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*Research Article*

# Antiperiodic Solutions for Quaternion-Valued Shunting Inhibitory Cellular Neural Networks with Distributed Delays and Impulses

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This paper is concerned with quaternion-valued shunting inhibitory cellular neural networks (QVSICNNs) with distributed delays and impulses. By using a new continuation theorem of the coincidence degree theory, the existence of antiperiodic solutions for QVSICNNs is obtained. By constructing a suitable Lyapunov function, some sufficient conditions are derived to guarantee the global exponential stability of antiperiodic solutions for QVSICNNs. Finally, an example is given to show the feasibility of obtained results.

## 1. Introduction

The shunting inhibitory cellular neural networks (SICNNs) [1, 2] have found many applications in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Since all of these applications heavily rely on the dynamics of SICNNs and time delays are unavoidable in a realistic system [3–13], there have been extensive results about the dynamical behaviors of SICNNs with time delays [4–10]. Besides, a wide variety of evolutionary processes which exist universally in nature and many signal transmission processes in neural networks are often subject to abrupt changes at certain moments due to instantaneous perturbations which lead to impulsive effects. Also, the existence of impulses is frequently a source of instability, bifurcation, and chaos for neural networks [11–13]. Therefore, many researchers have investigated various dynamical behaviors of SICNNs with time delays and impulses [12, 13].

On the other hand, quaternion-valued neural networks, which can be seen as a generic extension of complex-valued neural networks (CVNNs) or real-valued neural networks (RVNNs), are much more complicated than CVNNs for their quaternion-valued states, quaternion-valued connection weights, and quaternion-valued activation functions. In the past decades, QVNNs have found many practical applications in aerospace and satellite tracking, processing of polarized waves, image processing, 3D geometrical affine transformation, spatial rotation [14, 15], color night vision [16], and so on. Due to so many practical applications, it is necessary to study the dynamics of QVNNs. At present, only a few of the dynamical behaviors of QVNNs have been studied [17–24]. For example, in [21, 22], the global $\mu$-stability criteria for QVNNs were studied, respectively; in [23], based on Mawhin's continuation theorem of coincidence degree theory, the existence of periodic solutions for QVNNs was established; in [24], the multiplicity of periodic solutions for QVNNs was discussed by employing Brouwer's and Leray-Schauder's fixed point theorems. However, to the best of our knowledge, the antiperiodic oscillation of QVVNs with time-varying delays and impulses has not been reported. Since the

existence and stability of antiperiodic solutions are an important topic in nonlinear differential equations and the signal transmission process of neural networks can often be described as an antiperiodic process, the antiperiodic oscillation of neural networks have been considered by many authors, see [13, 25–33]. So, it is necessary to study the antiperiodic solutions for QVNNs.

Motivated by the above, in this paper, we are concerned with the following QVSICNN with distributed delays and impulsive effects:

$$\begin{cases} \dot{x}_{pq}(t) = -a_{pq}(t)x_{pq}(t) - \sum\limits_{C_{kl}\in N_r(p,q)} C_{pq}^{kl}(t)\displaystyle\int_0^{+\infty} K_{pq}(u)x_{pq}(t)f(x_{kl}(t-u))du + T_{pq}(t), \quad t\geq 0, t\neq t_h, h\in\mathbb{N}, \\ \Delta x_{pq}(t_h) = x_{pq}(t_h^+) - x_{pq}(t_h^-) = I_{pqh}\left(x_{pq}(t_h)\right), \quad t=t_h, p=1,\dots,m, q=1,\dots,n, \end{cases} \tag{1}$$

where $pq\in\{11,12,\dots,1n,\dots,m1,m2,\dots,mn\}:=\mathscr{B}$; $C_{pq}$ denotes the cell at the $(p,q)$ position of the lattice. The $r$ neighborhood $N_r(p,q)$ of $C_{pq}$ is given by

$$N_r(p,q) = \left\{C_{pq} : \max\left(|k-p|,|l-q|\right)\leq r, pq\in\mathscr{B}\right\}. \tag{2}$$

$x_{pq}\in\mathbb{Q}$ is the activity of the cell $C_{pq}$; $T_{pq}:\mathbb{Q}\to\mathbb{Q}$ is the external input to $C_{pq}$, $a_{pq}(t)>0$ represents the passive decay rate of the cell activity, $C_{pq}^{kl}(t)\geq 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell $C_{pq}$, and the activity function $f:\mathbb{Q}\to\mathbb{Q}$ is a continuous function representing the output or firing rate of the cell $C_{pq}$; $K_{pq}(t)$ corresponds to the transmission delay kernel; and $\{t_h, h\in\mathbb{N}\}$ is a sequence of real numbers such that $0<t_1<t_2<\cdots<t_h\to\infty$ as $h\to\infty$, there exists a $\mu\in\mathbb{N}$ such that $t_{h+\mu}=t_h+\omega/2$, $I_{pq(h+\mu)}(x_{pq}(t_{h+\mu}))=-I_{pqh}(-x_{pq}(t_h))$, $h\in\mathbb{N}$. Without loss of generality, we also assume that $[0,\omega/2)\cap\{t_h : h\in\mathbb{N}\}=\{t_1,t_2,\dots,t_\mu\}$. For convenience, we denote $\bar{g}=\sup_{t\in\mathbb{R}}|g(t)|$ and $\underline{g}=\inf_{t\in\mathbb{R}}|g(t)|$, where $g$ is a bounded function.

The main purpose of this paper is to establish the existence of antiperiodic solutions to (1) by using a new continuation theorem of coincidence degree theory and by constructing a suitable Lyapunov function to obtain the global exponential stability of the antiperiodic solution. The results of this paper are completely new and supplement the previously known results.

Throughout this paper, we assume Hypotheses 1 to 4.

(H1) Let $x_{pq}=x_{pq}^R+ix_{pq}^I+jx_{pq}^J+kx_{pq}^K\in\mathbb{Q}$, assume that the activity function $f:\mathbb{Q}\to\mathbb{Q}$ of (1) can be expressed as

$$\begin{aligned} f\left(x_{pq}\right) = {} & f^R\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right) + \mathrm{i}f^I\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right) \\ & + jf^J\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right) + kf^K\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right), \end{aligned} \tag{3}$$

where $f^\nu\in C(\mathbb{R}^4,\mathbb{R})$, $\nu\in\{R,I,J,K\}:=E$; the impulsive operators $I_{pqh}:\mathbb{Q}\to\mathbb{Q}$ can be expressed as

$$\begin{aligned} I_{pqh}\left(x_{pq}\right) = {} & I_{pqh}^R\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right) \\ & + iI_{pqh}^I\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right) \\ & + jI_{pqh}^J\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right) \\ & + kI_{pqh}^K\left(x_{pq}^R,x_{pq}^I,x_{pq}^J,x_{pq}^K\right), \end{aligned} \tag{4}$$

where $I_{pqh}^\nu\in C(\mathbb{R}^4,\mathbb{R})$, $\nu\in E$, $pq\in\mathscr{B}$, and $h\in\mathbb{N}$, and the external inputs $T_{pq}:\mathbb{R}\to\mathbb{Q}$ can be expressed as

$$T_{pq}(t) = T_{pq}^R(t) + iT_{pq}^I(t) + jT_{pq}^J(t) + kT_{pq}^K(t), \tag{5}$$

where $T_{pq}^\nu\in C(\mathbb{R},\mathbb{R})$, $\nu\in E$, and $pq\in\mathscr{B}$.

