Almost Periodic Synchronization for Quaternion-Valued Neural Networks with Time-Varying Delays

Yongkun Li,1 Xiaofang Meng,1 and Yuan Ye2

1Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China
2Graduate School, Yunnan University, Kunming, Yunnan 650091, China

Correspondence should be addressed to Yuan Ye; yye_yd@aliyun.com

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This paper focuses on the global exponential almost periodic synchronization of quaternion-valued neural networks with time-varying delays. By virtue of the exponential dichotomy of linear differential equations, Banach’s fixed point theorem, Lyapunov functional method, and differential inequality technique, some sufficient conditions are established for assuring the existence and global exponential synchronization of almost periodic solutions of the delayed quaternion-valued neural networks, which are completely new. Finally, we give one example with simulation to show the applicability and effectiveness of our main results.

1. Introduction

It is well known that the quaternion is a mathematical concept discovered by Hamilton in 1843, which did not arouse much attention for quite a long time, let alone real world applications. Due to the noncommutativity of quaternion multiplication, the study of quaternion is much more difficult than that of plurality, which is one of the reasons for the slow development of quaternion. Fortunately, over the past two decades, the quaternion theory has achieved a rapid development, especially in algebra, and found many applications in the real world, like attitude control, quantum mechanics, robotics, computer graphics, and so on [1–5]. For example, in the application of color image compression technology, one can apply the quaternion theory to encode and improve the color image; see [5]. In recent years, the quaternion-valued neural networks, as an extension of the real-valued neural networks and the complex-valued neural networks [6, 7], research has become a hot topic. It should be pointed out that, at present, almost all the investigations on quaternion-valued neural networks are mainly dealing with the stability, robustness, or dissipation of the equilibrium of the neural networks; see [8–17].

On the one hand, since the concept of drive-response synchronization for coupled chaotic systems was proposed by Pecora and Carroll in [18], chaos synchronization has become a hot research topic due to its potential applications in secure communication, automatic control, biological systems, and information science; see, for instance, [19–26].

On the other hand, the almost periodicity is a generalization of periodicity and is much more common than the periodicity, and so, almost periodic oscillation is a very important dynamic phenomenon for nonautonomous neural networks [27–30]. However, so far, few results have been available for the almost periodicity of complex-valued neural networks [31] and quaternion-valued neural networks. Moreover, in the past few years, although the problem of periodic synchronization for real-valued systems has been extensively studied by many scholars (see, for instance, [32–43]), there are few articles dealing with the almost periodic synchronization for real-valued systems, let alone dealing with the almost periodic synchronization for complex-valued or quaternion-valued neural networks. Very recently, some authors study the synchronization for complex-valued neural networks with time delays [44, 45]. Based on the LMI method and mathematical analysis technique, they obtain some sufficient conditions for synchronization of complex-valued neural networks. However, to the best of our knowledge, there has been no paper published on the almost periodic synchronization of the quaternion-valued neural networks,
which remains as an open challenge. So, it is necessary to investigate the synchronization of quaternion-valued neural networks.

Motivated by the above statement, the main purpose of this paper is to study the synchronization of quaternion-valued neural networks. The main contributions of this paper can be summarized as follows: (1) this is the first time to investigate the almost periodic synchronization of neural networks. (2) Compared with other results, the results of this paper are the ones about quaternion value and with time delays. Therefore, the results are less conservative and more general. (3) Our methods used in this paper can easily apply to study the almost periodic synchronization for other types of neural networks with delays.

This paper is organized as follows: In Section 2, we give the model description and introduce some definitions and preliminary lemmas and transform the quaternion-valued system (9) into an equivalent real-valued system. In Section 3, we establish some sufficient conditions for the existence and global exponential synchronization of almost periodic solutions of (9). In Section 4, we give an example to demonstrate the feasibility of our results. This paper ends with a brief conclusion in Section 5.

2. Model Description and Preliminaries

In this section, we shall first recall some fundamental definitions and lemmas which are used in what follows.

First, we give some notations of this paper. Let \( \mathbb{R} \) and \( \mathbb{Q} \) stand for the real field and the skew field of quaternions, respectively. \( \mathbb{R}^{n \times n}, \mathbb{Q}^{n \times n} \) denote the set of all \( n \times n \) real-valued and quaternion-valued matrices, respectively. The skew field of quaternion is denoted by

\[
\mathbb{Q} = \{ x = x^R + ix^I + jx^J + kx^K \},
\]

where \( x^R, x^I, x^J, x^K \) are real numbers and the elements \( i, j \) and \( k \) obey Hamilton’s multiplication rules:

\[
\begin{align*}
ij &= -ji = k, \\
jk &= -kj = i, \\
ki &= -ik = j, \\
i^2 &= j^2 = k^2 = ijk = -1.
\end{align*}
\]

Definition 1 (see [46], let \( f \in BC(\mathbb{R}, \mathbb{R}^n) \)). Function \( f \) is said to be almost periodic if, for any \( \varepsilon > 0 \), it is possible to find a real number \( l(\varepsilon) > 0 \), for any interval with length \( l(\varepsilon) \); there exists a number \( \tau = \tau(\varepsilon) \) in this interval such that \( |f(t + \tau) - f(t)| < \varepsilon \), for all \( t \in \mathbb{R} \).

