Research Article

Moderate Deviations for Stochastic Fractional Heat Equation Driven by Fractional Noise

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We consider a class of stochastic fractional heat equations driven by fractional noises. A central limit theorem is given, and a moderate deviation principle is established.

1. Introduction

Since the work of Freidlin and Wentzell [1], the large deviation principle (LDP) has been extensively developed for small noise systems and other types of models (such as interacting particle systems) (see [2–7]). Cardon-Weber [2] proved a LDP for a Burgers-type SPDE driven by white noise. Marquez-Carreras and Sarra [3] proved a LDP for a stochastic heat equation with spatially correlated noise, and Mellali and Mellouk [4] extended Marquez-Carreras and Sarra's [3] to a fractional operator. Jiang et al. [5] proved a LDP for a fourth-order stochastic heat equation driven by fractional noise. Budhiraja et al. [6] studied large deviation properties of systems of weakly interacting particles. Budhiraja et al. [7] proved a large deviation for Brownian particle systems with killing.

Similar to the large deviation, the moderate deviation problems also come from the theory of statistical inference. Using the moderate deviation principle (MDP), we can get the rate of convergence and an important method to construct asymptotic confidence intervals, for example, Liming [8], Guillin and Liptser [9], Cattani and Ciancio [10], and other references therein. There are also many works about MDP about stochastic (partial) differential equations; some surveys and literatures could be found in Budhiraja et al. [11], Wang and Zhang [12], Li et al. [13], Yang and Jiang [14], and the references therein. On

the other hand, fractional equations have attracted many physicists and mathematicians due to various applications in risk management, image analysis, and statistical mechanics (see Droniou and Imbert [15], Bakhoun and Toma [16], Levy and Pinchas [17], Mardani et al. [18], Niculescu et al. [19], Paun [20], and Pinchas [21] for a survey of applications). Stochastic partial differential equations involving a fractional Laplacian operator have been studied by many authors; see Mueller [22], Wu [23], Liu et al. [24], Wu [25], and the references therein.

Motivated above, we investigated the moderate deviations about the stochastic fractional heat equation with fractional noise as follows:

$$\begin{aligned} \frac{\partial v^\varepsilon}{\partial t}(t, x) &= D_{\delta, \alpha} v^\varepsilon(t, x) + f(v^\varepsilon(t, x)) + \sqrt{\varepsilon} B^H(dt, dx), \\ v^\varepsilon(o, x) &= 0, \end{aligned} \quad (1)$$

where $t \in [0, T]$, $x \in D = [0, 1]$, $D_{\delta, \alpha}$ is the fractional Laplacian operator which is defined in Appendix, and $B^H(dt, dx)$ denotes a fractional noise which is fractional in time and white in space with Hurst parameter $H \in (1/2, 1)$; that is, B^H is a mean zero Gaussian random field on $[0, T] \times D$ with covariance.

$$\text{Cov}(B^H(t, x)B^H(s, y)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})(x \wedge y). \quad (2)$$

Assume that the coefficients satisfy the following.

Assumption 1. Function f is Lipschitz; that is, there exist an $m > 0$ satisfying

$$|f(y) - f(x)| \leq m|y - x|, \quad \forall x, y \in D. \quad (3)$$

Under the conditions of Assumption 1, (1) possesses a unique solution in the sense of Walsh [26] as follows:

$$\begin{aligned} v^\varepsilon(t, x) = & \int_0^t \int_D G_\alpha^\delta(t - s, x - z) f(v^\varepsilon(s, z)) dz ds \\ & + \sqrt{\varepsilon} \int_0^t \int_D G_\alpha^\delta(t - s, x - z) B^H(ds, dz). \end{aligned} \quad (4)$$

As the parameter $\varepsilon \rightarrow 0$, the solution $v^\varepsilon(t, x)$ of (1) will tend to $v^0(t, x)$ which is the solution to the following equation:

$$\begin{aligned} \frac{\partial v^0}{\partial t}(t, x) &= D_{\delta, \alpha} v^0(t, x) + f(v^0(t, x)), \\ v^0(0, x) &= 0. \end{aligned} \quad (5)$$

This paper mainly devotes to investigate the deviations of v^ε from the deterministic solution v^0 , as $\varepsilon \rightarrow 0$, that is, the asymptotic behavior of the trajectories.

$$V^\varepsilon(t, x) := \frac{1}{\sqrt{\varepsilon} a(\varepsilon)} (v^\varepsilon - v^0)(t, x), \quad (t, x) \in [0, T] \times D, \quad (6)$$

where $a(\varepsilon)$ is the same deviation scale that strongly influences the asymptotic behavior of V^ε .

If $a(\varepsilon) = 1/\sqrt{\varepsilon}$, we are in the domain of large deviation estimate, which can be proved similarly to Jiang et al. [5].

The case $a(\varepsilon) \equiv 1$ provides the central limit theorem. As $\varepsilon \downarrow 0$, we will prove that $(v^\varepsilon - v^0)/\sqrt{\varepsilon}$ converges to a random field in this paper.

To fill the gap between scale $a(\varepsilon) = 1$ and scale $a(\varepsilon) = 1/\sqrt{\varepsilon}$, we mainly devote to the moderate deviation when the scale satisfies the following:

$$a(\varepsilon) \rightarrow +\infty, \quad \sqrt{\varepsilon} a(\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (7)$$

This paper is organized as follows. In Section 2, the definition of the fractional noise $B^H(ds, dz)$ is given. In Section 3, the main result is given and proved. In Appendix, some results about the Green kernel are given.

2. Fractional Noise

Let $H \in (1/2, 1)$, and $\{B^H([0, t] \times A)\}_{(t, A) \in [0, T] \times \mathcal{B}(\mathbb{R})}$ is a centered Gaussian family of random variables with the covariance satisfying

$$\mathbb{E}[B^H(t, A)B^H(s, B)] = |A \cap B| R_H(t, s), \quad (8)$$

with $s, t \in [0, T]$, $A, B \in \mathcal{B}(\mathbb{R})$ and covariance kernel $R_H(t, s) = (1/2)(t^{2H} + s^{2H} - |t - s|^{2H})$, where $|A|$ denotes the Lebesgue measure of the set $A \in \mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R})$ denotes the class of Borel sets in $\mathbb{R}$.

We denote φ as the set of step functions on $[0, T] \times \mathbb{R}$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of φ with respect to the scalar product.

$$\left\langle \mathbb{1}_{[0, t] \times A}, \mathbb{1}_{[0, s] \times B} \right\rangle_{\mathcal{H}} = \mathbb{E}[B^H(t, A)B^H(s, B)]. \quad (9)$$

According to Nualart and Ouknine [27], the mapping $\mathbb{1}_{[0, t] \times A} \rightarrow B^H(t, A)$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B^H)$ associated with B^H and denoted by

$$\phi \mapsto B^H(\phi) := \int_0^t \int_A \phi(s, x) B^H(ds, dx). \quad (10)$$

Define the linear operator $K_H^* : \varphi \mapsto L^2([0, T])$ by

$$K_H^*(\phi) = K_H(T, s)\phi(s, y) + \int_s^T (\phi(r, y) - \phi(s, y)) \frac{\partial K_H}{\partial r}(r, s) dr, \quad (11)$$

where K_H is defined by

$$K_H(t, s) := \begin{cases} C_H s^{(1/2)-H} \int_s^t (u-s)^{H-(3/2)} \left(1 - \left(\frac{s}{u}\right)^{(1/2)-H}\right) du + C_H (t-s)^{H-(1/2)}, & 0 < s \leq t, \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

with $C_H = (2\alpha\Gamma((3/2) - \alpha)/\Gamma(\alpha + (1/2))\Gamma(2 - 2\alpha))^{1/2}$, and one can get

$$\frac{\partial K_H}{\partial t}(t, s) = C_H \left(\frac{1}{2} - H\right) (t-s)^{H-(3/2)} \left(\frac{t}{s}\right)^{H-(1/2)}. \quad (13)$$

Moreover, K_H satisfies the following:

$$\int_0^{s \wedge t} K_H(s, r) K_H(t, r) dr = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}]. \quad (14)$$

Then, since

$$\left(K_H^* 1_{[0,t] \times A}\right)(s, x) = K_H(t, s) 1_{[0,t] \times A}(s, x), \quad (15)$$

one can get

$$\langle K_H^*(\phi), K_H^*(\psi) \rangle_{L^2([0,T] \times D)} = \langle \phi, \psi \rangle_{\mathcal{H}}, \quad (16)$$

where ϕ and ψ in φ are any step functions. So the operator K_H^* gives an isometry between the Hilbert space $\mathcal{H}$ and $L^2([0, T] \times [0, 1])$. Hence, $\{W(t, A), t \in [0, T], A \in \mathcal{B}[0, 1]\}$ defined by

$$W(t, A) = B^H \left((K_H^*)^{-1} \left(1_{[0,t] \times A} \right) \right) \quad (17)$$

is a space-time white noise, and B^H has the following form:

$$B^H(t, x) = \int_0^t \int_0^x K_H(t, s) W(ds, dy). \quad (18)$$

Therefore, the mild formulation of (4) has the following form:

$$\begin{aligned} v^\varepsilon(t, x) &= \int_0^t \int_D G_\alpha^\delta(t-s, x-z) f(v^\varepsilon(s, z)) dz ds \\ &\quad + \sqrt{\varepsilon} \int_0^t \int_D K_H^* G_\alpha^\delta(t-s, x-z) W(ds, dz). \end{aligned} \quad (19)$$

That is, the last term of (4) is equal to

$$\begin{aligned} &\int_0^t \int_D G_\alpha^\delta(t-s, x-z) B^H(ds, dz) \\ &= \int_0^t \int_D K_H^* G_\alpha^\delta(t-s, x-z) W(ds, dz). \end{aligned} \quad (20)$$

The following embedding proposition is given by Nualart and Ouknine [27].