(H2) Functions $a_{pq}\in C((\mathbb{R},(0,+\infty))$, $C_{pq}^{kl}\in C(\mathbb{R},\mathbb{R})$, and $T_{pq}\in C(\mathbb{R},\mathbb{Q})$, where $pq\in\mathscr{B}$, and there exists a constant $\omega>0$ such that

$$\begin{aligned} a_{pq}\left(t+\frac{\omega}{2}\right) &= a_{pq}(t), \\ C_{pq}^{kl}\left(t+\frac{\omega}{2}\right) &= C_{pq}^{kl}(t), \\ T_{pq}\left(t+\frac{\omega}{2}\right) &= -T_{pq}(t), \\ t &\in\mathbb{R}, \end{aligned} \tag{6}$$

and functions $f, I_{pqh}\in C(\mathbb{Q},\mathbb{Q})$, where $pq\in\mathscr{B}$ and $h\in\mathbb{N}$, satisfying

$$\begin{aligned} f(-x) &= f(x), \\ I_{pqh}(-x) &= -I_{pqh}(x), \\ x &\in\mathbb{Q}. \end{aligned} \tag{7}$$

(H3) For $pq \in \mathscr{B}$, $\int_0^{+\infty} |K_{pq}(u)| \mathrm{d}u < +\infty$.

(H4) For $pq \in \mathscr{B}$, $u^\nu$, and $v^\nu \in \mathbb{R}$, there exist positive constants $L_f^\nu$, $M_f^\nu$, and $I$ such that

$$\begin{aligned} &\left|f^\nu\left(u^R, u^I, u^J, u^K\right) - f^\nu\left(v^R, v^I, v^J, v^K\right)\right| \\ &\quad \le \left|L_f^R\right|u^R - v^R| + L_f^I|u^I - v^I| \\ &\qquad + L_f^J|u^J - v^J| + L_f^K|u^K - v^K|, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\left|f^\nu\left(u^R, u^I, u^J, u^K\right)\right| \le M_f^\nu, \\ &\left|I_{pqh}^\nu\left(u^R, u^I, u^J, \mathrm{u}^K\right)\right| \le I, \end{aligned} \tag{9}$$

where $f^\nu$ and $I_{pqh}^\nu(u)$ and $\nu \in E$ are mentioned in H1.

We will adopt the following notations:

$$\begin{aligned} T &= \max_{pq\in\mathscr{B}}\left\{\sup_{t\in[0,\omega]}\left|T_{pq}^\nu(t)\right|, \nu \in E\right\}, \\ a^- &= \min_{pq\in\mathscr{B}}\left\{\inf_{t\in[0,\omega]} a_{pq}(t)\right\}, \\ a^+ &= \max_{pq\in\mathscr{B}}\left\{\sup_{t\in[0,\omega]} a_{pq}(t)\right\}. \end{aligned} \tag{10}$$

The initial value of (1) is given by

$$x_{pq}(s) = \varphi_{pq}(s) \in \mathbb{Q}, \quad s \in (-\infty, 0], \tag{11}$$

where

$$\begin{aligned} &\varphi_{pq}(s) = \varphi_{pq}^R(s) + i\varphi_{pq}^I(s) + j\varphi_{pq}^J(s) + k\varphi_{pq}^K(s), \\ &\varphi_{pq}^\nu \in BC((-\infty, 0], \mathbb{R}), \\ &pq \in \mathscr{B}. \end{aligned} \tag{12}$$

The remaining part of this paper is organized as follows. In Section 3, some definitions are given. In Section 4, we obtain sufficient conditions for the existence of antiperiodic solutions of (1). In Section 5, the global exponential stability of the antiperiodic solution is studied. In Section 6, we give an example to illustrate the feasibility of the obtained results.

## 2. Preliminaries

The quaternion was invented in 1843 by Hamilton [34]. The skew field of the quaternion is denoted by

$$\mathbb{Q} := \{q = q_0 + iq_1 + jq_2 + kq_3\}, \tag{13}$$

where $q_0$, $q_1$, $q_2$, and $q_3$ are real numbers and the elements $i$,$j$, and $k$ obey Hamilton's multiplication rules:

$$\begin{aligned} &ij = -ji = k, \\ &jk = -kj = i, \\ &ki = -ik = j, \\ &i^2 = j^2 = k^2 = ijk = -1. \end{aligned} \tag{14}$$

In order to avoid the difficulty resulting from the noncommutativity of the quaternion multiplication, by Hamilton's rules and H1, we decompose (1) into the following systems:

$$\begin{cases} \dot{x}_{pq}^R(t) = -a_{pq}(t)x_{pq}^R(t) - \sum\limits_{C_{kl}\in N_r(p,q)} C_{pq}^{kl}(t)\left(\int_0^{+\infty} K_{pq}(u)f^R[t,u,x]dux_{pq}^R(t) - \int_0^{+\infty} K_{pq}(u)f^I[t,u,x]dux_{pq}^I(t)\right. \\ \qquad \left. - \int_0^{+\infty} K_{pq}(u)f^J[t,u,x]dux_{pq}^J(t) - \int_0^{+\infty} K_{pq}(u)f^K[t,u,x]dux_{pq}^K(t)\right) + T_{pq}^R(t) \triangleq F_{pqR}(t,x), \quad t \ne t_h, h \in \mathbb{N}, \\ \Delta x_{pq}^R(t_h) = I_{pqh}^R\left(x_{pq}(t_h)\right), \quad pq \in \mathscr{B}, h \in \mathbb{N}, \end{cases}$$

$$\begin{cases} \dot{x}_{pq}^I(t) = -a_{pq}(t)x_{pq}^I(t) - \sum\limits_{C_{kl}\in N_r(p,q)} C_{pq}^{kl}(t)\left(\int_0^{+\infty} K_{pq}(u)f^R[t,u,x]dux_{pq}^I(t) + \int_0^{+\infty} K_{pq}(u)f^I[t,u,x]dux_{pq}^R(t)\right. \\ \qquad \left. + \int_0^{+\infty} K_{pq}(u)f^J[t,u,x]dux_{pq}^K(t) - \int_0^{+\infty} K_{pq}(u)f^K[t,u,x]dux_{pq}^J(t)\right) + T_{pq}^I(t) \triangleq F_{pqI}(t,x), \quad t \ne t_h, h \in \mathbb{N}, \\ \Delta x_{pq}^I(t_h) = I_{pqh}^I\left(x_{pq}(t_h)\right), \quad pq \in \mathscr{B}, h \in \mathbb{N}, \end{cases}$$

$$\begin{cases} \dot{x}_{pq}^J(t) = -a_{pq}(t)x_{pq}^J(t) - \sum\limits_{C_{kl}\in N_r(p,q)} C_{pq}^{kl}(t)\left(\int_0^{+\infty} K_{pq}(u)f^R(t,u,x)dux_{pq}^J(t) - \int_0^{+\infty} K_{pq}(u)f^I[t,u,x]dux_{pq}^K(u)\right. \\ \qquad \left. + \int_0^{+\infty} K_{pq}(u)f^J[t,u,x]dux_{pq}^R(t) + \int_0^{+\infty} K_{pq}(u)f^K[t,u,x]dux_{pq}^I(t)\right) + T_{pq}^J(t) \triangleq F_{pqJ}(t,x), \quad t \ne t_h, h \in \mathbb{N}, \\ \Delta x_{pq}^J(t_h) = I_{pqh}^J\left(x_{pq}(t_h)\right), \quad pq \in \mathscr{B}, h \in \mathbb{N}, \end{cases}$$