We denote by \( \text{AP}(\mathbb{R}, \mathbb{R}^n) \) the set of all almost periodic functions from \( \mathbb{R} \) to \( \mathbb{R}^n \).

Definition 2. Quaternion-valued function \( f = f^R + if^I + jf^J + kf^K \) is an almost periodic function if and only if \( f^R, f^I, f^J \) and \( f^K \) are almost periodic functions.

Consider the following linear homogenous system

\[
x'(t) = A(t)x(t)
\]

and linear nonhomogenous system

\[
x'(t) = A(t)x(t) + f(t),
\]

where \( A(t) \) is an almost periodic matrix function and \( f(t) \) is an almost periodic vector function.

Definition 3 (see [46]). System (3) is said to admit an exponential dichotomy if there exist a projection \( P \) and positive constants \( \alpha, \beta \) such that the fundamental solution matrix \( X(t) \) satisfies

\[
\begin{align*}
|X(t)P^{-1}(s)| &\leq \beta e^{-\alpha(t-s)}, \quad t \geq s, \\
|X(t)(I - P)X^{-1}(s)| &\leq \beta e^{\alpha(s-t)}, \quad t \leq s.
\end{align*}
\]

Lemma 4 (see [46]). If the linear system (3) admits an exponential dichotomy, then system (4) has a unique almost periodic solution that can be expressed as

\[
x(t) = \int_{-\infty}^{t} X(t)P^{-1}(s)f(s)ds \\
- \int_{t}^{+\infty} X(t)(I - P)X^{-1}(s)f(s)ds,
\]

where \( X(t) \) is the fundamental solution matrix of (3).

Lemma 5 (see [46]). Let \( c_p \) be an almost periodic function on \( \mathbb{R} \) and

\[
M\{c_p\} = \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} c_p(s)ds > 0, \quad p = 1, 2, \ldots, n.
\]

Then the linear system

\[
x'(t) = \text{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t)
\]

admits an exponential dichotomy on \( \mathbb{R} \).

Consider the following quaternion-valued neural networks with time-varying delays:

\[
x'_p(t) = -d_p(t)x_p(t) + \sum_{q=1}^{n} a_{pq}(t)f_q(x_q(t)) \\
+ \sum_{q=1}^{n} b_{pq}(t)g_q(x_q(t - \tau_{pq}(t))) + u_p(t),
\]

where \( p \in \{1, 2, \ldots, n\} = T \) and \( x_p(t) \in \mathbb{Q} \) is the state of the \( p \)th neuron at time \( t \); \( d_p(t) > 0 \) is the self-feedback connection weight; \( a_{pq}(t) \in \mathbb{Q} \) and \( b_{pq}(t) \in \mathbb{Q} \) are the connection weight and the delay connection weight from neuron \( q \) to neuron \( p \) at time \( t \), respectively; \( u_p(t) \in \mathbb{Q} \) is an external input on the \( p \)th unit at time \( t \); \( f_q(x_q(t)) \in \mathbb{Q} \) and \( g_q(x_q(t - \tau_{pq}(t))) \in \mathbb{Q} \) denote the activation functions without
and with time-varying delays, respectively; $\tau_{pq}(t)$ represents the time-varying delay and satisfies $0 \leq \tau_{pq}(t) \leq \tau$.

The initial value of (9) is given by

$$x_p(s) = \phi_p(s), \quad s \in [-\tau, 0], \quad p \in T,$$

where $\phi_p \in C([-\tau, 0], \mathbb{Q})$.

The response system of (9) is designed as

$$y'_p(t) = -d_p(t) y_p(t) + \sum_{q=1}^{n} a_{pq}(t) f_q(y_q(t))$$

$$+ \sum_{q=1}^{n} b_{pq}(t) g_q(y_q(t) - \tau_{pq}(t)) + u_p(t) + \theta_p(t),$$

where $p \in T$, $y_p(t) \in \mathbb{Q}$ denotes the state of the $p$th neuron at time $t$ of the response system and $\theta_p(t) \in \mathbb{Q} \subset \mathbb{R}$ is a controller.

The initial condition associated with (11) is of the form

$$y_p(s) = \psi_p(s), \quad s \in [-\tau, 0], \quad p \in T,$$

where $\psi_p \in C([-\tau, 0], \mathbb{Q})$.

