Research Article

A Novel Approach to a Time-Dependent-Coefficient WBK System: Doubly Periodic Waves and Singular Nonlinear Dynamics

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1. Introduction

Nonlinear complex phenomena in natural world, for example, solitons first observed by Russell in 1834 [1], are often described by nonlinear PDEs. Usually, people restore to exact solutions of nonlinear PDEs to gain more insight into the essence behind these nonlinear phenomena for further applications. In the past several decades, many effective methods for exactly solving nonlinear PDEs have been presented like those in [2–22]. In 2003, Zhou et al. proposed the so-called $F$-expansion method [22] to construct different Jacobi elliptic doubly periodic solutions of nonlinear PDEs in a uniform way, which can be thought of as an overall generalization of the Jacobi elliptic function expansion method [23]. The $F$-expansion method has been widely used to a great many of nonlinear PDEs [24–26] and was improved in different manners [27–30]. In 2006, Zhang and Xia [30] generalized the $F$-expansion method by introducing a new and more general ansatz. The present paper is motivated by the desire to extend the generalized $F$-expansion method [30] to the new and more general tcWBK system [31, 32]:

$$u_t + \gamma_1 u u_x + \gamma_2 v_x + \gamma_3 u_{xx} = 0,$$

$$v_t + \gamma_4 u_x v + \gamma_5 v_x + \gamma_6 u_{xxx} = 0,$$

where $\gamma_i (i = 1, 2, \ldots, 6)$ are arbitrary smooth functions of $t$, which represent different dispersion and dissipation forces. Clearly, (1) includes some existing well-known important equations as special cases; they are the approximate equations for long water waves [33]:

$$u_t - uu_x - v_x + \frac{1}{2} u_{xx} = 0,$$

$$v_t - (uv)_x - \frac{1}{2} v_{xx} = 0,$$

which are the Boussinesq-Burgers equations, and the variant Boussinesq equations. To construct doubly periodic wave solutions, we extend the generalized $F$-expansion method for the first time to the tcWBK system. As a result, many new Jacobi elliptic doubly periodic solutions are obtained; the limit forms of which are the hyperbolic function solutions and trigonometric function solutions. It is shown that the original $F$-expansion method cannot derive Jacobi elliptic doubly periodic solutions of the tcWBK system, but the novel approach of this paper is valid. To gain more insight into the doubly periodic waves contained in the tcWBK system, we simulate the dynamical evolutions of some obtained Jacobi elliptic doubly periodic solutions. The simulations show that the doubly periodic waves possess time-varying amplitudes and velocities as well as singularities in the process of propagations.
the WBK equations in shallow water [34]:
\[ u_t + uu_x + v_x + \beta u_{xx} = 0, \]
\[ v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0, \]  
(3)

the Boussinesq-Burgers (BB) equations [35]:
\[ u_t + 2uu_x - \frac{1}{2} v_x = 0, \]
\[ v_t + 2(uv)_x - \frac{1}{2} u_{xxx} = 0, \]  
(4)

the variant Boussinesq equations [34]:
\[ u_t + uu_x + v_x = 0, \]
\[ v_t + (uv)_x + u_{xxx} = 0, \]  
(5)

It should be pointed out that the $F$-expansion method [22] cannot derive Jacobi elliptic doubly periodic solutions of (1). To be specific, according to the $F$-expansion method [22], we first suppose that (1) has exact solutions of the forms:
\[ u = a_0 + \sum_{i=1}^{n} (a_i F^i(\xi) + b_i F^{-i}(\xi)), \]
\[ v = A_0 + \sum_{i=1}^{m} (A_i F^i(\xi) + B_i F^{-i}(\xi)), \]  
(6)

where $\xi = kx + \eta$, the integers $n$ and $m$ and the constant $k$ are to be determined, while $a_0 = a_0(t)$, $a_i = a_i(t)$, $b_i = b_i(t)$, $A_0 = A_0(t)$, $A_i = A_i(t)$, $B_i = B_i(t)$, and $\eta = \eta(t)$ are all undetermined functions of the indicated variables; $F(\xi)$ satisfies
\[ F''(\xi) = PF^4(\xi) + QF^2(\xi) + R, \]  
(7)

and hence holds
\[ F'''(\xi) = 2PF^3(\xi) + QF(\xi), \]
\[ F^{(3)}(\xi) = (6PF^2(\xi) + Q)F'(\xi), \]
\[ F^{(4)}(\xi) = 24P^2F^3(\xi) + 20PQF^3(\xi) + (Q^2 + 12PR)F(\xi), \]  
(8)

