Research Article

Minor Prime Factorization for n -D Polynomial Matrices over Arbitrary Coefficient Field

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In this paper, we investigate two classes of multivariate (n -D) polynomial matrices whose coefficient field is arbitrary and the greatest common divisor of maximal order minors satisfy certain condition. Two tractable criteria are presented for the existence of minor prime factorization, which can be realized by programming and complexity computations. On the theory and application, we shall obtain some new and interesting results, giving some constructive computational methods for carrying out the minor prime factorization.

1. Introduction

In the polynomial approach, pioneered by Rosenbrock [1], matrices over $R[s]$, $s \equiv d/dt$ are used to represent linear systems of ordinary differential equations. For more general linear functional systems, e.g., partial differential systems or delay-differential systems, the resulting system matrices are multivariate. The problem of multivariate (n -D) polynomial matrix has attracted much attention over the past decades because of its diverse applications in n -D systems, network theory, Wiener-Hopf equations, controls, and signal processing ([2–8]); its factorizations have attracted much attention over the past decades because of their wide applications ([9–25]). Lin-Bose conjecture is one of the important problems of n -D polynomial matrix factorizations, which has been proved in [15–20]. It gave a sufficient condition for a matrix to have zero prime factorization or minor prime factorization. After that, Lin et al. proposed a tractable criterion for the existence of minor prime factorizations for a class of n -D polynomial matrices whose greatest common divisor of maximal order minors is $z_1 - f(z_2, \dots, z_n)$. They have presented a constructive method for carrying out the minor prime factorization when it exists [14], where the coefficient field is the complex number field. In this paper, we investigate the n -D polynomial matrices whose coefficient field is arbitrary and minor prime factorizations for

two classes of n -D polynomial matrices whose the greatest common divisor of maximal order minors is $z_1 - f(z_2, \dots, z_n)$ or $\prod_{k=1}^p (z_1 - z_{1k})$. In this case, because the coefficient field is an arbitrary field, the many previous results (lemmas) cannot be used directly. By giving some new key lemmas, two tractable criteria for the existence of minor prime factorization are obtained. These criteria can be judged by using “programming of reduced Gröbner basis.” Furthermore, we give constructive computational methods for carrying out the minor prime factorization.

2. Notations and Preliminaries

For an arbitrary field K , let $K[z_1, z_2, \dots, z_n]$ denote the set of polynomial ring in n variables $z_1, z_2, \dots, z_n$ with coefficients in K and $\bar{K}$ denote an algebraic closed field of K . For the convenience, $K[z_1, z_2, \dots, z_n]$ is denoted simply by $K[z]$; when $K = \mathbb{C}$ is the complex number field, $\mathbb{C}[z_1, z_2, \dots, z_n]$ is denoted simply by $\mathbb{C}[z]$. Let $K^{l \times m}[z]$ denote the set of $l \times m$ matrices with entries in $K[z]$ and $0_{r \times t}$ denote the $r \times t$ zero matrices. Throughout the paper, the argument (z) is omitted whenever its omission does not cause confusion.

Definition 2.1 (see [21]). Let $F(z) \in K^{l \times m}(z)$ be of normal rank r and $a_1(z), a_2(z), \dots, a_\beta(z)$ denote all the $r \times r$ minors

of the matrix $F(z)$. Extracting the greatest common divisor (g.c.d.), denoted by $d(F)$ or d , of $a_1(z), a_2(z), \dots, a_\beta(z)$ gives

$$a_i(z) = db_i(z), \quad i = 1, 2, \dots, \beta. \quad (1)$$

Then, $b_1(z), b_2(z), \dots, b_\beta(z)$ are called the reduced minors of $F(z)$.

Definition 2.2. Let $F(z) \in K^{l \times m}[z]$ be of normal full rank, then $F(z)$ is said to be zero left prime (zero right prime) if the $l \times l(m \times m)$ minors of $F(z)$ generate the unit ideal $K[z]$, which is simplified as that $F(z)$ is ZLP (ZRP).

First, the following two lemmas and a well-known result (Theorem 1 in [14]) are required.

Lemma 2.1. Let $g(z) \in K[z]$ and $f(z) \in K[z_2, \dots, z_n]$. Supposing that $g(f(z_2, \dots, z_n)) = 0$, then $z_1 - f(z_2, \dots, z_n)$ is a divisor of $g(z)$.

The proof is simple and omitted.

Lemma 2.2. Let $f_1(z), f_2(z), \dots, f_s(z) \in K[z]$ be polynomials, then they have no common zeros in $\bar{K}^n$ (are zero coprime) if and only if there are $u_1(z), u_2(z), \dots, u_s(z) \in K[z]$ such that

$$u_1(z)f_1(z) + u_2(z)f_2(z) + \dots + u_s(z)f_s(z) = 1, \quad (2)$$

i.e., $(f_1(z), f_2(z), \dots, f_s(z))$ is a ZLP row vector in $K^n[z]$, or $f_1(z), \dots, f_s(z)$ generate the unit ideal $K[z]$.

Proof 1. Necessity. It is clear that $f_i(z) \in \bar{K}[z]$, $i = 1, 2, \dots, s$. Let I_K and $I_{\bar{K}}$ denote the ideal generated by $f_1(z), f_2(z), \dots, f_s(z)$ in $K[z]$ and in $\bar{K}[z]$, respectively. From the assumption and Hilbert Zero Theorem, $I_{\bar{K}} = \bar{K}[z]$. Then the unit element 1 (in $\bar{K}$) may be represented by a Gröbner basis G of $I_{\bar{K}}$, i.e., 1 may be reduced to 0 by G . According to computational methods of G , we have that $G \subset K[z]$, then 1 can be represented by G in $K[z]$, i.e., there are $u_1(z), u_2(z), \dots, u_s(z) \in K[z]$ such that

$$u_1(z)f_1(z) + u_2(z)f_2(z) + \dots + u_s(z)f_s(z) = 1. \quad (3)$$

Sufficiency. We assume that $f_1(z), f_2(z), \dots, f_s(z)$ have common zeros in $\bar{K}^n$ and $z_0 = (z_{11}, z_{21}, \dots, z_{n1})$ is one of them. From (2), we have

$$u_1(z_0)f_1(z_0) + u_2(z_0)f_2(z_0) + \dots + u_s(z_0)f_s(z_0) = 0 \neq 1. \quad (4)$$

This is a contradiction. So $f_1(z), f_2(z), \dots, f_s(z)$ have no common zeros in $\bar{K}^n$.