Lemma 1. Set $H > (1/2)$, then, we have

$$L^{1/H}([0, T] \times \mathbb{R}) \subset \mathcal{H}. \quad (21)$$

3. Main Results and Their Proof

3.1. Main Results. For any function ϕ defined on $[0, T] \times D$, let

$$\begin{aligned} \|\phi\|_{t,\infty} &= \sup \{ |\phi(s, z)| : (s, z) \in [0, t] \times D \}, \\ \|\phi\|_{t,\gamma} &= \sup \left\{ \frac{|\phi(s_1, z_1) - \phi(s_2, z_2)|}{(|s_1 - s_2|^\mu + |z_1 - z_2|^\theta)^\gamma} : (s_1, z_1) \neq (s_2, z_2) \in [0, t] \times D \right\}, \end{aligned} \quad (22)$$

where $\mu \in (0, (\alpha + 1)H - 1/\alpha)$, $\theta \in (0, \min \{1, (\alpha + 1)H - 1\})$, and $\gamma \in (0, 1)$. Let

$$\|\phi\|_\gamma = \|\phi\|_{T,\infty} + \|\phi\|_{T,\gamma}. \quad (23)$$

Let $\mathcal{E}^\gamma([0, T] \times D)$ be the functions $\phi : [0, T] \times D$ which satisfy $\|\phi\|_\gamma < \infty$, endowed with the $\|\phi\|_\gamma$ -norm.

Define

$$\mathcal{E} = \left\{ e(x, t) = \int_0^t \int_0^x \bar{e}(s, z) ds dz, \quad \bar{e} \in L^2([0, T] \times D) \right\}, \quad (24)$$

which is a Cameron-Martin space endowed with the norm

$$\|e\|_{\mathcal{E}} = \left(\int_0^t \int_0^x \bar{e}^2(s, z) ds dz \right)^{1/2}. \quad (25)$$

Suppose $e \in \mathcal{E}$. Now, let v^e be the solution of the following deterministic equation:

$$\begin{aligned} \frac{\partial v^e}{\partial t}(t, x) &= D_{\delta,\alpha} v^e(t, x) + f(v^e(t, x)) + F(\bar{e}(t, x)), \\ v^e(0, x) &= 0, \end{aligned} \quad (26)$$

where

$$F(\bar{e}(x, t)) := \frac{\partial^2}{\partial x \partial t} \int_0^t \int_0^x K_H(t, s) \bar{e}(z, s) dz ds \quad (27)$$

satisfies the following:

$$\begin{aligned} &\int_0^t \int_D G_\alpha^\delta(t-s, x-z) F(\bar{e}(z, e)) dz ds \\ &= \int_0^t \int_D K_H^* G_\alpha^\delta(t-s, x-z) \bar{e}(z, s) dz ds. \end{aligned} \quad (28)$$

The above (26) shows a unique mild solution.

$$\begin{aligned} Z^e(x, t) &= \int_0^t \int_D G_\alpha^\delta(t-s, x-z) f(s, z, Z^e(z, s)) dz ds \\ &\quad + \int_0^t \int_D K_H^* G_\alpha^\delta(t-s, x-z) \bar{e}(z, s) dz ds. \end{aligned} \quad (29)$$

Similar to Jiang et al. [5], one can get the following:

Theorem 1. Let $H \in (1/2, 1)$. Under Assumption 1, the law of the solution to (1) satisfies a deviation principle on $\mathcal{C}^\gamma(D \times [0, T])$ with the good rate function:

$$I(\varphi) = \inf \left\{ \frac{1}{2} \|e\|_{\mathcal{G}}^2; Z^e = \varphi \right\}, \quad (30)$$

with the convention $\inf \emptyset = \infty$, where $0 < \gamma < 1$.

More precisely, for any Borel measurable subset B of $\mathcal{C}^\gamma([0, T] \times D)$,

$$\begin{aligned} -\inf_{\varphi \in B^\circ} I(\varphi) &\leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \log P(Z^e \in B) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(Z^e \in B) \leq -\inf_{\varphi \in \bar{B}} I(\varphi), \end{aligned} \quad (31)$$

where B° and $\bar{B}$ denote the interior and the closure of B , respectively.

We furthermore suppose that the coefficients satisfy the following.

Assumption 2. f is differentiable, and the derivative f' of f is Lipschitz. That is to say, there exist positive constant m and m' which satisfy the following:

$$|f'(y) - f'(x)| \leq m'|y - x|, \quad \forall x, y \in D. \quad (32)$$

Together with the Lipschitz of f , we conclude that

$$|f'(x)| \leq m. \quad (33)$$

Now, we give the following central limit theorem.

Theorem 2. Let f and its derivative f' satisfy Assumptions 1 and 2. Then, for $p \geq 1$, $(v^\varepsilon - v^0)/\sqrt{\varepsilon}$ converges in L^p to a random field U on $\mathcal{C}^\gamma([0, T] \times D)$ with $0 < \gamma < 1$, determined by

$$\begin{aligned} \frac{\partial U(t, x)}{\partial t} &= D_{\delta, \alpha} U(t, x) + f'(v^0(t, x)) U(t, x) + B^H(dt, dx), \\ U(0, x) &= 0, \end{aligned} \quad (34)$$

for all, $(t, x) \in [0, T] \times D$.

Let the function U^e be the solution to the following partial differential equation:

$$\begin{aligned} \frac{\partial U^e(t, x)}{\partial t} &= D_{\delta, \alpha} U^e(t, x) + f'(v^0(t, x)) U^e(t, x) + F(\bar{v}(t, x)), \\ U^e(0, x) &= 0. \end{aligned} \quad (35)$$

Under Assumptions 1 and 2, by Theorem 1, one can get $U/a(\varepsilon)$ which satisfies large deviation principles on $\mathcal{C}^\gamma([0, T] \times D)$ with the speed $\varepsilon^2(\varepsilon)$ and the good rate function satisfies the following:

$$I(\varphi) \begin{cases} \inf \left\{ \frac{1}{2} \|e\|_{\mathcal{G}}^2, & U^e = \varphi \right\}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (36)$$

Now, the second result is given as follows:

Theorem 3. In moderate deviation principle, let $H \in (1/2, 1)$. Under the Assumptions 1 and 2, then, the random field $(1/\sqrt{\varepsilon}a(\varepsilon))(v^\varepsilon - v^0)$ satisfies a large deviation principle on the space $\mathcal{C}^\gamma([0, T] \times D)$ with speed $a^2(\varepsilon)$ and the good rate function $I(\varphi)$ defined by (36), where $0 < \gamma < 1$.

3.2. Convergence of the Solution

Lemma 2. Let $H \in (1/2, 1)$. Under Assumption 1, then, there exists a unique solution to (1). Moreover, for any $p \in [1, \infty)$, $T > 0$,

$$\sup_{\varepsilon \geq 1} \sup_{(t, x) \in [0, T] \times D} E|v^\varepsilon(t, x)|^p < \infty. \quad (37)$$

As $\varepsilon \rightarrow 0$, we get the convergence of v^ε as follows:

Proposition 1. Let $H \in (1/2, 1)$ and $p \geq 2$. By Assumption 1, there exists a constant $c(m, p, H, T)$ which satisfies the following:

$$E\left(|v^\varepsilon - v^0|_{t, \infty}\right)^p \leq \varepsilon^{p/2} c(m, p, H, T). \quad (38)$$

Proof. Note that

$$\begin{aligned} v^\varepsilon(t, x) - v^0(t, x) &= \int_0^t \int_D G_\alpha^\delta(t-s, x-z) [f(v^\varepsilon(s, z)) - f(v^0(s, z))] dz ds \\ &\quad + \sqrt{\varepsilon} \int_0^t \int_D G_\alpha^\delta(t-s, x-z) B^H(ds dz). \end{aligned} \quad (39)$$

One can get

$$\begin{aligned} \left(|v^\varepsilon(t, x) - v^0(t, x)|_{t, \infty}\right)^p &\leq 2^{p-1} \left(\sup_{(t, x) \in [0, T] \times D} \left| \int_0^t \int_D G_\alpha^\delta(t-s, x-z) [f(v^\varepsilon(s, z)) - f(v^0(s, z))] dz ds \right| \right)^p \\ &\quad + \varepsilon^{p/2} 2^{p-1} \left(\sup_{(t, x) \in [0, T] \times D} \left| \int_0^t \int_D G_\alpha^\delta(t-s, x-z) B^H(ds dz) \right| \right)^p. \end{aligned} \quad (40)$$

Set

$$N_1^\varepsilon(t, x) = \int_0^t \int_D G_\alpha^\delta(t-s, x-z) [f(v^\varepsilon(s, z)) - f(v^0(s, z))] dz ds,$$

$$N_2^\varepsilon(t, x) = \int_0^t \int_D G_\alpha^\delta(t-s, x-z) B^H(dsdz). \quad (41)$$

Together with Hölder's inequality, the Lipschitz condition (C) and (5) of Lemma A.1, for $1/\alpha < q < \alpha$, we have

$$\begin{aligned} E(|N_1^\varepsilon|_{T, \infty})^p &\leq m^p \left(\sup_{(t, x) \in [0, T] \times D} G_\alpha^\delta(t-s, x-z) dz ds \right)^{p/q} \\ &\quad \times E \int_0^T (|v^\varepsilon - v^0|_{T, \infty})^p dt \\ &\leq C(p, m, T) E \int_0^T (|v^\varepsilon - v^0|_{T, \infty})^p dt, \end{aligned} \quad (42)$$

where $(1/p) + (1/q) = 1$.