where $P$, $Q$, and $R$ are parameters. In [30], Jacobi elliptic function solutions and their degenerated solutions of (7) are listed, which depend on the values of parameters $P$, $Q$, and $R$. Secondly, substituting (6) along with (7) and (8) into (1) and then balancing the highest order partial derivative $uu_x$ and the highest order nonlinear term $u_{xxx}$ yield the integer $2n + 1 = n + 2$ which gives $n = 1$. Similarly, we determine the integer $m = 2$ by balancing $uv_x$ and $u_{xxx}$. Thirdly, we substitute (6) given the values of $n = 1$ and $m = 2$ along with (7) and (8) into (1) and collect all terms with the same order of $F^j(\xi)F^{s'}(\xi)$ ($j = 0, \pm 1, \pm 2, \ldots$, $s = 0, 1$) together, we have
\[ -k(b_1^2y_1^2 + 2b_2y_3)F^2(\xi)F^{-3}(\xi) \]
\[ - \left( ka_0b_1y_1 + kB_1y_2 + b_1\eta \right) F'(\xi)F^{-2}(\xi) \]
\[ + \left( ka_0a_1y_1 + kA_1y_2 + a_1\eta \right) F'(\xi) \]
\[ + k(a_1^2y_1 + 2A_2y_2)F^2(\xi)F(\xi) \]
\[ + 2k^2RB_1y_3F^{-3}(\xi) + \left( k^2QB_1y_3 + b_1^2 \right) F^{-1}(\xi) \]
\[ + a_0^2 + \left( k^2QA_1y_3 + a_1^2 \right) F(\xi) \]
\[ + 2k^2PA_1y_3F^3(\xi) = 0, \]
\[ -3kb_1(3B_1y_4 + 2k^2Ry_6)F^2(\xi)F^{-4}(\xi) \]
\[ -2 \left( k^2B_1y_4 + ka_0B_2y_4 + B_1^2\eta \right) F'(\xi)F^{-3}(\xi) \]
\[ - \left( kA_0B_1y_4 + ka_0B_2y_4 + kb_1y_4 + k^3QB_1y_6 + B_1^2\eta \right) F^{3}(\xi) \]
\[ \cdot F^{-2}(\xi) + \left( ka_0a_1y_4 + kb_1A_2y_4 + a_1^2 \right) F'(\xi) \]
\[ + k^2QA_1y_6 + A_1\eta \right) F^2(\xi) \]
\[ + 2 \left( ka_1A_2y_4 + ka_0A_2y_4 + a_2^2 \right) F'(\xi)F(\xi) \]
\[ + 3ka_1(3A_2y_4 + 2k^2P_6y_4)F^2(\xi)F(\xi) \]
\[ - 6k^2RB_1y_5F^{-4}(\xi) - 2k^2RB_1y_5F^{-3}(\xi) \]
\[ - 4k^2QB_2y_5 - B_2^2 \right) F^2(\xi) - \left( k^2QB_1y_5 - B_1^2 \right) F^{-1}(\xi) \]
\[ - 2k^2RA_2y_5 - 2k^2PB_2y_5 + A_0^2 - \left( k^2QA_1y_5 - A_1^2 \right) F(\xi) \]
\[ - \left( 4k^2QA_2y_5 - A_2^2 \right) F^2(\xi) - 2k^2PA_1y_5F^3(\xi) \]
\[ - 6k^2PA_2y_5F^4(\xi) = 0. \]  
(9)

Since $PQR \neq 0$ is the necessary condition that (7) exists Jacobi elliptic doubly periodic, without loss of generality, we have
\[ a_0 = a_1 = b_1 = A_0' = A_1 = A_2 = B_1 = B_2 = 0, \]  
(10)

when setting each coefficient of $F^j(\xi)F^{s'}(\xi)$ of (9) to zeros. This tells that (6) let (1) has only constant solutions but not Jacobi elliptic doubly periodic solutions as expected.

