Redundancy Optimization of an Uncertain Parallel-Series System with Warm Standby Elements

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The redundancy optimization problem is formulated for an uncertain parallel-series system with warm standby elements. The lifetimes and costs of elements are considered uncertain variables, and the weights and volumes of elements are random variables. The uncertain measure optimization model (UMOM), the uncertain optimistic value optimization model (UOVOM), and the uncertain cost optimization model (UCOM) are developed through reliability maximization, lifetime maximization, and cost minimization, respectively. An efficient simulation optimization algorithm is provided to calculate the objective values and optimal solutions of the UMOM, UOVOM, and UCOM. A numerical example is presented to illustrate the rationality of the models and the feasibility of the optimization algorithm.

1. Introduction

The primary goal of reliability design is to improve the reliability of a system. To maintain the reliability to a higher level, the redundancy allocation is an effective method in the system design phase. While improving system reliability by a redundancy method, the cost, weight, and volume also increase. Thus, it is an important topic for system decision-makers to determine the optimal number of redundant elements under certain system constraints.

In the traditional redundancy optimization problem, various kinds of optimization models have been proposed under the assumption that the lifetimes of the elements are random variables. Due to imprecision of data for element lifetimes in certain situations, fuzzy redundancy optimization models [1–4] are then developed based on fuzzy set theory [5, 6]. Furthermore, Zhao and Liu [7] proposed three redundancy optimization models under the assumption that the lifetimes of the elements are presented as fuzzy variables. Wang and Watada [8] developed two fuzzy random redundancy allocation models for a parallel-series system when the lifetimes of the elements are treated as fuzzy random variables. Recently, some researchers have addressed reliability optimization designs of some systems by considering interval-valued component reliability in an uncertain environment. Roy et al. [9] applied the symmetrical form of interval numbers by interval-valued parametric functional form to evaluate the optimum system reliability and system cost of the redundancy allocation problem. Zhang and Chen [10] investigated an interval multiobjective optimization problem for reliability redundancy allocation of a series-parallel system. Moreover, some researchers concentrated on some hybrid uncertainty optimization problems for system reliability [11, 12].

The probability, interval, and fuzzy theories have been widely used to handle the high level of uncertainty in various real-world applications. With the development of the research on the uncertainty phenomena, the mathematical model based on the probability, interval, and fuzzy theories is not enough to solve all problems, especially when we have no available samples but belief degree from the experts. Belief degree function is a type of distribution function for indeterminate quantity. Since it usually deviates far from the frequency, using probability theory may lead to counterintuitive results. In this case, we should use uncertainty theory. The uncertainty theory provides a useful tool to study reliability modeling and optimization problem of systems.
with human uncertainty phenomena. The basic uncertainty theory was founded by Liu [13] in 2007. It was refined by Liu [14] in 2010 based on normality axiom, duality axiom, subadditivity axiom, and product axiom. Nowadays, uncertainty theory has become a branch of mathematics for modeling human uncertainty. Many theories and applications have been done based on uncertainty theory, for example, uncertain statistics [15, 16], uncertain programming [17, 18], uncertain logic [19–21], uncertain inference [22–24], uncertain process [25–27], uncertain differential equations [28, 29], and uncertain graph [30].

The system reliability via uncertainty measure was first studied by Liu [31]. Afterwards, Liu [32] investigated reliability of redundant systems with cold and warm redundant elements based on uncertainty theory. Liu et al. [33] studied the reliability and MTTF of unrepairable systems with uncertain lifetimes. Wen and Kang [34] analyzed an uncertain random system based on chance theory which is a generalization of both probability theory and uncertainty theory. Gao et al. [35] studied the reliability of k-out-of-n systems with uncertain random lifetimes. Gao et al. [36] studied the reliability of the k-out-of-n system with uncertain weights. Zeng et al. [37] defined belief reliability as an uncertain measure due to the explicit representation of epistemic uncertainty and investigated the belief reliability for coherent systems based on minimal cut sets.

Making use of an uncertain variable as a tool to characterize the lifetimes and costs of elements, we will discuss the redundancy optimization problems for a parallel-series system with warm standby elements in this paper. In this work, three uncertain optimization models are developed, and an efficient simulation optimization algorithm is given to solve these models. In Section 2, some basic concepts and theorems concerning uncertainty theory are presented. The problem formulation of an uncertain parallel-series system is considered in Section 3. Section 4 shows the three uncertain optimization models and gives a solution approach to these models. A numerical example is provided in Section 5, and Section 6 presents a general conclusion.

2. Preliminaries

Definition 1 (see [13, 38]). Let \( \Gamma \) be a \( \sigma \)-algebra on a non-empty set \( \Gamma \). A set function \( \mathcal{M} : \mathcal{L} \rightarrow [0, 1] \) is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (normality axiom). \( \mathcal{M}\{\Gamma\} = 1 \) for the universal set \( \Gamma \).