Theorem 2.1 (see [14]). Let $F(z) \in C^{m \times l}[z]$, $d(F) = z_1 - f(z_2, \dots, z_n)$. Then a necessary and sufficient condition for $F(z)$ to admit a minor prime factorization $F(z) = F_0(z)G_0(z)$ with $F_0(z) \in K^{m \times l}[z]$, $G_0(z) \in K^{l \times l}[z]$, and $\det G_0(z) = d(F)$ is that there exists a full rank $(l-1) \times l$ submatrix $F_1(z_2, \dots, z_n) \triangleq F_1(f(z_2, \dots, z_n), z_2, \dots, z_n)$ of $F(f(z_2, \dots, z_n), z_2, \dots, z_n)$ such that the reduced minors of $F_1(z_2, \dots, z_n)$ are zero coprime.

3. Main Results

For the complex number field C and an n -D polynomial matrix $F(z_1, z_2, \dots, z_n) \in C[z]$, from Lemma 2.2, we see that the reduced minors of $F(z_1, z_2, \dots, z_n)$ are zero coprime if and only if the reduced minors of $F(z_1, z_2, \dots, z_n)$ generate the unit ideal $C[z]$. In the following, we investigate the case in which $F(z_1, z_2, \dots, z_n) \in K[z]$, where K is an arbitrary field (may be a finite field); we shall extend the range of applicability of Theorem 2.1 and present our main results. First, several key lemmas are presented.

Lemma 3.1. Let $Q \in K^{(l-1) \times l}[z]$ be of normal full rank, if the reduced minors of Q generate $K[z]$, then there is a ZLP matrix $w \in K^{l \times 1}[z]$ such that $Q \cdot w = 0_{(l-1) \times 1}$.

Proof 2. Since the reduced minors of Q generate the unit ideal $K[z]$, by Lin-Bose Theorem (Theorem 3.3 in [18]), Q can be factorized as $Q = Q_1 B_1$ such that $Q_1 \in R^{(l-1) \times (l-1)}$ is of normal full rank, and $B_1 \in K^{(l-1) \times l}$ is ZRP. By Quillen-Suslin Theorem [26], there is a unimodular $U_1 \in K^{l \times l}[z]$ such that B_1 is its first $l-1$ rows, i.e., $U_1 = (B_1^T \ V_1^T)^T$, where $V_1 \in K^{1 \times l}$. Let $U_1^{-1} = (V_2 \ w)$, where $V_2 \in R^{(l-1) \times l}$, $w \in R^{1 \times l}$, then $B_1 w = 0_{(l-1) \times 1}$. Obviously, w is ZLP. We have that $Q \cdot w = Q_1 \cdot B_1 \cdot w = Q_1 \cdot 0_{(l-1) \times l} = 0_{(l-1) \times 1}$.

Lemma 3.2. Let $F(z) \in K^{m \times l}[z]$ ($l \leq m$) and $d(F) = \prod_{s=1}^q (z_1 - f_s(z_2, \dots, z_n))$, where $f_i - f_j \neq 0$ for $i \neq j$. Then rank $F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$ is $l-1$, where $p = 1, 2, \dots, q$.

Proof 3. Let $F(z) = (f_{ij})_{m \times l}$, for $p \in \{1, 2, \dots, q\}$, we may write f_{ij} as

$$f_{ij} = q_{ij}(z_1 - f_p(z_2, \dots, z_n)) + b_{ij}, \quad (5)$$

where $q_{ij} \in K[z]$, $b_{ij} \in K[z_2, \dots, z_n]$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, l$.

Setting $A \triangleq (a_{ij})_{m \times l} \triangleq (q_{ij}(z_1 - f_p(z_2, \dots, z_n)))_{m \times l}$, $B \triangleq (b_{ij})_{m \times l}$, we have that $F = A + B$. Since $d(F) = \prod_{s=1}^q (z_1 - f_s(z_2, \dots, z_n))$, then there is an $l \times l$ minor Δ of $F(z)$ such that $z_1 - f_p(z_2, \dots, z_n)$ is a simple divisor of Δ . Let $F(j_1, j_2, \dots, j_l)$ denote an $l \times l$ submatrix consisting of the $j_1, j_2, \dots, j_l$ rows of F such that $z_1 - f_p(z_2, \dots, z_n)$ is a simple divisor of $\det F(j_1, j_2, \dots, j_l)$. Then

$$F(j_1, j_2, \dots, j_l) = \begin{pmatrix} a_{j_1 1} + b_{j_1 1} & a_{j_1 2} + b_{j_1 2} & \dots & a_{j_1 l} + b_{j_1 l} \\ a_{j_2 1} + b_{j_2 1} & a_{j_2 2} + b_{j_2 2} & \dots & a_{j_2 l} + b_{j_2 l} \\ \dots & \dots & \dots & \dots \\ a_{j_l 1} + b_{j_l 1} & a_{j_l 2} + b_{j_l 2} & \dots & a_{j_l l} + b_{j_l l} \end{pmatrix}. \quad (6)$$

Since $(z_1 - f_p(z_2, \dots, z_n)) | a_{ij}$, where $i = 1, 2, \dots, m, j = 1, 2, \dots, l$, b_{ij} is independent of z_1 , and $z_1 - f_p(z_2, \dots, z_n)$ is a simple divisor of $\det F(j_1, j_2, \dots, j_l)$, by the properties of determinant, some $k(1 \leq k \leq l)$ exists such that

$$\begin{vmatrix} b_{j_1 1} & \cdots & b_{j_1 k-1} & a_{j_1 k} & b_{j_1 k+1} & \cdots & b_{j_1 l} \\ b_{j_2 1} & \cdots & b_{j_2 k-1} & a_{j_2 k} & b_{j_2 k+1} & \cdots & b_{j_2 l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{j_l 1} & \cdots & b_{j_l k-1} & a_{j_l k} & b_{j_l k+1} & \cdots & b_{j_l l} \end{vmatrix} \neq 0. \quad (7)$$

From (7) and Laplace Theorem, there is a permutation $i_1 i_2 \cdots i_{l-1}$ of $j_1 j_2 \cdots j_l$ such that

$$\Delta_k \triangleq \begin{vmatrix} b_{i_1 1} & \cdots & b_{i_1 k-1} & b_{i_1 k+1} & \cdots & b_{i_1 l} \\ b_{i_2 1} & \cdots & b_{i_2 k-1} & b_{i_2 k+1} & \cdots & b_{i_2 l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i_{l-1} 1} & \cdots & b_{i_{l-1} k-1} & b_{i_{l-1} k+1} & \cdots & b_{i_{l-1} l} \end{vmatrix} \neq 0. \quad (8)$$

It is obvious that $F(f_p(z_2, \dots, z_n), z_2, \dots, z_n) = B$, then Δ_k is an $(l-1) \times (l-1)$ minor of $F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$. Since $d(F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)) = 0$, combined with (8), we have that $\text{rank } F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$ is $l-1$.