The present paper is motivated by the desire to investigate Jacobi elliptic doubly periodic solutions of (1). The rest of the paper is organized as follows. In Section 2, the generalized $F$-expansion method [30] is extended to (1) for constructing Jacobi elliptic doubly periodic solutions. In Section 3, we simulate the dynamical evolutions of some obtained Jacobi elliptic doubly periodic solutions to gain more insight into the doubly periodic waves contained in (1). In Section 4, we conclude this paper.
2. Doubly Periodic Waves

To extend the generalized $F$-expansion method [30] to (1), in this section, we suppose that (1) has Jacobi elliptic doubly periodic solutions of the forms:

\[
\begin{align*}
    u &= a_0 + a_1 F(\xi) + b_1 F^{-1}(\xi) + c_1 F'(\xi) + d_1 F^{-1}(\xi), \\
    v &= A_0 + A_1 F(\xi) + A_2 F^2(\xi) + B_1 F^{-1}(\xi) + B_2 F^2(\xi) + C_1 F' F'(\xi) + D_1 F^{-1}(\xi) F'(\xi) + D_2 F^{-2}(\xi) F'(\xi).
\end{align*}
\]

Substituting (11) along with (7) and (8) into (1) and collecting all terms with the same order of $F(\xi) F'(\xi)$, $F^2(\xi)$, together, we derive a set of nonlinear PDEs for $a_0, a_1, b_1, c_1, d_1, A_0, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, \eta$, and $k$. Solving this set of nonlinear PDEs, we have three cases.

Case 1.

\[
\begin{align*}
    a_1 &= \pm \frac{2k}{2} \sqrt{P(\gamma_2^2 + c_0 \gamma_1 \gamma_6)} \gamma_1, \\
    b_1 &= \pm \frac{2k}{2} \sqrt{Q(\gamma_2^2 + c_0 \gamma_1 \gamma_6)} \gamma_1, \\
    a_0 &= A_0 = \text{const},
\end{align*}
\]

where $k$ is an arbitrary constant; the signs “$\pm$” and “$\mp$” in (12) and (13) mean that all possible combinations of “$+$” and “$-$” can be taken. If it is taken the same sign in $a_1$ and $b_1$, then it must be taken “$-$” in $\gamma_2^2$ and $\gamma_6'$. If it is taken the different signs in $a_1$ and $b_1$, then it must be taken “$+$” in $\gamma_2^2$ and $\gamma_6'$. At the same time, it must be taken the different signs in $a_1$ and $C_1$ and the same sign in $b_1$ and $D_2$. While $\gamma_i (i = 1, 3, 6)$ in (12) satisfy the constraints:

\[
\begin{align*}
    y_4 &= y_1, \\
    y_5 &= y_1, \\
    y_2 &= c_0 y_1, \\
    y_3' &= c_0 A_0 y_1^2 + 2k \left( \frac{Q + 2 \sqrt{P}}{\gamma_2^2 + c_0 \gamma_1 \gamma_6} \right) (\gamma_2^2 + c_0 \gamma_1 \gamma_6) + \frac{y_3 y_1'}{y_1}, \\
    y_6' &= 2c_0 y_1 y_1' y_2' + \frac{c_0 y_1^2 y_1' y_6 - 2c_0 y_1^2 y_3 \left[ c_0 A_0 y_1^2 + 2k \left( \frac{Q + 2 \sqrt{P}}{\gamma_2^2 + c_0 \gamma_1 \gamma_6} \right) (\gamma_2^2 + c_0 \gamma_1 \gamma_6) - 2c_0 y_1 y_1' \right]}{c_0 y_1^2}.
\end{align*}
\]

Case 2.

\[
\begin{align*}
    a_1 &= \pm \frac{2k}{2} \sqrt{P(\gamma_2^2 + c_0 \gamma_1 \gamma_6)} \gamma_1, \\
    a_0 &= A_0 = \text{const},
\end{align*}
\]

where $k$ is an arbitrary constant; the signs “$\pm$” and “$\mp$” in (14) and (15) mean that it must be taken the different signs in $a_1$.
and $C_i$, while $\gamma_i (i = 1, 3, 6)$ in (14), (15), and (16) satisfy
the constraints:

\[
\begin{align*}
\gamma_4 &= \gamma_1, \\
\gamma_5 &= \gamma_3, \\
\gamma_2 &= c_0 \gamma_1, \\
\gamma'_4 &= c_0 A_0 \gamma_1^2 + k^2 Q(\gamma_3^2 + c_0 \gamma_1 \gamma_6) + \frac{y_3 y'_1}{y_1}, \\
\gamma'_5 &= \frac{2 c_0 y_1 y'_1 y'_6 + c_0 y_1 y_6}{c_0 y_1^2} 2 c_0 A_0 \gamma_1^2 + k^2 Q(\gamma_3^2 + c_0 \gamma_1 \gamma_6) - 2 c_0 y_1 y_3 \gamma_1^3 [c_0 A_0 \gamma_1^2 + k^2 Q(\gamma_3^2 + c_0 y_1 \gamma_6) - 2 c_0 y_1 y_3].
\end{align*}
\]