In order to overcome the inconvenience of the noncommutativity of quaternion multiplication, in the following, we always assume the following:

$$(H_1) \quad \text{Let } x_q = q^R + ix_q^I + jx_q^J + kx_q^K, x_q^R, x_q^I, x_q^J, x_q^K \in \mathbb{R}.$$  

Then $f_q(x_q)$ and $g_q(x_q)$ can be expressed as

$$f_q(x_q) = f_q^R(x_q^R, x_q^I, x_q^J, x_q^K) + if_q^I(x_q^R, x_q^I, x_q^J, x_q^K) + kf_q^K(x_q^R, x_q^I, x_q^J, x_q^K),$$

$$g_q(x_q) = g_q^R(x_q^R, x_q^I, x_q^J, x_q^K) + ig_q^I(x_q^R, x_q^I, x_q^J, x_q^K) + kg_q^K(x_q^R, x_q^I, x_q^J, x_q^K),$$

where $f_q^\nu, g_q^\nu : \mathbb{R}^4 \to \mathbb{R}, q \in T, \nu \in \{R, I, J, K\} = E$.

According to $(H_1)$, system (9) can be transformed into the following real-valued system:

$$\begin{align*}
(x_q^R)'(t) &= -d_q(t) x_q^R(t) + \sum_{q=1}^{n} (a^R_{pq}(t) f^R_q[t,x] + a^I_{pq}(t) f^I_q[t,x] + a^K_{pq}(t) f^K_q[t,x]), \\
-x^R_{pq}(t) f^R_q[t,x] - x^I_{pq}(t) f^I_q[t,x] - x^K_{pq}(t) f^K_q[t,x] + \sum_{q=1}^{n} (b^R_{pq}(t) g^R_q[t - \tau_{pq}(t), x] + b^I_{pq}(t) g^I_q[t - \tau_{pq}(t), x] + b^K_{pq}(t) g^K_q[t - \tau_{pq}(t), x]) + u_p(t),
\end{align*}$$

where $f^\nu_q[t,x] = f^\nu_q(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t)), g^\nu_q[t - \tau_{pq}(t), x] = g^\nu_q(x_q^R(t - \tau_{pq}(t)), x_q^I(t - \tau_{pq}(t)), x_q^J(t - \tau_{pq}(t)), x_q^K(t - \tau_{pq}(t)))$ for $t \in T, \nu \in E,$ and

$$\begin{align*}
 a_{pq}(t) &= a^R_{pq}(t) + ia^I_{pq}(t) + ja^K_{pq}(t), \\
b_{pq}(t) &= b^R_{pq}(t) + ib^I_{pq}(t) + jb^K_{pq}(t), \\
u_p(t) &= u^R_p(t) + iu^I_p(t) + ju^K_p(t),
\end{align*}$$

with $q \in T, p \in T.$
It follows from (14) that

\[ X_p^I(t) = -d_p X_p(t) + \sum_{q=1}^{n} A_{pq} (t) F_q(t, x) \]

\[ + \sum_{q=1}^{n} B_{pq} (t) G_q[t - \tau_{pq}(t), x] + U_p(t), \]

where

\[ A_{pq}(t) = \begin{bmatrix}
    a^R_{pq}(t) & -a^I_{pq}(t) & -a^K_{pq}(t) \\
    a^I_{pq}(t) & a^R_{pq}(t) & -a^K_{pq}(t) \\
    a^K_{pq}(t) & a^K_{pq}(t) & a^R_{pq}(t) \\
    a^R_{pq}(t) & a^I_{pq}(t) & a^K_{pq}(t)
\end{bmatrix}, \]

\[ B_{pq}(t) = \begin{bmatrix}
    b^R_{pq}(t) & -b^I_{pq}(t) & -b^K_{pq}(t) \\
    b^I_{pq}(t) & b^R_{pq}(t) & -b^K_{pq}(t) \\
    b^K_{pq}(t) & b^K_{pq}(t) & b^R_{pq}(t) \\
    b^R_{pq}(t) & b^I_{pq}(t) & b^K_{pq}(t)
\end{bmatrix}, \]

\[ F_q(t, x) = \begin{bmatrix}
    f^R_q(t, x) \\
    f^I_q(t, x) \\
    f^I_q(t, x) \\
    f^R_q(t, x)
\end{bmatrix}, \]

\[ G_q[t - \tau_{pq}(t), x] = \begin{bmatrix}
    g^R_q[t - \tau_{pq}(t), x] \\
    g^I_q[t - \tau_{pq}(t), x] \\
    g^I_q[t - \tau_{pq}(t), x] \\
    g^R_q[t - \tau_{pq}(t), x]
\end{bmatrix}, \]

\[ X_p(t) = (x^R_p(t), x^I_p(t), x^K_p(t))^T, \]

\[ U_p(t) = (u^R_p(t), u^I_p(t), u^K_p(t))^T, \]