$$\begin{cases} \dot{x}_{pq}^{K}(t) = -a_{pq}(t)x_{pq}^{K}(t) - \sum_{C_{kl}\in N_r(p,q)} C_{pq}^{kl}(t)\left(\int_0^{+\infty} K_{pq}(u)f^R[t,u,x]dux_{pq}^{K}(t) + \int_0^{+\infty} K_{pq}(u)f^I[t,u,x]dux_{pq}^{J}(t)\right. \\ \qquad \left. - \int_0^{+\infty} K_{pq}(u)f^J[t,u,x]dux_{pq}^{I}(t) + \int_0^{+\infty} K_{pq}(u)f^K[t,u,x]dux_{pq}^{R}(t)\right) + T_{pq}^{K}(t) \triangleq F_{pqK}(t,x), \quad t \neq t_h, h \in \mathbb{N}, \\ \Delta x_{pq}^{K}(t_h) = I_{pqh}^{K}\left(x_{pq}(t_h)\right), \quad pq \in \mathscr{B}, h \in \mathbb{N}, \end{cases} \tag{15}$$

where $x_{pq}^R + ix_{pq}^I + jx_{pq}^J + kx_{pq}^K = x_{pq}$, $f^\nu[t,u,x] \triangleq f^\nu(x_{kl}^R(t-u), x_{kl}^I(t-u), x_{kl}^J(t-u), x_{kl}^K(t-u))$, and $I_{pqh}^\nu(x_{pq}(t_h)) \triangleq I_{pqh}^\nu(x_{pq}^R(t_h), x_{pq}^I(t_h), x_{pq}^J(t_h), x_{pq}^K(t_h))$, $pq \in \mathscr{B}$, $h \in \mathbb{N}$, and $\nu \in E$. That is, (1) can be decomposed as the following real-valued system:

$$\begin{cases} \dot{x}_{pq}^{\nu}(t) = F_{pq\nu}(t,x), \quad t \neq t_h, h \in \mathbb{N}, \\ \Delta x_{pq}^{\nu}(t_h) = I_{pqh}^{\nu}\left(x_{pq}(t_h)\right), \quad pq \in \mathscr{B}, h \in \mathbb{N}, \nu \in E, \end{cases} \tag{16}$$

with the initial values

$$x_{pq}^{\nu}(s) = \varphi_{pq}^{\nu}(s), \quad s \in (-\infty, 0], h \in \mathbb{N}, pq \in \mathscr{B}, \nu \in E, \tag{17}$$

where $\varphi_{pq}^{\nu} : (-\infty, 0] \to \mathbb{R}$ is continuous and bounded.

*Definition 1.* A piecewise continuous function $x = (x_{11}^R, x_{11}^I, x_{11}^J, x_{11}^K, \ldots, x_{mn}^R, x_{mn}^I, x_{mn}^J, x_{mn}^K)^T : \mathbb{R} \to \mathbb{R}^{4nm}$ is said to be a solution of (16), if

(i) $x(s) = \varphi(s)$ for $s \in (-\infty, 0]$, where $\varphi = (\varphi_{11}^R, \varphi_{11}^I, \varphi_{11}^J, \varphi_{11}^K, \ldots, \varphi_{mn}^R, \varphi_{mn}^I, \varphi_{mn}^J, \varphi_{mn}^K)^T$, $\varphi_{pq}^{\nu} \in C((-\infty, 0], \mathbb{R})$, $\nu \in E$, and $pq \in \mathscr{B}$;

(i) $x(t)$ satisfies (16) for $t \geq 0$;

(ii) $x(t)$ is continuous everywhere except for some $t_h$ and left continuous at $t = t_h$, and the right limit $x(t_h^+)$ exists for $h \in \mathbb{N}$.

*Definition 2.* A solution $x$ of system (16) is said to be the $\omega/2$-antiperiodic solution of (16), if

$$\begin{cases} x\left(t + \frac{\omega}{2}\right) = -x(t), \quad t \neq t_h, \\ x\left(\left(t_h + \frac{\omega}{2}\right)^+\right) = -x(t_h^+), \quad h \in \mathbb{N}. \end{cases} \tag{18}$$

*Definition 3.* Let $x = (x_{11}^R, x_{11}^I, x_{11}^J, x_{11}^K, \ldots, x_{mn}^R, x_{mn}^I, x_{mn}^J, x_{mn}^K)^T$ be a solution of (16) with the initial value $\varphi = (\varphi_{11}^R, \varphi_{11}^I, \varphi_{11}^J, \varphi_{11}^K, \ldots, \varphi_{mn}^R, \varphi_{mn}^I, \varphi_{mn}^J, \varphi_{mn}^K)^T$ and $y = (y_{11}^R, y_{11}^I, y_{11}^J, y_{11}^K, \ldots, y_{mn}^R, y_{mn}^I, y_{mn}^J, x_{mn}^K)^T$ be an arbitrary solution of (16) with the initial value $\psi = (\psi_{11}^R, \psi_{11}^I, \psi_{11}^J, \psi_{11}^K, \ldots, \psi_{mn}^R, \varphi_{mn}^I, \varphi_{mn}^J, \varphi_{mn}^K)^T$. If there exist constants $\lambda > 0$ and $M > 0$ such that

$$\|x(t) - y(t)\|_0 \leq M\|\varphi - \psi\|e^{-\lambda t}, \quad \forall t > 0, \tag{19}$$

then the solution $x$ of (16) is said to be globally exponentially stable, where

$$\begin{aligned} \|y(t) - x(t)\|_0 &= \max_{pq\in\mathscr{B}}\left\{\max_{\nu\in E}\left|y_{pq}^{\nu}(t) - x_{pq}^{\nu}(t)\right|\right\}, \\ \|\varphi - \psi\| &= \sum_{pq\in\mathscr{B}} \sup_{s\in(-\infty,0]}\left(\left|\varphi_{pq}^R(s) - \psi_{pq}^R(s)\right|\right. \\ &\quad + \left|\varphi_{pq}^I(s) - \psi_{pq}^I(s)\right| + \left|\varphi_{pq}^J(s) - \psi_{pq}^J(s)\right| \\ &\quad \left. + \left|\varphi_{pq}^K(s) - \psi_{pq}^K(s)\right|\right). \end{aligned} \tag{20}$$

*Remark 1.* If $x = (x_{11}^R, x_{11}^I, x_{11}^J, x_{11}^K, \ldots, x_{mn}^R, x_{mn}^I, x_{mn}^J, x_{mn}^K)^T$ is an $\omega/2$-antiperiodic solution to (16), then $u = (x_{11}, \ldots, x_{mn})^T$, where $x_{pq} = x_{pq}^R + ix_{pq}^I + jx_{pq}^J + kx_{pq}^K, pq \in E$ must be an $\omega/2$-antiperiodic solution to (1). Thus, the problem of finding an $\omega/2$-antiperiodic solution for (1) reduces to finding one for the system of (16). For considering the stability of the solution of (1), we just need to consider the stability of the solutions of (16).

## 3. The Existence of Antiperiodic Solutions

In this section, based on a new continuation theorem of coincidence degree theory, we shall study the existence of antiperiodic solutions of (1).