Axiom 2 (duality axiom). \( \mathcal{M}\{A\} + \mathcal{M}\{A'\} = 1 \) for any event \( A \).

Axiom 3 (subadditivity axiom). For any countable sequence of events \( A_1, A_2, \ldots \), we have

\[
\mathcal{M}\left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \mathcal{M}\{A_i\},
\]

the triple \( (\Gamma, \mathcal{L}, \mathcal{M}) \) is called an uncertainty space.

Axiom 4 (product axiom). Let \( (\Gamma_k, \mathcal{L}_k, \mathcal{M}_k) \) be the uncertainty space for \( k = 1, 2, \ldots \). Then, the product uncertain measure \( \mathcal{M} \) is an uncertain measure satisfying

\[
\mathcal{M}\left( \prod_{k=1}^{\infty} A_k \right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{A_k\},
\]

where \( A_k \) are arbitrarily chosen events from \( \mathcal{L} \) for \( k = 1, 2, \ldots \), respectively.

Definition 2 (see [13]). An uncertain variable is a measure function \( \xi \) from an uncertainty space \( (\Gamma, \mathcal{L}, \mathcal{M}) \) to the set \( \mathcal{R} \) of real numbers, that is, for any Borel set \( B \) of real numbers, the set

\[
\{ \xi \in B \} = \{ y \in \Gamma | \xi(y) \in B \}
\]

is an event.

Definition 3 (see [13]). The uncertainty distribution \( \Phi(x) \) of an uncertain variable \( \xi \) is defined as

\[
\Phi(x) = \mathcal{M}\{\xi \leq x\},
\]

for any real number \( x \).

Definition 4 (see [13]). An uncertainty distribution \( \Phi(x) \) is said to be regular if it is a continuous and strictly increasing function with respect to \( x \) at which \( 0 < \Phi(x) < 1 \), and

\[
\lim_{x \rightarrow +\infty} \Phi(x) = 0,
\]

\[
\lim_{x \rightarrow -\infty} \Phi(x) = 1.
\]

In addition, the inverse function \( \Phi^{-1}(\alpha) \) is called the inverse uncertainty distribution of \( \xi \).

Definition 5 (see [17]). A variable \( \xi_{\sup}(\alpha) \) is said to be an \( \alpha \)-optimistic value if \( \xi \) is an uncertain variable and

\[
\xi_{\sup}(\alpha) = \sup \{ r | \mathcal{M}\{\xi \geq r\} \geq \alpha \},
\]

where \( \alpha \in (0, 1] \).

Definition 6 (see [13]). An uncertain variable \( \xi \) is said to be linear if it has a linear uncertainty distribution.

\[
\Phi(x) = \begin{cases} 0, & \text{if } x \leq \alpha, \\ \frac{x - \alpha}{b - \alpha}, & \text{if } \alpha < x \leq b, \\ 1, & \text{if } x > b, \end{cases}
\]

which is denoted by \( \mathcal{L}(a, b) \). Apparently, the linear uncertain variable \( \xi \) is regular and has an inverse uncertainty distribution \( \Phi^{-1}(\alpha) = (1 - \alpha)a + ab \).
Definition 7 (see [13]). An uncertain variable $\xi$ is said to be lognormal if $\ln \xi$ has a normal uncertainty distribution.

$$\Phi(x) = \left(1 + \exp \left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R},$$  \hspace{1cm} (8)

denoted by $\mathcal{LN}(e, \sigma)$, where $e$ and $\sigma$ are real numbers with $\sigma > 0$. The uncertain variable is regular, and its inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = \exp\left(e\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{3}}\right).$$  \hspace{1cm} (9)

Theorem 1 (see [13]). Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution $\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))$.

Theorem 2 (see [13]). Let $\xi$ be an uncertain variable with a regular uncertainty distribution $\Phi$. Then, $E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha$.

Theorem 3 (see [13]). Let $\xi$ and $\eta$ be independent uncertain variables with finite expected values. Then, for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$  \hspace{1cm} (10)

Theorem 4 (see [13]). Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Then, $\xi_1 + \xi_2 + \cdots + \xi_n$, $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$ and $\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n$ have uncertainty distributions $\sup \{\min \Phi_1(x_i), \Phi_1(x) \vee \Phi_2(x) \vee \cdots \vee \Phi_n(x), \Phi_1(x) \wedge \Phi_2(x) \wedge \cdots \wedge \Phi_n(x)\}$, respectively.

3. Problem Formulation of an Uncertain Parallel-Series System

Consider a warm standby redundant parallel-series system composed of $m$ subsystems $A_1, A_2, \ldots, A_m$, and subsystem $A_i$ consists of $n_i$ components connected in series, as shown in Figure 1. The component $j$ in subsystem $A_i$ contains one original element and $x_{ij} - 1$ warm standby redundant elements, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n_i$.