The initial condition associated with (16) is of the form

\[ X_p(s) = \Phi_p(s), \quad s \in [-\tau, 0], \quad p \in T, \]

where \( \Phi_p(s) = (\psi^R_p(s), \psi^I_p(s), \psi^K_p(s))^T, \psi^R_p(s) \in C([-\tau, 0], \mathbb{R}), \psi^I_p(s) \in E. \)

Similarly, the response system (11) can be transformed into the following real-valued system:

\[ Y_p^I(t) = -d_p Y_p(t) + \sum_{q=1}^{n} A_{pq} (t) F_q(t, y) \]

\[ + \sum_{q=1}^{n} B_{pq} (t) G_q[t - \tau_{pq}(t), y] + U_p(t) \]

\[ + \Theta_p(t), \quad p \in T, \]

where \( Y_p(t) = (y^R_p(t), y^I_p(t), y^K_p(t))^T, \Theta_p(t) = (\theta^R_p(t), \theta^I_p(t), \theta^K_p(t))^T. \)

The initial condition associated with (19) is of the form

\[ Y_p(s) = \Psi_p(s), \quad s \in [-\tau, 0], \quad p \in T, \]

where \( \Psi_p = (\psi^R_p(s), \psi^I_p(s), \psi^K_p(s))^T, \psi^R_p(s) \in C([-\tau, 0], \mathbb{R}), \psi^I_p(s) \in E. \)

Let \( Z_p(t) = Y_p(t) - X_p(t), Z_p(t) = (z^R_p(t), z^I_p(t), z^K_p(t)), p \in T. \) Then, from (16) and (19), we obtain the following error system:

\[ Z'_p(t) = -d_p Z_p(t) + \sum_{q=1}^{n} A_{pq} (t) \cdot (F_q[t, y] - F_q[t, x]) + \sum_{q=1}^{n} B_{pq} (t) \cdot (G_q[t - \tau_{pq}(t), y] - G_q[t - \tau_{pq}(t), x]) + \Theta_p(t), \quad p \in T. \]

In order to realize the almost periodic synchronization of the drive-response system, we choose the controller

\[ \theta^R_p(t) = \epsilon_p z^R_p(t), \]

\[ \theta^I_p(t) = \epsilon_p z^I_p(t), \]

\[ \theta^K_p(t) = \rho_p z^K_p(t), \]

where \( \epsilon_p, \epsilon_p, \rho_p, \phi_p \in \mathbb{R}^+. \)

Definition 6. The response system (19) and the drive system (16) can be globally exponentially synchronized, if there exist positive constants \( M \geq 1, \lambda > 0 \) such that

\[ \|Z(t)\| \leq \|\psi - \Phi\| Me^{-\lambda t}, \quad t \geq 0, \]

where \( \|Z(t)\| = \max_{1 \leq p \leq n} \{\max_{x \in E} \|z^R_p(t)\|, \|z^I_p(t)\|, \|z^K_p(t)\|\} \).

Let \( \mathbb{X} = \{x \mid x = (x^R_1, x^I_1, x^K_1, x^R_2, x^I_2, x^K_2, \ldots, x^R_n, x^I_n, x^K_n)^T \in AP(\mathbb{R}^n, \mathbb{R}^n)\} \) with the norm \( \|x\|_{\mathbb{X}} = \max_{p \in T} \max_{x \in E} \|x^R_p(t)\| \). Then \( \mathbb{X} \) is a Banach space.

Throughout this paper, we assume that the following conditions hold:

(H2) Function \( d_p \in C(\mathbb{R}, \mathbb{R}^+) \) with \( M[d_p] > 0, U_p \in C(\mathbb{R}, \mathbb{R}^{4x1}), A_{pq}, B_{pq} \in C(\mathbb{R}, \mathbb{R}^{4x4}), \) and \( \tau_{pq} \in C(\mathbb{R}, \mathbb{R}^+) \) with \( \sup_{t \in \mathbb{R}} \tau_{pq}(t) = \beta < 1 \) are almost periodic, where \( p, q \in T. \)
(H₃) There exist positive constants \( l_q^p, m_q^p \) such that for all \( x^p, y^p \in \mathbb{R} \),

\[
\left| f_q^y(x_q^p, x_q^f, x_q^K) - f_q^y(y_q^p, y_q^f, y_q^K) \right| \\
\leq l_q^R |x_q^p - y_q^p| + l_q^I |x_q^f - y_q^f| + l_q^K |x_q^K - y_q^K|,
\]

\[
g_q^y(x_q^p, x_q^f, x_q^K) - g_q^y(y_q^p, y_q^f, y_q^K) \\
\leq m_q^R |x_q^p - y_q^p| + m_q^I |x_q^f - y_q^f| + m_q^K |x_q^K - y_q^K|,
\]

and \( f_q^y(0, 0, 0, 0) = g_q^y(0, 0, 0, 0) = 0 \), where \( q \in T \), \( v \in E \).

