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Research Article

Fast Consensus Seeking on Networks with Antagonistic Interactions

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It is well known that all agents in a multiagent system can asymptotically converge to a common value based on consensus protocols. Besides, the associated convergence rate depends on the magnitude of the smallest nonzero eigenvalue of Laplacian matrix L. In this paper, we introduce a superposition system to superpose to the original system and study how to change the convergence rate without destroying the connectivity of undirected communication graphs. And we find the result if the eigenvector x of eigenvalue λ has two identical entries $x_i = x_j$, then the weight and existence of the edge e_{ij} do not affect the magnitude of λ, which is the argument of this paper. By taking advantage of the inequality of eigenvalues, conditions are derived to achieve the largest convergence rate with the largest delay margin, and, at the same time, the corresponding topology structure is characterized in detail. In addition, a method of constructing invalid algebraic connectivity weights is proposed to keep the convergence rate unchanged. Finally, simulations are given to demonstrate the effectiveness of the results.

1. Introduction

In recent years, many issues related to consensus have been studied [1–10]. This is due to the large number of potential applications of consensus, ranging from engineering and computer science to biology, ecology, and social science. In [1], Jadbabaie proposed a nearest neighbor law to coordinate the control of the whole system, so that agents can reach a common final value. Olfati-Saber and Murray proposed a simple consensus protocol to achieve average consensus for undirected graphs and balanced digraphs in [2]. Based on quasi-consensus, Cai proposed an approach for clustering in [3]. And Liu proposed several necessary and sufficient conditions for consensus of second-order multiagent systems under directed topologies [4]. For high-order linear time-invariant singular multiagent systems with constant time delays, admissible output consensus design problem was investigated in [5]. For hybrid multiagent systems, necessary and sufficient conditions were also developed for the consensus [6, 7].

The smallest nonzero eigenvalue $\lambda_2(L)$ of Laplacian L is the algebraic connectivity, which was first proposed by Fiedler [11]. Olfati-Saber and Murray conducted a preliminary discussion on the convergence rate of consensus and proposed the concept of communication cost [2]. The algebraic connectivity increases as the communication cost goes up. An approach of minimizing the guaranteed cost was given in [12]. For a stable system, we always expect that the system has a larger convergence rate. Based on the consensus protocol, the convergence rate becomes larger as the algebraic connectivity of the system increases.

The convergence rate problem of consensus had been studied in [13–20]. The fast consensus and the optimization problem of convergence rate have been studied extensively [13]. The convergence rate of consensus was studied under the condition of weighted average in [14]. Jin proposed a multihop relay protocol to make a larger algebraic connectivity [15]. In [16], Olfati-Saber studied the fast consensus for small-world networks through randomly rewiring edges. In [18], Yang studied the optimization problem of convergence rate for second-order systems with time delays by the frequency-domain method. Yang proposed a new variable not only to measure the situation of optimization, but also to cope with the tradeoff between $\lambda_2(L)$ and $\lambda_n(L)$. Xiao developed the results of delay margin by analyzing $\lambda_n(L)$ in [17]. An optimal consensus protocol minimizing team cost function was proposed in [19], and an optimal synchronization protocol was designed for the largest convergence rate and minimal steady state error when the protocol is perturbed by an additive noise in [20].

To improve the convergence rate of consensus in multiagent systems, we need to change the algebraic connectivity $\lambda_2(L)$. The variation of algebraic connectivity depends on the variation of weights and topologies of communication graphs. The algebraic connectivity can be improved by taking advantage of algorithms and graph theory, which was explored in [21–24]. The convergence rate is susceptible to the perturbation, which will further affect the stability of the system [25–28]. Based on former researchers' works [29–37], we propose a concept of invalid algebraic connectivity weights (IACW), which is shown not only to be resistant to the perturbation but also can avoid unnecessary waste of costs. There are three methods to get a larger convergence rate: (1) changing the protocol; (2) changing the weights of a system; (3) changing the topology of a system. In this paper, we take methods (2) and (3). In particular, we investigate how the convergence rate varies when a superposition system is joined to the original system and present a detailed characterization for the variation of convergence rate. The results are developed by analyzing the variation of eigenvalues and eigenvectors of Laplacian matrix. Under fixed costs, we give the most optimal case, which can make the convergence rate and delay margin the largest. For a complete graph, if the cost is fixed, then $\lambda_2(L)$ reaches maximum and $\lambda_n(L)$ reaches minimum. Thus, the convergence rate and delay margin achieve the biggest magnitude. Since the topology is unique, this achieves the optimization of convergence rate and delay margin. In addition, the method and conditions are put forward for the construction of invalid algebraic connectivity weights.

In a multiagent network system, not all agents are cooperative. There are some agents which are competing. The cooperative and competitive relationship is represented, respectively, by positive weights and negative weights. Positive and negative weights have opposite effects on system performance. In some circumstances, negative weights are also needed. For example, in a system with time delay, the negative weights can reduce the magnitude of $\lambda_n(\widehat{L})$, which results in a larger delay margin. Hence, the fast consensus with antagonistic interactions is considered in the paper.

The rest of the paper is organized as follows. In Section 2, basic definitions, properties, and system model are addressed. In Section 3, we study the variation of convergence rate when a superposition system is superposed on the original system, and characterize the most optimal case of convergence rate under fixed cost. In Section 4, results are derived for the identification of invalid algebraic connectivity weights and the associated construction of IACW. In Section 5, simulation results are presented to show the effectiveness of the approach. Finally, conclusions are given in Section 6.

2. Consensus Protocol and Consensus State

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$ be a weighted undirected graph, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ denotes a set of n nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges, and $A \in \mathbb{R}^{n\times n}$ is the adjacency matrix of undirected graph $\mathcal{G}$. In this paper, if v_i has a communication link with v_j, then there is an edge $e_{ij} \in \mathcal{E}$ between nodes v_i and v_j, with $a_{ij} = a_{ji} \neq 0$, $i, j \in \{1, \ldots, n\}$; and a_{ij} is the weight between nodes v_i and v_j. Here, the self-loop of $\mathcal{V}$, under which $a_{ii} = 0, \forall i = 1, \ldots, n$, is not considered.

The first-order multiagent system is given by

$$\dot{x}_i = -\sum_{j\in\mathcal{N}(i)} a_{ij} \left(x_i - x_j\right) \tag{1}$$

where x_i denotes the state of agent i, which is the ith component of x; $\mathcal{N}(i)$ denotes the set of adjacent agents of agent i. The Laplacian matrix of $\mathcal{G}$ is represented by $L = D - A$, where $D \in \mathbb{R}^{n\times n}$ is the diagonal connectivity degree matrix of A. So (1) can be written as

$$\dot{x} = -Lx \tag{2}$$

The entries of L can be written as

$$l_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \sum_{j\in\mathcal{N}(i)} a_{ij} & i = j. \end{cases} \tag{3}$$

D can be written as

$$D = \operatorname{diag}\left\{\sum_{j\in\mathcal{N}(1)} a_{1j}, \sum_{j\in\mathcal{N}(2)} a_{2j}, \ldots, \sum_{j\in\mathcal{N}(n)} a_{nj}\right\}. \tag{4}$$

Assume that the communication graph consists of n nodes, and the spectrum of eigenvalues is $\lambda_1(L) \leq \lambda_2(L) \cdots \leq \lambda_n(L)$. The spectrum means that the eigenvalues are arranged according to a certain order. Let $S = \operatorname{span}\{u_1, \ldots, u_n\}$ denote a vector space, in which any vector can be represented as a linear combination of $u_1, \ldots, u_n$; i.e., $S = \{\eta \mid \eta = k_1 u_1 + \cdots + k_n u_n\}$. Below $A \cup \widetilde{A}$ means that the entries of matrices A and $\widetilde{A}$ plus together. For $\widehat{A} = A \cup \widetilde{A}$, the entries of $\widehat{A}$ are $\widehat{a}_{ij} = a_{ij} + \widetilde{a}_{ij}$, where a_{ij}, $\widetilde{a}_{ij}$ are the entries of A, $\widetilde{A}$, respectively.

Definition 1. The system associated with graph $\widetilde{\mathcal{G}} = \{\mathcal{V}, \widetilde{\mathcal{E}}, \widetilde{A}\}$, which has the same node set $\mathcal{V}$ with the original graph $\mathcal{G}$, is called a superposition system. Let $\widetilde{\mathcal{G}}$ be superposed to $\mathcal{G}$. We get a connected graph $\widehat{\mathcal{G}} = \{\mathcal{V}, \mathcal{E} \cup \widetilde{\mathcal{E}}, A \cup \widetilde{A}\}$. Then the system associated with $\widehat{\mathcal{G}}$ is called a superposed system.

We give the definition of superposition system to represent the situation of the adding edges such as $\operatorname{rank}(\widetilde{L})$, numbers, and weights of edges, so that we study the variation of superposed systems under different situations of adding edges.

In this paper, graph $\widehat{\mathcal{G}}$ is connected and the spectrums of eigenvalues associated with both $\widetilde{L}$ and $\widehat{L}$ are identical with

L, where $\widetilde{L}$ denotes the Laplacian matrix of $\widetilde{\mathcal{G}}$, $1 \leq \operatorname{rank}(\widetilde{L}) \leq n-1$ and $\widehat{L}$ denote the Laplacian matrix of $\widehat{\mathcal{G}}$. In what follows, $L - \widetilde{L} \geq 0$ means that $L - \widetilde{L}$ is positive semidefinite. Although there are negative weights, $\widetilde{L}$ can still be positive semidefinite or negative semidefinite.

3. Variation of Convergence Rate

The changing of algebraic connectivity relies on the variation of eigenvalues of Laplacian L. So the following lemma is introduced.

Lemma 2 (Weyl theorem [38]). *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric and the eigenvalues of A, B and $A+B$ be ordered as that of L, respectively. Then*

$$\lambda_i(A+B) \leq \lambda_{i+j}(A) + \lambda_{n-j}(B), \quad j = 0, 1, \ldots, n-i \tag{5}$$

for each $i = 1, \ldots, n$, with equality holding for some pair i, j if and only if there is a nonzero vector x such that $Ax = \lambda_{i+j}(A)x$, $Bx = \lambda_{n-j}(B)x$, $(A+B)x = \lambda_i(A+B)x$. Also,

$$\lambda_{i-j+1}(A) + \lambda_j(B) \leq \lambda_i(A+B), \quad j = 1, \ldots, i \tag{6}$$

for each $i = 1, \ldots, n$, with equality holding for some pair i, j if and only if there is a nonzero vector x such that $Ax = \lambda_{i-j+1}(A)x$, $Bx = \lambda_j(B)x$, $(A+B)x = \lambda_i(A+B)x$. If A and B share no common eigenvectors, then (5) and (6) are strict inequalities.

