\lim_{t\longrightarrow\infty} \langle x(t)\rangle = 0, \quad a.s. \quad a - 0.5\sigma_1^2 = 0, \tag{67}$$
$$\limsup_{t\longrightarrow\infty} \langle x(t)\rangle \le \frac{a - 0.5\sigma_1^2}{b}, \quad a.s. \quad a - 0.5\sigma_1^2 > 0.$$

(ii) For population y_1, applying Itô formula, we have

$$\ln y_1(t) = \int_0^t \left[\left(d - 0.5\sigma_2^2\right) - \hat{e}x(s) - f\left(y_1(s) + y_2(s)\right) - \delta y_2(s)\right] \mathrm{d}s + \sigma_2 w_2(t) + \ln y_{10} \le \left(d - 0.5\sigma_2^2\right) t - f \int_0^t y_1(s)\,\mathrm{d}s + \sigma_2 w_2(t) + \ln y_{10}. \quad (68)$$

A similar discussion to that in the above for x, we also have

$$\begin{aligned} &\lim_{t\to\infty} y_1(t) = 0, \ \ a.s. \quad d - 0.5\sigma_2^2 < 0, \\ &\lim_{t\to\infty} \langle y_1(t)\rangle = 0, \ \ a.s. \quad d - 0.5\sigma_2^2 = 0, \\ &\limsup_{t\to\infty} \langle y_1(t)\rangle \le \frac{d - 0.5\sigma_2^2}{f}, \ \ a.s. \quad d - 0.5\sigma_2^2 > 0. \end{aligned} \quad (69)$$

(iii) For population y_2, using Itô formula results in

$$\mathrm{d}\ln y_2(t) = \left[(\delta - f) y_1(t) - gx(t) - fy_2(t)\right] \mathrm{d}t. \quad (70)$$

It follows from $d - 0.5\sigma_2^2 < 0$ that $\lim_{t\to\infty} y_1(t) = 0$ a.s. Let $\Omega_2 = \{\omega \in \Omega : \lim_{t\to\infty} y_1(t,\omega) = 0\}$; then $\lim_{t\to\infty} y_1(t) = 0$ a.s. implies $\mathbb{P}(\Omega_2) = 1$. Hence, for any $\omega \in \Omega_2$ and any constant $\varepsilon > 0$, there exists a constant $t_0 = t_0(\omega, \varepsilon) > 0$ such that, for any $t \ge t_0$,

$$-\varepsilon \le y_1(t,\omega) \le \varepsilon. \quad (71)$$

Thus, for every, $t \ge t_0$,

$$\ln y_2(t) = \int_{t_0}^t \left[(\delta - f) y_1(s) - gx(s) - fy_2(s)\right] \mathrm{d}s + \ln y_2(t_0). \quad (72)$$

Consequently, if $\delta \le f$, then

$$\ln y_2(t) \le (f - \delta)\varepsilon(t - t_0) - f \int_{t_0}^t y_2(s)\,\mathrm{d}s + \ln y_2(t_0), \quad (73)$$

whereas if $\delta > f$, then

$$\ln y_2(t) \le (\delta - f)\varepsilon(t - t_0) - f \int_{t_0}^t y_2(s)\,\mathrm{d}s + \ln y_2(t_0). \quad (74)$$

Making use of Lemma 8 and the arbitrariness of ε, we have $\limsup_{t\to\infty}\langle y_2(t)\rangle \le 0$ a.s. Note that $\liminf_{t\to\infty}\langle y_2(t)\rangle \ge 0$ a.s. Thus,

$$\lim_{t\to\infty} \langle y_2(t)\rangle = 0 \ \ a.s. \quad (75)$$

The proof is therefore complete. □

The following theorem tells us that competition coefficients play an important role in determining extinction of species for stochastic model. Denote

$$\begin{aligned} \Delta &= bf - c\hat{e}, \\ \Delta_1 &= \left(a - 0.5\sigma_1^2\right) f - \left(d - 0.5\sigma_2^2\right) c, \\ \Delta_2 &= c(f + \delta) - f\eta, \\ \Delta_3 &= \eta(f - \delta) - cf, \\ \Delta_4 &= \left(d - 0.5\sigma_2^2\right) b - \left(a - 0.5\sigma_1^2\right)\hat{e}, \\ \Delta_5 &= \hat{e}\eta - b(f + \delta), \\ \Delta_6 &= f\hat{e} - (f + \delta) g. \end{aligned} \quad (76)$$

Theorem 10. *Let $(x(t), y_1(t), y_2(t))$ be the solution of model (6) with initial value $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$. Assume that $a - 0.5\sigma_1^2 > 0$ and $d - 0.5\sigma_2^2 > 0$. Suppose $\Delta \ge 0$ (it is easy to see that $\Delta_1 < 0$ and $\Delta_4 < 0$ cannot simultaneously hold in this case).*

(i) If $\Delta_1 < 0$ and $\Delta_2 \le 0$, then $\lim_{t\to\infty} x(t) = 0$ a.s. Furthermore, if $\Delta_3 \ge 0$, then

$$\begin{aligned} &\lim_{t\to\infty} y_2(t) = 0 \ \ a.s., \\ &\lim_{t\to\infty} \langle y_1(t)\rangle = \frac{d - 0.5\sigma_2^2}{f} \ \ a.s. \end{aligned} \quad (77)$$

(ii) If $\Delta_4 < 0$ and $\Delta_5 \le 0$ then $\lim_{t\to\infty} y_1(t) = 0$ a.s. Moreover, if $\Delta_6 \ge 0$, then

$$\begin{aligned} &\lim_{t\to\infty} y_2(t) = 0 \ \ a.s., \\ &\lim_{t\to\infty} \langle x(t)\rangle = \frac{a - 0.5\sigma_1^2}{b} \ \ a.s. \end{aligned} \quad (78)$$