Lemma 3.3. Let $F(z) \in K^{l \times l}[z]$ and $\det(F) = \prod_{s=1}^q (z_1 - f_s(z_2, \dots, z_n))$, where $0 \neq f_i - f_j \in K$ for $i \neq j$. Then $\text{rank } F(f_p(z_2, \dots, z_n), z_2, \dots, z_n) = l-1$ for every $(z_2, \dots, z_n) \in \bar{K}^{n-1}$, and $\text{rank } F(f_p(z_2, \dots, z_n), z_2, \dots, z_n) = l-1$; furthermore, the $(l-1) \times (l-1)$ minors of $F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$ generate the unit ideal $K[z]$, where $p = 1, 2, \dots, q$.

Proof 4. For arbitrary but fixed $z_2 = z_{20}, \dots, z_n = z_{n0}$, where $(z_{20}, \dots, z_{n0}) \in \bar{K}^{n-1}$, let $z_{1p} = f_p(z_{20}, \dots, z_{n0})$, then $z_{11}, z_{12}, \dots, z_{1q}$ are distinguished. Assuming that $F(z_1, z_{20}, \dots, z_{n0}) = (f_{ij})_{l \times l}$, for $p \in \{1, 2, \dots, q\}$, we may write f_{ij} as

$$f_{ij} = q_{ij}(z_1 - z_{1p}) + b_{ij}, \quad (9)$$

where $q_{ij} \in \bar{K}[z_1]$, $b_{ij} \in \bar{K}$, $i, j = 1, 2, \dots, l$.

Setting $A \triangleq (a_{ij})_{l \times l} \triangleq (q_{ij}(z_1 - z_{1p}))_{l \times l}$, $B \triangleq (b_{ij})_{l \times l}$, then we have that $F(z_1, z_{20}, \dots, z_{n0}) = A + B$,

$$\det F(z_1, z_{20}, \dots, z_{n0}) = \begin{vmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1l} + b_{1l} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2l} + b_{2l} \\ \cdots & \cdots & \cdots & \cdots \\ a_{l1} + b_{l1} & a_{l2} + b_{l2} & \cdots & a_{ll} + b_{ll} \end{vmatrix}. \quad (10)$$

Since $\det F(z_1, z_{20}, \dots, z_{n0}) = \prod_{s=1}^q (z_1 - f_s(z_2, \dots, z_n)) | a_{ij}$, where $i, j = 1, 2, \dots, l$, and $b_{ij} \in \bar{K}$, $z_1 - z_{1p}$ is a simple divisor

of $\det F(z_1, z_{20}, \dots, z_{n0})$, by the properties of determinant, there is some $k(1 \leq k \leq l)$ such that

$$\begin{vmatrix} b_{11} & \cdots & b_{1k-1} & a_{1k} & b_{1k+1} & \cdots & b_{1l} \\ b_{21} & \cdots & b_{2k-1} & a_{2k} & b_{2k+1} & \cdots & b_{2l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{l1} & \cdots & b_{lk-1} & a_{lk} & b_{lk+1} & \cdots & b_{ll} \end{vmatrix} \neq 0. \quad (11)$$

From (11) and Laplace Theorem, there is a permutation $i_1 i_2 \cdots i_{l-1}$ of $12 \cdots l$ such that

$$\Delta_k \triangleq \begin{vmatrix} b_{i_1 1} & \cdots & b_{i_1 k-1} & b_{i_1 k+1} & \cdots & b_{i_1 l} \\ b_{i_2 1} & \cdots & b_{i_2 k-1} & b_{i_2 k+1} & \cdots & b_{i_2 l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i_{l-1} 1} & \cdots & b_{i_{l-1} k-1} & b_{i_{l-1} k+1} & \cdots & b_{i_{l-1} l} \end{vmatrix} \neq 0. \quad (12)$$

It is obvious that $F(f_p(z_{20}, \dots, z_{n0}), z_{20}, \dots, z_{n0}) = B$, then Δ_k is an $(l-1) \times (l-1)$ minor of $F(f_p(z_{20}, \dots, z_{n0}), z_{20}, \dots, z_{n0})$. Since $\det(F(f_p(z_{20}, \dots, z_{n0}), z_{20}, \dots, z_{n0})) = 0$, combined with (12), we have that $\text{rank } F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$ is $l-1$ for every $(z_2, \dots, z_n) \in \bar{K}^{n-1}$, i.e., the $(l-1) \times (l-1)$ minor of $F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$ have no common zeros in $\bar{K}^{n-1}$. Noting that $\det F(f_p(z_2, \dots, z_n), z_2, \dots, z_n) = 0$, then $\text{rank } F(f_p(z_2, \dots, z_n), z_2, \dots, z_n) = l-1$. Furthermore, by Lemma 2.2, we obtain that the $(l-1) \times (l-1)$ minors of $F(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$ generate the unit ideal $K[z]$.

Corollary 3.1. Let $F(z) \in K^{l \times l}[z]$ with $\det F(z) = \prod_{k=1}^q (z_1 - z_{1k})$, where $z_{11}, z_{12}, \dots, z_{1q} \in K$ are distinguished. Then $\text{rank } F(z_{1k}, z_2, \dots, z_n) = l$, the $(l-1) \times (l-1)$ minors of $F(z_{1k}, z_2, \dots, z_n)$ generate the unit ideal $K[z]$, where $k = 1, 2, \dots, q$.

Lemma 3.4. Let $G(z) \in K^{l \times l}[z]$, $\det G(z) = \prod_{s=1}^q (z_1 - f_s(z_2, \dots, z_n))$, where $0 \neq f_i - f_j \in K$ for $i \neq j$. Then, for $p \in \{1, 2, \dots, q\}$,

$$G(f_p(z_2, \dots, z_n), z_2, \dots, z_n) = \bar{G}(z) \cdot \text{diag}\{0, 1, \dots, 1\} \cdot V_p, \quad (13)$$

where $\bar{G}(z) \in K^{l \times l}[z_2, \dots, z_n]$, $V_p \in K^{l \times l}[z_2, \dots, z_n]$ is an invertible matrix.