**Lemma 1.** *[35] Let $X$ and $Y$ be Banach spaces, and let $L : \mathrm{Dom}\, L \subset X \to Y$ be linear and $N : X \to Y$ be continuous. Assume that $L$ is one-to-one and $K := L^{-1}N$ is compact. Furthermore, assume that there exists a bounded and open subset $\Omega \subset X$ with $0 \in \Omega$ such that equation $Lu = \lambda Nu$ has no solutions in $\partial\Omega \cap \mathrm{Dom}\, L$ for any $\lambda \in (0, 1)$. Then the problem $Lu = Nu$ has at least one solution in $\bar{\Omega}$.*

**Theorem 1.** *Assume that H1–H4 holds. Furthermore, suppose that*

*A1. If one has*

$$\Gamma = \frac{a^{-}\omega(1-a^{+}\omega)}{1+a^{-}\omega} - \left(M_f^R + M_f^I + M_f^J + M_f^K\right) \cdot \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \int_0^{+\infty} \left|K_{pq}(u)\right| du > 0, \tag{21}$$

*then (1) has at least one ω/2-antiperiodic solution.*

*Proof 1.* One has

$$PC\left(\mathbb{R}, \mathbb{R}^{4mn}\right) = \left\{ x = \left(x_{11}^R, x_{11}^I, x_{11}^J, x_{11}^K, \dots, x_{mn}^R, x_{mn}^I, x_{mn}^J, x_{mn}^K\right)^T : \mathbb{R} \to \mathbb{R}^{4mn}, x\Big|_{(t_h, t_{h+1}]} \in C\left((t_h, t_{h+1}, ]\mathbb{R}^{4n}\right), x(t_h^+), x(t_h) \text{exist and } x(t_h^-) = x(t_h), h \in \mathbb{N} \right\}. \tag{22}$$

Set

$$X = \left\{ x : x = \left(x_{11}^R, x_{11}^I, x_{11}^J, x_{11}^K, \dots, x_{mn}^R, x_{mn}^I, x_{mn}^J, x_{mn}^K\right)^T \in PC\left(\mathbb{R}, \mathbb{R}^{4mn}\right), x\left(t + \frac{\omega}{2}\right) = -x(t) \right\}, \tag{23}$$

and

$$Y = X \times \mathbb{R}^{4mn\times\mu}. \tag{24}$$

Then $X$ is a Banach space with the norm $\|x\|_X = \max_{pq\in\mathscr{B}}\{\max_{\nu\in E}\{\sup_{t\in\mathbb{R}}|x_{pq}^{\nu}(t)|\}\}$, and $Y$ is also a Banach space with the norm $\|y\|_Y = \|x\|_X + \|u\|, x \in X, u \in \mathbb{R}^{4mn\times\mu}$, where $\|\cdot\|$ is any norm of $\mathbb{R}^{4mn\times\mu}$.

Define a linear operator $L : \operatorname{Dom} L \subset X \to Y$ by

$$Lx = \left(\dot{x}, \Delta x(t_1), \Delta x(t_2), \dots, \Delta x\left(t_\mu\right)\right), \tag{25}$$

where $\operatorname{Dom} L = \{x : x \in X, \dot{x} \in X\}$ and a continuous operator $N : X \to Y$ by

$$Nx = \left( \begin{pmatrix} F_{11R}(t,x) \\ F_{11I}(t,x) \\ F_{11J}(t,x) \\ F_{11K}(t,x) \\ \vdots \\ F_{mnR}(t,x) \\ F_{mnI}(t,x) \\ F_{mnJ}(t,x) \\ F_{mnK}(t,x) \end{pmatrix}, \begin{pmatrix} I_{111}^R(x_{11}(t_1)) \\ I_{111}^I(x_{11}(t_1)) \\ I_{111}^J(x_{11}(t_1)) \\ I_{111}^K(x_{11}(t_1)) \\ \vdots \\ I_{mn1}^R(x_{mn}(t_1)) \\ I_{mn1}^I(x_{mn}(t_1)) \\ I_{mn1}^J(x_{mn}(t_1)) \\ I_{mn1}^K(x_{mn}(t_1)) \end{pmatrix}, \begin{pmatrix} I_{112}^R(x_{11}(t_2)) \\ I_{112}^I(x_{11}(t_2)) \\ I_{112}^J(x_{11}(t_2)) \\ I_{112}^K(x_{11}(t_2)) \\ \vdots \\ I_{mn2}^R(x_{mn}(t_2)) \\ I_{mn2}^I(x_{mn}(t_2)) \\ I_{mn2}^J(x_{mn}(t_2)) \\ I_{mn2}^K(x_{mn}(t_2)) \end{pmatrix}, \dots, \begin{pmatrix} I_{11\mu}^R(x_{11}(t_\mu)) \\ I_{11\mu}^I(x_{11}(t_\mu)) \\ I_{11\mu}^J(x_{11}(t_\mu)) \\ I_{11\mu}^K(x_{11}(t_\mu)) \\ \vdots \\ I_{mn\mu}^R(x_{mn}(t_\mu)) \\ I_{mn\mu}^I(x_{mn}(t_\mu)) \\ I_{mn\mu}^J(x_{mn}(t_\mu)) \\ I_{mn\mu}^K(x_{mn}(t_\mu)) \end{pmatrix} \right). \tag{26}$$

It is easy to see that

$$\ker L = \{0\},$$
$$\operatorname{Im} L = \left\{ y = (g, d_{1,}, \dots, d_{11})^T \in \mathbb{Y} : \int_0^\omega g(s)ds = 0 \right\} = \mathbb{Y}. \tag{27}$$

Hence, $L$ is reversible. Denote by $L^{-1} : \operatorname{Im} L \to \operatorname{Dom} L$ the inverse of $L$, one has

$$\left(L^{-1}y\right)(t) = \int_0^t g(s)ds - \frac{1}{2}\int_0^{\omega/2} g(s)ds + \sum_{t_h<t} d_h - \frac{1}{2}\sum_{h=1}^{u} d_h, \quad t \in \mathbb{R}, \tag{28}$$

where $d_{\mu+h} = d_h$ for all $1 \le h \le \mu$. Let $K := L^{-1}N$, by applying the Arzela-Ascoli theorem, we know that $K$ is compact. Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have

$$\begin{cases} \dot{x}_{pq}^{\nu}(t) = \lambda F_{pq\nu}(t,x), & t \ge 0, t \ne t_h, \\ \Delta x_{pq}^{\nu}(t_h) = \lambda I_{pqh}^{\nu}\left(x_{pq}(t_h)\right), & pq \in \mathscr{B}, h \in \mathbb{N}, \nu \in E. \end{cases} \tag{29}$$

Suppose that $x = (x_{11}^R, x_{11}^I, x_{11}^J, x_{11}^K, \dots, x_{mn}^R, x_{mn}^I, x_{mn}^J, x_{mn}^K)^T \in X$ is a solution of (29) for a certain $\lambda \in (0,1)$, set $t_0 = t_0^+ = 0, t_{2q+1} = \omega$, then we have