Proof. It follows from Itô formula that

$$\ln x(t) = \left(a - \frac{\sigma_1^2}{2}\right) t - b \int_0^t x(s)\,\mathrm{d}s - c \int_0^t y_1(s)\,\mathrm{d}s - \eta \int_0^t y_2(s)\,\mathrm{d}s + \sigma_1 w_1(t) + \ln x_0, \quad (79)$$

$$\ln y_1(t) = \left(d - \frac{\sigma_2^2}{2}\right) t - \hat{e} \int_0^t x(s)\,\mathrm{d}s - f \int_0^t y_1(s)\,\mathrm{d}s - (f + \delta) \int_0^t y_2(s)\,\mathrm{d}s + \sigma_2 w_2(t) + \ln y_{10}, \quad (80)$$

$$\ln y_2(t) = -g \int_0^t x(s)\,\mathrm{d}s - (f - \delta) \int_0^t y_1(s)\,\mathrm{d}s - f \int_0^t y_2(s)\,\mathrm{d}s + \ln y_{20}. \quad (81)$$

(i) First, computing (79) × f − (80) × c leads to

$$\begin{aligned} f\ln x(t)-c\ln y_1(t) &= f\ln x_0 - c\ln y_{10} + \Delta_1 t \\ &\quad - \Delta\int_0^t x(s)\,\mathrm{d}s \\ &\quad + \Delta_2\int_0^t y_2(s)\,\mathrm{d}s + f\sigma_1 w_1(t) \\ &\quad - c\sigma_2 w_2(t), \end{aligned} \tag{82}$$

which, together with the conditions of Theorem 10, yields

$$\begin{aligned} \frac{\ln x(t)}{t} &\le \frac{\Delta_1}{f} + \frac{1}{ft}\left[c\ln y_1(t) + f\sigma_1 w_1(t)\right. \\ &\quad \left. - c\sigma_2 w_2(t) + f\ln x_0 - c\ln y_{10}\right]. \end{aligned} \tag{83}$$

Then by applying the strong law of large numbers and Remark 3, we have

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \le \frac{\Delta_1}{f} < 0. \tag{84}$$

Thus,

$$\lim_{t\to\infty} x(t) = 0 \ a.s. \tag{85}$$

Next, computing (81) × η − (79) × f leads to

$$\begin{aligned} \eta\ln y_2(t) - f\ln x(t) &= -f\left(a-\frac{\sigma_1^2}{2}\right)t \\ &\quad - (g\eta - bf)\int_0^t x(s)\,\mathrm{d}s \\ &\quad - \Delta_3\int_0^t y_1(s)\,\mathrm{d}s - f\sigma_1 w_1(t) \\ &\quad + \eta\ln y_{20} - f\ln x_0, \end{aligned} \tag{86}$$

which, together with the conditions of Theorem 10, yields

$$\begin{aligned} \frac{\eta\ln y_2(t)}{t} &\le -f\left(a-\frac{\sigma_1^2}{2}\right) - \frac{(g\eta-bf)}{t}\int_0^t x(s)\,\mathrm{d}s \\ &\quad + \frac{f\ln x(t)}{t} - \frac{f\sigma_1 w_1(t)}{t} + \frac{\eta\ln y_{20}}{t} \\ &\quad - \frac{f\ln x_0}{t}. \end{aligned} \tag{87}$$

Let $\Omega_3 = \{\omega\in\Omega : \lim_{t\to\infty} x(t,\omega) = 0\}$; then (85) implies $\mathbb{P}(\Omega_3) = 1$. Hence, for any $\omega\in\Omega_3$ and any constant $\varepsilon > 0$, there exists a constant $T = T(\omega,\varepsilon) > 0$ such that, for any $t \ge T$,

$$x(t,\omega) \le \varepsilon. \tag{88}$$

Now, we introduce two cases.

1° Suppose that $g\eta - bf < 0$. From (87), it follows that, for any $t \ge T$,

$$\begin{aligned} \frac{\eta\ln y_2(t)}{t} &\le -f\left(a-\frac{\sigma_1^2}{2}\right) - \frac{(g\eta-bf)\varepsilon(t-T)}{t} \\ &\quad + \frac{f\ln x(t)}{t} - \frac{(g\eta-bf)}{t}\int_0^T x(s)\,\mathrm{d}s \\ &\quad - \frac{f\sigma_1 w_1(t)}{t} + \frac{\eta\ln y_{20}}{t} - \frac{f\ln x_0}{t}. \end{aligned} \tag{89}$$

Thus, from Remark 3 and the strong law of large numbers, it follows that

$$\limsup_{t\to\infty}\frac{\ln y_2(t)}{t} \le \frac{1}{\eta}\left[-f\left(a-\frac{\sigma_1^2}{2}\right) - (g\eta-bf)\varepsilon\right]. \tag{90}$$

Then, by the arbitrariness of ε, we have

$$\limsup_{t\to\infty}\frac{\ln y_2(t)}{t} \le -\frac{f}{\eta}\left(a-\frac{\sigma_1^2}{2}\right) < 0. \tag{91}$$

2° Suppose that $g\eta - bf \ge 0$. Then

$$\begin{aligned} \frac{\eta\ln y_2(t)}{t} &\le -f\left(a-\frac{\sigma_1^2}{2}\right) + \frac{f\ln x(t)}{t} - \frac{f\sigma_1 w_1(t)}{t} \\ &\quad + \frac{\eta\ln y_{20}}{t} - \frac{f\ln x_0}{t}. \end{aligned} \tag{92}$$

Thus, from Remark 3 and the strong law of large numbers, it follows that

$$\limsup_{t\to\infty}\frac{\ln y_2(t)}{t} \le -\frac{f}{\eta}\left(a-\frac{\sigma_1^2}{2}\right) < 0. \tag{93}$$

Therefore,

$$\lim_{t\to\infty} y_2(t) = 0 \ a.s. \tag{94}$$

Finally, it follows from (80) that

$$\begin{aligned} \ln y_1(t) &\le \left(d-\frac{\sigma_2^2}{2}\right)t - f\int_0^t y_1(s)\,\mathrm{d}s + \sigma_2 w_2(t) \\ &\quad + \ln y_{10}. \end{aligned} \tag{95}$$