Throughout the paper, we assume the following for the warm standby redundant parallel-series system under consideration.

1. The system, components, and elements are only in one of two states (up state or down state) at any time.
2. All the elements are independent.
3. The system starts to work at time 0.
4. For each component, there is one element available.
5. The system or element is nonrepairable.
6. The standby redundant elements may deteriorate in the standby period.
7. The lifetime of each element is an uncertain variable, and the standby deterioration rates of all the elements in the standby period are the same constant.
(8) Conversion switches of the standby system are absolutely reliable, and the conversion is instantaneous.

(9) The costs of all the elements are independent uncertain variables, and the weights and volumes of all the elements are independent random variables.

3.1. Component Lifetime. According to the assumptions, the lifetime of the component \( j \) in the subsystem \( A_i \) can be expressed as

\[
T_{ij}(x, \xi) = \bar{\xi}_{ij1} + \bar{\xi}_{ij2} + \cdots + \bar{\xi}_{ijx_j},
\]

where

\[
x = (x_{11}, x_{12}, \ldots, x_{1n_1}, \ldots, x_{m1}, x_{m2}, \ldots, x_{mn_m}),
\]

\[
\xi = (\bar{\xi}_{111}, \bar{\xi}_{112}, \ldots, \bar{\xi}_{11n_1}, \bar{\xi}_{ij1}, \bar{\xi}_{ij2}, \ldots, \bar{\xi}_{ijx_j}, \ldots, \bar{\xi}_{mn_1}, \ldots, \bar{\xi}_{mn_2}, \ldots, \bar{\xi}_{mn_m}),
\]

where \( x_{ij} \) represents the number of the elements in the component \( j \) for the subsystem \( A_i \) and \( \bar{\xi}_{ijx_j} \) represents the working lifetime of the \( x_{ij} \)th element in the component \( j \) for the subsystem \( A_i \), \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n_j \).

For simplicity, it is assumed that the deterioration rates of all the warm standby elements are the same constant, denoted by \( \lambda \) (very small number). According to [32], the working lifetime of the \( x_{ij} \)th element in the component \( j \) for the subsystem \( A_i \) is expressed as

\[
\bar{\xi}_{ijx_j} = \xi_{ijx_j} - \lambda \left( \bar{\xi}_{ij1} + \bar{\xi}_{ij2} + \cdots + \bar{\xi}_{ij(x_j-1)} \right),
\]

where \( \xi_{ijx_j} \) represents the uncertain lifetime of the \( x_{ij} \)th element in the component \( j \) for the subsystem \( A_i \) from time 0 and \( \bar{\xi}_{ij} = \xi_{ij} \) (i.e., the first element starts to work at time 0). Thus, the working lifetime of all elements in the component \( j \) for the subsystem \( A_i \) can be expressed as follows:

\[
\begin{align*}
\bar{\xi}_{ij1} &= \xi_{ij1}, \\
\bar{\xi}_{ij2} &= \xi_{ij2} - \lambda \bar{\xi}_{ij1}, \\
\bar{\xi}_{ij3} &= \xi_{ij3} - \lambda \bar{\xi}_{ij1} + \bar{\xi}_{ij2}, \\
&\vdots \\
\bar{\xi}_{ijx_j} &= \xi_{ijx_j} - \lambda \left( \bar{\xi}_{ij1} + \bar{\xi}_{ij2} + \cdots + \bar{\xi}_{ij(x_j-1)} \right).
\end{align*}
\]

By (11) and (14), the lifetime of component \( j \) in the subsystem \( A_i \) can be expressed as

\[
T_{ij}(x, \xi) = (1 - \lambda)^{x_j-1} \xi_{ij1} + (1 - \lambda)^{x_j-2} \xi_{ij2} + \cdots + \xi_{ijx_j}.
\]

3.2. System Lifetime. The lifetime of subsystem \( A_i \) can be obtained by

\[
T_A(x, \xi) = \bigwedge_{j=1,2,\ldots,n_j} T_{ij}(x, \xi)
\]

\[
= \bigwedge_{j=1,2,\ldots,n_j} \sum_{l=1,2,\ldots,n_{ij}} (1 - \lambda)^{x_{ijl}-k} \bar{\xi}_{ijl}.
\]

The lifetime of the system can be determined as

\[
T(x, \xi) = \max \left\{ \bigwedge_{j=1,2,\ldots,n_j} \sum_{l=1,2,\ldots,n_{ij}} (1 - \lambda)^{x_{ijl}-k} \bar{\xi}_{ijl} \right\}.
\]

Using (15), (16), and (17), \( T(x, \xi) \) can be written as

\[
T(x, \xi) = \max_{1\leq i\leq m, 1\leq j\leq n_i} \sum_{l=1,2,\ldots,n_{ij}} (1 - \lambda)^{x_{ijl}-k} \bar{\xi}_{ijl}.
\]

4. The System Optimization Model

The purpose of redundancy optimization is to find the optimal solution \( x = (x_{11}, x_{12}, \ldots, x_{1n_1}, \ldots, x_{m1}, x_{m2}, \ldots, x_{mn_m}) \) (the numbers of elements in each component for each subsystem) for improving the system performances under certain constraints. In this section, three different standby redundancy optimization models are considered for different management purposes. In these models, the costs of all the elements are presented as uncertain variables and the weights and volumes of all the elements are presented as random variables.