(H₄) There exists a positive constant \( \kappa \) such that

\[
\max \left\{ \frac{\Omega_p \kappa + u_p^0}{d_p^0} \right\} \leq \kappa,
\]

\[
\max \left\{ \frac{\Omega_p}{d_p^0} \right\} = r < 1, \quad p \in T, \quad v \in E,
\]

where

\[
\Omega_p = \Lambda_p + \Delta_p, \quad p \in T,
\]

\[
\Lambda_p = \sum_{q=1}^{n} \left( l_{pq}^R + l_{pq}^I + l_{pq}^K \right), \quad p \in T,
\]

\[
\Delta_p = \sum_{q=1}^{n} \left( m_{pq}^R + m_{pq}^I + m_{pq}^K \right), \quad p \in T.
\]

(3. Main Results)

In this section, we will study the existence of almost periodic solutions of system (16).

**Theorem 7.** Let (H₃)–(H₄) hold. Then system (16) has a unique almost periodic solution in the region \( \Xi^* = \{ \varphi \mid \varphi \in \mathcal{X}, \| \varphi \|_\mathcal{X} \leq \kappa \} \).

**Proof.** For any \( \varphi \in \mathcal{X} \), we consider the linear almost periodic differential equation system

\[
X_p^y(t) = -d_p X_p(t) + \sum_{q=1}^{n} A_{pq}(t) F_q [t, \varphi] \\
+ \sum_{q=1}^{n} B_{pq}(t) G_q [t - \tau_{pq}(t), \varphi] + U_p(t),
\]

where

\[
X_p^y(t) = (X_1^y(t), X_2^y(t), \ldots, X_n^y(t))^{T},
\]

and

\[
X_p^y(t) = \int_{-\infty}^{t} e^{-\int_{s}^{t} d_p(u) du} \left( \sum_{q=1}^{n} A_{pq}(s) F_q [s, \varphi] \\
+ \sum_{q=1}^{n} B_{pq}(s) G_q [s - \tau_{pq}(s), \varphi] + U_p(s) \right) ds,
\]

Now, we define an operator \( \Phi : \mathcal{X}^* \to \mathcal{X} \) as follows:

\[
\Phi \varphi = X_p^y.
\]

First, we show that, for any \( \varphi \in \mathcal{X}^* \), we have \( \Phi \varphi \in \mathcal{X}^* \). From (32), we have
\[ \left| (\Phi \varphi)^R_p (t) \right| = \int_{-\infty}^{t} e^{-\int_{s}^{t} \nu_{\alpha}(u) \, du} \left( \sum_{q=1}^{n} (a_{pq} \varphi(s) f^R_q [s, \varphi] - a_{pq}^I \varphi(s) f^I_q [s, \varphi] - a_{pq}^K \varphi(s) f^K_q [s, \varphi]) \right) \] 

\[ + \sum_{q=1}^{n} \left( b_{pq} \varphi(s) \varphi(s) g^R_q [s - \tau_{pq}(s), \varphi] - b_{pq}^I \varphi(s) g^I_q [s - \tau_{pq}(s), \varphi] - b_{pq}^K \varphi(s) g^K_q [s - \tau_{pq}(s), \varphi] \right) \] 

\[ + u^R_p (s) \right) ds \leq \int_{-\infty}^{t} e^{-\int_{s}^{t} \nu_{\alpha}(u) \, du} \left( \sum_{q=1}^{n} (a_{pq}^R | f^R_q [s, \varphi]| + a_{pq}^I | f^I_q [s, \varphi]| + a_{pq}^K | f^K_q [s, \varphi]|) \right) \] 

\[ + \sum_{q=1}^{n} \left( b_{pq}^R \varphi(s) g^R_q [s - \tau_{pq}(s), \varphi] + b_{pq}^I \varphi(s) g^I_q [s - \tau_{pq}(s), \varphi] + b_{pq}^K \varphi(s) g^K_q [s - \tau_{pq}(s), \varphi] \right) \] 

\[ + u^R_p (s) \right) ds \leq \int_{-\infty}^{t} e^{-\int_{s}^{t} \nu_{\alpha}(u) \, du} \left( \sum_{q=1}^{n} (a_{pq}^R + a_{pq}^I + a_{pq}^K) \right) \] 

\[ + \sum_{q=1}^{n} \left( b_{pq}^R \varphi(s) g^R_q [s - \tau_{pq}(s), \varphi] + b_{pq}^I \varphi(s) g^I_q [s - \tau_{pq}(s), \varphi] + b_{pq}^K \varphi(s) g^K_q [s - \tau_{pq}(s), \varphi] \right) \] 

\[ + u^R_p (s) \right) ds \leq \int_{-\infty}^{t} e^{-\int_{s}^{t} \nu_{\alpha}(u) \, du} \left( \sum_{q=1}^{n} (a_{pq}^R + a_{pq}^I + a_{pq}^K) \right) \] 

\[ + \sum_{q=1}^{n} \left( b_{pq}^R \varphi(s) g^R_q [s - \tau_{pq}(s), \varphi] + b_{pq}^I \varphi(s) g^I_q [s - \tau_{pq}(s), \varphi] + b_{pq}^K \varphi(s) g^K_q [s - \tau_{pq}(s), \varphi] \right) \] 

\[ + u^R_p (s) \right) ds \leq \frac{1}{d_p} \left( \Lambda \rho \kappa + \Lambda \rho \kappa + u^R_p \right) = \frac{\Omega_p \rho + u^R_p}{d_p}, \] 

\[ p \in T. \]