Note that $\lim_{t\to\infty}((\sigma_2 w_2(t) + \ln y_{10})/t) = 0$ and $d - \sigma_2^2/2 > 0$. Thus, from (I) in Lemma 8, it follows that

$$\limsup_{t\to\infty}\langle y_1(t)\rangle \le \frac{d-0.5\sigma_2^2}{f}. \tag{96}$$

On the other hand, it follows from (85) and (94) that, for any constant $\varepsilon > 0$, there is a positive constant t_1 such that $t \ge t_1$

$$\begin{aligned} \hat{e}x(t) &\le \frac{\varepsilon}{2}, \\ (f+\delta)\,y_2(t) &\le \frac{\varepsilon}{2}. \end{aligned} \tag{97}$$

Substituting this into (80) yields

$$\begin{aligned}\ln y_1(t) \geq{}& \left(d - \frac{\sigma_2^2}{2}\right)t - \varepsilon(t - t_1) - f\int_0^t y_1(s)\,\mathrm{d}s \\ &- \int_0^{t_1}\left[\widehat{e}x(s) + (f+\delta)y_2(s)\right]\mathrm{d}s \\ &+ \sigma_2 w_2(t) + \ln y_{10} \\ \geq{}& \left(d - \frac{\sigma_2^2}{2} - \varepsilon\right)t - f\int_0^t y_1(s)\,\mathrm{d}s \\ &- \int_0^{t_1}\left[\widehat{e}x(s) + (f+\delta)y_2(s)\right]\mathrm{d}s \\ &+ \sigma_2 w_2(t) + \ln y_{10},\end{aligned} \tag{98}$$

for any $t \geq t_1$. Note that

$$\begin{aligned}&\lim_{t\longrightarrow\infty}\frac{1}{t}\left[\int_0^{t_1}\left[\widehat{e}x + (f+\delta)y_2\right]\mathrm{d}s - \sigma_2 w_2(t) - \ln y_{10}\right] \\ &\quad = 0.\end{aligned} \tag{99}$$

Then by (II) in Lemma 8 and the arbitrariness of ε, one can observe that

$$\liminf_{t\longrightarrow\infty}\langle y_1(t)\rangle \geq \frac{d - 0.5\sigma_2^2}{f}. \tag{100}$$

Therefore, we can derive that

$$\lim_{t\longrightarrow\infty}\langle y_1(t)\rangle = \frac{d - 0.5\sigma_2^2}{f}. \tag{101}$$

(ii) A similar discussion to that in the above for (i), we also have the desired assertion (ii). This completes the proof. □

If we do not consider the effect of disease, then model (6) can be degraded into the following model

$$\begin{aligned}\mathrm{d}x(t) &= x(t)\left[a - bx(t) - cy_1(t)\right]\mathrm{d}t \\ &\quad + \sigma_1 x(t)\,\mathrm{d}w_1(t), \\ \mathrm{d}y_1(t) &= y_1(t)\left[d - \widehat{e}x(t) - fy_1(t)\right]\mathrm{d}t \\ &\quad + \sigma_2 y_1(\mathrm{t})\,\mathrm{d}w_2(t),\end{aligned} \tag{102}$$

with initial value $x(0) = x_0$ and $y_1(0) = y_{10}$. Model (102) is the same as the stochastic competitive population model (SM_0) discussed in [15].

From Theorems 9 and 10, we can get the following corollaries with the proof being omitted.

Corollary 11. *Let $(x(t), y_1(t))$ be the solution of degradation model (102) with any initial value $(x_0, y_{10}) \in \mathbb{R}_+^2$.*

(i) *If $a < 0.5\sigma_1^2$, then the population x will go to extinction almost surely;*

(ii) *If $d < 0.5\sigma_2^2$, then the population y_1 will go to extinction almost surely.*

Corollary 12. *Let $(x(t), y_1(t))$ be the solution of degradation model (102) with any initial value $(x_0, y_{10}) \in \mathbb{R}_+^2$. Suppose that $a > 0.5\sigma_1^2$, $d > 0.5\sigma_2^2$, and $\Delta \geq 0$.*

(i) *If $\Delta_1 < 0$, then the population x will go to extinction almost surely; if $\Delta_4 < 0$, then the population y_1 will go to extinction almost surely.*

(ii) *If $\Delta_1 > 0$ and $\Delta_4 < 0$, then $\lim_{t\longrightarrow\infty}\langle x(t)\rangle = (a - 0.5\sigma_1^2)/b$ a.s.; if $\Delta_1 < 0$ and $\Delta_4 > 0$, then $\lim_{t\longrightarrow\infty}\langle y_1(t)\rangle = (d - 0.5\sigma_2^2)/f$ a.s.*

Remark 13. Corollaries 11 and 12 are consistent with Theorems 9 and 10 in [15]. Moreover, if one considers the effects of the disease, from Theorem 10 we know that the conditions for population extinction and persistence will be more complicated. Therefore, our work can be seen as the extension of [15].

6. Asymptotic Behavior around the Interior Equilibrium $\widehat{E}$

From [4], it follows that the interior equilibrium $\widehat{E} = (\widehat{x}, \widehat{y}_1, \widehat{y}_2)$ of deterministic model (2) is feasible when $\delta > f$, $g(f+\delta) > \widehat{e}f$, $a > (ca_1 + \eta a_3)$, and $(\delta - f)(gf + g\delta - \widehat{e}f) > g\delta^2$. Moreover, from Theorem 3.2 in [4], if there exist positive constants k_1 and k_2 such that

$$\begin{aligned}(c + k_1\widehat{e})^2 &< bk_1 f, \\ (\eta + k_2 g)^2 &< bk_2 f, \\ (f(k_1 + k_2) + \delta(k_1 - k_2))^2 &< k_1 k_2 f,\end{aligned} \tag{103}$$

then the interior equilibrium $\widehat{E}$ of deterministic model (2) is globally asymptotically stable. Obviously, the interior equilibrium $\widehat{E}$ is not the solution of stochastic model (6); the following theorem demonstrates the asymptotic behavior of the solution of model (6) around the equilibrium $\widehat{E}$. It follows from Theorem 9 that if $a - \sigma_1^2/2 < 0$ and $d - \sigma_2^2/2 < 0$, then $\lim_{t\longrightarrow\infty}x(t) = 0$, $\lim_{t\longrightarrow\infty}y_i(t) = 0$ a.s. Therefore, in this section, we assume that $a - \sigma_1^2/2 > 0$ and $d - \sigma_2^2/2 > 0$. Moreover, we have the following result.