4.1. The Uncertain Measure Optimization Model. The uncertain measure is used to define the system reliability, and the optimization objective is to maximize the uncertainty measure \( M\{T(x, \xi) > t_0\} \) that the system lifetime is greater than or equal to the given time \( t_0 \) under the expected cost, expected weight, and expected volume constraints. In this way, the general form of the uncertain measure optimization model (UMOM) is as follows:

\[
\max_x \quad M\{T(x, \xi) > t_0\}
\]

s.t. \( E \left\{ \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n_i} x_{ij} c_{ij} \right\} \leq c_0 \)

\[
E \left\{ \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n_i} x_{ij} w_{ij} \right\} \leq w_0
\]
where \( c_{ij}, w_{ij}, \) and \( v_{ij} \) represent the uncertain cost, random weight, and random volume of each element in the component \( j \) for the subsystem \( A_i \), respectively.

Assume that the uncertain lifetimes \( \xi_{ij1}, \xi_{ij2}, \ldots, \xi_{ijx_{ij}} \) for subsystem \( A_i \) have the same uncertainty distribution \( \Phi_{ij}(t) \). According to Theorem 4 and [32], the uncertain lifetime distribution functions of the subsystems \( A_1, A_2, \ldots, A_m \), denoted by \( \psi_{A_1}(x, t), \psi_{A_2}(x, t), \ldots, \psi_{A_m}(x, t) \), respectively, can be obtained as

\[
\begin{align*}
\psi_{A_1}(x, t) &= \mathcal{M}\{T_{A_1}(x, \xi) \leq t\} \\
&= \bigvee_{j=1,2,\ldots,n_i} \Phi_{ij}(x, t) \\
&= \bigvee_{j=1,2,\ldots,n_i} \Phi_{ij} \left( \frac{\lambda t}{1 - (1 - \lambda)^{v_{ij}}} \right), \\
\psi_{A_2}(x, t) &= \mathcal{M}\{T_{A_2}(x, \xi) \leq t\} \\
&= \bigvee_{j=1,2,\ldots,n_i} \Phi_{2j}(x, t) \\
&= \bigvee_{j=1,2,\ldots,n_i} \Phi_{2j} \left( \frac{\lambda t}{1 - (1 - \lambda)^{v_{ij}}} \right), \\
&\vdots \\
\psi_{A_m}(x, t) &= \mathcal{M}\{T_{A_m}(x, \xi) \leq t\} \\
&= \bigvee_{j=1,2,\ldots,n_i} \Phi_{mj}(x, t) \\
&= \bigvee_{j=1,2,\ldots,n_i} \Phi_{mj} \left( \frac{\lambda t}{1 - (1 - \lambda)^{v_{ij}}} \right),
\end{align*}
\]

where \( \Phi_{ij}(x, t) \) is the uncertain lifetime distribution function of the component \( j \) in subsystem \( A_i \). The lifetime of the system has an uncertainty distribution function.

\[
\Phi(x, t) = \min_{1 \leq i \leq m} \psi_{A_i}(x, t) \\
= \min_{1 \leq i \leq m} \bigvee_{j=1,2,\ldots,n_i} \Phi_{ij} \left( \frac{\lambda t}{1 - (1 - \lambda)^{v_{ij}}} \right).
\]

According to the system reliability definition and duality of uncertain measure, we have

\[
\mathcal{M}\{T(x, \xi) > t\} = 1 - \Phi(x, t) \\
= 1 - \min_{1 \leq i \leq m, j=1,2,\ldots,n_i} \Phi_{ij} \left( \frac{\lambda t}{1 - (1 - \lambda)^{v_{ij}}} \right).
\]

According to Theorem 3 and the linear operation of the expected value of the random variable, we have

\[
\begin{align*}
E \left[ \sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}v_{ij} \right] &\leq v_0 \\
E \left[ \sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}c_{ij} \right] &\leq c_0 \\
E \left[ \sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}w_{ij} \right] &\leq w_0 \\
E \left[ \sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}v_{ij} \right] &\leq v_0
\end{align*}
\]