$$\begin{aligned}
\int_0^{\omega}\left|\dot{x}_{pq}^{R}(t)\right|dt &= \sum_{k=1}^{2q+1}\int_{t_{h-1}^{+}}^{t_h}\left|\dot{x}_{pq}^{R}(t)\right|dt + \sum_{h=1}^{2q}\left|I_{pqh}^{R}\left(x_{pq}(t_h)\right)\right| \\
&\le \int_0^{\omega}\left|a_{pq}(t)x_{pq}^{R}(t)\right|dt + \int_0^{\omega}\left|\sum_{C_{kl}\in N_r(p,q)} C_{pq}^{kl}(t)\right. \\
&\quad \cdot \left(\int_0^{+\infty} K_{pq}(u)x_{pq}^{R}(t)f^{R}[t,u,x]du\right. \\
&\quad - \int_0^{+\infty} K_{pq}(u)x_{pq}^{I}(t)f^{I}[t,u,x]du \\
&\quad - \int_0^{+\infty} K_{pq}(u)x_{pq}^{J}(t)f^{J}[t,u,x]du\Big| \\
&\quad \left. - \int_0^{+\infty} K_{pq}(u)x_{pq}^{K}(t)f^{K}[t,u,x]du\right)\Bigg|dt \\
&\quad + \int_0^{+\infty}\left|T_{pq}^{R}(t)\right|dt + \sum_{h=1}^{2q}\left|I_{pqh}^{R}\left(x_{pq}(t_h)\right)\right| \\
&\le \int_0^{\omega}\left|a_{pq}(t)x_{pq}^{R}(t)\right|dt + \int_0^{\omega}\sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl} \\
&\quad \cdot \left(\int_0^{+\infty}\left|K_{pq}(u)\right|\left|x_{pq}^{R}(t)\right|\left|f^{R}[t,u,x]\right|du\right. \\
&\quad + \int_0^{+\infty}\left|K_{pq}(u)\right|\left|x_{pq}^{I}(t)\right|\left|f^{I}[t,u,x]\right|du \\
&\quad + \int_0^{+\infty}\left|K_{pq}(u)\right|\left|x_{pq}^{J}(t)\right|\left|f^{J}[t,u,x]\right|du \\
&\quad \left. + \int_0^{+\infty}\left|K_{pq}(u)\right|\left|x_{pq}^{K}(t)\right|\left|f^{K}[t,u,x]\right|du\right)dt \\
&\quad + \int_0^{\omega}\left|T_{pq}^{R}(t)\right|dt + \sum_{h=1}^{2q}\left|I_{pqh}^{R}\left(x_{pq}(t_h)\right)\right| \\
&\le a^{+}\omega\left|x_{pq}^{R}\right|_{\infty} + \int_0^{\omega}\sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl} \\
&\quad \cdot \left(\int_0^{+\infty}\left|K_{pq}(u)\right|du\|x\|_X M_f^{R}\right. \\
&\quad + \int_0^{+\infty}\left|K_{pq}(u)du\|x\|_X M_f^{I}\right. \\
&\quad + \int_0^{+\infty}\left|K_{pq}(u)\right|du\|x\|_X M_f^{J} \\
&\quad \left. + \int_0^{+\infty}\left|K_{pq}(u)\right|du\|x\|_X M_f^{K}\right)dt \\
&\quad + \int_0^{\omega}\left|T_{pq}^{R}(t)\right|dt + \sum_{h=1}^{2q}\left|I_{pqh}^{R}\left(x_{pq}(t_h)\right)\right| \\
&\le a^{+}\omega\left|x_{pq}^{R}\right|_{\infty} + \omega\left(M_f^{R}+M_f^{I}+M_f^{J}+M_f^{K}\right) \\
&\quad \|x\|_X \sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl}\int_0^{+\infty}\left|K_{pq}(u)\right|du + \omega T \\
&\quad + 2qI, \quad pq\in\mathscr{B}.
\end{aligned} \tag{30}$$

Repeating the above procession, for $\nu = I, J, K$, we can obtain that

$$\begin{aligned}
\int_0^{\omega}\left|\dot{x}_{pq}^{\nu}(t)\right|dt &\le a^{+}\omega\left|x_{pq}^{\nu}\right|_{\infty} + \omega\left(M_f^{R}+M_f^{I}+M_f^{J}+M_f^{K}\right) \\
&\quad \cdot \|x\|_X \sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl}\int_0^{+\infty}\left|K_{pq}(u)\right|du \\
&\quad + \omega T + 2qI, \quad pq\in\mathscr{B}.
\end{aligned} \tag{31}$$

Integrating both sides of (29) over the interval $[0,\omega]$, we can obtain

$$\int_0^{\omega}F_{pq\nu}(t,x)dt + \sum_{h=1}^{2q}I_{pqh}\left(x_{pqh}(t_h)\right) = 0, \quad pq\in\mathscr{B}, \nu\in E. \tag{32}$$

Hence,

$$\begin{aligned}
&\left|\int_0^{\omega}a_{pq}(t)x_{pq}^{R}(t)dt\right| \\
&= \left|\int_0^{\omega}\left[\sum_{C_{kl}\in N_r(p,q)}C_{pq}^{kl}(t)\left(\int_0^{+\infty}K_{pq}(u)f^{R}[t,u,x]dux_{pq}^{R}(t)\right.\right.\right. \\
&\quad - \int_0^{+\infty}K_{pq}(u)f^{I}[t,u,x]dux_{pq}^{I}(t) \\
&\quad - \int_0^{+\infty}K_{pq}(u)f^{J}[t,u,x]dux_{pq}^{J}(t) \\
&\quad \left.\left. - \int_0^{+\infty}K_{pq}(u)f^{K}[t,u,x]dux_{pq}^{K}(t)\right) \right. \\
&\quad \left.\left. + T_{pq}^{R}(t)\right]dt + \sum_{h=1}^{2q}I_{pqh}^{R}\left(x_{pq}(t_h)\right)\right| \\
&\le \int_0^{\omega}\left|\sum_{C_{kl}\in N_r(p,q)}C_{pq}^{kl}(t)\left(\int_0^{+\infty}K_{pq}(u)f^{R}[t,u,x]dux_{pq}^{R}(t)\right.\right. \\
&\quad - \int_0^{+\infty}K_{pq}(u)f^{I}[t,u,x]dux_{pq}^{I}(t) \\
&\quad - \int_0^{+\infty}K_{pq}(u)f^{J}[t,u,x]dux_{pq}^{J}(t)\Big| \\
&\quad \left.\left. - \int_0^{+\infty}K_{pq}(u)f^{K}[t,u,x]dux_{pq}^{K}(t)\right)\right|dt \\
&\quad + \int_0^{\omega}\left|T_{pq}^{R}(t)\right|dt + \sum_{h=1}^{2q}\left|I_{pqh}^{R}\left(x_{pq}(t_h)\right)\right| \\
&\le \int_0^{\omega}\sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl}\left(\int_0^{+\infty}\left|K_{pq}(u)\right|du\left(M_f^{R}+M_f^{I}+M_f^{J}\right.\right. \\
&\quad \left.\left. + M_f^{K}\right)\|x\|_X\right)dt + \int_0^{\omega}\left|T_{pq}^{R}(t)\right|dt + \sum_{h=1}^{2q}\left|I_{pqh}^{R}\left(x_{pq}(t_h)\right)\right| \\
&\le \omega\left(M_f^{R}+M_f^{I}+M_f^{J}+M_f^{K}\right)\|x\|_X \sum_{C_{kl}\in N_r(p,q)} \\
&\quad \cdot \bar{C}_{pq}^{kl}\int_0^{+\infty}\left|K_{pq}(u)\right|du + \omega T + 2qI, \quad pq\in\mathscr{B}.
\end{aligned} \tag{33}$$

Repeating the above procession, for $\nu = I, J, K$, we have

$$\left|\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt\right| \le \omega\left(M_f^R + M_f^I + M_f^J + M_f^K\right) \cdot \|x\|_X \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl}\int_0^{+\infty}\left|K_{pq}(u)\right|du + \omega T + 2qI, \quad pq \in \mathscr{B}. \tag{34}$$

Since for any $t_1, t_2 \in [0,\omega]$, and $pq \in \mathscr{B}$, we have

$$\begin{aligned} x_{pq}^\nu(t) &\le x_{pq}^\nu(t_1) + \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt, \\ x_{pq}^\nu(t) &\ge x_{pq}^\nu(t_2) - \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt, \end{aligned} \tag{35}$$

then for any $\xi_{pq}^\nu, \eta_{pq}^\nu \in [0,\omega]$, $pq \in \mathscr{B}$, and $\nu \in E$, we obtain

$$\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt \le \int_0^\omega a_{pq}(t)x_{pq}^\nu\left(\xi_{pq}^\nu\right)dt + \int_0^\omega a_{pq}(t)\left(\int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt\right)dt, \tag{36}$$

$$\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt \ge \int_0^\omega a_{pq}(t)x_{pq}^\nu\left(\eta_{pq}^\nu\right)dt - \int_0^\omega a_{pq}(t)\left(\int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt\right)dt. \tag{37}$$

Dividing by $\int_0^\omega a_{pq}(t)dt$ on the both sides of (36) and (37), respectively, we have

$$\begin{aligned} x_{pq}^\nu\left(\xi_{pq}^\nu\right) &\ge \frac{1}{\int_0^\omega a_{pq}(t)dt}\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt - \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt, \quad pq \in \mathscr{B}, \nu \in E, \\ x_{pq}^\nu\left(\eta_{pq}^\nu\right) &\le \frac{1}{\int_0^\omega a_{pq}(t)dt}\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt + \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt, \quad pq \in \mathscr{B}, \nu \in E. \end{aligned} \tag{38}$$