Theorem 14. *Assume that $\delta > f$, $g(f+\delta) > \widehat{e}f$, $a > (ca_1 + \eta a_3)$, and $(\delta - f)(gf + g\delta - \widehat{e}f) > g\delta^2$. Let $(x(t), y_1(t), y_2(t))$ be the solution of model (6) with initial value $(x_0, y_{10}, y_{20}) \in \mathbb{R}_+^3$. If $a - \sigma_1^2/2 > 0$ and $d - \sigma_2^2/2 > 0$, then there exist positive constants $l_i > 0$ and $m_i > 0$ $(i = 1, 2)$ such that*

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[(x - \widehat{x})^2 + (y_1 - \widehat{y}_1)^2 + (y_2 - \widehat{y}_2)^2\right]\mathrm{d}s \\ &\quad \leq \frac{(1/2)\sigma_1^2\widehat{x} + (1/2)\sigma_2^2\widehat{y}_1 + D(m_1 + 2m_2)/(l_1 \wedge l_2)}{b \wedge f},\end{aligned} \tag{104}$$

where $a_1 = df/\delta^2$, $a_3 = (\delta - f)/\delta^2$, and $D = \max\{(c + \widehat{e})\widehat{y}_1 + (\eta + g)\widehat{y}_2, (c + \widehat{e})\widehat{x} + 2f\widehat{y}_2, (\eta + g)\widehat{x} + 2f\widehat{y}_1\}$.

Proof. From [4], we know that the interior equilibrium $\widehat{E}$ of deterministic model (2) is feasible when $\delta > f$, $g(f+\delta) > \widehat{e}f$, $a > (ca_1 + \eta a_3)$, and $(\delta - f)(gf + g\delta - \widehat{e}f) > g\delta^2$. And if (103) holds, the interior equilibrium $\widehat{E}$ of model (2) is globally asymptotically stable. The coordinates of $\widehat{E}$ satisfy

$$\begin{aligned} b\widehat{x} + c\widehat{y}_1 + \eta\widehat{y}_2 &= a, \\ \widehat{e}\widehat{x} + f\widehat{y}_1 + (f+\delta)\widehat{y}_2 &= d, \\ (\delta - f)\widehat{y}_1 - g\widehat{x} - f\widehat{y}_2 &= 0. \end{aligned} \tag{105}$$

Define a function

$$\begin{aligned} V_1(x, y_1, y_2) = {} & x - \widehat{x} - \widehat{x}\ln\frac{x}{\widehat{x}} + y_1 - \widehat{y}_1 - \widehat{y}_1\ln\frac{y_1}{\widehat{y}_1} \\ & + y_2 - \widehat{y}_2 - \widehat{y}_2\ln\frac{y_2}{\widehat{y}_2}, \end{aligned} \tag{106}$$

and then using Itô's formula and (105), we have

$$\begin{aligned} \mathrm{d}V_1 = {} & \Big[(x - \widehat{x})(a - bx - cy_1 - \eta y_2) + (y_1 - \widehat{y}_1) \\ & \cdot (d - \widehat{e}x - fy_1 - (f+\delta)y_2) + (y_2 - \widehat{y}_2) \\ & \cdot ((\delta - f)y_1 - gx - fy_2) + \frac{1}{2}\sigma_1^2\widehat{x} + \frac{1}{2}\sigma_2^2\widehat{y}_1\Big]\mathrm{d}t \\ & + \sigma_1(x - \widehat{x})\mathrm{d}w_1(t) + \sigma_2(y_1 - \widehat{y}_1)\mathrm{d}w_2(t) \\ = {} & \Big\{(x - \widehat{x}) \\ & \cdot [-b(x - \widehat{x}) - c(y_1 - \widehat{y}_1) - \eta(y_2 - \widehat{y}_2)] \\ & + (y_1 - \widehat{y}_1) \\ & \cdot [-\widehat{e}(x - \widehat{x}) - f(y_1 - \widehat{y}_1) - (f+\delta)(y_2 - \widehat{y}_2)] \\ & + (y_2 - \widehat{y}_2) \\ & \cdot [(\delta - f)(y_1 - \widehat{y}_1) - g(x - \widehat{x}) - f(y_2 - \widehat{y}_2)] \\ & + \frac{1}{2}\sigma_1^2\widehat{x} + \frac{1}{2}\sigma_2^2\widehat{y}_1\Big\}\mathrm{d}t + \sigma_1(x - \widehat{x})\mathrm{d}w_1(t) \\ & + \sigma_2(y_1 - \widehat{y}_1)\mathrm{d}w_2(t) \le \Big[-b(x - \widehat{x})^2 \\ & - f(y_1 - \widehat{y}_1)^2 - f(y_2 - \widehat{y}_2)^2 \\ & + [(c + \widehat{e})\widehat{y}_1 + (\eta + g)\widehat{y}_2]x + [(c + \widehat{e})\widehat{x} + 2f\widehat{y}_2] \\ & \cdot y_1 + [(\eta + g)\widehat{x} + 2f\widehat{y}_1]y_2 + \frac{1}{2}\sigma_1^2\widehat{x} + \frac{1}{2} \\ & \cdot \sigma_2^2\widehat{y}_1\Big]\mathrm{d}t + \sigma_1(x - \widehat{x})\mathrm{d}w_1(t) + \sigma_2(y_1 \\ & - \widehat{y}_1)\mathrm{d}w_2(t) \le \Big[-b(x - \widehat{x})^2 \\ & - f(y_1 - \widehat{y}_1)^2 - f(y_2 - \widehat{y}_2)^2 + D(x + y_1 + y_2) \\ & + \frac{1}{2}\sigma_1^2\widehat{x} + \frac{1}{2}\sigma_2^2\widehat{y}_1\Big]\mathrm{d}t + \sigma_1(x - \widehat{x})\mathrm{d}w_1(t) \\ & + \sigma_2(y_1 - \widehat{y}_1)\mathrm{d}w_2(t), \end{aligned} \tag{107}$$

where $D = \max\{(c + \widehat{e})\widehat{y}_1 + (\eta + g)\widehat{y}_2, (c + \widehat{e})\widehat{x} + 2f\widehat{y}_2, (\eta + g)\widehat{x} + 2f\widehat{y}_1\}$.