Therefore, the UMOM (19) is equivalent to

\[
\begin{align*}
&\max_{x} \left( 1 - \min_{1 \leq i \leq m, j=1,2,\ldots,n_i} \Phi_{ij} \left( \frac{\lambda t_0}{1 - (1 - \lambda)^{v_{ij}}} \right) \right) \\
&\text{s.t. } \sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}E[v_{ij}] \leq v_0 \\
&\sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}E[c_{ij}] \leq c_0 \\
&\sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}E[w_{ij}] \leq w_0 \\
&x_{ij} \geq 1, x_{ij} \in N^+, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n_i.
\end{align*}
\]

4.2. The Uncertain Optimistic Value Optimization Model

If we use the system lifetime to measure the system performance, then the optimization objective is to maximize the \( \alpha \)-optimistic value of the system lifetime \( T(x, \xi) \) when \( \alpha \) is equal to the given value \( \alpha_0 \) under the same constraints as model (19). In this way, the form of the uncertain optimistic value optimization model (UOVOM) can be expressed as

\[
\begin{align*}
&\max_{x} t_x \\
&\text{s.t. } \mathcal{M}\{T(x, \xi) > t_x\} \geq \alpha_0 \\
&\sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}c_{ij} \leq c_0 \\
&\sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}w_{ij} \leq w_0 \\
&\sum_{1 \leq i \leq m, 1 \leq j \leq n_i} x_{ij}v_{ij} \leq v_0
\end{align*}
\]

where \( t_x \) represents the time related to decision vector \( x \). By Theorem 1, the inverse uncertainty distribution of the system lifetime \( T(x, \xi) \) is

\[
\Phi^{-1}(\alpha) = \max_{1 \leq i \leq m, j=1,2,\ldots,n_i} \frac{1 - (1 - \lambda)^{v_{ij}}}{\lambda} \Phi_{ij}^{-1}(\alpha),
\]

where \( 0 < \alpha \leq 1 \).
where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of the lifetime of the component $j$ in the subsystem $A_i$. By using the duality of uncertain measure, $\mathcal{M}\{T(x, \xi) > t_x\} \geq \alpha$ can be replaced by $\Phi(x, t_x) \leq 1 - \alpha$ which is equivalent to $t_x \leq \Phi^{-1}(1 - \alpha)$. According to the monotonicity of $\Phi(x, t_x)$, $t_x$ can reach its maximum when $t_x = \Phi^{-1}(1 - \alpha)$. Therefore, the UOVOM (25) is equivalent to

$$\max \ x \quad \Phi^{-1}(1 - \alpha)$$

s.t. \begin{align*}
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[c_{ij}] & \leq c_0 \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[w_{ij}] & \leq w_0 \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[v_{ij}] & \leq v_0
\end{align*}

(27)

$x_{ij} \geq 1, x_{ij} \in N^+, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n_i$.

4.3. The Uncertain Cost Optimization Model. In the following, we give an uncertain cost optimization model with system reliability, expected weight, and expected volume constraints. Since the cost of the system is an uncertain variable, it cannot be directly minimized. Based on the expected value of the system cost, we may minimize its expected value. Then, we have the uncertain cost optimization model (UCOM) as follows:

$$\min \ x \quad E \left[ \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} c_{ij} \right]$$

s.t. \begin{align*}
\mathcal{M}\{T(x, \xi) > t_0\} & \geq \alpha_0 \\
E \left[ \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} w_{ij} \right] & \leq w_0 \\
E \left[ \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} v_{ij} \right] & \leq v_0
\end{align*}

(28)

$x_{ij} \geq 1, x_{ij} \in N^+, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n_i$.

The UCOM (28) is equivalent to

$$\min \ x \quad \Phi^{-1}(1 - \alpha_0)$$

s.t. \begin{align*}
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[c_{ij}] & \leq c_0 \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[w_{ij}] & \leq w_0 \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[v_{ij}] & \leq v_0
\end{align*}

(29)

$x_{ij} \geq 1, x_{ij} \in N^+, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n_i$.

4.4. Solution Approach for the Optimization Model. It is easy to see that the models (24), (27), and (29) which are equivalent to the UMOM, UOVOM, and UCOM, respectively, are crisp nonlinear integer programming models. An efficient simulation optimization algorithm is provided to calculate the objective values and optimal solutions of the UMOM, UOVOM, and UCOM. The main steps of the proposed algorithm are as follows:

Step 1. Set the range of the integer decision variable $x_{ij}$ to be $[l, u]$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n_i$.

Step 2. Randomly generate $r$ integer vectors $x_k = (x_{k11}, x_{k12}, \ldots, x_{k1n_i}, x_{k21}, x_{k22}, \ldots, x_{kmn_m})$ ($k = 1, 2, \ldots, r$) from the interval $[l, u]$, and construct a matrix with $r$ rows and $n_1 + n_2 + \ldots + n_m$ columns.

\[
X = \begin{pmatrix}
  x_{111} & x_{112} & \cdots & x_{11n_i} & x_{1m1} & x_{1m2} & \cdots & x_{1mn_m} \\
  x_{211} & x_{212} & \cdots & x_{21n_i} & x_{2m1} & x_{2m2} & \cdots & x_{2mn_m} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{r11} & x_{r12} & \cdots & x_{r1n_i} & x_{rm1} & x_{rm2} & \cdots & x_{rmn_m}
\end{pmatrix}
\]

(30)

where the row vector $x_k = (x_{k11}, x_{k12}, \ldots, x_{k1n_i}, \ldots, x_{km1}, x_{km2}, \ldots, x_{kmn_m})$ is the possible solution vector.