For any $0 \leq t \leq T$, $x, z \in D$, $p > 2$, by (21) and (A.9), there exist $\theta \in (0, \min \{1, (\alpha+1)H-1\})$ which satisfies the following:

$$\begin{aligned} E |N_2^\varepsilon(t, x) - N_2^\varepsilon(t, x')|^p &= E \left| \int_0^t \int_D [G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t-s, x'-z)] B^H(dsdz) \right|^p \\ &\leq \left| \int_0^t \int_D [K_H^* (G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t-s, x'-z))]^2 dsdz \right|^{p/2} \\ &\leq \left(\int_0^t \int_D |G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t-s, x'-z)|^{1/H} dsdz \right)^{Hp} \\ &\leq m |x - x'|^{p\theta}. \end{aligned} \quad (43)$$

Similarly,

$$\begin{aligned} E |N_2^\varepsilon(t, x) - N_2^\varepsilon(t', x)|^p &= E \left| \int_0^t \int_D [G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z)] B^H(dsdz) \right|^p \\ &\leq \left| \int_0^t \int_D [K_H^* (G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z))]^2 dsdz \right|^{p/2} \\ &\leq C(p, m, T) \left| \int_{t'}^t \int_D [K_H^* (G_\alpha^\delta(t-s, x-z))]^2 dsdz \right|^{p/2} \\ &\quad + C(p, m, T) \left| \int_0^{t'} \int_D [K_H^* (G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z))]^2 dsdz \right|^{p/2}. \end{aligned} \quad (44)$$

By (21), (A.10), and (A.11), for $\mu \in (0, ((\alpha+1)H-1)/\alpha)$, one can get

$$\begin{aligned} &\left| \int_{t'}^t \int_D [K_H^* (G_\alpha^\delta(t-s, x-z))]^2 dsdz \right|^{p/2} \\ &\leq C_H \left(\int_{t'}^t \int_D |G_\alpha^\delta(t-s, x-z)|^{1/H} dsdz \right)^{Hp} \\ &\leq C_H (t - t')^{p((\alpha+1)H-1)/\alpha} \end{aligned} \quad (45)$$

and

$$\begin{aligned} &\left| \int_0^{t'} \int_D [K_H^* (G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z))]^2 dsdz \right|^{p/2} \\ &\leq C_H \left(\int_{t'}^t \int_D |K_H^* (G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z))|^{1/H} dsdz \right)^{Hp} \\ &\leq C_H (t - t')^{p\mu}. \end{aligned} \quad (46)$$

Together with (43), (44), (45), and (46), one can get for any $p \geq 2$ and $(t', x'), (t, x) \in [0, T] \times D$, there exists a constant $\beta > 0$ such that

$$E |N_2^\varepsilon(t, x) - N_2^\varepsilon(t, x')|^p \leq C(p, m, H, T) (|x - x'|^\theta + |t - t'|^\mu)^{p\beta}, \quad (47)$$

where $C(p, m, H, T)$ is independent of ε . For $p > 2/\beta$, by Garsis-Rodemich-Rumser's Lemma, there exist a constant C and a random variable $M_{p, \varepsilon}(\omega)$ satisfying

$$\begin{aligned} &|N_2^\varepsilon(t, x) - N_2^\varepsilon(t, x')|^p \\ &\leq m_{p, \varepsilon}(\omega) (|x - x'|^\theta + |t - t'|^\mu)^{p\beta-2} \\ &\quad \cdot \left(\log \left(\frac{c}{|x - x'|^\theta + |t - t'|^\mu} \right) \right)^2 \end{aligned} \quad (48)$$

and

$$\sup_{\epsilon \in (0,1)} E[m_{p,\epsilon}] < \infty. \quad (49)$$

If we choose $t' = 0$ in (48), we can get

$$\begin{aligned} E \left(\sup_{(t,x) \in [0,T] \times D} \int_0^t \int_D G_\alpha^\delta(t-s, x-z) B^H(ds dz) \right)^p \\ \leq c(m, p, H, T) \sup_{\epsilon \in (0,1)} E[m_{p,\epsilon}] < \infty. \end{aligned} \quad (50)$$

By Gronwall's inequality and (40), (42), and (48), there exists a constant $C(p, m, H, T)$ satisfying

$$E \left(|v^\epsilon - v^0|_{t,\infty} \right)^p \leq \epsilon^{p/2} c(m, p, H, T). \quad (51)$$

The proof is completed.

3.3. The Proof of the Main Results. We first prove Theorem 1. Our proof is based on the following proposition (see Doss and Priouret [28]).

Proposition 2. Suppose $\{X_j^\epsilon, \epsilon > 0\}$ and $(j = 1, 2)$ are two families of random variables and (E_j, d_j) and $(j = 1, 2)$ are two Polish spaces. Suppose that

- (1) There exists a map $K : \{I_1 < \infty\} \rightarrow E_1$ such that $\{I_1 \leq a\} \rightarrow E_1$ is continuous for any $a < \infty$.
- (2) $\{X_1^\epsilon, \epsilon > 0\}$ satisfies with the rate functions $I_1 : E_1 \rightarrow [0, \infty]$.
- (3) For any $R, \delta, a > 0$, there exist $\rho > 0$ and $\epsilon_0 > 0$ satisfying $I_1 \leq a$ and $\epsilon \leq \epsilon_0$ for all $h \in E_1$, and

$$P(d_1(X_1^\epsilon) \leq \rho, d_2(x_2^\epsilon, K(e)) \geq \delta) \leq \exp \left(-\frac{R}{\epsilon^2} \right). \quad (52)$$

Then, $\{X_2^\epsilon, \epsilon > 0\}$ satisfies a LDP with the rate functions

$$I(h) = \inf \{I_1(e) : K(e) = h\}. \quad (53)$$

To prove Theorem 1, one only needs to prove

- (i) Under some topology, $Z(e) : \{e : \|e\|_{\mathcal{G}} \leq a\} \rightarrow C([0, T], L^p(D))$ is continuous for any $a > 0$.
- (ii) In Freidin-Wentzell inequality, for any $R > 0, \eta > 0$, and $e \in \mathcal{E}$, there exist a $\delta > 0$ satisfying

$$\lim_{\epsilon \rightarrow 0} \sup \epsilon^2 \log P \left(\|\epsilon W - e\|_\infty < \delta, \sup_{t \in [0, T]} u^\epsilon(t, \cdot) - Z(e)(t, \cdot) \|_2 \geq \rho \right) \leq -R. \quad (54)$$

Theorem 4. When the level set $\{\|e\|_{\mathcal{G}} \leq a\}$ is endowed with the topology of uniform convergence on $[0, T] \times D$,

$$Z(e) : \{e : \|e\|_{\mathcal{G}} \leq a\} \rightarrow C([0, T], L^p(D)) \quad (55)$$

is the continuous map for any $a \in [0, +\infty)$.

Proof. One only needs to prove that for fixed $a > 0, e, h \in \{\|e\|_{\mathcal{G}} \leq a\}$,

$$\sup_{t \in [0, T]} \int_{\mathbb{R}} |Z(e)(t, x) - Z(h)(t, x)|^2 dx \leq C \|e - h\|_{\mathcal{G}}. \quad (56)$$

Note that

$$\begin{aligned} Z(e)(t, x) - Z(h)(t, x) &= \int_0^t \int_{\mathbb{R}} G_\alpha^\delta(t-s, x-z) (f(s, z, Z(e)(s, z)) - f(s, z, Z(h)(s, z))) dz ds \\ &\quad + \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, x-z) \bar{e}(s, z) dz ds := I(t, x) + II(t, x). \end{aligned} \quad (57)$$

Using (A.5) in Appendix with $p = q = 2$, then $\rho = 1$, one can get

$$\|I(t, x)\|_2^2 \leq C \int_0^t (t-s)^{-(1/\alpha)+(1/\alpha\rho)} \|f(s, z, Z(e)(s, z)) - f(s, z, Z(h)(s, z))\|_2 ds \leq C \sup_{0 \leq s \leq t} \|Z(e)(s, \cdot) - Z(h)(s, \cdot)\|_2. \quad (58)$$

Now, we deal with $\Pi(t, x)$, together with

$$\begin{aligned} \|\Pi(t, x)\|_2^2 &= \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, x-z) (\bar{e}(s, z) - \bar{h}(s, z)) dz ds \right|^2 dx \\ &\leq C \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \left| K_H^* G_\alpha^\delta(t-s, x-z) \right|^2 dz ds dx \times \|e - h\|_E^2 := CM(t) \times \|e - h\|_{\mathcal{E}}^2. \end{aligned} \quad (59)$$

Recalling K_H^* is defined by (11), one can get

$$(K_H^* \phi)(s, x) \leq \max_{w \in [s, T]} \phi(w, x) K_H(T, s),$$

$$\begin{aligned} M(t) &= \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \left| K_H^* G_\alpha^\delta(t-s, x-z) \right|^2 dz ds dx \\ &\leq \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \left| \max_{w \in [s, t]} G_\alpha^\delta(t-s, x-z) \right|^2 K_H^2(t, s) dz dx ds \\ &\leq C \int_0^t (t-s)^{-(1/\alpha)} (t-s)^{2H-1} ds < \infty. \end{aligned} \quad (60)$$

Using Gronwall's inequality, we can get

$$\sup_{t \in [0, T]} \int_{\mathbb{R}} |Z(e)(t, x) - Z(h)(t, x)|^2 dx \leq C \|e - h\|_{\mathcal{E}}. \quad (61)$$

The proof of the theorem is completed.