Define

$$V_2(x, y_1, y_2) = x + y_1 + y_2. \tag{108}$$

Note that there exist constants $l_1 > 0$, $l_2 > 0$, $m_1 > 0$, and $m_2 > 0$ such that for any $z > 0$

$$\begin{aligned} az - bz^2 &\le -l_1 z + m_1, \\ dz - fz^2 &\le -l_2 z + m_2. \end{aligned} \tag{109}$$

Thus, by Itô's formula, we have

$$\begin{aligned} \mathrm{d}V_2 = {} & \big[ax - bx^2 - cxy_1 - \eta xy_2 + dy_1 - fy_1^2 - \widehat{e}xy_1 \\ & - 2fy_1y_2 - gxy_2 - fy_2^2\big]\mathrm{d}t + \sigma_1 x\mathrm{d}w_1(t) \\ & + \sigma_2 y_1\mathrm{d}w_2(t) \le \big[ax - bx^2 + dy_1 - fy_1^2 + dy_2 \\ & - fy_2^2\big]\mathrm{d}t + \sigma_1 x\mathrm{d}w_1(t) + \sigma_2 y_1\mathrm{d}w_2(t) \le [-l_1 x \\ & + m_1 - l_2 y_1 + m_2 - l_2 y_2 + m_2]\mathrm{d}t + \sigma_1 x\mathrm{d}w_1(t) \\ & + \sigma_2 y_1\mathrm{d}w_2(t) \le [-(l_1 \wedge l_2)(x + y_1 + y_2) \\ & + (m_1 + 2m_2)]\mathrm{d}t + \sigma_1 x\mathrm{d}w_1(t) + \sigma_2 y_1\mathrm{d}w_2(t). \end{aligned} \tag{110}$$

Then define

$$\begin{aligned} V_3(x, y_1, y_2) = {} & V_1(x, y_1, y_2) \\ & + \frac{D}{(l_1 \wedge l_2)}V_2(x, y_1, y_2). \end{aligned} \tag{111}$$

Using Itô's formula, and noting that (107) and (110) and $b \wedge f > 0$, one can get

$$\begin{aligned} \mathrm{d}V_3 = {} & \mathrm{d}V_1 + \frac{D}{(l_1 \wedge l_2)}\mathrm{d}V_2 \le \Big[-b(x - \widehat{x})^2 \\ & - f(y_1 - \widehat{y}_1)^2 - f(y_2 - \widehat{y}_2)^2 + \frac{1}{2}\sigma_1^2\widehat{x} + \frac{1}{2}\sigma_2^2\widehat{y}_1 \\ & + \frac{D(m_1 + 2m_2)}{(l_1 \wedge l_2)}\Big]\mathrm{d}t + \sigma_1(x - \widehat{x})\mathrm{d}w_1(t) \\ & + \sigma_2(y_1 - \widehat{y}_1)\mathrm{d}w_2(t) + \frac{D\sigma_1}{(l_1 \wedge l_2)}x\mathrm{d}w_1(t) \\ & + \frac{D\sigma_2}{(l_1 \wedge l_2)}y_1\mathrm{d}w_2(t) \le \Big\{-(b \wedge f) \end{aligned}$$

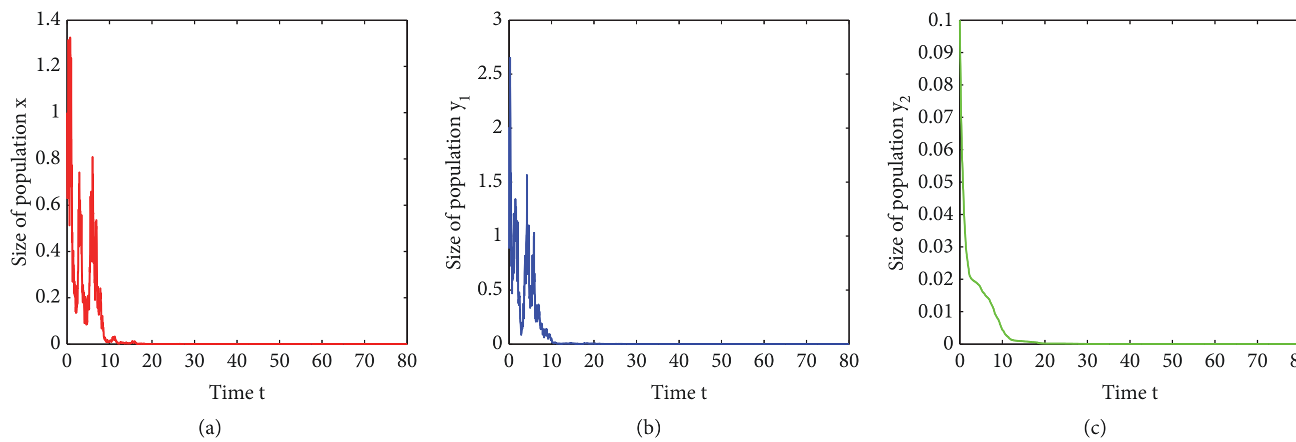


Figure 1: Numerical simulation of the solution of model (6). The parameters of model (6) are $a = 0.3$, $b = 0.15$, $c = 0.2$, $\eta = 0.2$, $d = 0.3$, $\widehat{e} = 0.15$, $f = 0.5$, $\delta = 0.1$, $g = 0.5$, $\sigma_1^2 = 1$, and $\sigma_2^2 = 1$.