Proof 5. Since $\det G(z) = \prod_{s=1}^q (z_1 - f_s(z_2, \dots, z_n))$, for $p \in \{1, 2, \dots, q\}$, by Lemma 3.3, $\text{rank } G(f_p(z_2, \dots, z_n), z_2, \dots, z_n) = l-1$ and the $(l-1) \times (l-1)$ minors of $G(f_p(z_2, \dots, z_n), z_2, \dots, z_n)$ generate $K[z]$. By Lemma 3.1, there is a ZLP vector $w_p \in K^{l \times 1}[z_2, \dots, z_n]$ such that $G \cdot F(f_p(z_2, \dots, z_n), z_2, \dots, z_n) \cdot w_p(z_2, \dots, z_n) = (0, \dots, 0)^T$. According to Quillen-

Suslin Theorem [26], a unimodular matrix $U_1 \in K^{l \times l}[z_2, \dots, z_n]$ can be constructed such that $w_p(z_2, \dots, z_n)$ is its first column, then

$$G(f_p(z_2, \dots, z_n), z_2, \dots, z_n) \cdot U_1 = \bar{G}(z) \cdot \text{diag} \{0, 1, \dots, 1\}, \quad (14)$$

where $\bar{G}(z) \in K^{l \times l}[z_2, \dots, z_n]$. Setting $V_p = U_1^{-1}$, then the desired results are obtained.

Lemma 3.5. *Let $F(z) \in K^{(l-1) \times l}[z]$, if $F(z)$ is of normal full rank and $F(z)$ has the following factorization:*

$$F(z) = A(z) \cdot \text{diag} \{0, 1, \dots, 1\} \cdot V(z), \quad (15)$$

where $A(z) \in K^{(l-1) \times l}[z]$, $V(z) \in K^{l \times l}[z]$ is a unimodular matrix. Then the reduced minors of $F(z)$ generate the unit ideal $K[z]$.

Proof 6. We partition $A(z)$, $V(z)$ as the following form, respectively,

$$\begin{aligned} A(z) &= (A_1(z) \quad A_2(z)), \\ V(z) &= (V_1^T(z) \quad V_2^T(z))^T, \end{aligned} \quad (16)$$

where $A_1(z) \in K^{(l-1) \times 1}[z]$, $A_2(z) \in K^{(l-1) \times (l-1)}[z]$, $V_1(z) \in K^{1 \times l}[z]$, $V_2(z) \in K^{(l-1) \times l}[z]$. Then $V_2(z)$ is ZLP, i.e., the reduced minors of $V_2(z)$ generate the unit ideal $K[z]$, and

$$F(z) = (0 \quad A_2(z)) \cdot (V_1^T(z) \quad V_2^T(z))^T = A_2(z) \cdot V_2(z). \quad (17)$$

So the reduced minors of $F(z)$ are the same as that of $V_2(z)$, which generate the unit ideal $K[z]$.

Theorem 3.1. *Let K be an arbitrary field, $F(z) \in K^{m \times l}[z]$ be of normal full rank ($l \leq m$), $d(F) = z_1 - f(z_2, \dots, z_n) \triangleq d(z)$. Then a necessary and sufficient condition for $F(z)$ to admit a minor prime factorization $F(z) = F_0(z)G_0(z)$ with $F_0(z) \in K^{m \times l}[z]$, $G_0(z) \in K^{l \times l}[z]$, and $\det G_0(z) = d(z)$ is that there exists a normal full rank $(l-1) \times l$ submatrix $F_1(z_2, \dots, z_n) \triangleq F_1(f(z_2, \dots, z_n), z_2, \dots, z_n)$ of $F(f(z_2, \dots, z_n), z_2, \dots, z_n)$ such that the reduced minors of $F_1(z_2, \dots, z_n)$ generate the unit ideal $K[z]$.*

Proof 7. Necessity. Supposing that $F(z)$ admits a minor prime factorization

$$F(z) = F_0(z)G_0(z), \quad (18)$$

with $F_0(z) \in K^{m \times l}[z]$, $G_0(z) \in K^{l \times l}[z]$, and $\det G_0(z) = z_1 - f(z_2, \dots, z_n) = d(z)$. From Lemma 3.4, $G_0(f(z_2, \dots, z_n), z_2,$

$\dots, z_n) = \bar{G}_0(z) \cdot \text{diag} \{0, 1, \dots, 1\} \cdot V$, where $V \in K[z_2, \dots, z_n]$ is unimodular. Combined with (18), we have

$$\begin{aligned} F(f(z_2, \dots, z_n), z_2, \dots, z_n) &= F_0(f(z_2, \dots, z_n), z_2, \dots, z_n) \\ &\quad \cdot G_0(f(z_2, \dots, z_n), z_2, \dots, z_n) \\ &= F_0(f(z_2, \dots, z_n), z_2, \dots, z_n) \\ &\quad \cdot \bar{G}_0(z) \cdot \text{diag} (0, 1, \dots, 1) \cdot V. \end{aligned} \quad (19)$$

By Lemma 3.2, we see that the normal rank of $F(f(z_2, \dots, z_n), z_2, \dots, z_n)$ is $l-1$. Hence, from (19), there exists a normal full rank $(l-1) \times l$ submatrix $F_1(z_2, \dots, z_n) \triangleq F_1(f(z_2, \dots, z_n), z_2, \dots, z_n)$ of $F(f(z_2, \dots, z_n), z_2, \dots, z_n)$ such that

$$F_1(z_2, \dots, z_n) = A(z) \cdot \text{diag} (0, 1, \dots, 1) \cdot V, \quad (20)$$

where $A(z)$ is a $(l-1) \times l$ submatrix of $F_0(f(z_2, \dots, z_n), z_2, \dots, z_n) \cdot \bar{G}_0(z)$. By Lemma 3.5, the reduced minors of $F_1(z_2, \dots, z_n)$ generate the unit ideal $K[z]$.