(17)

Case 3.

\[
b_1 = \pm \frac{2 k^2 R y_3 (y_3^2 + c_0 \gamma_1 \gamma_6)}{c_0 y_1},
\]

(18)

\[
a_0 = A_0 = \text{const.},
\]

\[
B_2 = -\frac{2 k^2 R y_3 (y_3^2 + c_0 \gamma_1 \gamma_6)}{c_0 y_1^2},
\]

\[
D_2 = \pm \frac{2 k^2 y_3 \sqrt{R (y_3^2 + c_0 \gamma_1 \gamma_6)}}{c_0 y_1^2},
\]

(19)

where $b_1$, $a_0$, $A_0$, and $k$ are arbitrary constants, $\gamma_i = k x - k a_0 \int y_1 \, dt$, and $\gamma_i (i = 1, 2, 3, 6)$ in (21) satisfy the constraints in (13).

From Cases 1–3, we obtain three formulae of fundamental solutions of (1) as follows:

\[
u = A_0 - \frac{2 k^2 P (y_3^2 + c_0 \gamma_1 \gamma_6)}{c_0 y_1} F^2(\xi) - \frac{2 k^2 R (y_3^2 + c_0 \gamma_1 \gamma_6)}{c_0 y_1^2} F^2(\xi),
\]

\[
u = \frac{2 k^2 y_3 \sqrt{R (y_3^2 + c_0 \gamma_1 \gamma_6)}}{c_0 y_1} F(\xi).
\]

(21)

where $a_0$, $A_0$, and $k$ are arbitrary constants, $\xi = k x - k a_0 \int y_1 \, dt$, and $\gamma_i (i = 1, 3, 6)$ in (22) satisfy the constraints in (17).

\[
u = A_0 - \frac{2 k^2 P (y_3^2 + c_0 \gamma_1 \gamma_6)}{c_0 y_1} F^2(\xi)
\]

\[
u = \frac{2 k^2 y_3 \sqrt{R (y_3^2 + c_0 \gamma_1 \gamma_6)}}{c_0 y_1} F(\xi)
\]

(22)

\[
u = A_0 - \frac{2 k^2 R (y_3^2 + c_0 \gamma_1 \gamma_6)}{c_0 y_1^2} F^2(\xi)
\]

\[
u = \frac{2 k^2 y_3 \sqrt{R (y_3^2 + c_0 \gamma_1 \gamma_6)}}{c_0 y_1} F(\xi)
\]

(23)
where $a_0$, $A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int y_i \, dt$, and $y_i (i = 1, 2, 3, 6)$ in (24) and (25) satisfy the constraints in (17).

With the help of (22), (23), (24), and (25) and (Appendices A, B, and C [30]), we obtain many exact solutions of (1). For example, selecting $P = 1$, $Q = -(1 + m^2)$, $R = m^2$, and $F(\xi) = \text{ns}^2 \xi$, from (21), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$u = a_0 \pm \frac{2k\sqrt{y_3^2 + c_0y_1y_6}}{y_1} \text{sn} \xi, \quad v = A_0 - \frac{2k^2(y_3^2 + c_0y_1y_6)}{c_0y_1^2} \text{ns}^2 \xi$$

(24)

$$\pm \frac{2k m\sqrt{y_3^2 + c_0y_1y_6}}{c_0y_1^2} \text{cs} \xi \text{ds} \xi,$$

where $a_0$, $A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int y_i \, dt$, and $y_i (i = 1, 2, 3, 6)$ in (24) and (25) satisfy the constraints $y_4 = y_1, y_5 = y_3, y_2 = c_0y_1$, and

$$y_i' = c_0A_0y_i' + k^2(-1 - m^2 + 2)\left(y_i' + c_0y_1y_6\right) + \frac{y_3y_i'}{y_1},$$

$$y_i'' = \frac{2c_0y_1y_iy_3' + c_0y_1y_iy_6' - 2c_0y_1y_3y_6' + k^2(-1 - m^2 + 2)(y_3 + c_0y_1y_6) - 2c_0y_1y_3y_5'}{c_0y_1^2}.$$

(26)