We now prove the Freidin-Wentzell inequality as follows:

Suppose $\{W(t, x), (t, x) \in [0, T] \times \mathbb{R}\}$ is a Brownian sheet. For $e \in \mathcal{E}$ and $\epsilon > 0$, we define

$$\tilde{W}(t, x) = W(t, x) - \frac{e(x, t)}{\epsilon} \quad (62)$$

and

$$\frac{\tilde{d}p}{dp} = \exp \left(\frac{1}{\epsilon} \int_0^T \int_{\mathbb{R}} \bar{e}(t, x) W(dt, dx) - \frac{1}{2\epsilon^2} \int_0^T \int_{\mathbb{R}} \bar{e}^2(t, x) dx dt \right) := Z_T. \quad (63)$$

Using Girsanov's theorem, the process $\tilde{W}$ is a Brownian sheet under $\tilde{P}$. Suppose $\tilde{v}^\epsilon(t, x)$ is a solution of (1) under $\tilde{p}$. Then,

$$\begin{aligned} \tilde{v}^\epsilon(t, x) &= \int_{\mathbb{R}} G_\alpha^\delta(t, x-z) v_0(z) dz \\ &+ \int_0^t \int_{\mathbb{R}} G_\alpha^\delta(t-s, x-z) f(t, x, \tilde{v}^\epsilon(s, z)) dz ds \\ &+ \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, x-z) \bar{e}(s, z) dz ds \\ &+ \epsilon \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, x-z) \tilde{W}(ds, dz). \end{aligned} \quad (64)$$

Now, one can prove (36). Note that, under $\tilde{p}$, then,

$$\begin{aligned} \|\tilde{u}^\epsilon(t, \cdot) - Z(e)(t, \cdot)\|_2 &= \left\| \int_0^t \int_{\mathbb{R}} G_\alpha^\delta(t-s, \cdot-z) (f(t, x, \tilde{u}^\epsilon(s, z)) - f(t, x, Z(e)(s, z))) dz ds \right\|_2 \\ &+ \left\| \epsilon \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, \cdot-z) \tilde{W}(ds, dz) \right\|_2 \\ &\leq \int_0^t \int_{\mathbb{R}} \|\tilde{u}^\epsilon(s, \cdot) - Z(e)(s, \cdot)\|_2 dz ds + \left\| \epsilon \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, \cdot-z) \tilde{W}(ds, dz) \right\|_2. \end{aligned} \quad (65)$$

So under $\tilde{p}$, by Gronwall's Lemma, one can get

$$\sup_{t \in [0, T]} \|\tilde{u}^\epsilon(s, \cdot) - Z(h)(s, \cdot)\|_2 \leq C \sup_{t \in [0, T]} \left\| \epsilon \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, x-z) \tilde{W}(ds, dz) \right\|_2. \quad (66)$$

Now, one can change (34) the proof to the following theorem.

Theorem 5. Suppose $e \in \mathcal{E}$ and $e \in \mathcal{E} \leq a$. For each $R > 0$, $\eta > 0$, and $e \in \mathcal{E}$, there exists a constant $\delta > 0$ satisfying

$$\lim_{\epsilon \rightarrow 0} \sup \epsilon^2 \log \tilde{P} \left(\sup_{t \in [0, T]} \left\| \epsilon \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, x-z) \tilde{W}(ds, dz) \right\|_2 \geq \rho, \|\epsilon \tilde{W}\|_\infty < \delta \right) \leq -R. \quad (67)$$

In the following, we give a key Lemma to prove Theorem 1, which is similar to Candon-Weber [2], and the proof is omitted.

Lemma 3. Suppose $F : ([0, T] \times D)^2 \rightarrow \mathbb{R}$, $C_F > 0$ and $\beta_0 > 0$ satisfying

$$\int_0^T \int_D |F(t, x, v, z) - F(s, y, v, z)|^2 dv dz \leq C_F \left(|t-s|^\mu + |x-y|^\theta \right)^{\beta_0}, \quad (68)$$

for any (s, z) and $(t, x) \in [0, T] \times D$. Suppose $N : \Omega \times [0, T] \times [0, 1] \rightarrow \mathbb{R}$ which is an almost surely continuous, $\mathcal{F}_t$ -adapted process satisfying $\sup \{|N(t, x)| : (t, x) \in [0, T] \times D\} \leq \rho$, a, s , and for $(t, x) \in [0, T] \times D$, suppose

$$\mathfrak{F}(t, x) = \int_0^T \int_D F(t, x, v, z) N(v, z) W(dv dz). \quad (69)$$

Then, for any $0 < \beta < \beta_0/2$, there exist a positive constant C , $\zeta(\beta, \beta_0)$, and $C(\beta, \beta_0)$ such that for all $M \geq \rho C_F^{1/2} \zeta(\beta, \beta_0)$ $C(\beta, \beta_0)$,

$$P \left(\|\mathfrak{F}\|_{t, \beta} \geq M \right) \leq C \exp \left(- \frac{M^2}{\rho^2 C_F C \beta, \beta_0 C^2(\beta, \beta_0)} \right). \quad (70)$$

Proof of Theorem 5. Suppose

$$F(t, x, v, z) = \epsilon K_H^* G_\alpha^\delta(t-v, x-z). \quad (71)$$

Then, there exists $\beta_0 > 0$ satisfying

$$\begin{aligned} & \int_0^t \int_D \left| K_H^* \left(G_\alpha^\delta(t-v, x-z) - G_\alpha^\delta(s-v, y-z) \right) \right|^2 dv dz \\ & \leq C \left(\int_0^t \int_{\mathbb{R}} \left| K_H^* \left(G_\alpha^\delta(t-v, x-z) - G_\alpha^\delta(s-v, x-z) \right) \right|^2 dv dz + \int_0^t \int_{\mathbb{R}} \left| K_H^* \left(G_\alpha^\delta(s-v, x-z) - G_\alpha^\delta(s-v, y-z) \right) \right|^2 dv ds \right) \\ & \leq C \left(\left(\int_0^t \int_{\mathbb{R}} \left| G_\alpha^\delta(t-v, x-z) - G_\alpha^\delta(s-v, x-z) \right|^{1/H} dv ds \right)^{2H} + \left(\int_0^t \int_{\mathbb{R}} \left| G_\alpha^\delta(s-v, x-z) - G_\alpha^\delta(s-v, y-z) \right|^{1/H} dv dz \right)^{2H} \right) \\ & \leq C(|t-s|^\mu + |x-z|^\theta)^{\beta_0}. \end{aligned} \quad (72)$$

If $R \leq (\rho^2/2C_F)C^2(V)$ and $\rho \leq K(v)C(v)\sqrt{C_F}$, by Lemma 2, we can get

$$\begin{aligned} & \tilde{P} \left(\sup_{t \in [0, T]} \left\| \epsilon \int_0^t \int_{\mathbb{R}} K_H^* G_\alpha^\delta(t-s, x-z) \tilde{W}(ds, dz) \right\|_2 \geq \rho \right) \\ & \leq \left(\sqrt{(2)T^2\pi^2 + 1} \right) \exp \left(- \frac{R}{\epsilon^2} \right). \end{aligned} \quad (73)$$

The proof of Theorem 3 is completed.

In the following, we first give Garsis-Rodemich-Rumser's Lemma in Bally et al. [29].

Lemma 4. Assume $p > 1$ and $\{U^\epsilon(t, x) : (t, x) \in [0, T] \times D\}$ are a family of real-valued stochastic processes. Suppose the following is true.

Assumption 3. One has

$$\lim_{\epsilon \rightarrow 0} E|U^\epsilon(t, x)|^p = 0 \quad (74)$$

as $(t, x) \in [0, T] \times D$.

Assumption 4. For any $(t, x), (s, z) \in [0, T] \times D$, there exists $\nu > 0$ satisfying

$$E|U^\varepsilon(t, x) - U^\varepsilon(s, z)|^p \leq C(|t - s| + |x - z|^2)^\nu. \quad (75)$$

Then, for any $r \in [1, p), \gamma \in (0, \beta/p)$,

$$\lim_{\varepsilon \rightarrow 0} E \left(\|U^\varepsilon\|_\gamma \right)^r = 0. \quad (76)$$

Now we can prove Theorem 2.