Sufficiency. Assuming that there exists a normal full rank $(l-1) \times l$ submatrix $F_1(z_2, \dots, z_n) \triangleq F_1(f(z_2, \dots, z_n), z_2, \dots, z_n)$ of $F(f(z_2, \dots, z_n), z_2, \dots, z_n)$ such that the reduced minors of $F_1(z_2, \dots, z_n)$ generate the unit ideal $K[z]$. According to Lemma 3.1, a ZRP vector $\omega \in K^{l \times 1}[z_2, \dots, z_n]$ can be constructed such that

$$F_1(z_2, \dots, z_n) \cdot \omega(z_2, \dots, z_n) = [0, \dots, 0]^T. \quad (21)$$

Since $F(f(z_2, \dots, z_n), z_2, \dots, z_n)$ is of normal rank $l-1$, by (21) and the fact that $F_1(z_2, \dots, z_n)$ is a normal full rank submatrix of $F(f(z_2, \dots, z_n), z_2, \dots, z_n)$, we have

$$F(f(z_2, \dots, z_n), z_2, \dots, z_n) \cdot \omega(z_2, \dots, z_n) = [0, \dots, 0]^T. \quad (22)$$

By Quillen-Suslin Theorem [26], a unimodular square matrix $V \in K^{l \times l}[z_2, \dots, z_n]$ can be constructed such that $\omega(z_2, \dots, z_n)$ is its first column; combined with Lemma 2.1, we have

$$F(z) \cdot V(z_2, \dots, z_n) = F_0(z) \cdot \text{diag} \{z_1 - f(z_2, \dots, z_n), 1, \dots, 1\}, \quad (23)$$

where $F_0(z) \in K^{m \times l}[z]$. It follows that

$$F(z) = F_0(z)G_0(z), \quad (24)$$

where $G_0(z) = \text{diag} \{z_1 - f(z_2, \dots, z_n), 1, \dots, 1\} \cdot V^{-1} \in K^{l \times l}[z]$ with $\det G_0(z) = (z_1 - f(z_2, \dots, z_n)) = d(F)$.

Remark 3.1. When $K = \mathbb{C}$ is the complex number field, according to Hilbert Zero Theorem, we see that the condition in Theorem 3.1 is the same as that of Theorem 2.1, and Theorem 3.1 is a generalization of Theorem 2.1.

According to Theorem 3.1, in order to check that $F(z)$ satisfies the conditions of Theorem 3.1, we first find all the $(l-1) \times l$ submatrices of $F(z_2, \dots, z_n, z_2, \dots, z_n)$, denoted by $F_1, F_2, \dots, F_s$, where $s = C_m^{l-1} = m(m-1) \cdots (m-l)/(l-1)!$. For every $F_i (1 \leq i \leq s)$, we compute the $(l-1) \times (l-1)$ minors of F_i , saying as $a_{i1}, a_{i2}, \dots, a_{il}$. Let the ideal $I_i = \langle a_{i1}, a_{i2}, \dots, a_{il} \rangle$, we compute the reduced Gröbner basis G_i of I_i . If G_i contains only a polynomial $d_i(z)$, by the following result (Theorem 3.2), the reduced minors of F_i generate the unit ideal $K[z]$.

Theorem 3.2. Let $d_i(z)$ be the greatest common divisor of $a_{i1}, a_{i2}, \dots, a_{il}$, i.e.,

$$a_{ij} = b_j d_i(z), j = 1, 2, \dots, l, \quad (25)$$

and G_i be the reduced Gröbner basis of the ideal $I_i = \langle a_{i1}, a_{i2}, \dots, a_{il} \rangle$. Then G_i contains only a polynomial $c_i(z)$ if and only if $b_1, b_2, \dots, b_l$ generate the unit $K[z]$.

Proof 8. Necessity. Assuming that $G_i = \{c_i(z)\}$, it is obvious that $c_i(z) | a_{ij}$, where $j = 1, 2, \dots, l$. Then $c_i(z) | d_i(z)$. Setting $d_i(z) = p_i(z) \cdot c_i(z)$, since $c_i(z) \in I_i$, we have that there are $u_1, u_2, \dots, u_l$ such that

$$\begin{aligned} c_i(z) &= u_1 a_{i1} + u_2 a_{i2} + \dots + u_l a_{il} = u_1 b_1 d_i(z) + u_2 b_2 d_i(z) \\ &\quad + \dots + u_l b_l d_i(z), \\ d_i(z) &= p_i(z) c_i(z) = p_i u_1 b_1 d_i(z) + p_i u_2 b_2 d_i(z) + \dots \\ &\quad + p_i u_l b_l d_i(z). \end{aligned} \quad (26)$$

So $1 = p_i u_1 b_1 + p_i u_2 b_2 + \dots + p_i u_l b_l$. Thus, $b_1, b_2, \dots, b_l$ generate the unit ideal $K[z]$.

Sufficiency. Assuming that $b_1, b_2, \dots, b_l$ generate the unit ideal $K[z]$, then the reduced Gröbner basis of the ideal $\langle b_1, b_2, \dots, b_l \rangle$ is $\{1\}$. Furthermore, the reduced Gröbner basis G_i of $I_i = \langle b_1 d_i(z), b_2 d_i(z), \dots, b_l d_i(z) \rangle$ is $\{d_i(z)\}$.

The reduced Gröbner basis algorithm of an ideal can be found in [16].

Remark 3.2. From the discussion above, we see that judging whether $F(z)$ satisfies the conditions of Theorem 3.1 or not, we only need to compute the reduced Gröbner basis of I_i , where $i = 1, 2, \dots, l$.

Lin et al. studied the matrix factorization for a class of matrix whose g.c.d. of maximal order minors has divisor $\prod_{k=1}^p (z_1 - z_{1k})$ ([13]); in the following, we study the minor prime factorization for this class of matrices and propose a tractable criterion.

Theorem 3.3. Let K be an arbitrary field, $F(z) \in K^{m \times l} (l \leq m)$ be of normal full rank, $d(F) = \prod_{k=1}^p (z_1 - z_{1k})$, where $z_{11}, z_{12}, \dots, z_{1p} \in K$ are distinguished. Then $F(z)$ admits a minor prime factorization $F(z) = F_0(z) G_0(z)$ with $F_0(z) \in K^{m \times l}[z]$,

$G_0(z) \in K^{l \times l}[z]$, and $\det G_0(z) = d(F)$ if and only if there are normal full rank $(l-1) \times l$ submatrices $F_k(z_2, \dots, z_n) \triangleq F_k(z_{1k}, z_2, \dots, z_n)$ of $F(z_{1k}, z_2, \dots, z_n)$ such that the reduced minors of $F_k(z_2, \dots, z_n)$ generate the unit ideal $K[z]$, where $k = 1, 2, \dots, p$.