In a similar way, we can get

\[ \left| (\Phi \varphi)^p (t) \right| \leq \frac{\Omega_p \rho + u^R_p}{d_p}, \quad p \in T, \; \nu = I, J, K. \] 

By (H\_4), we have

\[ \| \Phi \varphi \|_\infty \leq \kappa, \] 

which implies that \( \Phi \varphi \in \mathbb{X}^* \), so the mapping \( \Phi \) is a self-mapping from \( \mathbb{X}^* \) to \( \mathbb{X}^* \). Next, we shall prove that \( \Phi \) is a contraction mapping. In fact, for any \( \varphi, \psi \in \mathbb{X}^* \), we have

\[ \left| (\Phi \varphi - \Phi \psi)^p (t) \right| \leq \frac{\Omega_p \rho}{d_p} \| \varphi - \psi \|_\infty, \] 

\[ p \in T, \; \nu = I, J, K. \]

By (H\_4), we have

\[ \| \Phi \varphi - \Phi \psi \|_\infty \leq \rho \| \varphi - \psi \|_\infty. \] 

Hence, \( \Phi \) is a contraction mapping. Therefore, system (16) has a unique almost periodic solution in the region \( \mathbb{X}^* = \{ \varphi \in \mathbb{X} : \| \varphi \|_\infty \leq \kappa \} \). The proof is complete. \( \square \)

**Theorem 8.** Suppose that (H\_4)–(H\_5) hold. Then the drive system (16) and the response system (19) are globally exponentially synchronized based on the controller (22).
Similarly, we can get

\[
\begin{align*}
V^*_p(t) & \leq \sum_{p=1}^{n} \left( \lambda - d_p^* + \epsilon_p + \Lambda_p + \frac{\Delta_p}{1 - \beta} e^{\lambda t} \right) e^{\lambda t} \| Z(t) \|, \\
V'_p(t) & \leq \sum_{p=1}^{n} \left( \lambda - d_p^* + \epsilon_p + \Lambda_p + \frac{\Delta_p}{1 - \beta} e^{\lambda t} \right) e^{\lambda t} \| Z(t) \|.
\end{align*}
\]
\[ \begin{align*}
\leq \sum_{p=1}^{n} \left[ \lambda - d_p + \varrho_p + \Lambda_p + \frac{\Delta_p}{1 - \beta} e^{\lambda t} \right] e^{\lambda t} \| Z(t) \|
\end{align*} \] (44)

From (H5), we obtain

\[ V_1(0) \leq \sum_{p=1}^{n} \left[ \frac{1}{1 - \beta} \sum_{q=1}^{n} \left( b_{pq}^r + b_{pq}^i + b_{pq}^j + b_{pq}^k \right) \right] \times \int_{0}^{\tau_{pq}(0)} \left( m_q^r |z_q^r(s)| + m_q^i |z_q^i(s)| + m_q^j |z_q^j(s)| + m_q^k |z_q^k(s)| \right) e^{\lambda(s+\tau)} ds \leq \left[ 1 + \frac{1}{1 - \beta} \sum_{p=1}^{n} \frac{e^{\lambda t}}{\lambda} \Delta_p \right] \| \psi - \phi \|. \] (46)

Similarly, we can get

\[ \begin{align*}
V_2(0) & \leq \left[ 1 + \frac{1}{1 - \beta} \sum_{p=1}^{n} \frac{e^{\lambda t}}{\lambda} \Delta_p \right] \| \psi - \phi \|, \\
V_3(0) & \leq \left[ 1 + \frac{1}{1 - \beta} \sum_{p=1}^{n} \frac{e^{\lambda t}}{\lambda} \Delta_p \right] \| \psi - \phi \|, \\
V_4(0) & \leq \left[ 1 + \frac{1}{1 - \beta} \sum_{p=1}^{n} \frac{e^{\lambda t}}{\lambda} \Delta_p \right] \| \psi - \phi \|.
\end{align*} \] (47)

It is obvious that \( \| Z(t) \| e^{\lambda t} \leq V(t) \); thus, for \( t \geq 0 \), we have

\[ \| Z(t) \| e^{-\lambda t} \leq V(0) e^{-\lambda t} \leq M \| \psi - \phi \| e^{-\lambda t}, \] (48)

where

\[ M = \left[ 1 + \frac{1}{1 - \beta} \sum_{p=1}^{n} \frac{e^{\lambda t}}{\lambda} \Delta_p \right] > 1. \] (49)

Therefore, the drive system (16) and the response system (19) are globally exponentially synchronized based on the controller (22). The proof is complete. \( \square \)

### 4. An Example

In this section, we give an example to illustrate the feasibility and effectiveness of our results obtained in Section 3.

