Hindawi
Complexity
Volume 2018, Article ID 9531064, 24 pages
https://doi.org/10.1155/2018/9531064

Research Article

Multiattribute Group Decision-Making Based on Linguistic Pythagorean Fuzzy Interaction Partitioned Bonferroni Mean Aggregation Operators

Mingwei Lin,[1,2] **Jiuhan Wei**,[2] **Zeshui Xu**,[3,4] **and Riqing Chen**[5]

[1] *College of Mathematics and Informatics, Fujian Normal University, Fuzhou, Fujian 350117, China*
[2] *Digital Fujian Internet-of-Things Laboratory of Environmental Monitoring, Fujian Normal University, Fuzhou, Fujian 350117, China*
[3] *Business School, Sichuan University, Chengdu, Sichuan 610064, China*
[4] *School of Computer and Software, Nanjing University of Information Science & Technology, Nanjing 210044, China*
[5] *Digital Fujian Institute of the Big Data for Agriculture and Forestry, Fujian Agriculture and Forestry University, Fuzhou, Fujian 350002, China*

Correspondence should be addressed to Riqing Chen; riqing.chen@fafu.edu.cn

Received 28 August 2018; Accepted 9 October 2018; Published 1 November 2018

Academic Editor: Arturo Buscarino

Copyright © 2018 Mingwei Lin et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The partitioned Bonferroni mean (PBM) operator can efficiently aggregate inputs, which are divided into parts based on their interrelationships. To date, it has not been used to aggregate linguistic Pythagorean fuzzy numbers (LPFNs). In this paper, we extend the PBM operator and partitioned geometric Bonferroni mean (PGBM) operator to the linguistic Pythagorean fuzzy sets (LPFSs) and use them to develop a novel multiattribute group decision-making model under the linguistic Pythagorean fuzzy environment. We first define some novel operational laws for LPFNs, which take into consideration the interactions between the membership degree (MD) and nonmembership degree (NMD) from two different LPFNs. Based on these novel operational laws, we put forward the interaction PBM (LPFIPBM) operator, the weighted interaction PBM (LPFWIPBM) operator, the interaction PGBM (LPFIPGBM) operator, and the weighted interaction PGBM (LPFWIPGBM) operator. Then, we study some properties of these proposed operators and discuss their special cases. Based on the proposed LPFWIPBM and LPFWIPGBM operators, a novel multiattribute group decision-making model is developed to process the linguistic Pythagorean fuzzy information. Finally, some illustrative examples are introduced to compare our proposed methods with the existing ones.

1. Introduction

Multiattribute group decision-making refers to a process where a group of decision-makers are invited to participate in the assessment of some given alternatives and then the optimal one is selected or all of them are ranked based on their assessment information [1–3]. It can be commonly seen in our daily life. As the multiattribute group decision-making problems become more complex, it is very difficult for the decision-makers to assess the alternatives using real values [4]. Zadeh [5] built the theory of fuzzy sets (FSs), which provides the decision-makers with an efficient means to model the fuzzy information. Nevertheless, FSs are incapable of expressing the nonmembership degree (NMD) of each element in the universe of discourse belonging to a FS [6]. To deal with the shortcoming of FSs, Atanassov [7] gave the theory of intuitionistic fuzzy sets (IFSs) in which each intuitionistic fuzzy number (IFN) consists of MD and NMD and the sum of them is less than or equal to 1. Liu et al. [8] extended the Heronian mean to aggregate IFNs.