Proof 9. Necessity. Supposing that $F(z)$ admits a minor prime factorization as follows:

$$F(z) = F_0(z) G_0(z), \quad (27)$$

where $F_0(z) \in K^{m \times l}[z]$, $G_0(z) \in K^{l \times l}[z]$, and $\det G_0(z) = d(F)$. For arbitrary but fixed $k (1 \leq k \leq p)$, from (27) and Lemma 3.4, we have

$$\begin{aligned} F(z_{1k}, z_2, \dots, z_n) &= F_0(z_{1k}, z_2, \dots, z_n) \cdot G_0(z_{1k}, z_2, \dots, z_n) \\ &= F_0(z_{1k}, z_2, \dots, z_n) \cdot \bar{G}(z_{1k}, z_2, \dots, z_n) \\ &\quad \cdot \text{diag}\{0, 1, \dots, 1\} \cdot V. \end{aligned} \quad (28)$$

By Lemma 3.2, we obtain that rank $F(z_{1k}, z_2, \dots, z_n)$ is $l-1$. Hence, from (28), there is a normal full rank $(l-1) \times l$ submatrix $F_k(z_2, \dots, z_n) \triangleq F_k(z_{1k}, z_2, \dots, z_n)$ of $F(z_{1k}, z_2, \dots, z_n)$ such that

$$F_k(z_2, \dots, z_n) = A(z) \cdot \text{diag}\{0, 1, \dots, 1\} \cdot V, \quad (29)$$

where $A(z)$ is an $(l-1) \times l$ submatrix of $F_0(z_{1k}, z_2, \dots, z_n) \cdot \bar{G}(z_{1k}, z_2, \dots, z_n)$. By Lemma 3.5, the reduced minors of $F_k(z_2, \dots, z_n)$ also generate the unit ideal $K[z]$.

Sufficiency. If there exists a normal full rank $(l-1) \times l$ submatrix $F_1(z_2, \dots, z_n) \triangleq F_1(z_{11}, z_2, \dots, z_n)$ of $F(z_{11}, z_2, \dots, z_n)$ such that the reduced minors of $F_1(z_2, \dots, z_n)$ generate the unit ideal $K[z]$, according to Lemma 3.1, we can construct a ZRP vector $w_1 \in K^{l \times l}[z_2, \dots, z_n]$ such that

$$F_1(z_2, \dots, z_n) \cdot w_1(z_2, \dots, z_n) = [0, \dots, 0]^T. \quad (30)$$

By Lemma 3.2, $F(z_{11}, z_2, \dots, z_n)$ is of normal rank $l-1$, from (30), and the fact that $F_1(z_2, \dots, z_n)$ is a normal full rank submatrix of $F(z_{11}, z_2, \dots, z_n)$, we have

$$F(z_{11}, z_2, \dots, z_n) \cdot w_1(z_2, \dots, z_n) = [0, \dots, 0]^T. \quad (31)$$

By Quillen-Suslin Theorem [26], a unimodular square matrix $V_1 \in K^{l \times l}[z_2, \dots, z_n]$ can be constructed such that w_1 is its first columns; combined with Lemma 2.1, we have

$$F(z) \cdot V_1(z_2, \dots, z_n) = A_1(z) \cdot \text{diag}(z_1 - z_{11}, 1, \dots, 1), \quad (32)$$

where $A_1(z) \in K^{m \times l}[z]$. It follows that

$$F(z) = A_1(z) \cdot G_1(z), \quad (33)$$

where $G_1(z) = \text{diag}(z_1 - z_{11}, 1, \dots, 1) \cdot V_1^{-1}$.

It is obvious that $\det G_1(z) = z_1 - z_{11}$, $d(A_1) = \prod_{k=2}^p (z_1 - z_{1k})$, and from (33), we have

$$F(z_{12}, z_2, \dots, z_n) = A_1(z_{12}, z_2, \dots, z_n) \cdot \text{diag} \{z_{12} - z_{11}, 1, \dots, 1\} \cdot V_1^{-1}, \quad (34)$$

$$F_2(z_{12}, z_2, \dots, z_n) = A_{12}(z_{12}, z_2, \dots, z_n) \cdot \text{diag} \{z_{12} - z_{11}, 1, \dots, 1\} \cdot V_1^{-1}, \quad (35)$$

where $A_{12}(z_{12}, z_2, \dots, z_n)$ is an $(l-1) \times l$ submatrix of $A_1(z_{12}, z_2, \dots, z_n)$. By Lemma 3.2, rank $F(z_{12}, z_2, \dots, z_n)$ is $l-1$ and from (34), rank $A_1(z_{12}, z_2, \dots, z_n)$ is also $l-1$. By the assumption, the reduced minors of $F_2(z_{12}, z_2, \dots, z_n)$ generate the unit ideal $K[z]$. Since $G_1(z_{12}, 1, \dots, 1) = \text{diag} \{z_{12} - z_{11}, 1, \dots, 1\} \cdot V_1^{-1}$ is unimodular, by (35) and Cauchy-Binet formula, we obtain that the reduced minors of $A_{12}(z_{12}, z_2, \dots, z_n)$ also generate the unit ideal $K[z]$.

Repeating the procedure above for $A_1(z)$ with respect to $(z_1 - z_{12})$, we have

$$A_1(z) = A_2(z) \cdot G_2(z) \quad (36)$$

or

$$F(z) = A_2(z) \cdot G_2(z) \cdot G_1(z), \quad (37)$$

such that $A_2(z) \in K^{m \times l}[z]$, $G_2(z) = \text{diag} (z_1 - z_{12}, 1, \dots, 1) \cdot V_2^{-1}$ with $\det G_2(z) = z_1 - z_{12}$, V_2 is a unimodular square matrix, $d(A_2) = \prod_{k=3}^p (z_1 - z_{1k})$. And

$$F(z_1, z_2, z_3) = \begin{pmatrix} 2z_1^4 + 4z_1^3z_2^2 + 2z_1^3z_3 + 4z_1z_3 + 3z_2^2z_3 + 4z_2^2 + 4z_2z_3 & 3z_2 + 4z_3 \\ 3z_1^2 + z_1z_2^2 + 2z_1z_2^2 + 4z_1z_2 + 4z_2^4 + 2z_2^2z_3 & 3z_1 \\ 4z_1^2z_2 + z_1z_2^2z_3 + 3z_1z_3^2 + 2z_1z_3 + 4z_2^2z_3 + z_2z_3 & 3z_1^2 + 2z_3 \end{pmatrix}. \quad (41)$$