Let $\bar{t}_{pq}^\nu, \underline{t}_{pq}^\nu \in [0,\omega]$ such that $x_{pq}^\nu(\bar{t}_{pq}^\nu) = \sup_{t\in[0,\omega]}x_{pq}^\nu(t)$, $x_{pq}^\nu(\underline{t}_{pq}^\nu) = \inf_{t\in[0,\omega]}x_{pq}^\nu(t)$, $pq \in \mathscr{B}$, and $\nu \in E$; by the arbitrariness of $\xi_{pq}^\nu, \eta_{pq}^\nu$, we can obtain

$$\begin{aligned} x_{pq}^\nu\left(\underline{t}_{pq}^\nu\right) &\ge \frac{1}{\int_0^\omega a_{pq}(t)dt}\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt - \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt \\ &\ge -\frac{1}{\int_0^\omega a_{pq}(t)dt}\left|\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt\right| - \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt \\ &\ge -\frac{1}{a^-\omega}\left[\omega\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\|x\|_X \cdot \sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl}\int_0^{+\infty}\left|K_{pq}(u)\right|du + \omega T + 2qI\right] \\ &\quad - \left[a^+\omega\left|x_{pq}^\nu\right|_\infty + \omega\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\|x\|_X \cdot \sum_{C_{kl}\in N_r(i,j)}\bar{C}_{pq}^{kl}\int_0^{+\infty}\left|K_{pq}(u)\right|du + \omega T + 2qI\right] \\ &= -a^+\omega\left|x_{pq}^\nu\right|_\infty - \left(\frac{1}{a^-} + \omega\right) \cdot \left[\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\|x\|_X \sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl} \times \int_0^{+\infty}\left|K_{pq}(u)\right|du + T + \frac{2qI}{\omega}\right], \quad pq \in \mathscr{B}, \nu \in E, \end{aligned} \tag{39}$$

and

$$\begin{aligned} x_{pq}^\nu\left(\bar{t}_{pq}^\nu\right) &\le \frac{1}{\int_0^\omega a_{pq}(t)dt}\int_0^\omega a_{pq}(t)x_{pq}^\nu(t)dt + \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt \\ &\le \frac{1}{\int_0^\omega a_{pq}(t)dt}\int_0^\omega\left|a_{pq}(t)x_{pq}^\nu(t)dt\right| + \int_0^\omega\left|\dot{x}_{pq}^\nu(t)\right|dt \\ &\le a^+\omega\left|x_{pq}^\nu\right|_\infty + \left(\frac{1}{a^-} + \omega\right) \cdot \left[\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\|x\|_X \sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl} \times \int_0^{+\infty}\left|K_{pq}(u)\right|du + T + \frac{2qI}{\omega}\right], \quad pq \in \mathscr{B}, \nu \in E. \end{aligned} \tag{40}$$

Thus, from (39) and (40) we have

$$\left|x_{pq}^\nu\right|_\infty = \sup_{t\in[0,\omega]}\left|x_{pq}^\nu(t)\right| \le a^+\omega\left|x_{pq}^\nu\right|_\infty + \left(\frac{1}{a^-} + \omega\right) \cdot \left[\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\|x\|_X \sum_{C_{kl}\in N_r(p,q)}\bar{C}_{pq}^{kl} \times \int_0^{+\infty}\left|K_{pq}(u)\right|du + T + \frac{2qI}{\omega}\right], \quad pq \in \mathscr{B}, \nu \in E, \tag{41}$$

hence

$$\|x\|_X \le a^+\omega\|x\|_X + \left(\frac{1}{a^-} + \omega\right)\left[\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\|x\|_X \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \times \int_0^{+\infty} \left|K_{pq}(u)\right| du + T + \frac{2qI}{\omega}\right], \tag{42}$$

that is

$$\|x\|_X \le a^+\omega\|x\|_X + \left(\frac{1}{a^-} + \omega\right)\left[\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\|x\|_X \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \times \int_0^{+\infty} \left|K_{pq}(u)\right| du + T + \frac{2qI}{\omega}\right], \tag{42}$$

Obviously, $A$ is independent of $\lambda$. Take $\Omega = \{x \in X : \|x\|_X < A + 1\}$, then $\Omega \subset X$ with $0 \in \Omega$ such that equation $Lu = \lambda Nu$ has no solutions in $\partial\Omega \cap \operatorname{Dom} L$ for any $\lambda \in (0, 1)$. Thus, by Lemma 1 we conclude that $Lx = Nx$ has at least one $\omega/2$-antiperiodic solution in $X$, that is, (16) has at least one $\omega/2$-antiperiodic solution. By Remark 1, we see that (1) has at least one $\omega/2$-antiperiodic solution.

## 4. The Global Exponential Stability of Antiperiodic Solution

In this section, by constructing a suitable Lyapunov function we derive sufficient conditions ensuring the global exponential stability of antiperiodic solutions for (1).

**Theorem 2.** *Let H1–H4 and A1 hold. Furthermore, assume that*

*A2. The impulsive operators satisfy*

$$I_{pqh}\left(x_{pq}(t_h)\right) = -\gamma_{pqh} x_{pq}(t_h), \quad 0 \le \gamma_{pqh} \le 2, pq \in \mathscr{B}, h \in \mathbb{N}. \tag{44}$$

*A3. There exists a positive constant $\lambda > 0$ satisfying*

$$\delta = \max_{pq\in\mathscr{B}}\left\{\delta_{pq}\right\} < 0, \tag{45}$$

*where*

$$\begin{aligned}\delta_{pq} = (\lambda - a^-) &+ \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \int_0^{+\infty} \left|K_{pq}(u)\right| \\ &\cdot \left[\left(M_f^R + M_f^I + M_f^J + M_f^K\right)\right. \\ &\left. + 4\left(L_f^R + L_f^I + L_f^J + L_f^K\right)\right] du,\end{aligned} \tag{46}$$

*then, the $\omega/2$-antiperiodic solution of (1) is globally exponentially stable.*

*Proof 2.* By Theorem 1, (16) has an $\omega/2$-periodic solution. Let $x(t)$ be the $\omega/2$-periodic solution with the initial value $\varphi(t)$ and $y(t)$ be an arbitrary solution with the initial value $\psi(t)$. Set $z_{pq}^{\nu}(t) = x_{pq}^{\nu}(t) - y_{pq}^{\nu}(t)$, $pq \in \mathscr{B}$, and $\nu \in E$. By (16), for $t > 0$ and $t \ne t_h$, we have

$$\begin{aligned}D^+\left|z_{pq}^{\nu}(t)\right| \le -a^-\left|z_{pq}^{\nu}(t)\right| &+ \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \int_0^{+\infty} K_{pq}(u) \\ &\cdot \left[M_f^R\left|z_{pq}^R(t)\right| + M_f^I\left|z_{pq}^I(t)\right| + M_f^J\left|z_{pq}^J(t)\right|\right. \\ &+ M_f^K\left|z_{pq}^K(t)\right| + 4\left(L_f^R\left|z_{kl}^R(t-u)\right|\right. \\ &+ L_f^I\left|z_{kl}^I(t-u)\right| + L_f^J\left|z_{kl}^J(t-u)\right| \\ &\left.\left. + L_f^K\left|z_{kl}^K(t-u)\right|\right)\right] du, \quad pq \in \mathscr{B}, \nu \in E.\end{aligned} \tag{47}$$

For $t = t_h, h \in \mathbb{N}$, from A2 we can have the following:

$$\begin{aligned}\left|x_{pq}^{\nu}(t_h^+) - y_{pq}^{\nu}(t_h^+)\right| &= \left|1 - \gamma_{pqh}\right|\left|x_{pq}^{\nu}(t_h) - y_{pq}^{\nu}(t_h)\right| \\ &\le \left|x_{pq}^{\nu}(t_h) - y_{pq}^{\nu}(t_h)\right|,\end{aligned} \tag{48}$$

that is

$$\left|z_{pq}^{\nu}(t_h^+)\right| \le \left|z_{pq}^{\nu}(t_h)\right|, \tag{49}$$

where $pq \in \mathscr{B}$ and $\nu \in E$.