Proof of Theorem 2. Set $U^\varepsilon = (\nu^\varepsilon - \nu^0)/\sqrt{\varepsilon}$. We will prove that

$$\lim_{\varepsilon \rightarrow 0} E \left(\|U^\varepsilon - U\|_\gamma \right)^r = 0. \quad (77)$$

To this end, we need to prove that Assumptions 3 and 4 are satisfied for $W^\varepsilon = U^\varepsilon - U$. Note that

$$\begin{aligned} U^\varepsilon(t, x) - U(t, x) &= \int_0^t \int_D G_\alpha^\delta(t-s, x-z) \\ &\quad \cdot \left(\frac{f(\nu^\varepsilon(s, z)) - f(\nu^0(s, z))}{\sqrt{\varepsilon}} - f'(\nu^0(s, z))U(s, z) \right) dsdz \\ &=: \text{III}^\varepsilon(t, x) + \text{IV}^\varepsilon(t, x) \end{aligned} \quad (78)$$

where

$$\begin{aligned} \text{III}^\varepsilon(t, x) &= \int_0^t \int_D G_\alpha^\delta(t-s, x-z) \\ &\quad \cdot \left(\frac{f(\nu^\varepsilon(s, z)) - f(\nu^0(s, z))}{\sqrt{\varepsilon}} - f'(\nu^0(s, z))U^\varepsilon(s, z) \right) dsdz \end{aligned} \quad (79)$$

and

$$\text{IV}^\varepsilon(t, x) = \int_0^t \int_D G_\alpha^\delta(t-s, x-z) f'(\nu^0(s, z)) (U^\varepsilon(s, z) - U(s, z)) dsdz. \quad (80)$$

Using Taylor's formula, there exists a $\xi^\varepsilon(t, x)$ such that

$$\begin{aligned} f(\nu^\varepsilon(t, x)) - f(\nu^0(t, x)) &= f'(\nu^0(t, x) + \xi^\varepsilon(t, x))(\nu^\varepsilon(t, x) - \nu^0(t, x)) \\ &\quad \times (\nu^\varepsilon(t, x) - \nu^0(t, x)). \end{aligned} \quad (81)$$

Note that f' is Lipschitz continuous and $\xi^\varepsilon(t, x) \in (0, 1)$; one can get

$$\begin{aligned} |f'(\nu^0(t, x) + \xi^\varepsilon(t, x))(\nu^\varepsilon(t, x) - \nu^0(t, x))| &\times (\nu^\varepsilon(t, x) - \nu^0(t, x)) \\ &\leq m' \xi^\varepsilon(t, x) |\nu^\varepsilon(t, x) - \nu^0(t, x)| \leq m' |\nu^\varepsilon(t, x) - \nu^0(t, x)|. \end{aligned} \quad (82)$$

So

$$\begin{aligned} \text{III}^\varepsilon(t, x) &\leq m' \int_0^t \int_D G_\alpha^\delta(t-s, x-z) |\nu^\varepsilon(t, x) - \nu^0(t, x)| U^\varepsilon(s, z) dsdz \\ &= \sqrt{\varepsilon} m' \int_0^t \int_D G_\alpha^\delta(t-s, x-z) |U^\varepsilon(s, z)|^2 dsdz. \end{aligned} \quad (83)$$

Using Hölder's inequality, for $p < 2$ and $1/\alpha < q < \alpha$, one can get

$$\begin{aligned} E(|\text{III}^\varepsilon|_\infty^t)^p &\leq \varepsilon^{p/2} m'^p \left(\sup_{(t,x) \in [0,T] \times D} \left| \int_0^t \int_D (G_\alpha^\delta(t-s, x-z))^q dsdz \right| \right)^{p/q} \\ &\quad \times \int_0^t E(|U^\varepsilon|_\infty^s)^{2p} ds, \end{aligned} \quad (84)$$

where $(1/p) + (1/q) = 1$. Together with (5) of Lemma A.1 and Proposition 1, there exists a constant $C(p, m, m', T)$ only depending on p, m, m', T satisfying

$$E(|\text{III}^\varepsilon|_\infty^t)^p \leq \varepsilon^{p/2} C(p, M, M', T). \quad (85)$$

Since $|f'| \leq M$, for $p > 2$, together with Hölder's inequality and (5) of Lemma A.1, we can get

$$\begin{aligned} E(|\text{IV}^\varepsilon|_\infty^t)^p &\leq M^p \left(\sup_{(t,x) \in [0,T] \times D} \left| \int_0^t \int_D (G_\alpha^\delta(t-s, x-z))^q dsdz \right| \right)^{p/q} \\ &\quad \times E \int_0^t (|U^\varepsilon - U|_\infty^s)^p ds, \end{aligned} \quad (86)$$

where $(1/p) + (1/q) = 1$. Together with (52), (56), and (57), we can get

$$E(|U^\varepsilon - U|_\infty^t)^p \leq C(p, m, m', T) \left(\varepsilon^{p/2} + E \int_0^t (|U^\varepsilon - U|_\infty^s)^p ds \right). \quad (87)$$

Using Gronwall's inequality, one can get

$$E(|U^\varepsilon - U|_\infty^t)^p \leq \varepsilon^{p/2} C(p, m, m', T) \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (88)$$

which implies Assumption 3 in Lemma 4.

Now we prove Assumption 4 in Lemma 4. We will prove III^ε and IV^ε satisfy Assumption 4 in Lemma 4. Using Hölder's inequality and (A.9), one can get

$$\begin{aligned} E \left| \text{III}^\varepsilon(t, x) - \text{III}^\varepsilon(t, x') \right|^p &\leq E \left| \sqrt{\varepsilon} m' \int_0^t \int_D \left[G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t-s, x'-z) \right] |U^\varepsilon(s, z)|^2 ds dz \right|^p \\ &\leq \varepsilon^{p/2} M'^p \left(\left| \int_0^t \int_D \left| G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t-s, x'-z) \right|^q ds dz \right|^{p/q} \times \left(\int_0^t \int_D E(|Y^\varepsilon(s, z)|_\infty^s)^{2p} ds dz \right)^{1/p} \right) \\ &\leq \varepsilon^{p/2} C(p, m, m', T) |x - x'|^{p\theta'}, \end{aligned} \quad (89)$$

where $\theta' \in (0, \min \{1, (\alpha+1)/q-1\})$ and $1/p + 1/q = 1$. Similarly,

$$\begin{aligned} E \left| \text{IV}^\varepsilon(t, x) - \text{IV}^\varepsilon(t', x) \right|^p &\leq E \left| \sqrt{\varepsilon} m' \int_0^t \int_D \left[G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z) \right] |U^\varepsilon(s, z)|^2 ds dz \right|^p \\ &\leq \varepsilon^{p/2} C(p, m, T) \left| \int_0^t \int_D \left[G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z) \right] |U^\varepsilon(s, z)|^2 ds dz \right|^p \\ &\leq \varepsilon^{p/2} C(p, m, T) \left| \int_{t'}^t \int_D \left[G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z) \right] |U^\varepsilon(s, z)|^2 ds dz \right|^p \\ &\quad + \varepsilon^{p/2} C(p, m, T) \left| \int_0^{t'} \int_D G_\alpha^\delta(t-s, x-z) U^\varepsilon(s, z) \right|^2 ds dz \leq C(p, m, m', T) |t - t'|^{p\mu}, \end{aligned} \quad (90)$$

where $\mu' \in (0, (\alpha+1/q-1)/\alpha)$. Together with (89) and (90), we can get

$$E |\text{III}^\varepsilon(t, x) - \text{III}^\varepsilon(s, z)|^p \leq C(|t-s| + |x-z|^2)^\beta \quad (91)$$

Similarly, one can get

$$E |\text{IV}^\varepsilon(t, x) - \text{IV}^\varepsilon(s, z)|^p \leq C(|t-s| + |x-z|^2)^\beta. \quad (92)$$

Together with (91) and (92), we can get

$$\begin{aligned} E |(U^\varepsilon(t, x) - U(t, x)) - (U^\varepsilon(s, z) - U(s, z))|^p \\ \leq C(|t-s| + |x-z|^2)^\beta. \end{aligned} \quad (93)$$

Then, for any $p > 2$ and $q \in (1, \alpha)$ such that $\gamma \in (0, \beta/p)$ and $r \in [1, p)$, using Lemma 4,

$$\lim_{\varepsilon \rightarrow 0} E \left(\|V^\varepsilon\|_\gamma \right)^r = 0. \quad (94)$$

The proof of Theorem 2 is completed.

Proof of Theorem 3. By Theorem 1, $U/a(\varepsilon)$ obeys large deviation principles on $\mathcal{C}^\nu([0, T] \times D)$, with the rate function I given by (30) and the speed function $h^2(\varepsilon)$. Using Dembo and Zeitouni ([30] Theorem 4.2.13), to prove the large deviation principles of $U^\varepsilon/a(\varepsilon)$ is $e^2(\varepsilon)$ -exponentially equivalent to $U/a(\varepsilon)$, that is,

$$\lim_{\varepsilon \rightarrow 0} \sup e^{-2(\varepsilon)} \log P \left(\frac{\|U^\varepsilon - U\|_{\mu', \theta'}}{a(\varepsilon)} > \delta \right) = -\infty \quad (95)$$

holds for any $\delta > 0$.