Selecting $P = -1$, $Q = 2 - m^2$, $R = m^2 - 1$, and $F(\xi) = \text{dn} \xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$u = a_0 \pm \frac{2k\sqrt{-y_3^2 + c_0y_1y_6}}{y_1} \text{dn} \xi,$$

$$v = A_0 + \frac{2k^2(y_3^2 + c_0y_1y_6)}{c_0y_1^2} \text{dn}^2(\xi)$$

$$\pm \frac{2k m^2\sqrt{-y_3^2 + c_0y_1y_6}}{c_0y_1^2} \text{sn} \xi \text{cn} \xi, \quad (27)$$

where $a_0$, $A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int y_i \, dt$, and $y_i (i = 1, 3, 6)$ in (27) satisfy the constraints $y_4 = y_1, y_5 = y_3, y_2 = c_0y_1$, and

$$y_i' = c_0A_0y_i' + k^2(2 - m^2)(y_i' + c_0y_1y_6) + \frac{y_3y_i'}{y_1},$$

$$y_i'' = \frac{2c_0y_1y_iy_3' + c_0y_1y_iy_6' - 2c_0y_1y_3y_6' + k^2(2 - m^2)(y_3 + c_0y_1y_6) - 2c_0y_1y_3y_5'}{c_0y_1^2}.$$

(28)

Selecting $P = 1$, $Q = -(1 + m^2)$, $R = m^2$, and $F(\xi) = \text{ns} \xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

$$u = a_0 \pm \frac{2k\sqrt{y_3^2 + c_0y_1y_6}}{y_1} \text{ns} \xi,$$

$$v = A_0 - \frac{2k^2(y_3^2 + c_0y_1y_6)}{c_0y_1^2} \text{ns}^2(\xi) \pm \frac{2k^2\sqrt{y_3^2 + c_0y_1y_6}}{c_0y_1^2} \text{cs} \xi \text{ds} \xi,$$

(29)
where $a_0, A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int y_1 \, dt$, and $y_i (i = 1, 3, 6)$ in (29) satisfy the constraints $y_4 = y_1, y_5 = y_3, y_2 = c_0 y_1$, and

\[
y_3' = c_0 A_0 y_1^2 - k^2 (1 + m^2) (y_3^2 + c_0 y_1 y_6) + \frac{y_3 y_1'}{y_1},
\]

\[
y_6' = \frac{2c_0 y_1 y_3 y_4' + c_0 y_1 y_6' - 2c_0 y_1 y_3 \left[ c_0 A_0 y_1^2 - k^2 (1 + m^2) (y_3^2 + c_0 y_1 y_6) - 2c_0 y_1 y_3 \right]}{c_0 y_1^2}.
\] (30)

Selecting $P = 1 - m^2, Q = 2 - m^2, R = 1, \text{ and } F(\xi) = sc\xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

\[
u = a_0 - \frac{2k \sqrt{1 - m^2} (y_3^2 + c_0 y_1 y_6)}{y_1} \, sc\xi,
\]

\[
v = A_0 - \frac{2k^2 (1 - m^2) (y_3^2 + c_0 y_1 y_6)}{c_0 y_1^2} \, sc\xi n\xi + \frac{2k^2 y_3 \sqrt{1 - m^2} (y_3^2 + c_0 y_1 y_6)}{c_0 y_1^2} \, dc\xi n\xi,
\] (31)

where $a_0, A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int y_1 \, dt$, and $y_i (i = 1, 3, 6)$ in (31) satisfy the constraints $y_4 = y_1, y_5 = y_3, y_2 = c_0 y_1$, and

\[
y_3' = c_0 A_0 y_1^2 + k^2 (2 - m^2) (y_3^2 + c_0 y_1 y_6) + \frac{y_3 y_1'}{y_1},
\]

\[
y_6' = \frac{2c_0 y_1 y_3 y_4' + c_0 y_1 y_6' - 2c_0 y_1 y_3 \left[ c_0 A_0 y_1^2 + k^2 (2 - m^2) (y_3^2 + c_0 y_1 y_6) - 2c_0 y_1 y_3 \right]}{c_0 y_1^2}.
\] (32)

Selecting $P = 1, Q = 2m^2 - 1, R = -m^2 (1 - m^2)$, and $F(\xi) = ds\xi$, from (22), we obtain new Jacobi elliptic doubly periodic solutions of (1):

\[
u = A_0 - \frac{2k \sqrt{y_3^2 + c_0 y_1 y_6}}{c_0 y_1^2} \, ds^2 \xi \pm \frac{2k^2 y_3 \sqrt{y_3^2 + c_0 y_1 y_6}}{c_0 y_1^2} \, cs\xi ns\xi,
\] (33)

where $a_0, A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int y_1 \, dt$, and $y_i (i = 1, 3, 6)$ in (33) satisfy the constraints $y_4 = y_1, y_5 = y_3, y_2 = c_0 y_1$, and

