A Study on Lump Solutions to a Generalized Hirota-Satsuma-Ito Equation in (2+1)-Dimensions

Wen-Xiu Ma,1,2,3,4,5,6 Jie Li,7 and Chaudry Masood Khalique6

1Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China
2Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
3Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
4College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, China
5College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China
6International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa
7Jining No. 1 People’s Hospital, Shandong, Jining 272011, China

Correspondence should be addressed to Wen-Xiu Ma; mawx@cas.usf.edu

Received 9 August 2018; Accepted 25 October 2018; Published 2 December 2018

1. Introduction

In the classical theory of differential equations, the main question is to study the existence of solutions to given equations, including many nonlinear equations describing real-world problems. Cauchy problems are to deal with the existence, uniqueness, and stability of solutions satisfying initial data. Laplace’s method is developed for solving Cauchy problems for linear ordinary differential equations and the Fourier transform method for linear partial differential equations. In modern soliton theory, the isomonodromic transform method and the inverse scattering transform method have been designed to solve Cauchy problems for nonlinear ordinary and partial differential equations [1–3]. Explicitly solvable differential equations include various constant-coefficient and linear differential equations, but it is extremely difficult to compute exact solutions to variable-coefficient or nonlinear equations.

However, the Hirota bilinear method provides us with a working approach to soliton solutions, historically found for nonlinear integrable equations [4, 5]. Soliton solutions are analytic ones exponentially localized in all directions in space and time. Let a polynomial P determine a Hirota bilinear differential equation:

\[ P(D_x, D_y, D_t) f \cdot f = 0, \]  

(1)

in (2+1)-dimensions, where \( D_x, D_y, \) and \( D_t \) are Hirota’s bilinear derivatives. The corresponding partial differential equation with a dependent variable \( u \) is determined usually by one of the logarithmic transformations: \( u = 2(\ln f)_x \) and \( u = 2(\ln f)_{xx} \). Within the Hirota bilinear formulation, soliton solutions are expressed through

\[ f = \sum_{\mu=0,1} \exp \left( \sum_{i=1}^{N_1} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij} \right), \]  

(2)
where \( \sum_{\mu=0,1} \) means the sum over all possibilities for \( \mu_1, \mu_2, \ldots, \mu_N \) taking either 0 or 1, and the wave variables and the phase shifts are defined by
\[
\xi_i = k_i x + l_i y - \omega_j t + \xi_{i,0}, \quad 1 \leq i \leq N, \tag{3}
\]
and
\[
e^{\psi_i} = \frac{-P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, \quad 1 \leq i < j \leq N, \tag{4}
\]
in which \( k_i, l_i, \) and \( \omega_i, 1 \leq i \leq N, \) satisfy the corresponding dispersion relation and \( \xi_{i,0}, 1 \leq i \leq N, \) are arbitrary phase shifts.

Lump solutions are a class of analytic rational solutions which are localized in all directions in space, originated from solving integrable equations in (2+1)-dimensions (see, e.g., [6–8]). Taking long wave limits of \( N \)-soliton solutions can generate special lumps [9]. Many integrable equations in (2+1)-dimensions exhibit the remarkable richness of lump solutions (see, e.g., [6, 7]). Such equations contain the KPI equation [10], whose special lump solutions have been derived from \( N \)-soliton solutions [11], the three-dimensional three-wave resonant interaction [12], the BKP equation [13, 14], the Davey-Stewartson equation II [9], the Ishimori-I equation [15], and the KP equation with a self-consistent source [16]. An important step in the process of getting lumps is to determine positive quadratic function solutions to the Hirota–Satsuma shallow water wave equation [17, 18]. We will consider a general class of HSI type equations while keeping the existence of lump solutions. A general such generalized HSI equation in (2+1)-dimensions will be determined through symbolic computations with Maple. For a special presented lump solution, three-dimensional plots and contour plots will be made via the Maple plot tool, to shed light on the characteristic of the presented lump solutions. A few concluding remarks will be given in the last section.