Step 3. Substitute each row of the matrix $X$ into

\[
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[c_{ij}] \leq c_0, \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[w_{ij}] \leq w_0, \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[v_{ij}] \leq v_0
\]

(31)

and

\[
\mathcal{M}\{T(x, \xi) > t_x\} \geq \alpha_0, \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[c_{ij}] \leq c_0, \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[w_{ij}] \leq w_0, \\
\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq m} x_{ij} E[v_{ij}] \leq v_0
\]

(32)
The minimum are the best solutions based on the row vectors of the feasible solution matrix in Step 3 for the three different models, respectively. The feasible solutions that make \( \Phi_{ij}\left(\lambda t(1-(1-\lambda)^v_i)\right), -\Phi^{-1}\left(1-\alpha_0\right), \) and \( \sum_{1 \leq j \leq m} \sum_{1 \leq i \leq n} x_{ij} E_{ij} \) the minimum are the best solutions for the three different models, respectively.

**Step 5.** Change \( r \) in Step 2, repeat Step 2 to Step 4, observe the changes of the objective function values obtained under different simulation times, and determine the most appropriate number of simulation \( r \).

**Step 6.** Return the best solutions obtained under the value \( r \) in Step 5 as the final optimal solutions for the three different models.

### 5. Numerical Example

In this section, we give an illustration of the optimization models and their solutions for a warm standby redundant parallel-series system shown in Figure 2. The detailed data used in this example are those given in Table 1.

In the UMOM, we take \( \lambda = 0.01, t_0 = 100, c_0 = 350, w_0 = 300, \) and \( v_0 = 250 \). The lower and upper bounds of the number of the redundant elements are \( l = 1 \) and \( u = 10 \), respectively. Then, the model can be expressed as

\[
\begin{align*}
\text{max} & \quad \left(1 - \min_{1 \leq i \leq 12} \Phi_{ii}\left(\lambda t(1-(1-\lambda)^v_i)\right)\right) \\
\text{s.t.} & \quad 12.97 x_{11} + 3.09 x_{12} + 9.00 x_{21} + 11.00 x_{22} + 5.10 x_{23} \leq 350 \\
& \quad 8.50 x_{11} + 8.50 x_{12} + 5.00 x_{21} + 3.00 x_{22} + 9.50 x_{23} \leq 300, \\
& \quad 8.75 x_{11} + 5.25 x_{12} + 9.50 x_{21} + 9.00 x_{22} + 3.00 x_{23} \leq 250 \\
& \quad x_{11}, x_{12}, x_{21}, x_{22}, x_{23} \in \mathbb{N}^+, n_1 = 2, n_2 = 3.
\end{align*}
\]

By using an enumeration algorithm in MATLAB, the exact solution of the model (34) is \( (4, 7, 8, 10, 3) \) and the corresponding objective function value is 0.9173. That is, the components 11, 12, 21, 22, and 23 have 3, 6, 7, 9, and 2 redundant elements, respectively. They are solved in about 100 s. It is very difficult to use the enumeration algorithm to solve the optimization problem when the decision variables are very large. However, the proposed simulation optimization algorithm in Section 4.4 is very efficient. The effectiveness of the algorithm is illustrated below. Firstly, the simulation times are increased by a step size of 100. Table 2 shows the change of reliability, optimal solution, running time, and error with different simulation times, where the error represents the absolute value of the difference between the real value and the optimal value obtained by our algorithm. We can see from Table 2 that the fluctuation of the error is very obvious and the running time increases as the simulation times increase. Secondly, the simulation times are increased by a step size of 1000, and the data obtained are shown in Table 3. It can be seen from Table 3 that the system reliability is stable at 0.9173 and the error fluctuation is zero. In addition, we can see that completing

---

**Table 1: The data of the example.**

<table>
<thead>
<tr>
<th>Component</th>
<th>Element lifetime</th>
<th>Element cost</th>
<th>Element weight</th>
<th>Element volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(5, 1) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(12, 16) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(11, 15) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(5, 2) )</td>
</tr>
<tr>
<td>12</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(2, 1) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(1, 0.5) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(8, 10) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(10, 12) )</td>
</tr>
<tr>
<td>21</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(9, 10) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(9, 10) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
</tr>
<tr>
<td>22</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(9, 10) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(9, 10) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
</tr>
<tr>
<td>23</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
<td>( \mathcal{O} \mathcal{B} \mathcal{N}(4, 5.6) )</td>
</tr>
</tbody>
</table>