However, sometimes the sum of MD and NMD given by the decision-maker may be larger than 1, but their square sum is less than or equal to 1. Obviously, IFSs cannot model this kind of fuzzy information [9]. In this case, Yager [10] defined the concept of Pythagorean fuzzy sets (PFSs) in which each Pythagorean fuzzy number (PFN) consists of MD and NMD whose square sum is less than or equal to 1. Figure 1 shows the distribution areas for IFNs and PFNs. The distribution

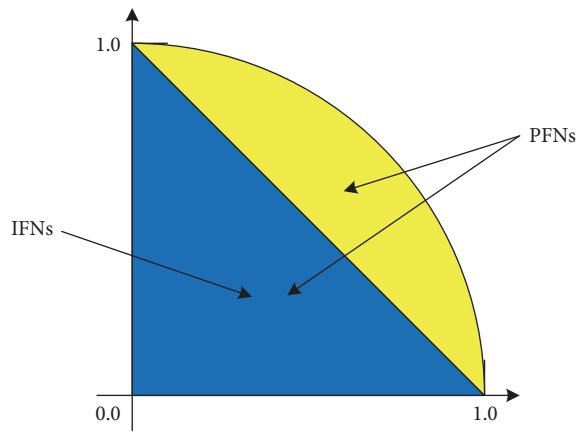

Figure 1: Possible distribution areas for IFNs and PFNs.

area refers to a set of all the possible IFNs or PFNs that are located in a two-dimensional coordinate system. It can be seen that all the possible IFNs only fall into the blue area and each PFN may appear in the blue area or yellow area. Namely, an IFN can also be a PFN, while a PFN may not be an IFN. For instance, when a decision-maker provides a pair of MD and NMD $\langle 0.6, 0.6\rangle$ to assess an alternative with respect to an attribute. It can be clearly seen that $0.6 + 0.6 > 1$, but $0.6^2 + 0.6^2 < 1$. Hence, $\langle 0.6, 0.6\rangle$ can be only modeled as a PFN. Both IFSs and PFSs are the special cases of q-rung orthopair fuzzy sets [11, 12].

Both IFSs and PFSs model the uncertain and vague information using the quantitative form. However, in some complex decision-making environments, the decision-makers may prefer to use the qualitative form to give their preference information. For example, when evaluating the running speed of a central processing unit (CPU), if the decision-makers do not know the detailed performance parameters of this CPU, they will prefer to utilize one of the linguistic terms such as "very slow," "slow," "medium," "fast," "very fast," and "perfect" to express their preference information [13]. The fuzzy linguistic approach put forward by Zadeh [14] can model this kind of qualitative preference information. Xu et al [15] also devised a virtual linguistic model to redefine the syntax and semantics of existing linguistic computational models. All of the linguistic computational models are built based on the concept of linguistic term sets (LTSs) [16–18].

To introduce the qualitative modeling capability into IFSs, Zhang [20] gave the concept of linguistic intuitionistic fuzzy sets (LIFSs), which combines the theory of IFSs and the concept of LTSs. A LIFS is composed of linguistic intuitionistic fuzzy numbers (LIFNs) whose MD and NMD are expressed using the linguistic terms. Inspired by Zhang's idea, Garg [19] also combined the theory of PFSs with the concept of LTSs and proposed a novel qualitative assessment model called the linguistic Pythagorean fuzzy set (LPFS). Any linguistic Pythagorean fuzzy number (LPFN) in the LPFS consists of MD and NMD that are modeled utilizing the linguistic terms. The sum of squares of subscripts of MD and NMD is less than or equal to the square of the cardinality of LTS.

Aggregation operator [21, 22], which is used to aggregate some pieces of data into one piece of data, plays an important role in the multiattribute decision-making models (MADMM) and multiattribute group decision-making models (MAGDMM). The functions and operations are two vital parts that have an impact on the aggregated results of aggregation operators. In the following part, the importance of functions and operations is discussed.