\[
y_3' = c_0 A_0 y_1^2 + k^2 (2m^2 - 1) (y_3^2 + c_0 y_1 y_6) + \frac{y_3 y_1'}{y_1},
\]

\[
y_6' = \frac{2c_0 y_1 y_3 y_4' + c_0 y_1 y_6' - 2c_0 y_1 y_3 \left[ c_0 A_0 y_1^2 + k^2 (2m^2 - 1) (y_3^2 + c_0 y_1 y_6) - 2c_0 y_1 y_3 \right]}{c_0 y_1^2}.
\] (34)
In the limits at \( m \to 1 \) and \( m \to 0 \), the above obtained Jacobi elliptic doubly periodic solutions degenerate into hyperbolic function solutions and trigonometric function solutions, respectively. When \( m \to 1 \), the Jacobi elliptic doubly periodic solutions (24) and (25) degenerate into hyperbolic function solutions:

\[
u = A_0 - \frac{2k^2 (y_3^2 + c_0 y_1 y_6)}{c_0 y_1^2} \cosh^2 \xi
- \frac{2k^2 (y_3^2 + c_0 y_1 y_6)}{c_0 y_1^2} \tanh^2 \xi
\pm \frac{2k^2 y_3 \sqrt{y_3^2 + c_0 y_1 y_6}}{c_0 y_1^2} \csc^2 \xi
\pm \frac{2k^2 y_3 \sqrt{y_3^2 + c_0 y_1 y_6}}{c_0 y_1^2} \sech^2 \xi,
\]

(36)

where \( a_0, A_0, \) and \( k \) are arbitrary constants, \( \xi = kx - ka_0 \int y_1 \, dt \), and \( y_{i_1}(i = 1, 3, 6) \) in (36) satisfy the constraints \( y_4 = y_1, y_5 = y_3, y_2 = c_0 y_1, \) and

When \( m \to 0 \), the Jacobi elliptic doubly periodic solutions (24) and (25) degenerate into trigonometric function solutions:

\[
u = A_0 - \frac{2k^2 (y_3^2 + c_0 y_1 y_6)}{c_0 y_1^2} \csc^2 \xi
\pm \frac{2k^2 y_3 \sqrt{y_3^2 + c_0 y_1 y_6}}{c_0 y_1^2} \cot \xi \csc \xi,
\]

(38)

where \( a_0, A_0, \) and \( k \) are arbitrary constants, \( \xi = kx - ka_0 \int y_1 \, dt \), and \( y_{i_1}(i = 1, 3, 6) \) in (38) satisfy the constraints \( y_4 = y_1, y_5 = y_3, y_2 = c_0 y_1, \) and

When \( m \to 1 \), the Jacobi elliptic doubly periodic solutions (27) degenerate into hyperbolic function solutions:

\[
u = A_0 - \frac{2k^2 (y_3^2 + c_0 y_1 y_6)}{c_0 y_1^2} \sech^2 \xi
\pm \frac{2k^2 y_3 \sqrt{y_3^2 + c_0 y_1 y_6}}{c_0 y_1^2} \tanh \xi \sech \xi,
\]

(40)
where \( a_0, A_0, \) and \( k \) are arbitrary constants, \( \xi = kx - ka_0 \int y_1 \, dt \), and \( y_i(i = 1, 3, 6) \) in (40) satisfy the constraints \( y_4 = y_1, y_5 = y_3, \) and

\[
y'_3 = c_0 A_0 y_1^2 + k^2 \left( y_3^2 + c_0 y_1 y_6 \right) + \frac{y_3y'_1}{y_1},
\]

\[
y'_6 = \frac{2c_0 y_1 y_1' y_3 + c_0 y_1^2 y_1 y_6 - 2c_0 y_1 y_6^2}{c_0 y_1^2} \left[ c_0 A_0 y_1^2 + k^2 \left( y_3^2 + c_0 y_1 y_6 \right) - 2c_0 y_1 y_1' y_3 \right].
\]