Since

$$\|U^\varepsilon - U\|_\gamma \leq C(T, \mu', \theta') \|U^\varepsilon - U\|_{T, \gamma}. \quad (96)$$

To prove (67), we only need to prove

$$\lim_{\varepsilon \rightarrow 0} \sup e^{-2(\varepsilon)} \log P \left(\frac{\|U^\varepsilon - U\|_{\mu', \theta'}}{a(\varepsilon)} > \delta \right) = -\infty. \quad (97)$$

Note the decomposition

$$U^\varepsilon(t, x) - U(t, x) = \text{III}^\varepsilon(t, x) + \text{IV}^\varepsilon(t, x). \quad (98)$$

For any $q \in (1, \alpha)$, $x, x' \in [0, 1]$, $(1/p) + (1/q) = 1$, and $0 \leq s \leq t \leq T$, by Hölder's inequality, (32), and (A.9), we can get

$$\begin{aligned} |IV^\varepsilon(t, x) - IV^\varepsilon(t, x')| &\leq m' \left| \int_0^t \int_D [G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t-s, x'-z)] (U^\varepsilon(s, z) - U(s, z)) ds dz \right| \\ &\leq m' \left(\left| \int_0^t \int_D |G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t-s, x'-z)|^q ds dz \right|^{1/q} \times \left(\int_0^t \int_D |U^\varepsilon(s, z) - U(s, z)|^p ds dz \right)^{1/p} \right) \\ &\leq M' |x - x'|^\theta \times \left(\int_0^t (|U^\varepsilon - U|_\infty^u)^p \right)^{1/p}. \end{aligned} \quad (99)$$

where $\theta' \in [0, \min \{1, (\alpha + 1)/q - 1\}]$. Similarly,

$$\begin{aligned} |IV^\varepsilon(t, x) - IV^\varepsilon(t', x)| &\leq m' \left| \int_0^{t'} \int_D [G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z)] (U^\varepsilon(s, z) - U(s, z)) ds dz \right| \\ &\leq m' \left(\left| \int_0^{t'} \int_D |G_\alpha^\delta(t-s, x-z) - G_\alpha^\delta(t'-s, x-z)|^q ds dz \right|^{1/q} \times \left(\int_0^{t'} \int_D |U^\varepsilon(s, z) - U(s, z)|^p ds dz \right)^{1/p} \right) \\ &\quad + m' \left(\left| \int_{t'}^t \int_D |G_\alpha^\delta(t-s, x-z)|^q ds dz \right|^{1/q} \times \left(\int_{t'}^t \int_D |U^\varepsilon(s, z) - U(s, z)|^p ds dz \right)^{1/p} \right) \\ &\leq m' |t - t'|^{\mu'} \times \left(\int_0^t (|U^\varepsilon - U|_\infty^u)^p \right)^{1/p}. \end{aligned} \quad (100)$$

where $\mu' \in (0, (\alpha + 1)/q - 1/\alpha)$. Together with (99) and (100), we can get

$$E|IV^\varepsilon(t, x) - IV^\varepsilon(s, z)|^p \leq C(|t - s|^{\mu'} + |x - z|^{\theta'}). \quad (101)$$

Note that $|U^\varepsilon - U|_{s, \infty} \leq C(s, \mu', \theta')|U^\varepsilon - U|_{s, \gamma}$, we have

$$|IV^\varepsilon(t, x)|_{t, \gamma} \leq C \left(\int_0^t (C(s, \mu', \theta')|U^\varepsilon - U|_{s, \gamma})^p ds \right). \quad (102)$$

Thus, for $t \in [0, T]$, we have

$$\left(|U^\varepsilon - U|_{t, \gamma} \right)^p \leq C(p, m, T) \left[\left(|III^\varepsilon|_{t, \gamma} \right) + \int_0^t \left(|U^\varepsilon - U|_{s, \gamma} \right)^p ds \right]. \quad (103)$$

Applying Gronwall's Lemma to $f(t) = (|U^\varepsilon - U|_{t, \gamma})^p$, we have

$$\left(|U^\varepsilon - U|_{T, \gamma} \right)^p \leq C(p, T, m) \left(|III^\varepsilon|_{T, \gamma} \right)^p. \quad (104)$$

By (64) to prove (67), we only need to prove that for any $\delta > 0$,

$$\limsup_{\varepsilon \rightarrow 0} e^{-2}(\varepsilon) \log P \left(\frac{|I^\varepsilon|_{T, \gamma}}{a(\varepsilon)} \geq \delta \right) = -\infty. \quad (105)$$

Note that

$$III^\varepsilon(t, x) = \int_0^t \int_D G_\alpha^\delta(t-s, x-z) \mathfrak{I}^\varepsilon(s, z) ds dz, \quad (106)$$

where

$$\mathfrak{I}^\varepsilon(s, z) = \left(\frac{f(v^\varepsilon(s, z)) - f(v^0(s, z))}{\sqrt{\varepsilon}} - f'(v^0(s, z)) U^\varepsilon(s, z) \right). \quad (107)$$

By the same method in the proof of (59), we have

$$\|III^\varepsilon\|_{T, \gamma} \leq C(T, \mu', \theta') \|\mathfrak{I}^\varepsilon\|_{T, \infty}. \quad (108)$$

Similar to the proof of (53), we have

$$|\mathfrak{A}^\varepsilon|_{T,\infty} \leq m' \frac{\left(|v^\varepsilon - v^0|_{T,\infty}\right)^2}{\sqrt{\varepsilon}}. \quad (109)$$

Together with (40) and (A.11), for any $t \in [0, T]$, we can get that

$$\begin{aligned} & |v^\varepsilon - v^0|_{t,\infty} \\ & \leq \sup_{(t,x) \in [0,T] \times D} \left| \int_0^t \int_D G_\alpha^\delta(t-s, x-z) m |v^\varepsilon(s, z) - v^0(s, z)| ds dz \right| \\ & \quad \cdot \sup_{(t,x) \in [0,T] \times D} \left| \sqrt{\varepsilon} \int_0^t \int_D G_\alpha^\delta(t-s, x-z) B^H(ds dz) \right| \\ & \leq C(m, T) \int_0^t |v^\varepsilon - v^0|_{s,\infty} ds + |\text{IIV}^\varepsilon|_{t,\infty}, \end{aligned} \quad (110)$$

where

$$\begin{aligned} \text{IIV}^\varepsilon(t, x) &= \sqrt{\varepsilon} \int_0^t \int_D G_\alpha^\delta(t-s, x-z) B^H(ds dz) \\ &= \sqrt{\varepsilon} \int_0^t \int_D K_H^* G_\alpha^\delta(t-s, x-z) W(ds dz). \end{aligned} \quad (111)$$

By Gronwall's Lemma, one can get

$$|v^\varepsilon - v^0|_{T,\infty} \leq C(M, T) |\text{IIV}^\varepsilon|_{T,\infty} \leq C(M, T) |\text{IIV}^\varepsilon|_\gamma. \quad (112)$$

Applying Lemma 4 with

$$\begin{aligned} F(t, x, s, z) &= K_H^* G(t-s, x-z) 1_{s \leq t}, \beta_0 = \frac{1}{2}, C_F = C, \\ \rho &= \sqrt{\varepsilon} M \left(1 + |v^0|_{T,\infty} + \theta\right), N(t, x) = \sqrt{\varepsilon} 1_{[|v^\varepsilon|_{T,\infty} < |v^0|_{T,\infty} + \theta]}, \end{aligned} \quad (113)$$

for any fixed $\theta > 0$, one can get that for any β , $G \geq \sqrt{\varepsilon} K(1 + |v^0|_{T,\infty} + \theta) C_F^{1/2} \zeta(\beta, \beta_0) C(\beta, \beta_0)$,

$$\begin{aligned} & P\left(|\text{IIV}^\varepsilon|_{\beta, \beta} \geq M, |v^\varepsilon|_{T,\infty} < |v^0|_{T,\infty} + \theta\right) \\ & \leq L \exp\left(-\frac{G^2}{\varepsilon M^2 C C^2(\beta, \beta_0(1 + |v^0|_{T,\infty} + \theta)^2)}\right). \end{aligned} \quad (114)$$

We can get

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} e^{-2}(\varepsilon) \log P\left(|v^\varepsilon|_{T,\infty} \geq |v^0|_{T,\infty} + \theta\right) \\ & \leq \limsup_{\varepsilon \rightarrow 0} e^{-2}(\varepsilon) \log P\left(|v^\varepsilon - v^0|_{T,\infty} \geq \theta\right) = -\infty. \end{aligned} \quad (115)$$

Together with (73), (78), (82), (83), and (85), we have

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} e^{-2}(\varepsilon) \log P\left(\left(\frac{\|\text{III}\|_{T, \mu', \theta'}}{a(\varepsilon)}\right) > \delta\right) \\ & \leq \limsup_{\varepsilon \rightarrow 0} e^{-2}(\varepsilon) \log P\left(\left(\frac{\|\text{III}\|_{T, \mu', \theta'}}{C(\beta, T, m)}\right)^2 > \frac{\sqrt{\varepsilon} a(\varepsilon) \delta}{C(\beta, T, m)}\right) \\ & \leq \limsup_{\varepsilon \rightarrow 0} e^{-2}(\varepsilon) \log \left[P\left(\left(\frac{\|\text{III}\|_{T, \mu', \theta'}}{C(\beta, T, m)}\right)^2 > \frac{\sqrt{\varepsilon} a(\varepsilon) \delta}{C(\beta, T, m)}, |v^\varepsilon|_{T,\infty} < |v^0|_{T,\infty} + \theta\right) + P\left(|v^\varepsilon|_{T,\infty} \geq |v^0|_{T,\infty} + \theta\right) \right] \\ & \leq \left(\limsup_{\varepsilon \rightarrow 0} \frac{-\delta}{\sqrt{\varepsilon} a(\varepsilon) C(\beta, T, m) m^2 C C^2(\beta, \beta_0) (1 + |v^0|_{T,\infty} + \theta)^2} \right) \vee \left(\limsup_{\varepsilon \rightarrow 0} e^{-2}(\varepsilon) \log P\left(|v^\varepsilon|_{T,\infty} \geq |v^0|_{T,\infty} + \theta\right) \right) = -\infty. \end{aligned} \quad (116)$$

This completes the proof.

Appendix

Green Function

The nonlocal fractional differential operator $D_{\delta, \alpha}$ is defined by

$$\mathcal{F}(D_{\delta, \alpha} \phi)(\xi) = -|\xi|^\alpha \exp\left(-i\delta \frac{\pi}{2} \text{sgn}(\xi)\right) \mathcal{F}(\phi)(\xi), \quad (\text{A.1})$$

where α is called the index of stability and δ improperly referred to as the skewness which satisfy $|\delta| \leq \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$ and $\delta = 0$ when $\delta \in 2\mathbb{N} + 1$.