By Hermam Weyl theorem, some significant inequalities can be derived. This is essential to the development of results. In what follows, let $A = L$, $B = \widetilde{L}$, $A + B = \widehat{L}$.

3.1. Convergence Rate of Undirected Graphs. For a system $\dot{x} = -Lx$ which can achieve consensus, its eigenvalues satisfy $\lambda_i(L) \geq 0$ and are arranged in an increasing order, $i = 1, \ldots, n$. The solution of the system is $x(t) = \exp(-Lt)x(0)$, where $x(0)$ is the initial state of agents. Let $-L = PJP^{-1}$, where J is the Jordan standard norm of $-L$, and the diagonal entries of J are the eigenvalues of $-L$. Then $x(t) = \exp(-Lt)x(0) = P\exp(Jt)P^{-1}x(0)$. Let $H = \operatorname{span}\{e^{-\lambda_1(L)t}, e^{-\lambda_2(L)t}, \ldots, e^{-\lambda_n(L)t}\}$. It follows that $x^* \in H$ if the multiplicity of all eigenvalues of L is 1, where x^* is the final value of the system. Therefore, the decay rate of the component $e^{-\lambda_2(L)t}x_2(0)$ in $x(t)$ is less than or equal to the components of $\lambda_3(L), \ldots, \lambda_n(L)$. If there is a repeated eigenvalue of L, x^* is a nonlinear generation of H. Assume that the nonlinear generation of $\lambda_2(L)$ in x^* is $\alpha_1 e^{\lambda_2(L)t} + \alpha_2 t e^{\lambda_2(L)t} + \cdots + \alpha_n t^{n-1} e^{\lambda_2(L)t}$, $\alpha_i \in \mathbb{R}^n$. When $t \longrightarrow \infty$, the derivative of its nonlinearity is less than or equal to the nonlinear generation corresponding to $\lambda_3(L), \ldots, \lambda_n(L)$. Thus, the rate that x converges to a steady final value x^* is determined by $\lambda_2(L)$.

The superposition system has the same node set with the original system, and its edges can be constructed arbitrarily. We aim to analyze the variations of convergence rate after joining different superposition systems. For a system with undirected graphs, the following theorem can be achieved after joining a superposition system.

Theorem 3. *For an undirected connected graph $\mathcal{G}$, if $\widetilde{L}$ is a positive semidefinite matrix, then*

$$\lambda_2(L) \leq \lambda_2(\widehat{L}). \tag{7}$$

If $\widetilde{L}$ is negative semidefinite, then

$$\lambda_2(L) \geq \lambda_2(\widehat{L}). \tag{8}$$

Only for $\lambda_2(L)$ with multiplicity 1, the necessary and sufficient condition for equalities in (7) and (8) to be true is that there is a vector x such that $Lx = \lambda_2(L)x$, $\widetilde{L}x = 0$, $\widehat{L}x = \lambda_2(\widehat{L})x$.

Proof. Let $x_1, \ldots, x_n$; $y_1, \ldots, y_n$; $z_1, \ldots, z_n$, be the eigenvectors of L, $\widetilde{L}$, and $\widehat{L}$, respectively, with $\|x_i\|_2 = \|y_i\|_2 = \|z_i\|_2 = 1$, $i \in \{1, \ldots, n\}$. Denote $S_1 = \operatorname{span}\{x_{i-j+1}, \ldots, x_n\}$, $S_2 = \operatorname{span}\{y_j, \ldots, y_n\}$, $S_3 = \operatorname{span}\{z_1, \ldots, z_i\}$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$. Then

$$\dim S_1 + \dim S_2 + \dim S_3 = 2n + 1, \tag{9}$$

which guarantees that there is a unit vector $x \in S_1 \cap S_2 \cap S_3$. So we obtain the following inequality:

$$\lambda_{i-j+1}(L) + \lambda_j(\widetilde{L}) \leq x^T L x + x^T \widetilde{L} x \leq \lambda_i(\widehat{L}), \quad j = 1, \ldots, n. \tag{10}$$

For $i = 2$, $j = 1$, let $\widetilde{L}$ be positive semidefinite; it follows that

$$\lambda_2(L) \leq \lambda_2(\widehat{L}). \tag{11}$$

Denote $S_1 = \operatorname{span}\{x_1, \ldots, x_{i+j}\}$, $S_2 = \operatorname{span}\{y_1, \ldots, y_{n-j}\}$, $S_3 = \operatorname{span}\{z_i, \ldots, z_n\}$, $i \in \{1, \ldots, n\}$, $j \in \{0, 1, \ldots, n-i\}$. Then

$$\dim S_1 + \dim S_2 + \dim S_3 = 2n + 1, \tag{12}$$

which guarantees that there is a unit vector $x \in S_1 \cap S_2 \cap S_3$. Thus,

$$\lambda_i(\widehat{L}) \leq x^T L x + x^T \widetilde{L} x \leq \lambda_{i+j}(L) + \lambda_{n-j}(\widetilde{L}), \quad j = 0, 1, \ldots, n-i \tag{13}$$

For $i = 2$, $j = 0$, let $\widetilde{L}$ be negative semidefinite; one has

$$\lambda_2(\widehat{L}) \leq \lambda_2(L). \tag{14}$$

If $\lambda_2(L)$ is not a repeated eigenvalue, then $\lambda_2(L) = \lambda_2(\widehat{L})$ only if there is an eigenvector x associated with $\lambda_2(L)$ satisfying $\widetilde{L}x = 0$. In case that there is no eigenvector x satisfying $\widetilde{L}x = 0$, then $\lambda_2(\widehat{L})$ is larger than $\lambda_2(L)$. So there are no vectors x, y, z with x=y=z such that $x^T L x + y^T \widetilde{L} y = \lambda_2(L) + 0 = \lambda_2(\widehat{L})$. Thus the necessary and sufficient condition for equalities in (7) and (8) to be true is that there is a vector x satisfying $Lx = \lambda_i(L)x$, $\widetilde{L}x = 0$, and $\widehat{L}x = \lambda_2(\widehat{L})x$.

Assume that $\lambda_2(L)$ is a repeated eigenvalue with multiplicity p. If $\lambda_2(L)$ or $\lambda_{1+p}(L)$ changes and the other eigenvalues remain unchanged, $\widehat{L}$ and L share $p-1$ eigenvalues, while the corresponding eigenvectors of $\widehat{L}$ are different from those of L. Since L has $p-1$ identical eigenvalues with $\widehat{L}$, there is an eigenvector z which has components satisfying $z_i = z_j$, such that $\widetilde{L}z = 0$, $i \neq j$, $i, j \in \{1, \ldots, n\}$. In this case, however, the components of the corresponding eigenvector of L are different from each other. That is, there are different vectors x, y, z such that $x^T Lx + y^T \widetilde{L} y = \lambda_2(L) + \lambda_2(\widetilde{L}) = \lambda_2(\widehat{L}) = z^T \widehat{L} z$. □

Theorem 3 shows that the convergence rate of a system can be increased or decreased by adding a superposition system. This is always true for the positive semidefinite matrix $\widetilde{L}$ or negative semidefinite matrix $\widetilde{L}$, no matter what is the sign of the weights of $\widetilde{L}$. It is based on the analysis whether the equalities in (7) and (8) hold, we can figure out the variations of the convergence rate. In what follows, more specific instructions will be given on how to affect the convergence rate with respect to the problems and situations encountered during the process.

In (10), take $j = 1$, we see that $\lambda_1(\widetilde{L}) = 0$. Then

$$\lambda_i(L) \leq \lambda_i(\widehat{L}), \quad i = 1, \ldots, n. \tag{15}$$

If $\widetilde{L}$ is positive semidefinite, (15) holds. Take $j = 0$ in (13). Then, based on (15), if $\widetilde{L}$ is negative semidefinite, we have

$$\lambda_i(\widehat{L}) \leq \lambda_i(L), \quad i = 1, \ldots, n. \tag{16}$$

Proposition 4. *For an undirected connected graph $\mathscr{G}$, let $\mathscr{X}$ represent the set of eigenvectors corresponding to $\lambda_{p+1}(L)$ with multiplicity p, $p \in \{1, \ldots, n-1\}$. If $\operatorname{rank}(\widetilde{L}) = 1$, and there exists a vector $x_p \in \mathscr{X}$ satisfying $\widetilde{L}x_p \neq 0$, then the multiplicity of $\lambda_{p+1}(L)$ corresponding to $\widehat{L}$ is $p-1$.*

Proof. Let $\widetilde{L}$ be a positive semidefinite matrix. For $j = 1$ in (13), and $\lambda_{n-1}(\widetilde{L}) = 0$, we have

$$\lambda_i(\widehat{L}) \leq \lambda_{i+1}(L), \quad i = 1, \ldots, n-1. \tag{17}$$

By (15) and (17),

$$\lambda_i(L) \leq \lambda_i(\widehat{L}) \leq \lambda_{i+1}(L), \quad i = 1, \ldots, n-1. \tag{18}$$

Assume that there are repeated eigenvalues of Laplacian L associated with graph $\mathscr{G}$, which are $\lambda_{i+1}(L) = \lambda_{i+2}(L) = \cdots = \lambda_{i+p}(L)$, $i = 1, \ldots, n-p$, and there exists one vector $x_p \in \mathscr{X}$ satisfying $\widetilde{L}x_p \neq 0$. Note that not for every vector $x \in \mathscr{X}$, $x^T Lx + x^T \widetilde{L} x = x^T Lx$ holds. So there is one of repeated eigenvalues, the value of which is improved; i.e., $\lambda_{i+1}(\widehat{L}) = \lambda_{i+2}(\widehat{L}) = \cdots = \lambda_{i+p-1}(\widehat{L}) < \lambda_{i+p}(\widehat{L})$, $i \in \{1, \ldots, n-p\}$.