When \( m \to 1 \), the Jacobi elliptic doubly periodic solutions (29) degenerate into hyperbolic function solutions:

\[
u = A_0 - \frac{2k^2 \left( y_3^2 + c_0 y_1 y_6 \right)}{c_0 y_1^2} \coth^2 (\xi)
\]

\[
\pm \frac{2k^2 y_3 \sqrt{y_3^2 + c_0 y_1 y_6} \csc h^2 \xi}{c_0 y_1^2},
\]

where \( a_0, A_0, \) and \( k \) are arbitrary constants, \( \xi = kx - ka_0 \int y_1 \, dt \), and \( y_i(i = 1, 3, 6) \) in (42) and (43) satisfy the constraints \( y_4 = y_1, y_5 = y_3, y_2 = c_0 y_1, \) and

\[
y'_3 = c_0 A_0 y_1^2 - 2k^2 \left( y_3^2 + c_0 y_1 y_6 \right) + \frac{y_3y'_1}{y_1},
\]

\[
y'_6 = \frac{2c_0 y_1 y_1' y_3 + c_0 y_1^2 y_1 y_6 - 2c_0 y_1 y_6^2}{c_0 y_1^2} \left[ c_0 A_0 y_1^2 - 2k^2 \left( y_3^2 + c_0 y_1 y_6 \right) - 2c_0 y_1 y_1' y_3 \right].
\]

When \( m \to 0 \), the Jacobi elliptic doubly periodic solutions (31) degenerate into trigonometric function solutions:

\[
u = A_0 - \frac{2k^2 \left( y_3^2 + c_0 y_1 y_6 \right)}{c_0 y_1^2} \tan^2 (\xi)
\]

\[
\pm \frac{2k^2 y_3 \sqrt{y_3^2 + c_0 y_1 y_6} \sec^2 \xi}{c_0 y_1^2},
\]

where \( a_0, A_0, \) and \( k \) are arbitrary constants, \( \xi = kx - ka_0 \int y_1 \, dt \), and \( y_i(i = 1, 3, 6) \) in (45) satisfy the constraints \( y_4 = y_1, y_5 = y_3, y_2 = c_0 y_1, \) and

\[
y'_3 = c_0 A_0 y_1^2 + 2k^2 \left( y_3^2 + c_0 y_1 y_6 \right) + \frac{y_3y'_1}{y_1},
\]

\[
y'_6 = \frac{2c_0 y_1 y_1' y_3 + c_0 y_1^2 y_1 y_6 - 2c_0 y_1 y_6^2}{c_0 y_1^2} \left[ c_0 A_0 y_1^2 + 2k^2 \left( y_3^2 + c_0 y_1 y_6 \right) - 2c_0 y_1 y_1' y_3 \right].
\]
When $m \to 1$, the Jacobi elliptic doubly periodic solutions (33) degenerate into trigonometric function solutions:

$$u = a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_4\gamma_6}}{\gamma_1} \csc \xi,$$

$$v = A_0 - \frac{2k^2(\gamma_3^2 + c_0\gamma_4\gamma_6)}{c_0\gamma_1^2} \csc^2 \xi \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_4\gamma_6}}{c_0\gamma_1^2} \coth \xi \csc \xi,$$

(47)

where $a_0$, $A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 dt$, and $\gamma_i (i = 1, 3, 6)$ in (47) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\gamma' = c_0A_0\gamma_1y_1^2 + k^2(\gamma_3^2 + c_0\gamma_4\gamma_6),$$

$$\gamma'' = \frac{2c_0\gamma_4\gamma_6y_2 + \gamma_3y_3'y_4'}{c_0\gamma_1^2}.$$ (48)

When $m \to 0$, the Jacobi elliptic doubly periodic solutions (33) degenerate into the following trigonometric function solutions which have the same expressions as solutions (38) but with different constraints (50):

$$u = a_0 \pm \frac{2k\sqrt{\gamma_3^2 + c_0\gamma_4\gamma_6}}{\gamma_1} \csc \xi,$$

$$v = A_0 - \frac{2k^2(\gamma_3^2 + c_0\gamma_4\gamma_6)}{c_0\gamma_1^2} \csc^2 \xi \pm \frac{2k^2\gamma_3\sqrt{\gamma_3^2 + c_0\gamma_4\gamma_6}}{c_0\gamma_1^2} \cot \xi \csc \xi,$$

(49)

where $a_0$, $A_0$, and $k$ are arbitrary constants, $\xi = kx - ka_0 \int \gamma_1 dt$, and $\gamma_i (i = 1, 3, 6)$ in (49) satisfy the constraints $\gamma_4 = \gamma_1$, $\gamma_5 = \gamma_3$, $\gamma_2 = c_0\gamma_1$, and

$$\gamma' = c_0A_0\gamma_1y_1^2 - k^2(\gamma_3^2 + c_0\gamma_4\gamma_6),$$

$$\gamma'' = \frac{2c_0\gamma_4\gamma_6y_2 + \gamma_3y_3'y_4'}{c_0\gamma_1^2}.$$ (50)

### 3. Singular Nonlinear Dynamics

In this section, we further investigate the nonlinear dynamics of (1) by means of Jacobi elliptic doubly periodic solutions.