$$\begin{aligned}
&\cdot \left[(x-\widehat{x})^2 + (y_1-\widehat{y}_1)^2 + (y_2-\widehat{y}_2)^2\right] + \frac{1}{2}\sigma_1^2\widehat{x} \\
&+ \frac{1}{2}\sigma_2^2\widehat{y}_1 + \frac{D(m_1+2m_2)}{(l_1\wedge l_2)}\Bigg\} dt + \sigma_1 (x \\
&- \widehat{x})\, dw_1(t) + \sigma_2 (y_1-\widehat{y}_1)\, dw_2(t) \\
&+ \frac{D\sigma_1}{(l_1\wedge l_2)} x\, dw_1(t) + \frac{D\sigma_2}{(l_1\wedge l_2)} y_1\, dw_2(t).
\end{aligned} \tag{112}$$

Integrating both sides of (112) from 0 to t and taking the expectation, we have

$$\begin{aligned}
0 \le \mathbb{E}V_3 &\le -(b\wedge f) \\
&\cdot \mathbb{E}\int_0^t \left[(x-\widehat{x})^2 + (y_1-\widehat{y}_1)^2 + (y_2-\widehat{y}_2)^2\right] ds \\
&+ \left[\frac{1}{2}\sigma_1^2\widehat{x} + \frac{1}{2}\sigma_2^2\widehat{y}_1 + \frac{D(m_1+2m_2)}{(l_1\wedge l_2)}\right] t.
\end{aligned} \tag{113}$$

Noting that $b\wedge f > 0$, so dividing both sides by $(b\wedge f)t$ and letting $t \longrightarrow \infty$, we get

$$\begin{aligned}
&\limsup_{t\longrightarrow\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[(x-\widehat{x})^2 + (y_1-\widehat{y}_1)^2 + (y_2-\widehat{y}_2)^2\right] ds \\
&\le \frac{(1/2)\sigma_1^2\widehat{x} + (1/2)\sigma_2^2\widehat{y}_1 + D(m_1+2m_2)/(l_1\wedge l_2)}{b\wedge f}.
\end{aligned} \tag{114}$$

The proof is therefore complete. □

7. Numerical Simulations

In this section, we make numerical simulations to illustrate our results by using the Milstein method (see, e.g., [27]). The numerical simulations of population dynamics are carried out for the academic tests with the arbitrary values of the vital rates and other parameters which do not correspond to some specific biological populations and exhibit only the theoretical properties of numerical solutions of the considered model. In the figures, the red lines, blue lines, and green lines represent the trajectories of populations $x(t)$, $y_1(t)$, and $y_2(t)$, respectively. Here we give numerical simulations of model (6) with the same initial values $x_0 = 1$, $y_{10} = 0.9$, and $y_{20} = 0.1$.

In Figure 1, we choose $a = 0.3$, $b = 0.15$, $c = 0.2$, $\eta = 0.2$, $d = 0.3$, $\widehat{e} = 0.15$, $f = 0.5$, $\delta = 0.1$, $g = 0.5$, $\sigma_1^2 = 1$, and $\sigma_2^2 = 1$. It is easy to see that $a < 0.5\sigma_1^2$, $d < 0.5\sigma_2^2$, and $\delta < f$. By Theorem 9, populations x, y_1, and y_2 will go to extinction.

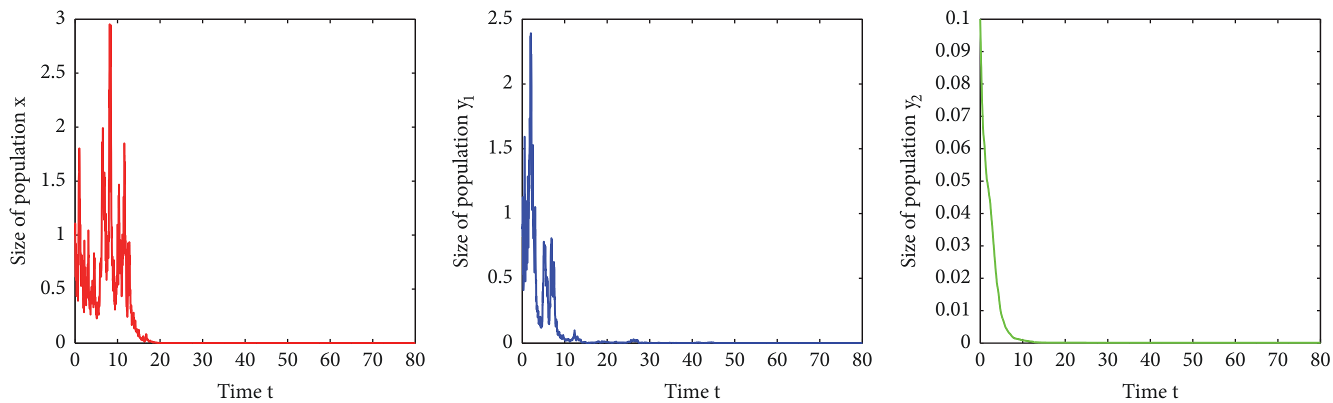
In Figure 2, we choose $a = 0.3$, $b = 0.15$, $c = 0.2$, $\eta = 0.2$, $d = 0.3$, $\widehat{e} = 0.15$, $f = 0.15$, $\delta = 0.2$, $g = 0.5$, $\sigma_1^2 = 1$, and $\sigma_2^2 = 1$. It is easy to see that $a < 0.5\sigma_1^2$, $d < 0.5\sigma_2^2$, and $\delta > f$. By Theorem 9, populations x, y_1, and y_2 will go to extinction.