Construct the Lyapunov function $V(t)$ as follows:

$$V(t) = V^R(t) + V^I(t) + V^J(t) + V^K(t), \tag{50}$$

where $V^{\nu}(t) = \sum_{pq\in\mathscr{B}} |z_{pq}^{\nu}(t)(t)| e^{\lambda t}$, $pq \in \mathscr{B}$, and $\nu \in E$.

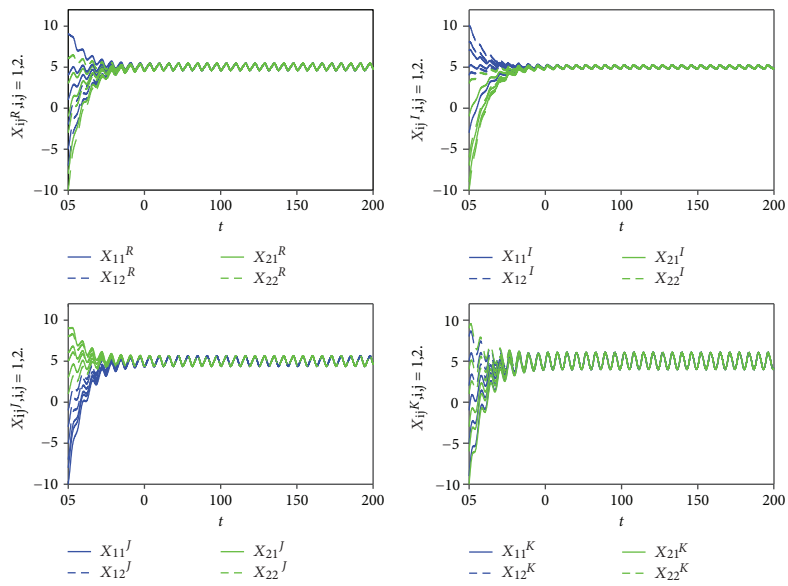


FIGURE 1: The states of four parts of $x_{11}$, $x_{12}$, $x_{21}$, and $x_{22}$.

Calculating the upper right derivative of $V^R(t)$ along the solutions of (16), for $t \neq t_h$, we obtain

$$\begin{aligned} D^+V^R(t) &= \sum_{pq\in\mathscr{B}} \left\{ \lambda e^{\lambda t}\left|z_{pq}^R(t)\right| + \operatorname{sign}\left(z_{pq}^R(t)\right) D^+\left(z_{pq}^R(t)\right) e^{\lambda t} \right\} \\ &\le \sum_{pq\in\mathscr{B}} \left\{ (\lambda - a^-)e^{\lambda t}\left|z_{pq}^R\right| + e^{\lambda t} \sum_{C_{kl}\in N_r(p,q)} \right. \\ &\quad \cdot \bar{C}_{pq}^{kl} \int_0^{+\infty} K_{pq}(u) \left[ \left( M_f^R \left|z_{pq}^R(t)\right| + M_f^I \left|z_{pq}^I(t)\right| \right.\right. \\ &\quad \left. + M_f^J \left|z_{pq}^J(t)\right| + M_f^K \left|z_{pq}^K(t)\right| \right) + 4\left( L_f^R \left|z_{kl}^R(t-u)\right| \right. \\ &\quad \left.\left. \left. + L_f^I \left|z_{kl}^I(t-u)\right| + L_f^J \left|z_{kl}^J(t-u)\right| \right) \right] du \right\} \\ &\le e^{\lambda t} \sum_{pq\in\mathscr{B}} \left\{ (\lambda - a^-) + \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \int_0^{+\infty} \left|K_{pq}(u)\right| \right. \\ &\quad \cdot \left[ \left( M_f^R + M_f^I + M_f^J + M_f^K \right) \right. \\ &\quad \left.\left. + 4\left( L_f^R + L_f^I + L_f^J + L_f^K \right) \right] du \right\} \|z\|. \end{aligned} \tag{51}$$

Repeat the same calculation and we can get

$$\begin{aligned} D^+V^\nu(t) &\le e^{\lambda t} \sum_{pq\in\mathscr{B}} \left\{ (\lambda - a^-) + \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \int_0^{+\infty} \left|K_{pq}(u)\right| \right. \\ &\quad \cdot \left[ \left( M_f^R + M_f^I + M_f^J + M_f^K \right) \right. \\ &\quad \left.\left. + 4\left( L_f^R + L_f^I + L_f^J + L_f^K \right) \right] du \right\} \|z\|, \quad \nu \in E, t \neq t_h. \end{aligned} \tag{52}$$

It follows from A3, (51), and (52) that for $t \neq t_h$,

$$D^+V(t) \le 0. \tag{53}$$

By (49), we also have

$$\begin{aligned} V(t_h^+) &= V^R(t_h^+) + V^I(t_h^+) + V^J(t_h^+) + V^K(t_h^+) \\ &= \sum_{pq\in\mathscr{B}} \left|z_{pq}^R(t_h^+)\right| e^{\lambda t_h^+} + \sum_{pq\in\mathscr{B}} \left|z_{pq}^I(t_h^+)\right| e^{\lambda t_h^+} \\ &\quad + \sum_{pq\in\mathscr{B}} \left|z_{pq}^J(t_h^+)\right| e^{\lambda t_h^+} + \sum_{pq\in\mathscr{B}} \left|z_{pq}^K(t_h^+)\right| e^{\lambda t_h^+} \\ &\le \sum_{pq\in\mathscr{B}} \left|z_{pq}^R(t_h)\right| e^{\lambda t_h} + \sum_{pq\in\mathscr{B}} \left|z_{pq}^I(t_h)\right| e^{\lambda t_h} \\ &\quad + \sum_{pq\in\mathscr{B}} \left|z_{pq}^J(t_h)\right| e^{\lambda t_h} + \sum_{pq\in\mathscr{B}} \left|z_{pq}^K(t_h)\right| e^{\lambda t_h} \\ &= V(t_h), \quad pq \in \mathscr{B}, h \in \mathbb{N}. \end{aligned} \tag{54}$$

Hence, $V(t) \le V(0)$ for all $t \ge 0$.
On the other hand, we have

$$\begin{aligned} V(0) &= V^R(0) + V^I(0) + V^J(0) + V^K(0) \\ &= \sum_{pq\in\mathscr{B}} \left( \left|z_{pq}^R(0)\right| + \left|z_{pq}^I(0)\right| + \left|z_{pq}^J(0)\right| + \left|z_{pq}^K(0)\right| \right) \\ &\le \sum_{pq\in\mathscr{B}} \left\{ \sup_{s\in(-\infty,0]} \left( \left|z_{pq}^R(s)\right| + \left|z_{pq}^I(s)\right| + \left|z_{pq}^J(s)\right| + \left|z_{pq}^K(s)\right| \right) \right\} \\ &\le \sum_{pq\in\mathscr{B}} \left\{ \sup_{s\in(-\infty,0]} \left( \left|\varphi_{pq}^R(s) - \psi_{pq}^R(s)\right| + \left|\varphi_{pq}^I(s) - \psi_{pq}^I(s)\right| \right.\right. \\ &\quad \left.\left. + \left|\varphi_{pq}^J(s) - \psi_{pq}^J(s)\right| + \left|\varphi_{pq}^K(s) - \psi_{pq}^K(s)\right| \right) \right\} = \|\varphi - \psi\|. \end{aligned} \tag{55}$$