Firstly, we consider solutions (24) and (25). To determine $\gamma_3$ and $\gamma_6$ with the sign “−” in (26), we select $\gamma_1 = e^t$ and then have

$$\gamma_3 = e^{-(\gamma_3^2/2A_0)c_0} \left( c_1 \int e^{-(\gamma_3^2/2A_0)c_0} dt + c_2 \right),$$

$$\gamma_6 = -e^{-(\gamma_3^2/2A_0)c_0} \left[ A_0c_0c_1 (-1 + m^2)^2 e^{(\gamma_3^2/2A_0)c_0+2t} + k^2 (-1 + m^2)^2 + k^2 (1 - m^2)^2 \left( c_1 \int e^{-(\gamma_3^2/2A_0)c_0} dt + c_2 \right)^2 \right].$$ (51)
where $c_1$ and $c_2$ are two integration constants.

In Figures 1 and 2, the spatial structures and contour lines of solutions (24) and (25) determined by (51) are shown by selecting the parameters as $a_0 = 2, A_0 = 0.5, c_0 = -4, c_1 = 3, c_2 = -1, k = 1.5, \text{ and } m = 0.8$, respectively. We shown the nonlinear dynamical evolutions of solutions (24) and (25) in Figures 3 and 4. It is easy to see from Figures 1–4 that the doubly periodic waves determined by solutions (24) and
(25) possess time-varying amplitudes and velocities as well as singularities in the process of propagations.

Secondly, we consider solutions (27). To determine $\gamma_3$ and $\gamma_6$ in (28), we let $\gamma_1 = e^t$ and then have

$$
\gamma_3 = c_1 e^t + c_2 e^{2t},
$$
$$
\gamma_6 = e^t \left[ -A_0 c_0 + c_2 - k^2 (2 - m^2) (c_1 + c_2 e^t) \right] \left/ c_0 k^2 (2 - m^2) \right., \tag{52}
$$

where $c_1$ and $c_2$ are two integration constants.

In Figure 5, the spatial structures of solutions (27) determined by (52) are shown by selecting the parameters $a_0 = 2$, $A_0 = 0.5$, $c_0 = -4$, $c_1 = 3$, $c_2 = -1$, $k = 1.5$, and $m = 0.8$, respectively. We shown the spatial structures of solutions (40) in Figure 6. It is easy to see from Figures 5 and 6 that both the doubly periodic waves determined by solutions (27) and the hyperbolic function solutions (40) possess time-varying amplitudes and velocities as well as singularities in the process of propagations.
Finally, we consider solutions (33). To determine $\gamma_3$ and $\gamma_6$ in (34), (35), we let $\gamma_1 = e^t$ and then have

$$\gamma_3 = c_1 e^t + c_2 e^{2t},$$
$$\gamma_6 = \frac{e^t \left[ -A_0 c_0 - c_2 - k^2 (1 - m^2) (c_1 + c_2 e^t) \right]}{c_0 k^2 (1 - m^2)},$$

(53)

where $c_1$ and $c_2$ are two integration constants.

In Figure 7, the spatial structures of solutions (33) determined by (53) are shown by selecting the parameters as $a_0 = 2$, $A_0 = 0.5$, $c_0 = -4$, $c_1 = 3$, $c_2 = -1$, $k = 1.5$, and $m = 0.8$, respectively. We can see from Figure 7 that the doubly periodic waves determined by solutions (33) possess time-varying amplitudes and velocities as well as singularities in the process of propagations.

4. Conclusion

In summary, new and more general Jacobi elliptic doubly periodic solutions of the tdcWBK system have been obtained, which degenerate into the hyperbolic function solutions and trigonometric function solutions in the limit cases. To the best of our knowledge, the obtained Jacobi elliptic doubly periodic solutions have not been reported in literatures. It is shown that the original $F$-expansion method cannot derive Jacobi elliptic doubly periodic solutions of the tdcWBK system but the novel approach of this paper is valid. In this sense, we would like to conclude that a novel approach of the generalized $F$-expansion method is extended to the tdcWBK system. The simulations show that the doubly periodic waves possess time-varying amplitudes and velocities as well as singularities in the process of propagations.

Recently, fractional-order differential calculus and its applications have attached much attention [36–51]. Constructing Jacobi elliptic doubly periodic solutions of nonlinear PDEs with fractional derivatives is worthy of the study. At the same time, constructing multisoliton solutions via the Riemann-Hilbert approach is also worthy of the study.

Data Availability

The data in the manuscript are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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