The operator $D_{\delta, \alpha}$ is a closed, densely defined operator on $L^2(\mathbb{R})$, and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. This operator is a generalization of various well-known operators, such as the Laplacian operator (when $\alpha = 2$), the inverse of the generalized Riesz-Feller potential (when $\alpha > 2$), and the

Riemann-Liouville differential operator (when $|\delta| = 2 + [\alpha]_2$ or $|\delta| = \alpha - [\alpha]$). It is self-adjoint only when $\delta = 0$, and in this case, it coincides with the fractional power of the Laplacian. We refer the readers to Debbi and Dozzi [31] for more details about this operator.

According to Komatsu [32], $D_{\delta,\alpha}$ can be represented for $1 < \alpha < 2$, by

$$\begin{aligned} D_{\delta,\alpha}\varphi(x) &= \int_0^1 \frac{\varphi(x+y) - \varphi(x) - y\varphi'(x)}{|y|^{1+\alpha}} \left(\kappa_-^\delta 1_{(-\infty,0)}(y) + \kappa_+^\delta 1_{(0,+\infty)}(y) \right) dy, \end{aligned} \quad (\text{A.2})$$

and for $0 < \alpha < 1$, by

$$D_{\delta,\alpha}\varphi(x) = \int_0^1 \frac{\varphi(x+y) - \varphi(x)}{|y|^{1+\alpha}} \left(\kappa_-^\delta 1_{(-\infty,0)}(y) + \kappa_+^\delta 1_{(0,+\infty)}(y) \right) dy, \quad (\text{A.3})$$

where κ_-^δ are κ_+^δ two nonnegative constants satisfying $\kappa_-^\delta + \kappa_+^\delta > 0$, φ is a smooth function for which the integral exists, and φ' is its derivative. This representation identifies it as the infinitesimal generator for a nonsymmetric α -stable Lévy process.

Suppose $G_\alpha^\delta(t, x)$ is the fundamental solution to the following equation:

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) &= D_{\delta,\alpha}(t, x), \\ v(0, x) &= \delta_0(x), \quad t > 0, x \in \mathbb{R}, \end{aligned} \quad (\text{A.4})$$

where $\delta_0(\cdot)$ is the Dirac distribution. Using Fourier transform, one can get $G_\alpha^\delta(t, x)$ which is given by

$$G_\alpha^\delta(t, x) = \frac{1}{2\pi} \int_D \exp \left(-izx - t|z|^\alpha \exp \left(-i\delta \frac{\pi}{2} \operatorname{sgn}(z) \right) \right) dz. \quad (\text{A.5})$$

Let us list some known facts on $G_\alpha^\delta(t, x)$ which will be used later on (see, e.g., Debbi and Dozzi [31]).

Lemma A.1. Suppose $\alpha \in (0, \infty) \setminus \{\mathbb{N}\}$, one can get the following:

- (1) $G_\alpha^\delta(t, \cdot)$ is not symmetric with respect to x .
- (2) For any $x \in \mathbb{R}$ and $s, t \in (0, \infty)$,

$$\frac{\partial^n}{\partial x^n} G_\alpha^\delta(t, x) = (s)^{-((n+t)/\alpha)} G_\alpha^\delta \left(s^{-1}t, s^{-(1/\alpha)}x \right) \quad (\text{A.6})$$

or equivalently

$$\frac{\partial^n}{\partial x^n} G_\alpha^\delta(t, x) = (t)^{-((n+t)/\alpha)} G_\alpha^\delta \left(1, (t)^{-(1/\alpha)}x \right). \quad (\text{A.7})$$

$$(3) \text{ For any } s, t \in (0, \infty), G_\alpha^\delta(s, \cdot) * G_\alpha^\delta(t, \cdot) = G_\alpha^\delta(s+t, \cdot)$$

$$(4) \text{ For } n \geq 1, \text{ there exist constants } C \text{ and } C_n > 0 \text{ satisfying}$$

$$\left| G_\alpha^\delta(1, x) \right| \leq C \frac{1}{1 + |x|^{1+\alpha}}, \quad \left| \frac{\partial^n}{\partial x^n} G_\alpha^\delta(1, x) \right| \leq C_n \frac{|x|^{\alpha+n-1}}{(1 + |x|^{\alpha+n})^2}. \quad (\text{A.8})$$

$$(5) \int_0^T \int_D |G_\alpha^\delta(t, x)|^\lambda dt dx < \infty \text{ if and only if } (1/\alpha) < \lambda < \alpha.$$

Lemma A.2. For $\theta \in [0, \min \{1, (\alpha+1)H-1\})$ and $\mu \in [0, ((\alpha+1)H-1)/\alpha]$, we have

$$\left[\int_0^t \int_{\mathbb{R}} \left| G_\alpha^\delta(t-v, y-w) - G_\alpha^\delta(t-v, y-w) \right|^{1/H} dw dv \right]^H \leq m|x-y|^\theta, \quad (\text{A.9})$$

$$\left[\int_0^t \int_{\mathbb{R}} \left| G_\alpha^\delta(t-v, x-w) - G_\alpha^\delta(s-v, x-w) \right|^{1/H} dw dv \right]^H \leq m|t-s|^\mu, \quad (\text{A.10})$$

$$\left[\int_s^t \int_{\mathbb{R}} \left| G_\alpha^\delta(t-v, x-w) \right|^{1/H} dw dv \right]^H \leq m|t-s|^{((\alpha+1)H-1)/\alpha}, \quad (\text{A.11})$$

where $m > 0$ is a constant.

Proof. For any $x, y \in \mathbb{R}$ and $t \in [0, T]$,

$$\begin{aligned} & \left[\int_0^t \int_{\mathbb{R}} \left| G_\alpha^\delta(t-v, x-w) - G_\alpha^\delta(t-v, y-w) \right|^{1/H} dw dv \right]^H \\ &= \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(t-*, y-\cdot) \right\|_{L^{1/H}([0,T] \times \mathbb{R})}^H \\ &= \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(t-*, y-\cdot) \right\|^\theta \\ & \quad \cdot \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(t-*, y-\cdot) \right\|^{1-\theta} \Big|_{L^{1/H}([0,T] \times \mathbb{R})} \\ &\leq \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(t-*, y-\cdot) \right\|^\theta \\ & \quad \cdot \left\| G_\alpha^\delta(t-*, x-\cdot) \right\|^{1-\theta} \Big|_{L^{1/H}([0,T] \times \mathbb{R})} \\ &+ \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(t-*, y-\cdot) \right\|^\theta \\ & \quad \cdot \left\| G_\alpha^\delta(t-*, y-\cdot) \right\|^{1-\theta} \Big|_{L^{1/H}([0,T] \times \mathbb{R})} := I_1 + I_2. \end{aligned} \quad (\text{A.12})$$

By the mean-value theorem, for $\theta \in [0, 1]$, one can get that

$$\begin{aligned} I_1 &= \left\| |x-y|^\theta \left| \frac{\partial}{\partial x} G_\alpha^\delta(t-*, \eta-\cdot) \right|^\theta \cdot \left| G_\alpha^\delta(t-*, x-\cdot) \right|^{1-\theta} \right\|_{L^{1/H}([0,T] \times \mathbb{R})} \\ &= \left(|x-y|^{\theta/H} \int_0^T \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} G_\alpha^\delta(t-u, \eta-w) \right|^{\theta/H} \left| G_\alpha^\delta(t-v, x-w) \right|^{(1-\theta)/H} dw du \right)^H \end{aligned} \quad (\text{A.13})$$

and

$$\begin{aligned} &\int_{\mathbb{R}} \left| \frac{\partial}{\partial x} G_\alpha^\delta(t-u, \eta-w) \right|^{\theta/H} \left| G_\alpha^\delta(t-u, x-w) \right|^{(1-\theta)/H} dw \\ &= (t-u)^{(1/\alpha)-((1+\theta)/\alpha H)} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} G_\alpha^\delta(1, \tilde{\eta}-w) \right|^{\theta/H} \left| G_\alpha^\delta(1, w) \right|^{(1-\theta)/H} dw \\ &\leq C_{H,\theta} (t-u)^{(1/\alpha)-((1+\theta)/\alpha H)}. \end{aligned} \quad (\text{A.14})$$

Therefore, if $(1/\alpha) - ((1+\theta)/\alpha H) + 1 > 0$, that is, $\theta < (\alpha+1)H-1$, then,

$$I_1 \leq C_{T,\theta,H} |x-w|^\theta. \quad (\text{A.15})$$

Similarly, we can check

$$I_2 \leq C_{T,\theta,H} |x-w|^\theta. \quad (\text{A.16})$$

Hence, the inequality (A.9) holds. As for the inequality (A.10), for any $x \in \mathbb{R}$ and $t, s \in [0, T]$,

$$\begin{aligned} &\left[\int_0^t \int_{\mathbb{R}} \left| G_\alpha^\delta(t-v, x-w) - G_\alpha^\delta(s-v, x-w) \right|^{1/H} dz dv \right]^H \\ &= \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(s-*, x-\cdot) \right\|_{L^{1/H}([0,T] \times \mathbb{R})} \\ &= \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(s-*, x-\cdot) \right\|^\mu \\ &\quad \cdot \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(s-*, x-\cdot) \right|^{1-\mu} \Big\|_{L^{1/H}([0,T] \times \mathbb{R})} \\ &\leq \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(s-*, x-\cdot) \right\|^\mu \\ &\quad \cdot \left\| G_\alpha^\delta(s-*, x-\cdot) \right|^{1-\mu} \Big\|_{L^{1/H}([0,T] \times \mathbb{R})} \\ &+ \left\| G_\alpha^\delta(t-*, x-\cdot) - G_\alpha^\delta(s-*, x-\cdot) \right\|^\mu \\ &\quad \cdot \left\| G_\alpha^\delta(s-*, x-\cdot) \right|^{1-\mu} \Big\|_{L^{1/H}([0,T] \times \mathbb{R})} := \Pi_1 + \Pi_2. \end{aligned} \quad (\text{A.17})$$