In Figure 3, we choose $a = 0.55$, $b = 0.3$, $c = 0.38$, $\eta = 0.6$, $d = 0.5$, $\widehat{e} = 0.23$, $f = 0.3$, $\delta = 0.1$, $g = 0.5$, and $\sigma_1^2 = \sigma_2^2 = 0.1$. It is easy to see that $\Delta = 0.0026 > 0$, $\Delta_1 = -0.021 < 0$, $\Delta_2 = -0.028 < 0$, and $\Delta_3 = 0.006 > 0$. By (i) of Theorem 10, populations x and y_2 will go to extinction and population y_1 will be weakly persistent in mean almost surely.

In Figure 4, we choose $a = 0.55$, $b = 0.3$, $c = 0.38$, $\eta = 0.5$, $d = 0.4$, $\widehat{e} = 0.23$, $f = 0.3$, $\delta = 0.1$, $g = 0.1$, and $\sigma_1^2 = \sigma_2^2 = 0.1$. It is easy to see that $\Delta = 0.0026 > 0$, $\Delta_4 = -0.01 < 0$, $\Delta_5 = -0.005 < 0$, and $\Delta_6 = 0.029 > 0$. By (ii) of Theorem 10, populations y_1 and y_2 will go to extinction and population x will be weakly persistent in mean almost surely.

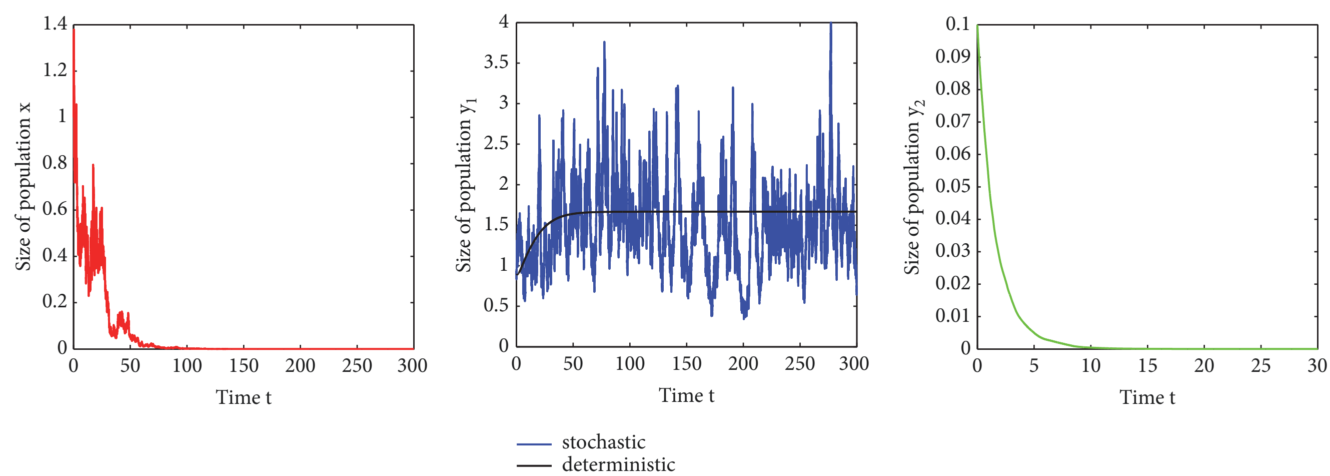
As done in [4], in Figure 5, we choose $a = 1$, $b = 1$, $c = 0.1$, $\eta = 0.1$, $d = 2$, $\widehat{e} = 0.1$, $f = 1$, $\delta = 2.5$, $g = 0.2$, and $\sigma_1 = \sigma_2 = 0.5$. For this example, we get the following interior equilibrium: $\widehat{E} = (0.9163, 0.4080, 0.4287)$. Moreover, we have $a - \sigma_1^2/2 > 0$ and $d - \sigma_2^2/2 > 0$. Thus, the conditions of Theorem 14 hold. Then, by Theorem 14, the solution of model (6) oscillates around the equilibrium $\widehat{E}$ of the corresponding deterministic model (2) (see Figure 5).

As can be seen from Figures 1 and 2, regardless of the size of the interspecific competition rate f and the rate of infection δ, as long as $a < 0.5\sigma_1^2$ and $d < 0.5\sigma_2^2$, we know that



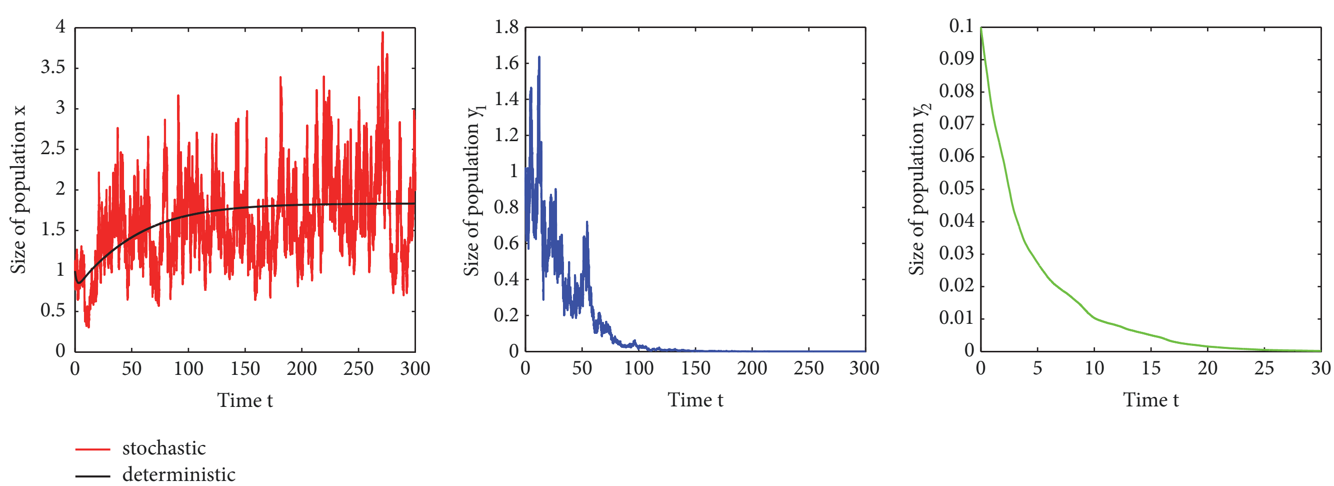
(a) (b) (c)

FIGURE 2: Numerical simulation of the solution of model (6). The parameters of model (6) are $a = 0.3$, $b = 0.15$, $c = 0.2$, $\eta = 0.2$, $d = 0.3$, $\hat{e} = 0.15$, $f = 0.15$, $\delta = 0.2$, $g = 0.5$, $\sigma_1^2 = 1$, and $\sigma_2^2 = 1$.