In case $\widetilde{L}$ is negative semidefinite, the same line of arguments yields that $\lambda_{i+1}(\widehat{L}) < \lambda_{i+2}(\widehat{L}) = \cdots = \lambda_{i+p-1}(\widehat{L}) = \lambda_{i+p}(\widehat{L})$, $i \in \{1, \ldots, n-p\}$. □

Proposition 4 shows if the eigenvector x of one eigenvalue λ has two identical entries $x_i = x_j$, adding an edge e_{ij} does not affect the eigenvalue λ, which means the weight, and even the existence of e_{ij} does not affect λ. For an eigenvalue λ with multiplicity p, the corresponding eigenvector has $p-1$ eigenvectors with two identical entries at any positions, so that if $\operatorname{rank}(L) = 1$, no matter the position of the adding edge, there is only one λ change; that is, the multiplicity of λ of superposed system is $p-1$. It is easy to know that if the multiplicity of λ is p, there is at least one eigenvector of λ that has $p-1$ pairs of identical entries; that is, if $\operatorname{rank}(\widetilde{L}) < p$, the superposed system has the eigenvalue λ.

Example 5. The graph of the original system is shown as Figure 1(a). Introducing the superposition system only establishes a connection between nodes 1 and 2. Thus, $\operatorname{rank}(\widetilde{L}) = 1$. The graph of the superposed system is shown as Figure 1(b). Below are L and $\widehat{L}$.

$$\begin{gathered} L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 7 & -5 & 0 \\ -1 & -1 & -5 & 12 & -5 \\ -1 & -1 & 0 & -5 & 7 \end{bmatrix}, \\ \widehat{L} = \begin{bmatrix} 9 & -6 & -1 & -1 & -1 \\ -6 & 9 & -1 & -1 & -1 \\ -1 & -1 & 7 & -5 & 0 \\ -1 & -1 & -5 & 12 & -5 \\ -1 & -1 & 0 & -5 & 7 \end{bmatrix}. \end{gathered} \tag{19}$$

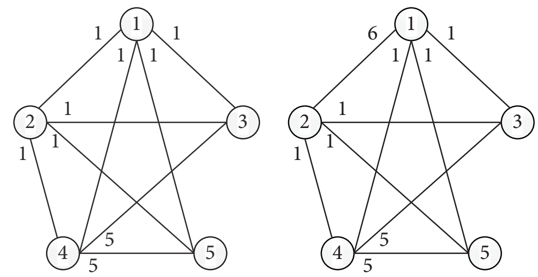
The nonzero eigenvalues of the original system are $\lambda_2(L) = 5$, $\lambda_3(L) = 5$, $\lambda_4(L) = 7$, and $\lambda_5(L) = 17$, and the nonzero eigenvalues of the superposed system are $\lambda_2(\widehat{L}) = 5$, $\lambda_3(\widehat{L}) = 7$, $\lambda_4(\widehat{L}) = 15$, and $\lambda_5(\widehat{L}) = 17$. The multiplicity of eigenvalue 5 is 2. Let us join the superposition system with only one connection; i.e., $\operatorname{rank}(\widetilde{L}) = 1$. Since the eigenvector x_p of the repeated eigenvalue satisfies $\widetilde{L}x_p \neq 0$, the multiplicity of $\lambda_2(\widehat{L})$ is 1.

3.2. Changes of Convergence Rate

Definition 6. For an entry a_{ij} of the adjacency matrix A, $i, j \in \{1, \ldots, n\}$, if the increase of its value does not affect the algebraic connectivity of the communication graph, we call these weighted entries invalid algebraic connectivity weights (IACW).

For an invalid algebraic connectivity weight a_{ij}, if there is a perturbation on one of associated nodes v_i and v_j so that the value of a_{ij} decreases, the magnitude of each eigenvalue except 0 and $\lambda_2(L)$ decreases firstly, while the delay margin of the system increases. Thus, in this case, the stability and the convergence of the system can be protected to a certain extent.

If there is an invalid algebraic connectivity weight in the system, then $\widetilde{L}x = 0$, where x is an eigenvector of $\lambda_2(L)$



(a) G (b) $\widehat{\mathcal{G}}$

Figure 1: (a) Original system; (b) superposed system.

satisfying $x^T L x + x^T \widetilde{L} x = \lambda_2(\widehat{L}) = x^T \widehat{L} x$. All the eigenvector $x's$ corresponding to the smallest nonzero eigenvalue must contain two components x_i and x_j with $x_i = x_j$, $i \neq j$, $i, j \in \{1, \ldots, n\}$. Below is a further explanation.

Due to the existence of repeated eigenvalue, the convergence rate does not necessarily change when a superposition system is superposed to the original system.

Theorem 7. *If a superposition system related undirected graph $\widetilde{\mathcal{G}}$ is superposed to an undirected connected graph $\mathcal{G}$, a composite connected graph $\widehat{\mathcal{G}}$ is generated for the corresponding superposed system. Then the convergence rate of the superposed system changes as follows.*

(i) *If there are invalid algebraic connectivity weights in $\mathcal{G}$, the adjacency matrix $\widetilde{A}$ of the superposition system only consists of invalid algebraic connectivity weights, and $\widetilde{L}$ is positive semidefinite, then the superposed system converges at a rate equal to the original system; i.e., $\lambda_2(\widehat{L}) = \lambda_2(L)$.*

(ii) *If $\lambda_2(L) \neq \lambda_3(L)$ and $\widetilde{L}x \neq 0$, where $\widetilde{L}$ is positive semidefinite and x is the eigenvector of $\lambda_2(L)$, then the superposed system converges faster than the original system; i.e., $\lambda_2(\widehat{L}) > \lambda_2(L)$.*

(iii) *If $\lambda_2(L) \neq \lambda_3(L)$, $\widetilde{L}x \neq 0$ and $D - \widetilde{D} \geq 0$, where $\widetilde{L}$ is negative semidefinite and x is the eigenvector of $\lambda_2(L)$, then the superposed system converges slower than the original system; i.e., $\lambda_2(\widehat{L}) < \lambda_2(L)$.*

Proof.

(i) If a system associated with the original graph $\mathcal{G}$ has an invalid algebraic connectivity weight, then all the eigenvectors corresponding to the smallest nonzero eigenvalue of L contain two components x_i, x_j with $x_i = x_j$, $i \neq j$, $i, j \in \{1, \ldots, n\}$. $\widetilde{A}$ only consists of invalid algebraic connectivity weights if the superposition system only establishes a connection between v_i and v_j, such that there is an eigenvector x corresponding to $\lambda_2(L)$ with $\widetilde{L}x = 0$, $x^T L x + x^T \widetilde{L} x = \lambda_2(L) + 0 = \lambda_2(\widehat{L}) = x^T \widehat{L} x$. Therefore, the convergence rate of the superposed system remains unchanged.

(ii) If $\lambda_2(L) \neq \lambda_3(L)$ and $\widetilde{L}x \neq 0$, where $\widetilde{L}$ is positive semidefinite, then $\lambda_2(L) \neq \lambda_2(\widehat{L})$, the multiplicity of $\lambda_2(L)$ in $\widehat{L}$ is 0. By (7), $\lambda_2(\widehat{L}) > \lambda_2(L)$ holds.

(iii) If $\lambda_2(L) \neq \lambda_3(L)$ and $\widetilde{L}x \neq 0$, where $\widetilde{L}$ is negative semidefinite, then $\lambda_2(L) \neq \lambda_2(\widehat{L})$ and the multiplicity of $\lambda_2(L)$ in $\widehat{L}$ is 0. By (8), $\lambda_2(\widehat{L}) < \lambda_2(L)$ and $D - \widetilde{D} \geq 0$, which yields that $\widehat{L}$ is positive semidefinite. □

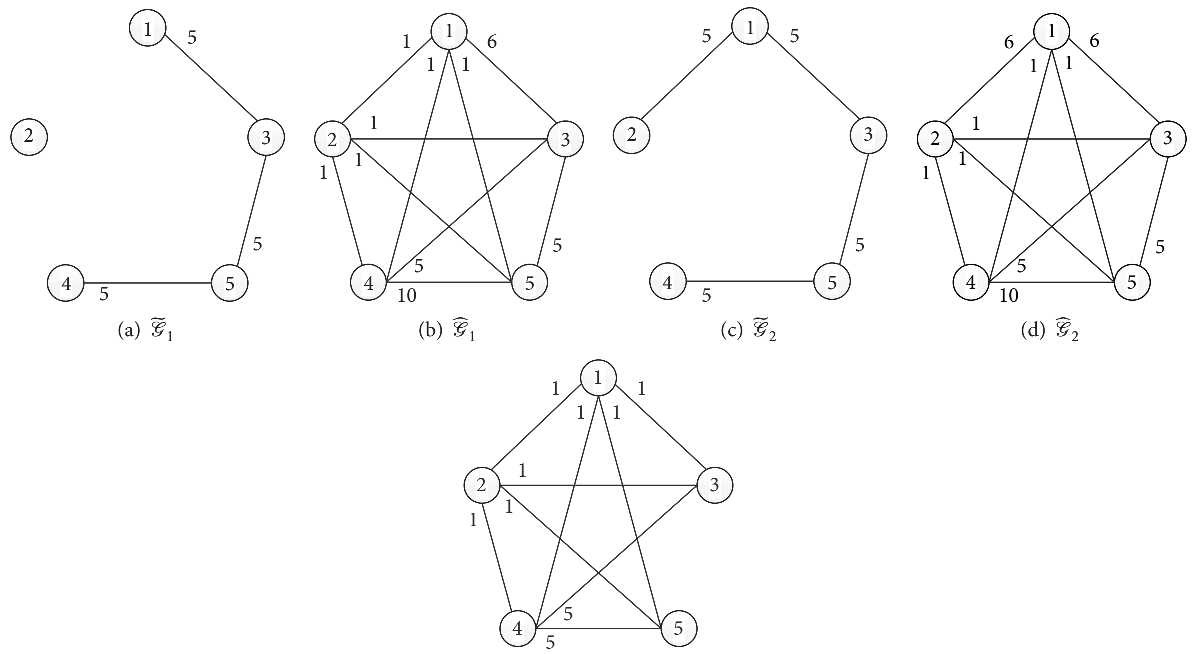
Example 8. The graph associated with the original system is shown as Figure 2(e). The superposition system $\widetilde{\mathcal{G}}_1$ only establishes connections among nodes 3, 4, and 5. Thus, $\operatorname{rank}(\widetilde{L}_1) = 3$. The superposed system $\widehat{\mathcal{G}}_1$ is shown as Figure 2(b). Superpose another superposition system $\widetilde{\mathcal{G}}_2$ to $\widehat{\mathcal{G}}_1$, where $\widetilde{\mathcal{G}}_2$ is shown as Figure 2(c) and only establishes the connection between nodes 1 and 2, which means $\operatorname{rank}(\widetilde{L}_2) = 1$. The superposed system $\widehat{\mathcal{G}}_2$ is shown as Figure 2(d). L, $\widehat{L}_1$, and $\widehat{L}_2$ are given as follows.