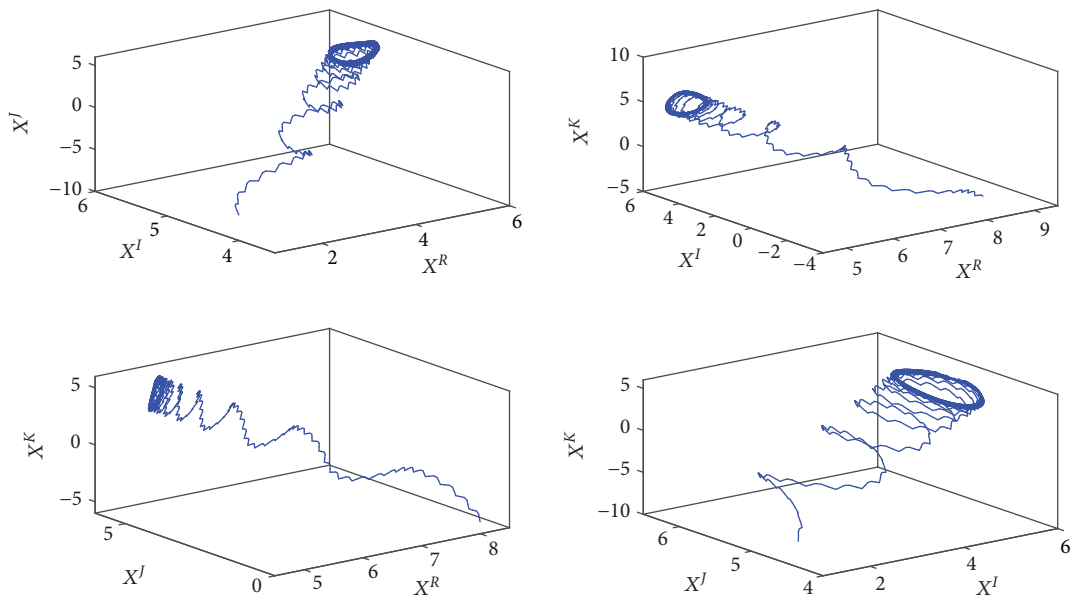


Figure 2: Curves of $x$ in 3-dimensional space for stable case.

Let $M = 1$, and we can easily obtain the following:

$$\|x(t) - y(t)\|_0 \le V(t)e^{-\lambda t} \le V(t)e^{-\lambda t} \le M\|\varphi - \psi\|e^{-\lambda t}, \quad t \ge 0. \tag{56}$$

Therefore, the $\omega/2$-antiperiodic solution of (16) is globally exponentially stable. According to Remark 1, we know that the $\omega/2$-antiperiodic solution of (16) is globally exponentially stable. The proof is complete.

## 5. An Illustrative Example

In this section, we give an example to show the feasibility and effectiveness of the results obtained in this paper.

*Example 1.* Consider the following QVSICNN:

$$\begin{cases} \dot{x}_{pq}(t) = -a_{pq}(t)x_{pq}(t) - \sum\limits_{C_{kl}\in N_r(p,q)} C_{pq}^{kl}(t)\displaystyle\int_0^{+\infty} K_{pq}(u)x_{pq}(t)f(x_{kl}(t-u))du + T_{pq}(t), \quad t \ge 0, t \ne t_h, \\ \Delta x_{pq}(t_h) = x_{pq}(t_h^+) - x_{pq}(t_h^-) = I_{pqh}(x_{pq}(t_h)), \quad t = t_h, \end{cases} \tag{57}$$

where $\dot{x}_{pq}(t) = \dot{x}_{pq}^R(t) + i\dot{x}_{pq}^I(t) + j\dot{x}_{pq}^J(t) + k\dot{x}_{pq}^K(t) \in \mathbb{Q}$, $K_{pq}(u) = e^{-u}$, $p, q = 1, 2$, and the coefficients are as follows:

$$\begin{aligned} f(x) &= 0.01|\sin x^R| + i0.02|\sin x^I| \\ &\quad + j0.03|\sin x^J| + k0.04|\sin x^K|, \\ a_{11}(t) &= a_{12}(t) = a_{21}(t) = a_{22}(t) = \frac{1}{8\pi}\sin^2 t + \frac{1}{4\pi}, \\ C_{11}(t) &= C_{12}(t) = C_{21}(t) = C_{22}(t) = 0.01\cos^2 t, \\ T_{11}(t) &= T_{12}(t) = T_{21}(t) = T_{22}(t) \\ &= 0.2\sin t + i0.1\cos t + j0.3\sin t \\ &\quad + k0.5\cos t, \end{aligned} \tag{58}$$

and the impulsive functions $I_{pqh}(x_{pq}(t_h)) = -0.001x_{pq}(t_h)$, $p, q = 1, 2$, $t_h = h\omega/2$, and $h \in \mathbb{N}$.

By a simple calculation, we can get

$$a^- = \frac{1}{4\pi},$$

$$a^+ = \frac{1}{8\pi} + \frac{1}{4\pi},$$

$$\omega = 2\pi,$$

$$\gamma_{pqh} = 0.001,$$

$$I = 0.001,$$

$$\int_0^{+\infty} K_{pq}(u)du = 1,$$

$$M_f^R = 0.01,$$

$$M_f^I = 0.02,$$

$$M_f^J = 0.03,$$

$$M_f^K = 0.04,$$

$$L_f^R = L_f^I = L_f^J = L_f^K = 0.04,$$

$$\sum_{C_{kl}\in N_1(1,1)} \bar{C}_{11}^{kl} = \sum_{C_{kl}\in N_1(1,2)} \bar{C}_{12}^{kl} = \sum_{C_{kl}\in N_1(2,1)} \bar{C}_{21}^{kl} = \sum_{C_{kl}\in N_1(2,2)} \bar{C}_{22}^{kl} = 0.04,$$

$$\Gamma = \frac{a^-\omega(1-a^+\omega)}{1+a^-\omega} - \left(M_f^R + M_f^I + M_f^J + M_f^K\right) \cdot \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \int_0^{+\infty} \left|K_{pq}(u)\right| du \approx 0.08 > 0. \tag{59}$$

Taking $\lambda = 0.01$, we obtain

$$\max_{1\le i,j\le 2}\left\{\delta_{pq}\right\} = (\lambda - a^-) + \sum_{C_{kl}\in N_r(p,q)} \bar{C}_{pq}^{kl} \int_0^{+\infty} \left|K_{pq}(u)\right| \cdot \left[\left(M_f^R + M_f^I + M_f^J + M_f^K\right) + 4\left(L_f^R + L_f^I + L_f^J + L_f^K\right)\right] \mathrm{d}u \approx -0.04 < 0. \tag{60}$$

Thus conditions H1–H4 and A1–A3 hold. Therefore, according to Theorems 1 and 2, (57) has at least one $\pi$-antiperiodic solution, which is globally exponentially stable (see Figures 1 and 2).

## 6. Conclusion

In this paper, we investigated the existence and global exponential stability of antiperiodic solutions for a class of QVSICNNs with impulsive effects. We introduce a new method different from all other antiperiodic solutions of neural networks in a previous work. By using a new continuation theorem of the coincidence degree theory and constructing a suitable Lyapunov function, we obtain the existence and global exponential stability results for an antiperiodic solution. However, in this paper, we only investigate the antiperiodic solution problem of QVSICNNs with impulsive effects. In future work, periodic solution, almost periodic solution, and pseudo almost periodic solution in the quaternion field can be considered.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflict of interest.

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