It is easy to check that the greatest common divisor of the 2×2 minors of $F(z)$ is $d(F) = z_1 + 2z_2^2 + z_3 = z_1 - (3z_2^2 + 4z_3)$. By computing, we have

$$F(3z_2^2 + 4z_3, z_2, z_3) = \begin{pmatrix} 4z_2^2 + 4z_2z_3 + z_3^2 & 3z_2 + 4z_3 \\ 2z_2^3 + z_2^2z_3 + z_2z_3 + 3z_3^2 & 4z_2^2 + 2z_3 \\ 4z_2^5 + 3z_2^4z_3 + 4z_2^3z_3 + 3z_2^2z_3^2 + z_2z_3^2 + z_2z_3 & 2z_2^4 + 2z_2^2z_3 + 3z_3^3 + 2z_3 \end{pmatrix}. \quad (42)$$

Obviously, $F(3z_2^2 + 4z_3, z_2, z_3)$ is of normal rank 1, the reduced minors of the first row of $F(3z_2^2 + 4z_3, z_2, z_3)$ (a 1×2 submatrix of $F(3z_2^2 + 4z_3, z_2, z_3)$) are $3z_2 + 4z_3$ and 1

$$F_3(z_{13}, z_2, \dots, z_n) = A_{23}(z_{13}, z_2, \dots, z_n) \cdot \text{diag} (z_{13} - z_{12}, 1, \dots, 1) \cdot V_2^{-1} \cdot \text{diag} (z_{13} - z_{11}, 1, \dots, 1) \cdot V_1^{-1}, \quad (38)$$

where $A_{23}(z_{13}, z_2, \dots, z_n)$ is an $(l-1) \times l$ submatrix of $A_2(z_{13}, z_2, \dots, z_n)$.

Furthermore, since $F_3(z_{13}, z_2, \dots, z_n)$ is of normal rank $l-1$ and the reduced minors of $F_3(z_{13}, z_2, \dots, z_n)$ generate the unit ideal $K[z]$, by (38), then $A_{23}(z_{13}, z_2, \dots, z_n)$ is of normal rank $l-1$, and the reduced minors of $A_{23}(z_{13}, z_2, \dots, z_n)$ also generate the unit ideal $K[z]$.

Repeating the same procedure continuously with respect to $(z_1 - z_{1k})$, $k = 3, \dots, p$, we obtain that

$$F(z) = A_p(z)G_p(z) \cdots G_2(z)G_1(z) \quad (39)$$

or

$$F(z) = F_0(z)G_0(z), \quad (40)$$

where $F_0(z) = A_p(z) \in K^{m \times l}[z]$, $G_0(z) = \prod_{k=1}^p G_p(z)$, $\det G_0(z) = \prod_{k=1}^p (z_1 - z_{1k})$.

Remark 3.3. Similar to Remark 3.2, judging whether $F(z)$ satisfies the conditions of Theorem 3.1 or not, we only need to compute the reduced Gröbner bases of some ideals.

Example 3.1. Let $\mathbf{K} \triangleq \mathbf{Z}/(5)$ be the finite field, $F(z_1, z_2, z_3) \in \mathbf{K}[z_1, z_2, z_3]$, and

(since $4z_2^2 + 4z_2z_3 + z_3^2 = (3z_2 + 4z_3)(3z_2 + 4z_3)$), which generate the unit ideal $\mathbf{K}[z]$. By Theorem 3.1, there is a minor prime factorization for $F(z_1, z_2, z_3)$.

To construct such a factorization, let

$$U = \begin{pmatrix} 3 & 0 \\ z_2 + 3z_3 & 2 \end{pmatrix}, \quad (43)$$

we have

$$F(3z_2^2 + 4z_3, z_2, z_3) \cdot U = \begin{pmatrix} 0 & z_2 + 3z_3 \\ 0 & 3z_2^2 + 4z_3 \\ 0 & 4z_2^4 + 4z_2^2z_3 + z_3^2 + 4z_3 \end{pmatrix}, \quad (44)$$

which gives

$$F(z_1, z_2, z_3) \cdot U = \begin{pmatrix} z_1^3 + 2z_3 & z_2 + 3z_3 \\ 4z_1 + z_2^2 & z_1 \\ 4z_1z_3 + z_3 & z_1^2 + 4z_3 \end{pmatrix} \cdot \begin{pmatrix} z_1 + 2z_2^2 + z_3 & 0 \\ 0 & 1 \end{pmatrix}. \quad (45)$$

Let

$$F_0(z) = \begin{pmatrix} z_1^3 + 2z_3 & z_2 + 3z_3 \\ 4z_1 + z_2^2 & z_1 \\ 4z_1z_3 + z_3 & z_1^2 + 4z_3 \end{pmatrix}, \quad (46)$$

$$G_0(z) = \begin{pmatrix} z_1 + 2z_2^2 + z_3 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 4z_2 + 2z_3 & 3 \end{pmatrix},$$

we have

$$F(z_1, z_2, z_3) = F_0(z_1, z_2, z_3) \cdot G_0(z_1, z_2, z_3). \quad (47)$$

Remark 3.4. In our two main theorems (Theorem 3.1 and 3.3), $F(z) \in K^{m \times l} (l \leq m)$ are of normal full rank. In fact, our methods can also deal with some matrices not of full normal rank. When the normal rank of $F(z)$ is r and $r < l$, if the reduce minors of $F(z)$ generate the unit ideal $K[z]$, one has that $F(z) = Q_1(z)B_1(z)$ such that $Q_1 \in K^{m \times r}$ of normal full rank, and $B_1 \in K^{r \times l}$ is ZRP. In order to consider the factorization of $F(z)$, one needs to consider the factorization of $Q_1(z)$. See the following example.