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 7 & -5 & 0 \\ -1 & -1 & -5 & 12 & -5 \\ -1 & -1 & 0 & -5 & 7 \end{bmatrix},$$

$$\widehat{L}_1 = \begin{bmatrix} 9 & -1 & -6 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -6 & -1 & 17 & -5 & -5 \\ -1 & -1 & -5 & 17 & -10 \\ -1 & -1 & -5 & -10 & 17 \end{bmatrix}, \tag{20}$$

$$\widehat{L}_2 = \begin{bmatrix} 14 & -6 & -6 & -1 & -1 \\ -6 & 9 & -1 & -1 & -1 \\ -6 & -1 & 17 & -5 & -5 \\ -1 & -1 & -5 & 17 & -10 \\ -1 & -1 & -5 & -10 & 17 \end{bmatrix}.$$

The nonzero eigenvalues of the original system are $\lambda_2(L) = 5$, $\lambda_3(L) = 5$, $\lambda_4(L) = 7$, and $\lambda_5(L) = 17$, and the



(a) $\widetilde{\mathcal{G}}_1$ (b) $\widehat{\mathcal{G}}_1$ (c) $\widetilde{\mathcal{G}}_2$ (d) $\widehat{\mathcal{G}}_2$ (e) $\mathcal{G}$

Figure 2: (a) Superposition system $\widetilde{\mathcal{G}}_1$; (b) superposed system $\widehat{\mathcal{G}}_1$; (c) superposition system $\widetilde{\mathcal{G}}_2$; (d) superposed system $\widehat{\mathcal{G}}_2$; (e) original system.

nonzero eigenvalues of the superposed system are $\lambda_2(\widehat{L}_1) = 5$, $\lambda_3(\widehat{L}_1) = 9.5969$, $\lambda_4(\widehat{L}_1) = 22.4031$, and $\lambda_5(\widehat{L}_1) = 27$. The invalid algebraic connectivity weights are $a_{34}(a_{43})$, $a_{35}(a_{53})$, and $a_{45}(a_{54})$. The established connections among nodes 1, 3, 4, and 5 mean $\operatorname{rank}(\widetilde{L}) = 3$. By Theorem 7, the convergence rate of the system still remains unchanged.

The nonzero eigenvalues of the superposed system $\mathcal{G}_2$ are $\lambda_2(\widehat{L}_2) = 7.4767$, $\lambda_3(\widehat{L}_2) = 15.7583$, $\lambda_4(\widehat{L}_2) = 23.765$, and $\lambda_5(\widehat{L}_2) = 27$. Let the superposition system only establish a connection between nodes 1 and 2; that is, $\operatorname{rank}(\widetilde{L}_2) = 1$. Then, there is no eigenvector x associated with $\lambda_2(L)$ satisfying $\widetilde{L}x = 0$. Therefore, the superposed system converges faster than the original one.

Corollary 9. *For an undirected connected graph $\mathcal{G}$ with the multiplicity of $\lambda_2(L)$ being r_2, let $\widehat{r}_2$ represent the multiplicity of $\lambda_2(L)$ of $\widehat{L}$, where the Laplacian $\widetilde{L}$ of the superposition system $\widetilde{\mathcal{G}}$ is positive semidefinite and* $\operatorname{rank}(\widetilde{L}) = q$, $q \in \{1, \ldots, n-1\}$. *Thus, if $r_2 - q = \widehat{r}_2 = 0$, then $\lambda_2(L) \neq \lambda_2(\widehat{L})$, and accordingly the convergence rate of $\widehat{\mathcal{G}}$ is larger than the original system.*

Proof. Set $\widetilde{L} = \widetilde{L}_1 + \widetilde{L}_2 + \cdots + \widetilde{L}_q$, $\operatorname{rank}(\widetilde{L}) = \operatorname{rank}(\widetilde{L}_1) + \operatorname{rank}(\widetilde{L}_2) + \cdots + \operatorname{rank}(\widetilde{L}_q)$, where $\widetilde{L}_j$ are different matrices, $\operatorname{rank}(\widetilde{L}_j) = 1$, $j = \{1, \ldots, q\}$, $q \in \{1, \ldots, n-1\}$. If a superposition system $\widetilde{L}_j$ is superposed to the original system, and there is an eigenvector x_j of $\lambda_2(\widehat{L}_j)$ associated with $\widehat{L}_j$ satisfying $\widetilde{L}_{j+1}x_j = 0$, then $r_2 - j < \widehat{r}_2$. Let x_j be an eigenvector corresponding to $\lambda_2(\widehat{L}_j)$, $j = 1, \ldots, q-1$. If $\widetilde{L}_{j+1}x_j = 0$ does not hold for every x_j, and there is no eigenvector x corresponding to $\lambda_2(L)$ satisfying $\widehat{L}_j x = 0$, then $r_2 - j = \widehat{r}_2$. Thus, if $r_2 - q = \widehat{r}_2 = 0$, then $\lambda_2(L)$ is not an eigenvalue of $\widehat{L}$ anymore. So $\lambda_2(L) \neq \lambda_2(\widehat{L})$, and accordingly the convergence rate of $\widehat{\mathcal{G}}$ is larger than that of $\mathcal{G}$. □

Remark 10. For any $\widetilde{L}$, if there is an eigenvector x of L such that $\widetilde{L}x = 0$, the multiplicity of eigenvalue $\lambda_i(L)$ associated with $\widehat{L}$ satisfies $\widehat{r} \geq 1$. That is to say, if $\|x\|_2 = 1$, then $x^T L x + x^T \widetilde{L} x = x^T \widehat{L} x = \lambda_i(L)$, $i \in \{1, 2, \ldots, n\}$.

Because the same elements may exist in the eigenvector x of $\lambda_2(L)$, superposition systems do not always enhance the convergence rate. In case $r_2 > 1$, the convergence rate changes only if the entries of x associated with the edges of supposition system are not all equal. For example, if only the entries x_3, x_4, x_5 of x are equal and the others are different from each other, then the superposition system with the connections among nodes 2, 3, 4, and 5 can enhance the convergence rate, while the superposition system with only the connections among nodes 3, 4, and 5 will not have such effect.

Algorithm 1 gives a method of finding a superposition system to enhance the convergence rate. In Algorithm, $i = i + 1$ changes the label of node i, which makes different nodes connect to node j or nodes $j, j+1, \ldots, j+m-1$ until $x_i \neq x_j$; or until $x_i, x_j, x_{j+1}, \ldots, x_{j+m-1}$ are not all equal. If $r_2 = 1$, there exists at least one pair of entries x_i, x_j with $x_i \neq x_j$ in the eigenvector x corresponding to $\lambda_2(L)$. If $r_2 = m$, there exists a set of elements $x_i, x_j, x_{j+1}, \ldots, x_{j+m-1}$ in eigenvector

```
r_2 := the multiplicity of λ_2(L)
L̃ := the Laplacian of superposition system
x_i, x_j, ..., x_{j+m-1} := the entries of the eigenvector of
λ_2(L)
if r_2 = 1 then
   Connecting node i to j in the superposition system
   while x_i = x_j do
      Connecting node i = i + 1 to j in the superposition
      system
   end while
end if
if r_2 = m > 1 then
   Connecting node i to j, j + 1, ..., j + m in the
   superposition system
   while x_i = x_j = x_{j+1} = ··· = x_{j+m-1} do
      Connecting node i = i + 1 to j, j + 1, ..., j + m in
      the superposition system
   end while
end if
```

Algorithm 1: Finding a superposition system to enhance the convergence rate.

x that are not all equal; otherwise, x is a zero vector. So there is a superposition system that can increase the rate of convergence.

Proposition 11. *For an undirected connected graph $\mathcal{G}$, if the smallest nonzero eigenvalue of L is repeated, the invalid algebraic connectivity weights can be constructed in any two nodes.*

Proof. The invalid algebraic connectivity weights exist if and only if, for an eigenvector x corresponding to the smallest nonzero eigenvalue of L, there are two components x_i, x_j of x satisfying $x_i = x_j$, $i \neq j, i, j \in \{1, \ldots, n\}$.

For the positive semidefinite matrix $\widetilde{L}_+$ or negative semidefinite matrix $\widetilde{L}_-$, let $\operatorname{rank}(\widetilde{L}_+) = \operatorname{rank}(\widetilde{L}_-) = 1$. If $\lambda_2(L)$ is a repeated eigenvalue, only one of repeated eigenvalues of $\widehat{L}$ will change. Then for the superposition system, the nonzero entries of $\widetilde{A}_+$ or $\widetilde{A}_-$ are invalid algebraic connectivity weights. All the eigenvectors corresponding to $\lambda_2(L)$ of $\widehat{L}$ contain components x_i, x_j with $x_i = x_j$, where (i, j) represents the position of each nonzero weights associated with $\widetilde{A}_+$ and $\widetilde{A}_-$. □

3.3. Optimization of Convergence Rate. In a stable system, it is always expected that there is a larger convergence rate to take less time to get the stable state. In case all the weights of a communication graph are increased, the values of nonzero eigenvalues will be improved as a whole, and accordingly the stability can be achieved with a larger rate. From a practical point of view, however, this will increase the cost of realization. Even so, we still hope to achieve the rapidity of convergence under fixed cost. Below, let $tr(L)$ denote the cost for achieving consensus. With $tr(L)$ being fixed, we hope to find the most optimal topology to achieve the largest convergence rate.