**Table 2: Simulation results with a step size of 100 in the UMOM.**

<table>
<thead>
<tr>
<th>Simulation times</th>
<th>Reliability</th>
<th>Error</th>
<th>Optimal solution</th>
<th>Running time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.8905</td>
<td>0.0268</td>
<td>(3, 8, 7, 9)</td>
<td>0.001629</td>
</tr>
<tr>
<td>200</td>
<td>0.8770</td>
<td>0.0403</td>
<td>(2, 7.7, 7.4)</td>
<td>0.001390</td>
</tr>
<tr>
<td>300</td>
<td>0.8581</td>
<td>0.0592</td>
<td>(8, 8.8, 5.2)</td>
<td>0.001055</td>
</tr>
<tr>
<td>400</td>
<td>0.9101</td>
<td>0.0027</td>
<td>(3.7, 7.9, 9)</td>
<td>0.001457</td>
</tr>
<tr>
<td>500</td>
<td>0.9173</td>
<td>0</td>
<td>(4, 7.8, 10, 3)</td>
<td>0.002060</td>
</tr>
<tr>
<td>600</td>
<td>0.9101</td>
<td>0.0027</td>
<td>(4.7, 7.9, 1)</td>
<td>0.002060</td>
</tr>
<tr>
<td>700</td>
<td>0.9101</td>
<td>0.0027</td>
<td>(5.8, 7.10, 1)</td>
<td>0.002025</td>
</tr>
<tr>
<td>800</td>
<td>0.9173</td>
<td>0</td>
<td>(5.8, 7.10, 1)</td>
<td>0.002624</td>
</tr>
<tr>
<td>900</td>
<td>0.9101</td>
<td>0.0027</td>
<td>(3.8, 9.9, 4)</td>
<td>0.002737</td>
</tr>
<tr>
<td>1000</td>
<td>0.9173</td>
<td>0</td>
<td>(4.7, 8, 10, 2)</td>
<td>0.002670</td>
</tr>
</tbody>
</table>
Table 3: Simulation results with a step size of 1000 in the UMOM.

<table>
<thead>
<tr>
<th>Simulation times</th>
<th>Reliability</th>
<th>Error</th>
<th>Optimal solution</th>
<th>Running time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.9173</td>
<td>0</td>
<td>(4, 7, 8, 10, 2)</td>
<td>0.002670</td>
</tr>
<tr>
<td>2000</td>
<td>0.9173</td>
<td>0</td>
<td>(3, 7, 7, 10)</td>
<td>0.004696</td>
</tr>
<tr>
<td>3000</td>
<td>0.9173</td>
<td>0</td>
<td>(4, 7, 7, 10, 4)</td>
<td>0.006890</td>
</tr>
<tr>
<td>4000</td>
<td>0.9173</td>
<td>0</td>
<td>(4, 7, 7, 10, 2)</td>
<td>0.009491</td>
</tr>
<tr>
<td>5000</td>
<td>0.9173</td>
<td>0</td>
<td>(3, 8, 8, 10, 1)</td>
<td>0.007648</td>
</tr>
<tr>
<td>6000</td>
<td>0.9173</td>
<td>0</td>
<td>(3, 8, 8, 10, 3)</td>
<td>0.009877</td>
</tr>
<tr>
<td>7000</td>
<td>0.9173</td>
<td>0</td>
<td>(5, 7, 7, 10, 1)</td>
<td>0.013472</td>
</tr>
<tr>
<td>8000</td>
<td>0.9173</td>
<td>0</td>
<td>(3, 8, 8, 10, 2)</td>
<td>0.009877</td>
</tr>
<tr>
<td>9000</td>
<td>0.9173</td>
<td>0</td>
<td>(3, 8, 8, 10, 3)</td>
<td>0.007648</td>
</tr>
<tr>
<td>10000</td>
<td>0.9173</td>
<td>0</td>
<td>(4, 7, 8, 10, 3)</td>
<td>0.015909</td>
</tr>
</tbody>
</table>

The computation takes very little time by using the simulation optimization algorithm. Figure 3 illustrates the objective function value (system reliability) versus simulation times. The optimal solutions for different deterioration rates are given in Table 4.