**Example 1.** Consider the following quaternion-valued neural networks with time-varying delay:

\[ \begin{align*}
x_p'(t) &= -d_p(t) x_p(t) + \sum_{q=1}^{2} a_{pq}(t) f_q(x_q(t)) \\
&\quad + \sum_{q=1}^{2} b_{pq}(t) g_q(x_q(t - \tau_{pq}(t))) + u_p(t),
\end{align*} \] (50)

\[ V'(t) \leq 0, \] (45)

which implies that \( V(t) \leq V(0) \) for all \( t \geq 0 \).

On the other hand, we have

\[ \begin{align*}
y_p'(t) &= -d_p(t) y_p(t) + \sum_{q=1}^{2} a_{pq}(t) f_q(y_q(t)) \\
&\quad + \sum_{q=1}^{2} b_{pq}(t) g_q(y_q(t - \tau_{pq}(t))) + u_p(t) + \theta_p(t).
\end{align*} \] (51)

The coefficients are taken as follows:

\[ \begin{align*}
d_1(t) &= 4 + |\cos(2t)|, \\
d_2(t) &= 6 - |\sin(\sqrt{3}t)|, \\
a_{11}(t) &= 0.1 \sin(\sqrt{2}t) + i0.3 \cos t + j0.2 \sin t + k0.4 \cos(\sqrt{2}t), \\
a_{12}(t) &= 0.3 \cos(\sqrt{3}t) + i0.1 \sin(2t) + j0.5 \sin(\sqrt{2}t) + k0.2 \sin t, \\
a_{21}(t) &= 0.4 \sin(2t) + i0.5 \cos t + j0.1 \cos(\sqrt{3}t) + k0.2 \sin t, \\
a_{22}(t) &= 0.3 \sin t + i0.4 \sin(\sqrt{3}t) + j0.1 \cos t - k0.2 \cos(3t), \\
b_{11}(t) &= 0.2 \cos t + i0.4 \sin t - j0.3 \sin(\sqrt{2}t) + k0.1 \sin(\sqrt{2}t), \\
b_{12}(t) &= 0.5 \sin t - i0.2 \cos t - j0.3 \cos(\sqrt{2}t) - k0.1 \cos t, \\
b_{21}(t) &= 0.2 \cos(2t) - i0.5 \sin(2t) + j0.3 \sin(3t) - k0.4 \cos(\sqrt{3}t), \\
b_{22}(t) &= 0.3 \sin(2t) - i0.4 \sin(\sqrt{3}t) - j0.1 \cos(\sqrt{3}t) + k0.2 \cos(3t).
\end{align*} \]
\[ b_{22}(t) = -0.6 \cos t - i0.1 \cos (\sqrt{3}t) + j0.2 \cos t - k0.5 \sin (3t), \]

\[ f_q(x_q) = \frac{1}{32} \tan \left( x_q^R + x_q^I \right) + i \frac{1}{32} \sin x_q^R + j \frac{1}{32} \left| x_q^R \right| + k \frac{1}{32} \sin x_q^R, \]

\[ g_q(x_q) = \frac{1}{24} \sin x_q^R + \frac{1}{24} \left| x_q^R + x_q^K \right| + j \frac{1}{24} \sin x_q^R + k \frac{1}{24} \left| x_q^R \right|, \]

\[ \tau_{11}(t) = \frac{1}{5} |\sin 2t|, \]

\[ \tau_{12}(t) = \tau_{21}(t) = \frac{1}{6} |\cos t|, \]

\[ \tau_{22}(t) = \frac{1}{4} |\sin t|, \]

\[ u_1(t) = 2 \sin \left( \sqrt{2}t \right) - i \cos t + j \sin t + k3 \cos t, \]

\[ u_2(t) = 2 \cos t + j3 \sin \left( \sqrt{3}t \right) - 5j \cos t - k4 \sin t. \]

(52)