Theorem 12. *For a system with fixed cost $\sum_{i=1}^{n} l_{ii} = c > 0$, $c \in \mathbb{R}$, if $L(t)$ is time-varying and $\lambda_2(L) = \sum_{i=1}^{n} l_{ii}/(n-1)$, then the system achieves the largest convergence rate.*

Proof. Assume $\lambda_2(L) = \sum_{i=1}^{n} l_{ii}/(n-1)$ and the system does not achieve the largest convergence rate. If $\lambda_2(L) > \sum_{i=1}^{n} l_{ii}/(n-1)$, then $\min \lambda_2(L) + \lambda_3(L) + \cdots + \lambda_n(L) > \sum_{i=1}^{n} l_{ii}$, which is in contradiction with the known condition. If $\lambda_2(L) < \sum_{i=1}^{n} l_{ii}/(n-1)$, it can be seen that the system does not achieve the largest convergence rate. The above arguments mean that the system achieves the largest convergence rate if $\lambda_2(L) = \lambda_3(L) = \cdots = \lambda_n(L) = \sum_{i=1}^{n} l_{ii}/(n-1)$. □

For the communication delay τ_{ij} between nodes v_i and v_j, we consider the case $\tau_{ij} = \tau$. Then system (2) along with protocol (1) can be written as

$$\dot{x}_i = -\sum_{j \in N(i)} a_{ij} \left(x_i (t-\tau) - x_j (t-\tau) \right) \tag{21}$$

$$\dot{x} = -Lx (t-\tau). \tag{22}$$

Lemma 13 (see [2]). *Consider a network of integrator agents with identical communication time delay τ in all links. Assume that the network topology is fixed, undirected, and connected. Then, protocol (22) with $\tau_{ij} = \tau$ globally asymptotically solves the average consensus problem if and only if either of the following equivalent conditions is satisfied.*

(i) *$\tau \in (0, \tau^*)$ with $\tau^* = \pi/2\lambda_n$.*

(ii) *The Nyquist plot of $\Gamma(s) = e^{-\tau s}/s$ has a zero encirclement around $-1/\lambda_k, \forall k > 1$.*

By Theorem 12, $\lambda_2(L)$ is determined as long as the system achieves the largest convergence rate under fixed cost. Since the value of $\lambda_2(L)$ is determined, it is expected to construct topologies to achieve the largest convergence rate, i.e., seeking L_{opt}, where L_{opt} is the Laplacian having an eigenvalue $\max_L \lambda_2(L)$. If the graph of a multiagent system is a complete graph and all of its weights are identical, then the system achieves the largest convergence rate under fixed cost $\sum_{i=1}^{n} l_{ii} = c > 0$, $\lambda_2(L) = \sum_{i=1}^{n} l_{ii}/(n-1)$. In this case, $\lambda_2(L)$ achieves the biggest magnitude and $\lambda_n(L)$ achieves the smallest magnitude. By Lemma 13, the delay margin τ^* is decided only by $\lambda_n(L)$. If there is a communication delay in the system and the $\lambda_n(L)$ achieves the smallest magnitude, then the delay margin achieves the biggest magnitude.

The Laplacian transformation of (22) is

$$X(s) = \frac{x(0)}{sI + e^{-\tau s}L} \tag{23}$$

Let $G(s) = sI + e^{-\tau s}L$. Then $det(G(s)) = det(sI + e^{-\tau s}L) = s\prod_{i=2}^{n}(s + \lambda_i(L)e^{-\tau s})$. In case $s + \lambda_i(L)e^{-\tau s} = 0$, we have $se^{\tau s} = -\lambda_i(L)$. Let $F(s) = se^{\tau s}$. The partial derivation of $F(s)$ is $\partial F(s)/\partial Re(s) = (1 + \tau s)e^{\tau s}$. If $F(s)$ is monotone for $Re(s)$ in $\mathbb{R}$, the root of $det(G(s))$ can only be one side of $-1/\tau$. As a consequence, the rightmost root of a system with time delays is related to $\lambda_2(L)$ or $\lambda_n(L)$.

Corollary 14. *If $tr(L) = c > 0$, $c \in \mathbb{R}$, and the communication topology is a complete graph with all weights being identical, then the delay margin τ^* achieves the biggest magnitude, and the topology is unique. And the system achieves largest convergence rate, if $F(s)$ is monotone for $Re(s)$.*

Proof. If the communication graph associated with the system is a complete graph, each node of the graph is connected with the others, and all the weights are identical. Hence, $\lambda_i(L) = \lambda_j(L) = c/(n-1) = l_{ii} - l_{ij}$, $i, j \in \{2, \ldots, n\}$. Then $\lambda_2(L)$ achieves the biggest magnitude. Thus, $\lambda_n(L)$ achieves the smallest magnitude; that is, $\tau^* = \pi/2\lambda_n(L)$ achieves the biggest magnitude.

If $F(s)$ is monotone increasing for $Re(s)$, then $\partial F(s)/\partial Re(s) > 0$. Thus, the convergence rate is determined by the upper bound $-\lambda_i(L)$; that is, $-\lambda_2(L)$, $i = \{2, \ldots, n\}$. $\min\{|Re(s) \| Re(s) \neq 0\}$ increases with $\lambda_2(L)$ increasing. If $F(s)$ is monotone reducing for $Re(s)$, then $\partial F(s)/\partial Re(s) < 0$. The convergence rate is determined by the lower bound $-\lambda_i(L)$; that is, $-\lambda_n(L)$, $i = \{2, \ldots, n\}$. $\min\{|Re(s) \| Re(s) \neq 0\}$ increases with $\lambda_n(L)$ reducing. If $\lambda_2(L)$ achieves the biggest magnitude, $\lambda_n(L)$ achieves the smallest magnitude, and $F(s)$ is monotone for $Re(s)$; then the system achieves the largest convergence rate. □

Remark 15. If a node v_i is connected with all the other nodes, and the weights between v_i and the others are identical, then $\lambda = l_{ii} - l_{ij}$ is an eigenvalue of L.

For a system with time delays τ, the convergence rate is also affected by time delays τ.

Corollary 16. *For a fixed topology, if the time delay τ increases and $\tau < \tau^*$, the convergence rate of the system reduces.*

Proof. The characteristic polynomial of the system is $\det(G(s)) = \det(sI + e^{-\tau s}L) = s\prod_{i=2}^{n}(s + \lambda_i(L)e^{-\tau s})$, $i = \{2, \ldots, n\}$. Let $Re(s) = c$, $Im(s) = d$; then the root of characteristic polynomial of the system can be solved from $c + jd + \lambda_i(L)e^{-\tau(c+jd)} = 0$. Thus, $c + \lambda_i(L)e^{-c\tau}\cos d\tau = 0$, $b - \lambda_i(L)e^{-c\tau}\sin d\tau = 0$, $\lambda_i(L) = -ce^{c\tau}/\cos d\tau = de^{c\tau}/\sin d\tau$, such that $c = -d/\tan d\tau$. $\partial c/\partial\tau = d^2/(\tan^2 d\tau \cdot \cos^2 d\tau) \geq 0$, and then c increases with τ increasing; that is, convergence rate reduces with τ increasing for $\tau < \tau^*$. □

Example 17. The original system is shown as Figure 3(a), and the system with the largest convergence rate under the same cost is shown as Figure 3(b). The corresponding Laplacian matrices are

$$L = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & 4 & -1 & -2 \\ -1 & 0 & -1 & 5 & -3 \\ -1 & -1 & -2 & -3 & 7 \end{bmatrix},$$
$$L_{opt} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}. \tag{24}$$

For the eigenvalues of original system, $\lambda_2(L) = 1.7273$, $\lambda_5(L) = 9.438$, and the delay margin is $\tau^* = 0.1661s$. All the nonzero eigenvalues of the system with the largest convergence rate are 5, and the delay margin of this system is $\tau^* = 0.3141s$.

4. Construction of Invalid Algebraic Connectivity Weights

Definition 18. $\pi = \{C_1, \ldots, C_r\}$ is called an equitable weights partition if each node in cell C_i has the same weight with the nodes in C_j, $\forall i, j \in \{1, \ldots, r\}$. If each node in C_i has the same weight with the nodes in C_j the partition π is called an almost equitable weights partition, where $\forall i \neq j, i, j \in \{1, \ldots, r\}$. We denote the cardinality of cell C_r with $|C_r|$.

A cell C_r is nontrivial if it contains more than one node; otherwise it is trivial. Equitable weights partition can be used in what follows to analyze the convergence rate of consensus, and we can construct the nontrivial cells to get the IACWs.

4.1. Convergence Rate under Equitable Weights Partition. For a nontrivial cell C_r in an equitable weights partition, suppose that the number of nodes in C_r is $|C_r| = m$. We have the following lemma.

Lemma 19. *If $\tilde{L}_{ij}, \tilde{L}_{kj}$ are superposition systems only constructed, respectively, by v_i and v_j and v_k and v_j; then $\lambda_2(\widehat{L}_{ij}) = \lambda_2(\widehat{L}_{kj})$, where $v_i, v_k \in C_r$, and $v_j \in \mathcal{V}$ is a node given in advance, $i \neq k$, $j \neq i$, $j \neq k$.*

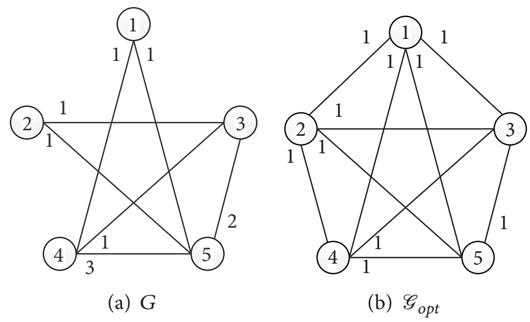


FIGURE 3: (a) Original system $\mathscr{G}$; (b) the largest convergence rate system $\mathscr{G}_{opt}$.