Example 3.2. Let K be the rational number field, $A(z) \in K^{3 \times 3}[z_1, z_2, z_3]$, and

$$A(z) = \begin{pmatrix} 3z_1^2 - 4z_1z_2 - 4z_2 & 3z_1^2z_2 - 4z_1^2z_2 + z_1z_2 - 1 & -3z_1^2z_2 + 4z_1z_2 - z_2 \\ 0 & z_1 - 3 & 0 \\ z_1^2z_2z_3 - z_1z_2z_3 & z_1^3z_2z_3 - z_1^2z_2z_3 + 3z_3 & -z_1^2z_2z_3 + 2z_1z_3 \end{pmatrix}. \quad (48)$$

By computing the minors of $A(z)$, we have that $\text{rank } A(z) = 2$, $d(A) = (z_1 - 1)(z_1 - 3)$, the reduced minors of $A(z)$ generate the unit ideal $K[z]$. Using Theorem 3.2 in [18], we have that $A(z) = F(z)B(z)$, where

$$F(z) = \begin{pmatrix} 4z_1z_2 - 4z_2 & z_1 - 1 \\ 3z_1z_2 - 9z_2 & z_1 - 3 \\ -z_1^2z_2z_3 + z_1z_2z_3 & 3z_3 \end{pmatrix}, \quad (49)$$

$$B(z) = \begin{pmatrix} -1 & z_1 & 1 \\ 3z_2 & 1 + 3z_1z_2 & z - 3z_2 \end{pmatrix},$$

and $B(z)$ is ZLP. It is easy to check that the greatest common divisor of the 2×2 minors of $F(z)$ is $d(F) = z_1^2 - 4z_1 + 3 = (z_1 - 1)(z_1 - 3)$. Obviously, $F(1, z_2, z_3)$ is of normal rank 1, the reduced minors of the second row of $F(1, z_2, z_3)$ (a 1×2 submatrix of $F(1, z_2, z_3)$) are $-6z_2$

and -2 , which generate the unit ideal $K[z]$. $F(3, z_2, z_3)$ is also of normal rank 1, the reduced minors of the first row of $F(3, z_2, z_3)$ are $8z_2$ and 8 , which also generate the unit ideal $K[z]$. By Theorem 3.3, there is a minor prime factorization for $F(z_1, z_2, z_3)$.

To construct such a factorization, let

$$U_1 = \begin{pmatrix} 1 & 0 \\ -3z_2 & 1 \end{pmatrix}, \quad (50)$$

we have

$$F(1, z_2, z_3) = \begin{pmatrix} 0 & 0 \\ -6z_2 & -2 \\ 9z_2z_3 & 3z_3 \end{pmatrix},$$

$$F(1, z_2, z_3) \cdot U_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad (51)$$

which gives

$$F(z_1, z_2, z_3) \cdot U_1 = \begin{pmatrix} -3z_1z_2 + z_2 & z_1^2 - 1 \\ 0 & z_1 - 3 \\ -z_1z_2z_3 & 3z_3 \end{pmatrix} \cdot \begin{pmatrix} z_1 - 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (52)$$

Thus,

$$F(z_1, z_2, z_3) = \begin{pmatrix} -3z_1z_2 + z_2 & z_1^2 - 1 \\ 0 & z_1 - 3 \\ -z_1z_2z_3 & 3z_3 \end{pmatrix} \cdot \begin{pmatrix} z_1 - 1 & 0 \\ 3z_2 & 1 \end{pmatrix}. \quad (53)$$

Let

$$F_1(z_1, z_2, z_3) = \begin{pmatrix} -3z_1z_2 + z_2 & z_1^2 - 1 \\ 0 & z_1 - 3 \\ -z_1z_2z_3 & 3z_3 \end{pmatrix}, \quad (54)$$

$$U_2 = \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix},$$

we have

$$F_1(3, z_2, z_3) = \begin{pmatrix} -8z_2 & 8 \\ 0 & 0 \\ -3z_2z_3 & 3z_3 \end{pmatrix}, \quad (55)$$

$$F_1(3, z_2, z_3) \cdot U_2 = \begin{pmatrix} 0 & 8 \\ 0 & 0 \\ 0 & 3z_3 \end{pmatrix},$$

which gives

$$F_1(z_1, z_2, z_3) \cdot U_2 = \begin{pmatrix} z_1z_2 & z_1^2 - 1 \\ z_2 & z_1 - 3 \\ -z_2z_3 & 3z_3 \end{pmatrix} \cdot \begin{pmatrix} z_1 - 3 & 0 \\ 0 & 1 \end{pmatrix} \quad (56)$$

Thus,

$$F_1(z_1, z_2, z_3) = \begin{pmatrix} z_1z_2 & z_1^2 - 1 \\ z_2 & z_1 - 3 \\ -z_2z_3 & 3z_3 \end{pmatrix} \cdot \begin{pmatrix} z_1 - 3 & 0 \\ -z_2 & 1 \end{pmatrix},$$

$$F(z_1, z_2, z_3) = \begin{pmatrix} z_1z_2 & z_1^2 - 1 \\ z_2 & z_1 - 3 \\ -z_2z_3 & 3z_3 \end{pmatrix} \cdot \begin{pmatrix} z_1 - 3 & 0 \\ -z_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} z_1 - 1 & 0 \\ 3z_2 & 1 \end{pmatrix}. \quad (57)$$

Let

$$F_0(z) = \begin{pmatrix} z_1z_2 & z_1^2 - 1 \\ z_2 & z_1 - 3 \\ -z_2z_3 & 3z_3 \end{pmatrix}, \quad (58)$$

$$G_0(z) = \begin{pmatrix} z_1 - 3 & 0 \\ -z_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} z_1 - 1 & 0 \\ 3z_2 & 1 \end{pmatrix},$$

it is clear that $\det G_0(z) = z_1^2 - 4z_1 + 3 = d(A)$. We have

$$F(z_1, z_2, z_3) = F_0(z_1, z_2, z_3) \cdot G_0(z_1, z_2, z_3),$$

$$A(z_1, z_2, z_3) = F_0(z_1, z_2, z_3) \cdot G_0(z_1, z_2, z_3) \cdot B(z_1, z_2, z_3), \quad (59)$$

where $B(z_1, z_2, z_3)$ is ZLP.

The desired minor prime factorization for $A(z_1, z_2, z_3)$ is obtained.

4. Conclusions

In this paper, we have investigated the minor prime factorization for two classes of n -D polynomial matrices whose g.c.d. of maximal order minors is $z_1 - f(z_2, \dots, z_n)$ or $\prod_{k=1}^p (z_1 - z_{1k})$ with their entries in $K[z]$, where K is an arbitrary field. We have presented some key lemmas and two tractable criterions for the existence of minor prime factorizations. These criterions are easily checked by computing reduced Gröbner basis of the ideal generated by lower order minors, and the minor prime factorization can be carried out by Lin's constructive methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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