By a simple calculation, we have

\[ d_1^R = 4, \]

\[ d_2^R = 5, \]

\[ a_{11}^R = 0.1, \]

\[ a_{11}^I = 0.3, \]

\[ a_{11}^T = 0.2, \]

\[ a_{12}^K = 0.4, \]

\[ a_{12}^R = 0.3, \]

\[ a_{12}^T = 0.1, \]

\[ a_{12}^R = 0.5, \]

\[ a_{12}^K = 0.2, \]

\[ a_{21}^R = 0.4, \]

\[ a_{21}^T = 0.5, \]

\[ a_{21}^R = 0.1, \]

\[ a_{21}^K = 0.2, \]

\[ a_{22}^R = 0.3, \]

\[ a_{22}^T = 0.4, \]

\[ a_{22}^R = 0.1, \]

\[ a_{22}^K = 0.2, \]

\[ b_{11}^R = 0.2, \]

\[ b_{11}^T = 0.4, \]

\[ b_{11}^R = 0.3, \]

\[ b_{11}^K = 0.1, \]

\[ b_{12}^R = 0.5, \]

\[ b_{12}^R = 0.2, \]

\[ b_{12}^K = 0.1, \]

\[ b_{21}^R = 0.2, \]

\[ b_{21}^R = 0.5, \]

\[ b_{21}^K = 0.1, \]

\[ b_{22}^R = 0.6, \]

\[ b_{22}^R = 0.1 \]

\[ u_1^R = 2, \]

\[ u_1^I = 1, \]

\[ u_1^J = 1, \]

\[ u_1^K = 3, \]

\[ u_2^R = 2, \]

\[ u_2^I = 3, \]

\[ u_2^J = 5, \]

\[ u_2^K = 4, \]

\[ \tau = \frac{1}{4}, \]

\[ \eta_q^R = \frac{1}{32}, \]

\[ \eta_q^R = \frac{1}{24}, \]

\[ q = 1, 2, \nu \in E, \]

\[ \Lambda_1 = 0.525, \]
Complexity

$\Lambda_2 = 0.55,$
$\Delta_1 = 0.7,$
$\Delta_2 \approx 0.9333.$

Take $\kappa = 2$; then we have $\Omega_1 = 1.225$, $\Omega_2 = 1.4833$,

$$\max_{w \in E} \left\{ \frac{\Omega_1 \kappa + u_1^i}{d_1}, \frac{\Omega_2 \kappa + u_2^i}{d_2} \right\} \approx \{1.3625, 1.5933\}$$

$$= 1.5933 < \kappa = 2,$$

$$\max \left\{ \frac{\Omega_1}{d_1}, \frac{\Omega_2}{d_2} \right\} \approx \{0.30625, 0.29666\} = 0.30625 < 1.$$ (53)

Moreover, take $\epsilon_1 = 1$, $\epsilon_2 = 1.5$, $\epsilon_1 = 0.8$, $\epsilon_2 = 1.8$, $\rho_1 = 1.2$, $\rho_2 = 1.6$, $\rho_1 = 0.5$, $\rho_2 = 1.9$, $\lambda = 1$, $\beta = 1/5$,

$$\lambda - d_1^e + \epsilon_1 + \Lambda_1 + \frac{\Delta_1}{1 - \beta} e^{\lambda \tau} \approx -0.3515 < 0,$$

$$\lambda - d_2^e + \epsilon_2 + \Lambda_2 + \frac{\Delta_2}{1 - \beta} e^{\lambda \tau} \approx -0.4520 < 0,$$

$$\lambda - d_1^e + \epsilon_1 + \Lambda_1 + \frac{\Delta_1}{1 - \beta} e^{\lambda \tau} \approx -0.5515 < 0,$$

$$\lambda - d_2^e + \epsilon_2 + \Lambda_2 + \frac{\Delta_2}{1 - \beta} e^{\lambda \tau} \approx -0.1520 < 0,$$

$$\lambda - d_1^e + \rho_1 + \Lambda_1 + \frac{\Delta_1}{1 - \beta} e^{\lambda \tau} \approx -0.1515 < 0,$$

$$\lambda - d_2^e + \rho_2 + \Lambda_2 + \frac{\Delta_2}{1 - \beta} e^{\lambda \tau} \approx -0.3520 < 0,$$

$$\lambda - d_1^e + q_1 + \Lambda_1 + \frac{\Delta_1}{1 - \beta} e^{\lambda \tau} \approx -0.8515 < 0,$$

$$\lambda - d_2^e + q_2 + \Lambda_2 + \frac{\Delta_2}{1 - \beta} e^{\lambda \tau} \approx -0.0520 < 0.$$ (55)

We can verify that all the assumptions of Theorems 7 and 8 are satisfied. Therefore, the drive system (50) and its response system (51) are globally exponential synchronized (see Figures 1–3).
Figure 2: The states of four parts of $y_1$ and $y_2$.

Figure 3: Synchronization.
5. Conclusion

In this paper, we consider the global exponential almost periodic synchronization of quaternion-valued neural networks with time-varying delays. By means of the exponential dichotomy of linear differential equations, Banach's fixed point theorem, Lyapunov functional method, and differential inequality technique, we establish the existence and global exponential synchronization of almost periodic solutions for system (9). To the best of our knowledge, this is the first time to study the almost periodic synchronization problem for neural networks. Furthermore, the method of this paper can be used to study other types of neural networks.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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