Proof. Since there is a nontrivial cell with m nodes and $v_j \in \mathscr{V}$ is fixed, we construct the superposition system by selecting any node v_i in the nontrivial cell to establish one communication link with the given v_j. The dynamic equations of the nodes in the nontrivial cell are

$$\begin{pmatrix} \dot{x}_{l+1} \\ \dot{x}_{l+2} \\ \vdots \\ \dot{x}_{l+m} \end{pmatrix} = \begin{bmatrix} l_{r1} & L_r & l_{r2} \end{bmatrix} x, \tag{25}$$

where the entries of the columns in $l_{r1} \in \mathbb{R}^{m\times l}$ and $l_{r2} \in \mathbb{R}^{m\times(n-m-l)}$ are all equal. In the superposition system, an arbitrary node $v_i \in C_r$ is selected to establish one communication link with the fixed v_j. Then the dynamic equations remain unchanged when the positions of $x_{l+1}, \ldots, x_{l+m}$ are exchanged, and the positions of $\dot{x}_{l+1}, \ldots, \dot{x}_{l+m}$ remain unchanged. In this situation, the topology of the system remains unchanged. The discussion above means that the variations of eigenvalues associated with L are equal since there is only one link between v_j and any node of C_r in the superposition system. Note that the variation of algebraic connectivity is also identical. Then, if $v_i \in C_r$, the variation of the algebraic connectivity is unrelated to the choosing of v_i. Thus, if $v_k \in C_r$ is selected to construct a superposition system, its convergence rate is identical with $\widetilde{L}_{ij}$, and accordingly $\lambda_2(\widehat{L}_{ij}) = \lambda_2(\widehat{L}_{kj})$, where $i \neq k$, $j \neq i$, $j \neq k$. □

Example 20. A graph with two nontrivial cells is shown as Figure 4. The corresponding Laplacian L is

$$L = \begin{bmatrix} 6 & -2 & -2 & -1 & -1 \\ -2 & 5 & -1 & -1 & -1 \\ -2 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & 5 & -2 \\ -1 & -1 & -1 & -2 & 5 \end{bmatrix}. \tag{26}$$

The expression of Laplacian L implies that there is a partition $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4, 5\}$. Consider the case that $\operatorname{rank}(\widetilde{L}) = 1$, which means that there is an edge

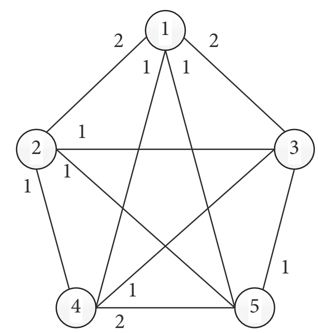


FIGURE 4: A graph with nontrivial cells.

with weight 1 in the graph associated with the superposition system. Let us establish the connection between any two nodes, and let $\widetilde{L}_{ij}$ denote the corresponding matrix which has only one link between nodes v_i and v_j. $\widehat{L}_{ij}$ denotes the Laplacian matrix of the superposed system corresponding to $\widetilde{L}_{ij}$. Then, $\lambda_2(\widehat{L}_{12}) = \lambda_2(\widehat{L}_{13}) = 5$, $\lambda_2(\widehat{L}_{14}) = \lambda_2(\widehat{L}_{15}) = 5.5188$, $\lambda_2(\widehat{L}_{24}) = \lambda_2(\widehat{L}_{25}) = \lambda_2(\widehat{L}_{34}) = \lambda_2(\widehat{L}_{35}) = 5.382$. The variations of the other eigenvalues of $\widehat{L}_{ij}$ are the same.

Let λ_{ij} denote an eigenvalue of a submatrix which is generated by selecting the rows and columns corresponding to the nodes of the nontrivial cell C_r. $\lambda_{ij} = l_{ii} - l_{ij} = l_{jj} - l_{ji}$, with i, j indicating that λ_{ij} only relies on $C_r, i, j \in C_r$. The *Faria vector* denotes the vector which only contains two nonzero entries, and the two entries are opposite from each other.

Corollary 21. *For an equitable weights partition π with a nontrivial cell C_r, the multiplicity of λ_{ij} satisfies $r(\lambda_{ij}) = n - \operatorname{rank}(\lambda_{ij}I - L)$, $m - 1 \leq r(\lambda_{ij})$, $|C_r| = m$.*

Proof. Since Laplacian L is symmetric, the geometrical and algebraic multiplicity of each eigenvalue of L is equal to each other. Hence, the number of linearly independent eigenvectors x of λ_{ij} is the multiplicity of λ_{ij}. Note that $n - \operatorname{rank}(\lambda_{ij}I - L)$ is equal to the number of linearly independent solutions of $(\lambda_{ij}I - L)x = 0$. It follows that $n - \operatorname{rank}(\lambda_{ij}I - L)$ is equal to the number of linearly independent vectors x which satisfies $Lx = \lambda_{ij}x$, $r(\lambda_{ij}) = n - \operatorname{rank}(\lambda_{ij}I - L)$. The structure form of $\lambda_{ij}I - L$ implies that the row entries of L corresponding to the nodes in C_r are equal. Therefore, $m - 1 \leq r(\lambda_{ij})$. □

Remark 22. The number of eigenvalues λ_{ij} is no less than the number of the nontrivial cells C_r.

It is worth mentioning that, for an equitable weights partition, the submatrix corresponding to a nontrivial cell C_r consisting of m nodes only has the eigenvalue λ_{ij} with its multiplicity satisfying $r(\lambda_{ij}) = m - 1$. There, however, will be such a situation $r(\lambda_{ij}) > m - 1$, which is the consequence of the nontrivial cell C_r working with the other cells.

Proposition 23. *For an equitable weights partition, if there is a nontrivial cell with m nodes, the number of linearly independent Faria vectors in the eigenvectors corresponding to λ_{ij} is $m - 1$.*

Proof. For a nontrivial cell with m nodes, the number of *Faria vectors* is C_m^2. Let $X_{C_m^2} = \{x_1, x_2, \ldots, x_{C_m^2}\}$, where x_i denotes a *Faria vector*, $i \in \{1, 2, \ldots, C_m^2\}$, and the nonzero entries in x_i is corresponding to the nodes in C_r. Since $X_{C_m^2}$ consists of C_m^2 *Faria vectors*, then $\operatorname{rank}(X_{C_m^2}) \leq m$. Let $X_{m-1} = \{x_1, x_2, x_3, \ldots, x_{m-1}\}$, where $x_1 = e_1 - e_2 = \{1, -1, 0, 0, \ldots, 0\}$, $x_2 = e_1 - e_3 = \{1, 0, -1, 0, \ldots, 0\}$, $x_3 = e_1 - e_4 = \{1, 0, 0, -1, \ldots, 0\}$, $\cdots$, $x_{m-1} = e_1 - e_m = \{1, 0, 0, 0, \ldots, 0, -1, 0, \ldots, 0\}$, and $e_i \in \mathbb{R}^n$ denotes a vector with the ith entry taking 1 and the others taking 0. $X_{C_m^2} \setminus X_{m-1}$ can be represented as a linear combination of $e_1, e_2, \ldots, e_{m-1}$, and the other $C_m^2 - (m-1)$ *Faria vectors* all can be represented by vectors x_1, x_2, x_3, $\cdots$, x_{m-1}. Hence, the number of linearly independent *Faria vectors* in the eigenvectors set of λ_{ij} is $m - 1$. □

Example 24. A graph with two nontrivial cells is shown in Figure 5. The corresponding Laplacian L is

$$L = \begin{bmatrix} 11 & -1 & -1 & -3 & -3 & -3 \\ -1 & 8 & -1 & -2 & -2 & -2 \\ -1 & -1 & 8 & -2 & -2 & -2 \\ -3 & -2 & -2 & 11 & -2 & -2 \\ -3 & -2 & -2 & -2 & 11 & -2 \\ -3 & -2 & -2 & -2 & -2 & 11 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 4 \\ 1 & 1 & -3 & 0 & 0 & 1 \\ 1 & -1 & -3 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & -2 \\ 1 & 0 & 1 & -1 & 0 & -2 \\ 1 & 0 & 1 & 0 & -1 & -2 \end{bmatrix} = [v_1, v_2, v_3, v_4, v_5, v_6], \tag{27}$$

$$D_\lambda = \operatorname{diag}\{0, 9, 10, 13, 13, 15\}.$$

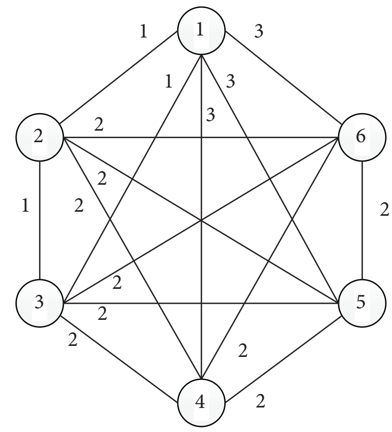


Figure 5: A graph with nontrivial cells.

The expression of L implies that there is a partition π with $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4, 5, 6\}$. The entries of diagonal matrix D_λ are the eigenvalues of L, and the columns of V are the eigenvectors corresponding to D_λ. Note that v_4 and v_5 are *Faria vectors*. It can be seen that 9 is an eigenvalue of the submatrix obtained by selecting the rows and columns corresponding to the nodes in C_2, and its corresponding eigenvector is a *Faria vector*. The eigenvalue 10 is not associated with any submatrix corresponding to any cell. The second and third entry of the eigenvector corresponding to eigenvalue 10 are equal, and the same thing happens to the fourth, fifth, and sixth entry. It is the same situation to eigenvalue 15 with the second and third entry of its eigenvector being equal as well as the fourth, fifth, and sixth entry. The two eigenvectors of eigenvalue 13 are both *Faria vectors*.

Lemma 25. *For a connected graph $\mathcal{G}$, if there exists a nontrivial cell C_r with m nodes, then the following claims hold under equitable weights partition.*

(i) *If $n - \operatorname{rank}(\lambda_{ij}I - L) = m - 1$, the eigenvectors of λ_{ij} are all Faria vectors.*

(ii) *If $n - \operatorname{rank}(\lambda_{ij}I - L) > m - 1$, the entries in the eigenvector x of λ_{ij} corresponding to any other nontrivial cell C_{r-1} are equal to each other as long as the eigenvector is not a Faria vetor.*

(iii) *If $\lambda(L) \neq \lambda_{ij}$, the entries in the eigenvectors of $\lambda(L)$ corresponding to the nontrivial cell C_r are equal to each other.*

Proof.