2. Lump Solutions

It is known that the Hirota-Satsuma shallow water wave equation [4],
\[
\begin{align*}
  u_t &= u_{xxx} + 3uu_x - 3u_x v_x - u_x, \\
  v_x &= -u_t,
\end{align*}
\tag{5}
\]
has a bilinear form,
\[
(D_x D_y^3 - D_x D_y - D_y^3) f \cdot f = 0, \tag{6}
\]
under the logarithmic transformations \( u = 2(\ln f)_x \) and \( v = 2(\ln f)_y \). An integrable \((2 + 1)\)-dimensional extension of the Hirota-Satsuma equation reads
\[
3(u_x u_y)_x + u_{xxx} + u_{yy} + u_x x = 0, \tag{7}
\]
which passes the Hirota three-soliton test [19], and has a bilinear form under the logarithmic transformation \( u = 2(\ln f)_x \): \[
(D_x^2 D_y + D_x D_y + D_y^2) f \cdot f = 0. \tag{8}
\]
Equation (7) is called the Hirota-Satsuma-Ito (HSI) equation in (2+1)-dimensions [19]. We would like to add three terms to generalize the abovementioned HSI equation to a new one which still possesses abundant interesting solution structures:
\[
P(u) = 3(u_x u_y)_x + u_{xxx} + \delta_1 u_{yy} + \delta_2 u_{xx} + \delta_3 u_{xy} + \delta_4 u_{x} + \delta_5 u_{y} = 0. \tag{9}
\]
This generalized HSI equation has a bilinear form under the logarithmic transformation \( u = 2(\ln f)_x \):
\[
B(f) = (D_x^2 D_x + \delta_1 D_y D_x + \delta_2 D_y^2 + \delta_3 D_x D_y + \delta_4 D_x + \delta_5 D_y) f \cdot f = 0. \tag{10}
\]
Precisely, under \( u = 2(\ln f)_x \), we have the relation \( P(u) = (B(f)/f^2)_x \). In what follows, we would like to determine lump solutions to the hHSI equation in (2+1)-dimensions (9), through symbolic computations with Maple.

We start to search for positive quadratic solutions to the hHSI bilinear equation (10) to generate lump solutions to the hHSI equation (9):
\[
f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9. \tag{11}
\]
Plugging this function into the hHSI bilinear equation (10) generates a system of nonlinear algebraic equations on the parameters \( a_i, 1 \leq i \leq 9 \). Conducting direct symbolic computation to solve this system gives a set of solutions for the parameters where
\[
\begin{align*}
a_3 &= \frac{-b_1}{(a_1 \delta_4 + a_2 \delta_1)^2 + (a_3 \delta_4 + a_6 \delta_1)^2}, \\
a_7 &= \frac{-b_2}{(a_1 \delta_4 + a_2 \delta_1)^2 + (a_7 \delta_4 + a_6 \delta_1)^2}, \\
a_9 &= \frac{3 \left( a_1^2 + a_2^2 \right) b_3}{(a_1 a_6 - a_2 a_5)^2 (\delta_1^2 \delta_2 - \delta_1 \delta_3 \delta_4 + \delta_2^2 \delta_5)},
\end{align*}
\tag{12}
\]
and all other $a_i$'s are arbitrary. The involved three constants are defined as follows:

\[
b_1 = \left[ (a_1^2 a_2 + 2a_1 a_3 a_6 - a_4 a_5^2) \delta_2 + a_1 (a_2^2 + a_6^2) \delta_3 \right. \\
+ a_2 (a_1^2 + a_6^2) \delta_1 + \left[ a_1 (a_1^2 + a_6^2) \delta_2 \\
+ a_2 (a_1^2 + a_6^2) \delta_3 + (a_1 a_4 + 2a_2 a_5 a_6 - a_4 a_5^2) \delta_5 \right] \delta_4, \\
b_2 = \left[ (-a_1 a_6 + 2a_1 a_2 a_5 + a_2^2 a_6) \delta_1 + a_5 (a_2^2 + a_6^2) \delta_3 \\
+ a_6 (a_1^2 + a_6^2) \delta_1 + \left[ a_5 (a_1^2 + a_6^2) \delta_2 \\
+ a_6 (a_1^2 + a_6^2) \delta_3 + (-a_2 a_5 + 2a_2 a_5 a_6 + a_5 a_6^2) \delta_5 \right] \delta_4 \right] \\
\cdot \delta_4, \\
b_3 = (a_1^2 + a_6^2) (a_2 a_3 + a_5 a_6) (\delta_1 \delta_2 + \delta_3 \delta_4) + (a_1^2 \\
+ a_6^2) (a_2^2 + a_6^2) \delta_1 \delta_3 + (a_1^2 + a_6^2) \delta_2 \delta_4 + (a_2^2 + a_6^2) \\
\cdot (a_1 a_2 + a_5 a_6) \delta_1 \delta_5 + \left[ (a_1 a_2 + a_5 a_6) \right]^2 \\
- (a_1 a_6 - a_2 a_5)^2 \right] \delta_4 \delta_5.
\]

Those formulas in (12) and (13) were obtained under a simplification process with Maple.

From (12), we can easily see that it is sufficient to guarantee $f > 0$ if we require

\[
\left( \delta_1^2 \delta_2 - \delta_1 \delta_3 \delta_4 + \delta_5^2 \right) b_2 > 0,
\]

and, thus, the function $f$ defined by (12) and (13) under the abovementioned condition and

\[
a_1 a_6 - a_2 a_5 \neq 0
\]

leads to lump solutions

\[
u = 2 \left( \ln f \right)_x = \frac{2 f_x}{f} 
\]

to the gHSI equation in (2+1)-dimensions (9).

When one takes

\[
\delta_1 = 1, \\
\delta_2 = 1, \\
\delta_3 = \delta_4 = \delta_5 = 0,
\]

one obtains the original HSI equation in (2+1)-dimensions (7), and the function $f$ by (12) and (13) presents a class of lump solutions to the HSI equation (7):

\[
u = 2 \left( \ln f \right)_x, \\
f = (a_1 x + a_2 y + a_3 t + a_4)^2 \\
+ (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9,
\]

where

\[
a_3 = -\frac{a_1^2 a_2 + 2a_1 a_3 a_6 - a_4 a_5^2}{a_2^2 + a_6^2}, \\
a_6 = \frac{a_2^2 a_6 - 2a_1 a_3 a_6 - a_4 a_5^2}{a_2^2 + a_6^2}, \\
a_9 = \frac{3 (a_1^2 + a_6^2) (a_1 a_2 + a_5 a_6)}{(a_2 a_6 - a_4 a_5)^2},
\]

and all other $a_i$'s are arbitrary. Solving the abovementioned parameter solutions on $a_3$ and $a_9$, for $a_2$ and $a_6$ and substituting the resulting expressions for $a_2$ and $a_6$ into the formula for $a_9$ in (19), we get

\[
a_2 = -\frac{a_1^2 a_2 + 2a_1 a_3 a_6 - a_4 a_5^2}{a_2^2 + a_6^2}, \\
a_6 = \frac{a_2^2 a_6 - 2a_1 a_3 a_6 - a_4 a_5^2}{a_2^2 + a_6^2}, \\
a_9 = -\frac{3 (a_1^2 + a_6^2) (a_1 a_2 + a_5 a_6)}{(a_2 a_6 - a_4 a_5)^2},
\]

It is easy to see that

\[
a_1 a_6 - a_2 a_5 = \frac{(a_1^2 + a_6^2) (a_1 a_2 - a_3 a_5)}{a_2^2 + a_6^2},
\]

and, thus, the conditions of

\[
a_1 a_3 + a_3 a_7 < 0, \\
a_1 a_2 - a_3 a_5 \neq 0
\]

guarantee that (16) with (11) and (20) will present lump solutions to the HSI equation (7) [20].