In the UOVOM, we take $\lambda = 0.01$, $\alpha_0 = 0.9$, $c_0 = 350$, $\omega_0 = 300$, and $v_0 = 250$. The lower and upper bounds of the number of the redundant elements are $I = 1$ and $u = 20$, respectively. Then, the model can be expressed as

$$
\max_x \left( \max_{1 \leq i \leq 23} \left( \frac{1 - (1 - 0.01)^{v_i}}{0.01} \Phi^{-1}_{ij} (1 - 0.9) \right) \right)
$$

s.t. \begin{align*}
12.97x_{11} + 3.09x_{12} + 9.00x_{21} + 11.00x_{22} + 5.10x_{23} &\leq 350 \\
8.50x_{11} + 8.50x_{12} + 5.00x_{21} + 3.00x_{22} + 9.50x_{23} &\leq 300 \\
8.75x_{11} + 5.25x_{12} + 9.50x_{21} + 9.00x_{22} + 3.00x_{23} &\leq 250 \\
x_{11}, x_{12}, x_{21}, x_{22}, x_{23} &\in N^+, n_1 = 2, n_2 = 3.
\end{align*}

(35)

The relationship between the simulation times and the $\alpha_0$-optimistic value of system lifetime is illustrated in Figure 4 when the step size of simulation times is 5000.

It can be seen that the $\alpha_0$-optimistic value of system lifetime has a common upper bound, whose value is the optimal value, and the corresponding solution is the optimal solution. The optimal solution is (6, 2, 1, 19, 1), and the maximum lifetime of the system is 228.7692. That is, the components 11, 12, 21, 22, and 23 have 5, 1, 0, 18, and 0 redundant elements, respectively. The optimal solutions and the maximum $\alpha_0$-optimistic value of system lifetime at different degradation rates are obtained in Table 5.

In the UCOM, we take $\lambda = 0.01$, $\alpha_0 = 0.9$, $t_0 = 100$, $\omega_0 = 300$, $v_0 = 250$, $l = 1$, and $u = 10$, then the model can be expressed as

$$
\min \left\{ 12.97x_{11} + 3.09x_{12} + 9.00x_{21} + 11.00x_{22} + 5.10x_{23} \right\}
$$

s.t. \begin{align*}
&\max_{1 \leq i \leq 23} \left( \frac{1 - (1 - 0.01)^{v_i}}{0.01} \Phi^{-1}_{ij} (1 - 0.9) \right) \geq 100 \\
&8.50x_{11} + 8.50x_{12} + 5.00x_{21} + 3.00x_{22} + 9.50x_{23} \leq 300 \\
&8.75x_{11} + 5.25x_{12} + 9.50x_{21} + 9.00x_{22} + 3.00x_{23} \leq 250 \\
x_{11}, x_{12}, x_{21}, x_{22}, x_{23} &\in N^+, n_1 = 2, n_2 = 3.
\end{align*}

(36)
In this paper, the authors present a numerical example (the data of the elements are assumed to be uncertain or random variables in the numerical example).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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References


The relationship between the simulation times and system expected cost is illustrated in Figure 5 when the step size of simulation times is 1000. From Figure 5, we can see that the system expected cost under different simulation times has common lower bound, whose value is the minimum expected cost of the system. The optimal solution is (3,1,1,8,1), and the corresponding minimum expected cost is 144.1000. That is, the components 11 and 22 have 2 and 7 redundant elements, respectively. The components 12, 21, and 23 have no redundant elements. The optimal solutions and the minimum expected costs at different degradation rates are given in Table 6.

6. Conclusion

In this paper, an uncertain parallel-series system with warm standby elements was investigated. Under the assumption that the element lifetime and cost are uncertain variables and the weight and volume of an element are random variables, the optimization model of warm standby redundancy for the uncertain parallel-series system was proposed. We formulated three different optimization models—UMOM, UOVOM, and UCOM—based on reliability maximization, lifetime maximization, and cost minimization, respectively. An efficient simulation optimization algorithm was designed to calculate the objective values and optimal solutions of the UMOM, UOVOM, and UCOM. The numerical example showed the rationality of the proposed models and the effectiveness of the simulation optimization algorithm. In the future, the redundancy optimization model with priority will be constructed according to the preference of the decision-maker. Also, the uncertain parallel-series system considered in this paper is a binary system; how to model the uncertain multistate parallel-series system is another future research direction.

Data Availability

In this paper, the authors present a numerical example (the data of the elements are assumed to be uncertain or random variables in the numerical example).

Table 6: Optimal solutions for different deterioration rates in the UCOM.

<table>
<thead>
<tr>
<th>Deterioration rate</th>
<th>System cost</th>
<th>Optimal solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>144.1000</td>
<td>(3,1,1,8,1)</td>
</tr>
<tr>
<td>0.02</td>
<td>155.1000</td>
<td>(3,1,1,9,1)</td>
</tr>
<tr>
<td>0.03</td>
<td>155.1000</td>
<td>(3,1,1,9,1)</td>
</tr>
<tr>
<td>0.04</td>
<td>155.1000</td>
<td>(3,1,1,9,1)</td>
</tr>
<tr>
<td>0.05</td>
<td>166.1000</td>
<td>(3,1,1,10,1)</td>
</tr>
</tbody>
</table>

Figure 5: System cost versus simulation times in the UCOM.