(a) (b) (c)

FIGURE 3: Numerical simulation of the solution of model (6). The parameters of model (6) are $a = 0.55$, $b = 0.3$, $c = 0.38$, $\eta = 0.6$, $d = 0.5$, $\hat{e} = 0.23$, $f = 0.3$, $\delta = 0.1$, $g = 0.5$, and $\sigma_1^2 = \sigma_2^2 = 0.1$.



(a) (b) (c)

FIGURE 4: Numerical simulation of the solution of model (6). The parameters of model (6) are $a = 0.55$, $b = 0.3$, $c = 0.38$, $\eta = 0.5$, $d = 0.4$, $\hat{e} = 0.23$, $f = 0.3$, $\delta = 0.1$, $g = 0.1$, and $\sigma_1^2 = \sigma_2^2 = 0.1$.

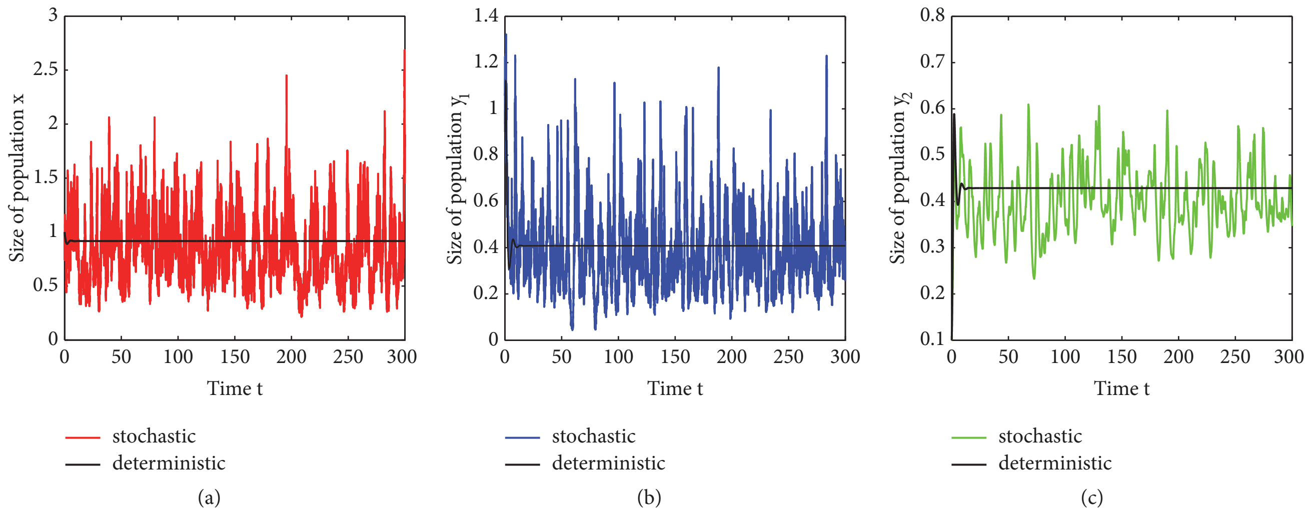


FIGURE 5: Numerical simulation of the solution of model (6). The parameters of model (6) are $a = 1$, $b = 1$, $c = 0.1$, $\eta = 0.1$, $d = 2$, $\hat{e} = 0.1$, $f = 1$, $\delta = 2.5$, $g = 0.2$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.5$.

the healthy individuals $x(t)$, the healthy individuals $y_1(t)$, and the infected individuals $y_2(t)$ will go to extinction.

From Figures 3 and 4, we can see that although the environmental noise is relatively small, population extinction can also occur when the competition coefficients satisfy certain conditions. It can be seen from Figure 2 that the population y is dominant in competition, while the population x is at a disadvantage, and the individuals with infectious diseases in population y will go to extinction. Moreover, populations x and y_2 will go to extinction and population y_1 will be weakly persistent in mean almost surely. However, from Figure 3, we can see that the population x is dominant in competition, while the population y is at a disadvantage, and the individuals with infectious diseases in population y will go to extinction. That is, populations y_1 and y_2 will go to extinction and population x will be weakly persistent in mean almost surely.

8. Conclusion and Discussion

In this paper, we consider a stochastic two-species competition model under the effect of disease. By the comparison theorem of stochastic differential equations, we prove the existence and uniqueness of global positive solution of the model. Then, an important asymptotic property of model is given via the exponential martingale inequality and Borel-Cantelli lemma. Next, we prove the boundedness of the pth moment of the global positive solution. Then, sufficient conditions for extinction and persistence in mean are obtained. Furthermore, by constructing suitable Lyapunov function, we investigate the asymptotic behavior of the stochastic system around the interior equilibrium of the deterministic model.

From Remark 13, we know that our work can be seen as the extension of [15]. Some interesting problems deserve further consideration. One may incorporate the Markovian switching into the stochastic population model (6). Since model (6) may be perturbed by the telegraph noise which can make the model switch from one environmental regime to another, one may also introduce the jumps into the stochastic model (6). Since population systems may suffer severe environmental perturbations, such as tsunami, volcanoes, avian influenza, SARS, floods, hurricanes, earthquakes, and toxic pollutants, we leave this for future consideration.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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