Particularly taking

\[
\delta_1 = 1, \\
\delta_2 = 1, \\
\delta_3 = -1, \\
\delta_4 = 1, \\
\delta_5 = -1,
\]

we obtain a special gHSI equation as follows:

\[
u_{xxxxt} + 3 (u_x u_t)_x + u_{yy} + u_{xx} - u_{xy} + u_{xt} - u_{yy} = 0, \\
\]

which has a Hirota bilinear form

\[
\left( D_x^2 D_y + D_y D_x + D_x^2 - D_x D_y + D_x D_t - D_y^2 \right) f \cdot f = 0,
\]
under the logarithmic transformation (16). Associated with

\[
\begin{align*}
  a_1 &= 1, \\
  a_2 &= -2, \\
  a_3 &= 2, \\
  a_4 &= 2, \\
  a_5 &= 2, \\
  a_6 &= -3, \\
  a_8 &= -5,
\end{align*}
\]

(26) with (11) and (22) present the lump solution to the special

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

Three three-dimensional plots and contour plots of this

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

lump solution are made via Maple plot tools, to shed light on the characteristic of the presented lump solutions, in

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

Figure 1. Profiles of \(u\) when \(t = 0, 3, 6\): 3D plots (top) and contour plots (bottom).

3. Concluding Remarks

We have studied a generalized (2+1)-dimensional Hirota-

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

Satsuma-Ito (HSI) equation to explore different equations which possess lump solutions, through symbolic computa-

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

tions with Maple. The results enrich the theory of lumps and

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

solitons, providing a new example of (2+1)-dimensional non-

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

linear equations, which possess beautiful lump structures.

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

Three-dimensional plots and contour plots of a specially

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

chosen lump solution were made by using the plot tool in

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

Maple.

Many nonlinear equations possess lump solutions, which include (2+1)-dimensional generalized KP, BKP,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

KP-Boussinesq, Sawada-Kotera, and Bogoyavlensky-Kono-

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

pelchenko equations [32–36]. Some recent studies also
demonstrate the strikingly high richness of lump solutions
to linear partial differential equations [37] and nonlinear

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

partial differential equations in (2+1)-dimensions (see,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

e.g., [38–41]) and (3+1)-dimensions (see, e.g., [42–48]).

Diversity of lump solutions supplements exact solutions
generated from different kinds of combinations (see, e.g.,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

[49–52]) and can yield various Lie-Bäcklund symmetries,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

which can be used to determine conservation laws by

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

symmetries and adjoint symmetries [53–55]. Moreover,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

diverse interaction solutions [35] have been exhibited

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

for many integrable equations in (2+1)-dimensions,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

including lump-soliton interaction solutions (see, e.g.,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

[56–58]) and lump-kink interaction solutions (see, e.g.,

\[
\begin{align*}
  u(x, y, t) &= \frac{2(10x - 16y - t - 16)}{(x - 2y - \frac{3}{2}t + 2)^2 + (2x - 3y + \frac{1}{2}t - 5)^2 + 15}.
\end{align*}
\]

[59–62]).
We finally remark that we could add one more term to the gHSI equation (9) to formulate a more generalized HSI bilinear equation,
\[
(D_x^2 + \delta_1 D_y^2 + \delta_2 D_x D_y + \delta_3 D_y^2 + \delta_4 D^2_x + \delta_5 D^2_y) f \cdot f = 0,
\]
where $\delta_i, 1 \leq i \leq 6$, are all constants, but we failed to drive any lump solution to the corresponding nonlinear equation on $u = 2(\ln f)$. The first term in the abovementioned bilinear equation is crucial in determining lump solutions but the last term brings the difficulty to work out lump solutions. There is no hint on how to solve any big system of resulting nonlinear algebraic equations. Nevertheless, some general considerations on the existence of lumps have been made for the Hirota bilinear case [6] and the generalized bilinear cases [7].

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**Acknowledgments**

The work was supported in part by NSFC under Grants nos. 11301454, 11301331, and 11371086, NSF under Grant no. DMS-1664561, the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17KJB110020), Emphasis Foundation of Special Science Research on Subject Frontiers of CUMT under Grant no. 2017XKZDII, and the Distinguished Professorships by Shanghai University of Electric Power, China, and North-West University, South Africa.

**References**