By mean-value theorem, it holds that

$$\begin{aligned} \Pi_1 &= |t-s|^\mu \left| \frac{\partial}{\partial t} G_\alpha^\delta(\rho-*, x-\cdot) \right|^\mu \cdot \left| G_\alpha^\delta(t-*, x-\cdot) \right|^{1-\mu} \Big\|_{L^{1/H}([0,T] \times \mathbb{R})} \\ &= \left(\int_0^T \int_{\mathbb{R}} \left| t-s \right|^{\mu/H} \left| \frac{\partial}{\partial t} G_\alpha^\delta(\rho-r, x-w) \right|^{\mu/H} \left| G_\alpha^\delta(t-r, x-w) \right|^{(1-\mu)/H} dw du \right)^H. \end{aligned} \quad (\text{A.18})$$

Note that

$$G_\alpha^\delta(t, y) = t^{-1/\alpha} G_\alpha^\delta\left(1, t^{-(1/\alpha)} y\right). \quad (\text{A.19})$$

Hence, one can attain

$$\begin{aligned} \frac{\partial}{\partial t} G_\alpha^\delta(t, y) &= \left(-\frac{1}{\alpha}\right) t^{-(1-(1/\alpha))} G_\alpha^\delta\left(1, t^{-(1/\alpha)} y\right) \\ &\quad + \left(-\frac{1}{\alpha}\right) t^{-(1-2(1/\alpha))} \left[y \frac{\partial}{\partial z} G_\alpha^\delta(1, w) \Big|_{w=t^{-(1/\alpha)} y} \right]. \end{aligned} \quad (\text{A.20})$$

So

$$\begin{aligned}
& \int_{\mathbb{R}} \left| \frac{\partial}{\partial t} G_{\alpha}^{\delta}(\rho - r, x - w) \right|^{\mu/H} \left| G_{\alpha}^{\delta}(t - r, x - w) \right|^{(1-\mu)/H} dw \\
& \leq C_{\alpha, H} \int_{\mathbb{R}} (t - r)^{(-1-(1/\alpha))(\mu/H) - (1/\alpha)((1-u)/H)} \left| G_{\alpha}^{\delta}\left(1, (\rho - u)^{-(1/\alpha)}(x - w)\right) \right|^{\mu/H} \cdot \left| G_{\alpha}^{\delta}\left(1, (t - u)^{-(1/\alpha)}(x - w)\right) \right|^{(1-\mu)/H} dw \\
& \quad + C_{\alpha, H} \int_{\mathbb{R}} (t - r)^{(-1-(2/\alpha))(\mu/H) - (1/\alpha)((1-u)/H)} \left[|x - w|^{\mu/H} \left| \frac{\partial}{\partial w} G_{\alpha}^{\delta}(1, w) \right|_{w=(\rho-r)^{-(1/\alpha)}(x-w)} \right]^{\mu/H} \cdot \left| G_{\alpha}^{\delta}\left(1, (t - u)^{-(1/\alpha)}(x - w)\right) \right|^{(1-\mu)/H} dw \\
& \leq C_{\alpha, H} (t - u)^{-((\alpha\mu+1)/H_{\alpha}) + (1/\alpha)} + C_{\alpha, H} (t - u)^{-((\alpha\mu+\mu+1)/H_{\alpha}) + (1/\alpha)}.
\end{aligned} \tag{A.21}$$

Therefore, if $-(\alpha\mu + 1)/H_{\alpha} + (1/\alpha) + (1 > 0)$ i.e. $\mu < ((\alpha + 1)H - 1)/\alpha$, then, for $\mu \in [0, ((\alpha + 1)H - 1)/\alpha]$,

$$\Pi_1 \leq C_{\alpha, H, T, \mu} |t - s|^{\mu}. \tag{A.22}$$

Similarly,

$$\Pi_2 \leq C_{\alpha, H, T, \mu} |t - s|^{\mu}. \tag{A.23}$$

So (A.10) holds. Now let us prove that (A.11) holds. For any $x \in \mathbb{R}$ and $s, t \in [0, T]$,

$$\begin{aligned}
& \left[\int_s^t \int_{\mathbb{R}} \left| G_{\alpha}^{\delta}(t - v, x - w) \right|^{1/H} dw dv \right]^H \\
& \leq \left[\int_s^t ((t - u))^{-(1/\alpha H) + (1/\alpha)} \left(\int_{\mathbb{R}} \left| G_{\alpha}^{\delta}(1, z) \right|^{1/H} dz \right) dv \right]^H \\
& \leq C_H (t - s)^{((\alpha+1)H-1)/\alpha}.
\end{aligned} \tag{A.24}$$

The proof of the lemma is completed.

Lemma A.3. Suppose $p \in [1, \infty)$, $q \in [1, p]$, and $\rho \in [1, \infty)$ satisfying

$$\frac{1}{\rho} = \frac{1}{p} - \frac{1}{q} + 1 \in [0, 1]. \tag{A.25}$$

Suppose $G_{\alpha}^{\delta} = G_{\alpha}^{\delta}(t, x - z)$ is the Green kernel, $Q = (\partial/\partial z)$ G_{α}^{δ} or $Q = G_{\alpha}^{\delta}$ with $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$. We define I by

$$I(u)(t, x) = \int_0^t \int_{\mathbb{R}} Q(t - s, x - z) u(s, z) ds dz, \tag{A.26}$$

with $u \in L^1([0, T]; L^q(\cdot))$. Then, $I : L^1([0, T]; L^q(\cdot)) \rightarrow L^{\infty}([0, T]; L^q(\cdot))$ is a bounded linear operator which satisfies the following:

(1) When $Q = (\partial/\partial y) G_{\alpha}^{\delta}$, then, for all $r \in [1, (1 + \alpha/2))$,

$$\begin{aligned}
& \|I(u)(t, \cdot)\|_p \\
& \leq C \int_0^t (t - s)^{-(2/\alpha) + (1/\alpha r)} \|u(x, \cdot)\|_q ds, \quad \forall t \in [0, T].
\end{aligned} \tag{A.27}$$

(2) When $Q = G_{\alpha}^{\delta}$, then, for all $\rho \in [1, 1 + \alpha)$,

$$\begin{aligned}
& \|I(u)(t, \cdot)\|_p \\
& \leq C \int_0^t (t - s)^{-(1/\alpha) + (1/\alpha r)} \|u(x, \cdot)\|_q ds, \quad \forall t \in [0, T].
\end{aligned} \tag{A.28}$$

Proof. Together with Minkowski's inequality, Young inequality and (4) of Lemma A.1. We can get

$$\begin{aligned}
\|I(u)(t, \cdot)\|_p &= \left\| \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial z} G_{\alpha}^{\delta}(t - s, \cdot - z) v(s, z) dz ds \right\|_p \\
&\leq \int_0^t \left\| \int_{\mathbb{R}} \frac{\partial}{\partial z} G_{\alpha}^{\delta}(t - s, \cdot - z) u(s, z) dz \right\|_p ds \\
&\leq C \int_0^t (t - s)^{-(2/a)} \\
&\quad \cdot \left\| \int_{\mathbb{R}} \frac{\partial}{\partial z} G_{\alpha}^{\delta}\left(1, (t - s)^{-(1/a)}(\cdot - z)\right) |u(s, z)| dz \right\|_p ds \\
&\leq C \int_0^t (t - s)^{-(2/a)} \\
&\quad \cdot \left\| \left(\frac{\partial}{\partial z} G_{\alpha}^{\delta}\left(1, (t - s)^{-(1/a)}(\cdot)\right) * |u(s, \cdot)| \right) (\cdot) \right\|_p ds \\
&\leq C \int_0^t (t - s)^{-(2/a)} \left\| \frac{\partial}{\partial y} G_{\alpha}^{\delta}\left(1, (t - s)^{-(1/a)}(\cdot)\right) \right\|_p \\
&\quad \cdot \|u(s, \cdot)\|_q ds \\
&\leq C \int_0^t (t - s)^{-(2/a) + (1/\alpha \rho)} \|u(s, \cdot)\|_q ds,
\end{aligned} \tag{A.29}$$

where one has used the result that, for $\rho \in [1, (1 + \alpha)/2)$,

$$\begin{aligned} \left\| G_{\alpha}^{\delta} \left(1, (t-s)^{-(1/\alpha)} \cdot \right) \right\|_{\rho} &= \left[\int_{\mathbb{R}} \frac{\partial}{\partial z} \left(G_{\alpha}^{\delta} \left(1, (t-s)^{-(1/\alpha)} z \right) \right)^{\rho} dz \right]^{1/\rho} \\ &\leq (t-s)^{1/\alpha\rho} \left[\int_{\mathbb{R}} \frac{\partial}{\partial y} \left(G_{\alpha}^{\delta} (1, y) \right)^{\rho} dy \right]^{1/\rho} \\ &\leq C(t-s)^{1/\alpha\rho}. \end{aligned} \quad (\text{A.30})$$

The proof of (2) is omitted since it is similar to case (1). The proof of this lemma is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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