(i) If there is a nontrivial cell C_r containing nodes v_i and v_j, let us set $\lambda_{ij} = l_{ii} - l_{ij}$. Then $\operatorname{rank}(\lambda_{ij}I - L) < n$, that is, $|\lambda_{ij}I - L| = 0$, where λ_{ij} is an eigenvalue of Laplacian L. In case $x = \{0, \ldots, 0, d_{ith}, 0 \ldots, 0, -d_{jth}, 0, \ldots, 0\}^T$, $Lx = \lambda_{ij}x$, where $d_{ith} = d_{jth}$, d_{ith} and d_{jth} are nonzero constants, which denote the ith and jth entries of vector x, respectively. If $n - \operatorname{rank}(\lambda_{ij}I - L) = m - 1$, it follows from Lemma 19 that the position exchanging of nodes in the nontrivial cell does not affect the structure of L. As a consequence, the eigenvectors of λ_{ij} are also unaffected. Thus, when d_{ith} exchanges its position with $-d_{jth}$, the original eigenvector x is replaced by $-x$, and the entries except the ith and jth entries in eigenvectors x are 0. That is, there are $m-1$ linearly independent eigenvectors of *Faria vector* corresponding to the $m-1$ eigenvalues λ_{ij}.

(ii) If $n-\text{rank}(\lambda_{ij}I-L) \geq m$, and the entries in eigenvector x corresponding to C_{r-1} are equal, then, there are more than $m-1$*Faria vectors*. Thus the entries of x's of λ_{ij} are equal.

(iii) If $\lambda(L) \neq l_{ii} - l_{ij}$, let us consider the entries of x's of $\lambda(L)$, where these entries correspond to the nodes in C_r. Assume that these entries are not equal. Then vectors x_e are still eigenvectors associated with $\lambda(L)$ after the positions of these unequal entries are exchanged. Let x_{lc} denote the linear combination of x and x_e. The entries of x_{lc} corresponding to the nodes in the nontrivial cell C_r are equal, and the others corresponding to the trivial cell $C_{\neg r}$ are 0, which contradicts with the assumption. Therefore, the entries in the eigenvectors corresponding to $\lambda(L) \neq l_{ii} - l_{ij}$ are equal.

□

The equitable weights partition leads to some special perspectives on the eigenvalues and eigenvectors of Laplacian matrix, which allows us to propose an explicit method for constructing IACW.

Theorem 26. *For an equitable weights partition of graph $\mathcal{G}$, suppose that there is a nontrivial cell C_r with m nodes. Then the following statements hold for IACW.*

(i) *If $\lambda_2(L) \neq \lambda_{ij}$, the weights between the nodes in C_r are IACW.*

(ii) *If $\lambda_2(L) = \lambda_{ij}$, and $n - \text{rank}(\lambda_{ij}I - L) = m - 1$, the weights except those between the nodes in the nontrivial cell C_r are IACW.*

Proof. By Lemma 25, if $\lambda_2(L) \neq \lambda_{ij}$, the entries in the eigenvector x of $\lambda_2(L)$ corresponding to the nodes in C_r are equal, and accordingly the weights between the nodes in C_r are IACW. If $\lambda_2(L) = \lambda_{ij}$, and $n - \text{rank}(\lambda_{ij}I - L) = m - 1$, then Lemma 25 means that the ith and the jth entry in the eigenvectors of $\lambda_2(L)$ are opposite, and the others are 0. Thus the weights except those in the nontrivial cell C_r are IACW. □

4.2. Convergence Rate under Almost Equitable Weights Partition. Different from the equitable weights partition, the weights in the nontrivial cell C_r are not equal if the almost equitable weights partition is taken into account. So the latter makes a part of properties lost compared to the equitable weights partition. Thus, this case is more complex. For the almost equitable weights partition, since the weights between the nodes in nontrivial cell C_r are different, $\lambda_{ij} = l_{ii} - l_{ij}$ is no longer true. As a consequence, λ_{ij} cannot be characterized exactly.

Lemma 27. *For an undirected graph $\mathcal{G}$, there is an eigenvector x of a nonzero eigenvalue of L so that $x^T 1_n = 0$, where $1_n \in \mathbb{R}^n$ denotes the column vector with all entries taking* 1.

Proof. For an undirected connected graph, 0 is a simple eigenvalue of L, and $L1_n = 0$. Let x denote an eigenvector of a nonzero eigenvalue of L; i.e., $Lx = \lambda(L)x$. The Laplacian L is symmetric. It follows that $x^T L = \lambda(L)x^T$, $x^T L 1_n = \lambda(L)x^T 1_n = 0$. Thus, $x^T 1_n = 0$. □

In case there is a partition with a nontrivial cell C_r, L can be decomposed in accordance with the nontrivial cell,

$$L = \begin{bmatrix} L_r & l_r \\ l_r^T & L_{\neg r} \end{bmatrix}, \tag{28}$$

where L_r corresponds to the nodes in the nontrivial cell, which indicates the weights between nodes in C_r and its degree. Each column of l_r is equal to the other columns, which indicates the weights between nodes in C_r and nodes in other cells. $L_{\neg r}$ corresponds to the trivial cells.

Remark 28. For an equitable weights partition, L_r has an eigenvalue $\lambda_{ij} = l_{ii} - l_{ij}$ with multiplicity $m-1$ if there is a nontrivial cell. For an almost equitable weights partition, L_r only has an eigenvalue with multiplicity 1 which can be characterized.

$L_r = l_{r1}^T 1_r I_r + L_{rr}$, where l_{r1} denotes the first column of l_r, and 1_r is the vector with proper dimension and all entries taking 1. L_{rr} is the Laplacian matrix of a system constructed by the nodes in the nontrivial cell C_r. For almost equitable weights partitions, L_{rr} is the general Laplacian matrix. So the nonzero eigenvalues cannot be characterized. Because of the existence of $l_{r1}^T 1_r I_r$, there is an eigenvalue $l_{r1}^T 1_r$ associated with L_r, and the entries in the eigenvector of $l_{r1}^T 1_r$ are all equal.

Lemma 29. *For a matrix $L_r \in \mathbb{R}^{m\times m}$, suppose that there is only one nontrivial cell associated with L_r. Then the Laplacian matrices L and L_r share $m-1$ common eigenvalues λ_{ij}. Moreover, for matrix L, its x's associated with λ_{ij} take x_r as a subvector, where x_r is the eigenvector of λ_{ij} associated with L_r; that is, $x^T = \{x_r^T, 0, \ldots, 0\}$.*

Proof. Assume that the nontrivial cell C_r contains m nodes, and $L_r = l_{r1}^T 1_r I_r + L_{rr}$. Then $L_r x_r = l_{r1}^T 1_r x_r + \lambda_{rr} x_r = \lambda_{ij} x_r$, where $L_{rr} x_r = \lambda_{rr} x_r$, $\lambda_{ij} = l_{r1}^T 1_r + \lambda_{rr}$. Let $x^T = \{x_r^T, 0, \ldots, 0\}$, with the positions of entries 0 corresponding to the nodes in $C_{\neg r}$. Since $x^T 1_n = 0$, it follows that $[L_r, l_r]x = \lambda_{ij} x_r$, $[l_r, L_{\neg r}]x = 0_{\neg r}$, $Lx = \lambda_{ij} x$. If $\lambda_{rr} = l_{r1}^T 1_r$, we see that x_r is a vector with all of its entries taking 1. Thus $x_r^T 1_r \neq 0$; that is, $l_r^T x_r \neq 0$, $[l_r, L_{\neg r}]x \neq 0_{\neg r}$. Therefore, L_r does not share the eigenvalue $l_{r1}^T 1_r$ with L. So L and L_r share $m-1$ common eigenvalues, and $x^T = \{x_r^T, 0, \ldots, 0\}$. □

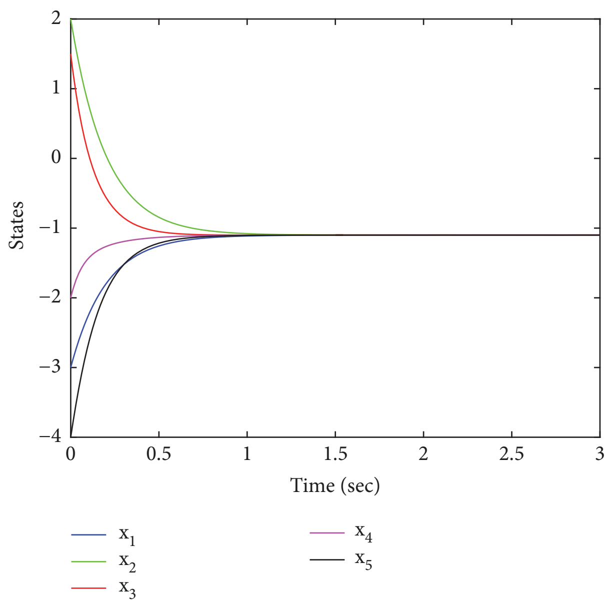


FIGURE 6: States of graph $\mathcal{G}$.

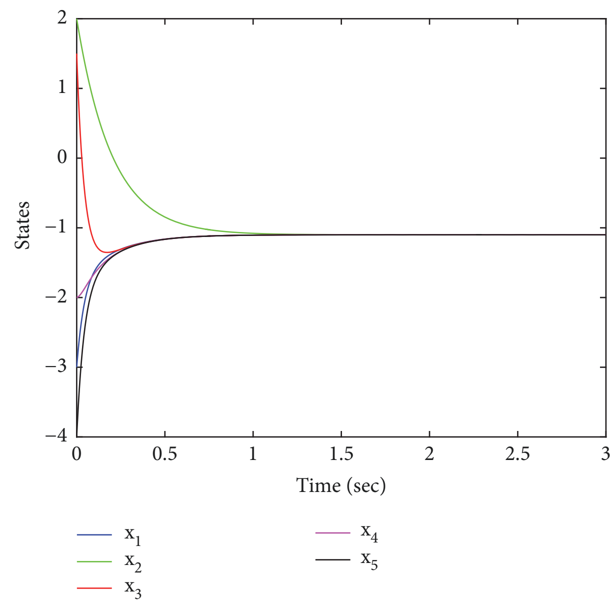


FIGURE 7: States of graph $\widehat{\mathcal{G}}_1$.

Lemma 29 explains the affections caused by nontrivial cells under equitable weights and almost equitable weights partition. The existence of nontrivial cells in L_r makes that part of eigenvalues and eigenvectors of L rely on L_r.

Theorem 30. *For an almost equitable weights partition of graph $\mathcal{G}$, suppose that there is a nontrivial cell C_r. If $\lambda_2(L) = \lambda_{ij}$, the weights except those in C_r are all IACW. If $\lambda_2(L) \neq \lambda_{ij}$, the weights in C_r are IACW.*

Proof. For an almost equitable weights partition with a nontrivial cell C_r, λ_{ij} is an eigenvalue of submatrix obtained by selecting the rows and columns corresponding to the nodes in C_r. By Lemma 29, if $\lambda_2(L) = \lambda_{ij}$, the entries of vector x except those corresponding to the nodes in C_r are 0, where x is the eigenvector corresponding to λ_{ij}. Therefore, the weights of $C_{\neg r}$ are IACW. Let $x^T = [x_r, x_{\neg r}]^T$, where x_r is the eigenvector of L_r. The entries of every column in l_r are equal to the others. Then $l_r x_{\neg r} = c\mathbf{1}_r$, where c is a constant. If $\lambda_2(L) \neq \lambda_{ij}$, and the entries of x_r are different, then $[L_r, l_r]x = L_r x_r + c\mathbf{1}_r = \lambda x_r$, $L_r x_r = \lambda x_r - c\mathbf{1}_r$. If the entries of x_r are different, the entries of $\lambda x_r - c\mathbf{1}_r$ are also different. Thus $\lambda = \lambda_{ij}$ and $\lambda_2(L) \neq \lambda_{ij}$ yield that the weights between the nodes in the nontrivial cell C_r are IACW. □

When the nontrivial cells existed, there are IACWs in a system. Thus, we can construct the nontrivial cell to get the IACWs. And it is easy to know that the IACWs are changed, when the case $\lambda_2(L) = \lambda_{ij}$ transforms into $\lambda_2(L) \neq \lambda_{ij}$. The cases of the IACWs are identical for equitable weights partition and almost equitable weights partition.

5. Simulation Results

When a superposition system is superposed to an original system, some weights of the original graph are changed

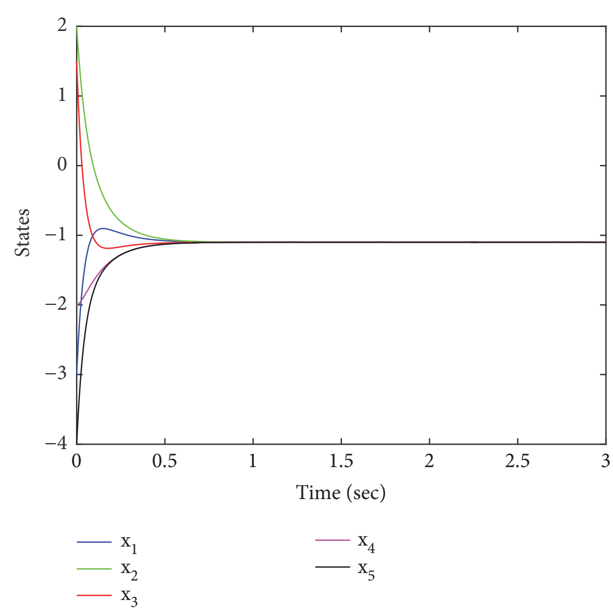


FIGURE 8: States of graph $\widehat{\mathcal{G}}_2$.

accordingly. In order to verify the variation of the convergence rate on topologies and weights, we do simulations for Example 8.

The convergence rate of a system depends on the magnitude of the smallest nonzero eigenvalue $\lambda_2(L)$. When a system gets a larger $\lambda_2(L)$, it can converge faster. For the system in Example 8, we choose initial states $x_1(0) = -3$, $x_2(0) = 2$, $x_3(0) = 1.5$, $x_4(0) = -2$, $x_5(0) = -4$. Therefore, $x^* = -1.1$ when the system achieves consensus. Figures 6–8 show the simulation results of $\mathcal{G}$, $\widehat{\mathcal{G}}_1$, $\widehat{\mathcal{G}}_2$, respectively, which show that the system associated with graph $\widehat{\mathcal{G}}_2$ achieves consensus

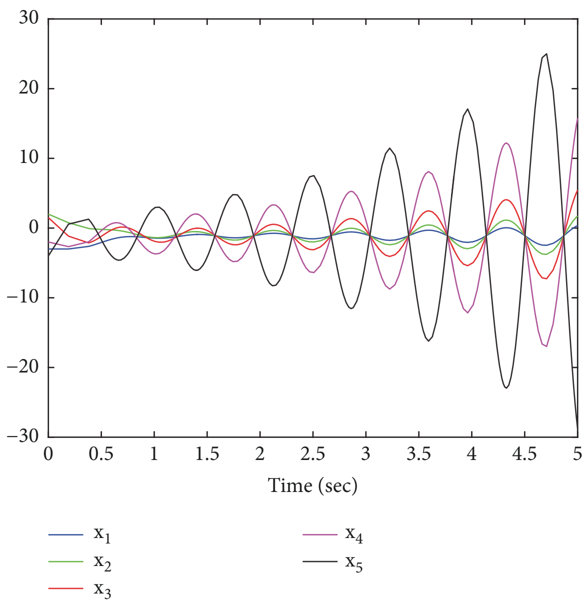


Figure 9: Sates of graph $\mathscr{G}$, $\tau = 0.19s$.

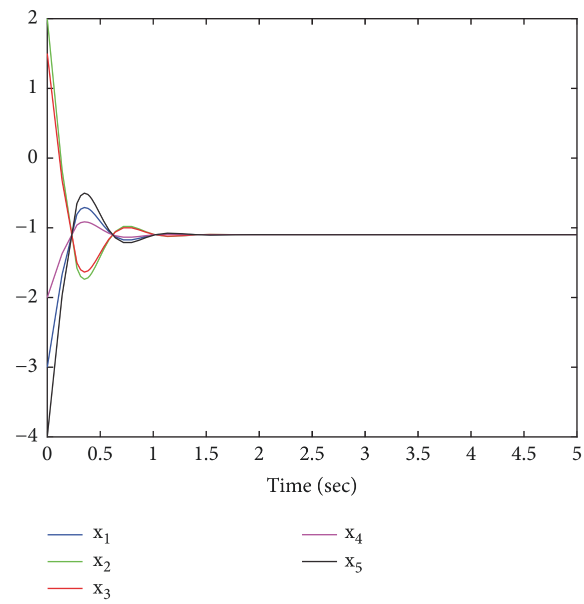


Figure 11: Sates of graph $\mathscr{G}_{opt}$, $\tau = 0.14s$.

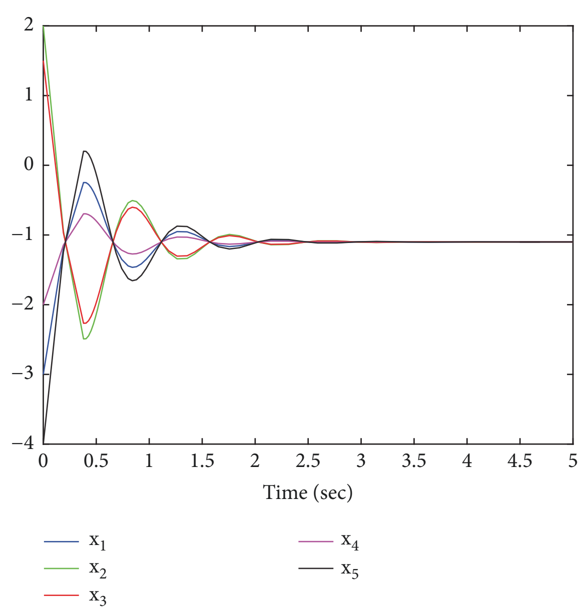


Figure 10: Sates of graph $\mathscr{G}_{opt}$, $\tau = 0.19s$.

faster than $\mathscr{G}$ and $\widehat{\mathscr{G}}_1$. In addition, the system associated with $\widehat{\mathscr{G}}_1$ converges as fast as $\mathscr{G}$ because of the identical λ_2.

Figures 9, 10, and 11 are the simulations of Example 17, where the initial states are the same as Example 8. For $\tau = 0.19s$, the state response is shown in Figure 9. It can be seen that the system is unstable since the time delay τ is larger than the delay margin $\tau^* = 0.1661s$. For graph $\mathscr{G}_{opt}$, however, the system can achieve consensus with $\tau = 0.19s$ as shown in Figure 10, which means that the system with graph $\mathscr{G}_{opt}$ has a larger delay margin τ^* than the system with graph $\mathscr{G}$ in Example 17. The corresponding topological structures in Figures 9 and 10 are, respectively, (a) and (b) in Figure 3. Precisely because the topology corresponding to Figure 9 is different from that corresponding to Figure 10, the system associated with Figure 9 is unstable at the delay τ while Figure 10 is stable at the same delay. When $\tau = 0.14s$, the system associated with graph $\mathscr{G}_{opt}$ achieves a faster consensus than the system at $\tau = 0.19s$, which is shown in Figure 11.

6. Conclusions and Future Work

In this paper, we proposed a superposition system which was superposed to the original system to explore the variation of convergence rate. By analyzing the eigenvector of $\lambda_2(L)$, results were derived on checking whether the convergence rate can be changed. When the Laplacian $\widetilde{L}$ of the superposition system only consists of invalid algebraic connectivity weights, it was proven that the convergence rate remains unchanged. Otherwise, the convergence rate changes. We gave the most optimal case of the convergence rate under fixed cost, which makes the convergence rate the largest and the system more stable. Finally, we proposed a method of constructing invalid algebraic connectivity weights to make systems resistant in a certain extent to the perturbation. In addition, based on the equitable weights partition and almost equitable weights partition, we analyzed the changes of eigenvalues and eigenvectors to discover the variation of convergence rate. In future work, the optimization of convergence rate and convergence rate on directed graphs will be studied. Although, explicit results have been derived for the convergence rate by taking advantage of the proposed concept of superposition systems, the optimization of convergence rate still needs further study. In the future work, convergence rate on directed graphs will also be studied.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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