(1) Regarding the functions of aggregation operators, the early aggregation operators are designed to fuse several real numbers into a single real number. To enhance the capability of fusing the complex data, various aggregation operators have been designed considering the importance of values or/and the relations among values. For example, the Choquet integral was introduced in [23] to study the correlated averaging operators and correlated geometric operators for aggregating the interval-valued intuitionistic fuzzy sets. To reduce the impact of unreasonable values on the decision results, several power aggregation operators [24] and dependent aggregation operators [25] have been proposed to fuse IFNs. In recent years, the Bonferroni mean (BM) operator [26] considering the interrelationships among input values is extended into the fuzzy environment [27, 28]. The BM operator can capture the interrelationship between each attribute value and the other ones. However, some attribute values may not have any relationship with the other ones in some practical decision-making problems [29]. For instance, when a house is assessed, four common attributes are usually considered: the price (attribute A_1), the geographical position (attribute A_2), the daylighting (attribute A_3), and the apartment layout (attribute A_4). In this case, there exists the relationship between A_1 and A_2, while A_3 is related with A_4. These four attributes can be divided into two sets: $P_1 = \{A_1, A_2\}$ and $P_2 = \{A_3, A_4\}$. The elements in the same set are related with each other, but the elements in the different sets do not have any relationship. Hence, the BM operator cannot capture this kind of relationship between the attribute values. To deal with this complex situation, the partitioned Bonferroni mean (PBM) operator is put forward in [30] to extend the capability of capturing relationships for BM operator.

(2) Regarding the operational rules of aggregation operators, until now, there have been very few studies reporting the aggregation operators for LPFNs. In [19], some averaging operators and geometric operators are put forward to fuse a set of LPFNs. However, the above study is put forward based on the operational rules, which are proposed by Garg [19]. The operational rules for calculating two LPFNs do not contain the interactions between the MD and NMD from different LPFNs, which may result in unreasonable decision results. To the best of our knowledge, the study reporting the operators in [19] is the only one for LPFNs.

Through the above analysis, it can be seen that designing excellent aggregation operators should take the functions and operational rules into account at the same time [31, 32]. However, previous aggregation operators [19] put forward to fuse LPFNs have some drawbacks, which can be summarized as follows.

(1) Previous aggregation operators cannot deal with the situation that the attributes in the same set are related with each other, while the attributes in the different sets have no relationship with each other. They simply calculate the averaging value or geometric value of a set of LPFNs and do not consider the complex relationships among the attributes.

(2) Previous aggregation operators are put forward using the operational rules proposed by Garg [19], which may lead to unreasonable decision results, especially when the subscript of NMD in one of the LPFNs is zero. If the subscript of NMD in one of the LPFNs is zero, then the aggregated NMD would always be zero regardless of the value of the NMD of other LPFNs.

To deal with the above drawbacks, in this paper, we focus on proposing some linguistic Pythagorean fuzzy interaction partitioned Bonferroni mean (LPFIPBM) aggregation operators to aggregate LPFNs. Our contributions can be listed as follows:

(1) The drawback of the operational rules for calculating two LPFNs is analyzed and the interactional operational rules considering the interaction between the MD and NMD from different LPFNs are devised for LPFNs.

(2) Some interaction PBM operators are developed for LPFNs based on the interactional operational rules.

(3) Several special cases of the proposed LPFIPBM aggregation operators are given and the desirable properties are also studied.

(4) The proposed LPFIPBM aggregation operators are used to develop a novel MAGDMM to fuse the LPFNs under the situation that the attributes in the same set are related with each other and the ones in the different sets are not related.

(5) An illustrative example concerning the selection of solid state drive (SSD) production is utilized to show the implementation process of the proposed MAGDMM based on the LPFIPBM aggregation operators and it is compared with the previous study under the linguistic Pythagorean fuzzy environment.

The remainder of this paper is arranged as follows: the concept of LPFSs, some operational rules of LPFSs, and the definitions of BM and PBM are briefly reviewed in Section 2. Several novel interactional operational rules are given for computing two LPFNs and then their properties are discussed in Section 3. In Section 4, the LPFIPBM, LPFWIPBM, LPFIPGBM, and LPFWIPGBM operators are devised to aggregate LPFNs. In Section 5, the LPFWIPBM and LPFWIPGBM operators are used to devise a novel MAGDMM with LPFNs. In Section 6, an illustrative example concerning the selection of SSD production is provided to demonstrate our proposed operators and a detailed comparison is performed between our proposed operators and previous study presented in [19]. Finally, Section 7 presents the conclusions of our wok.

2. Preliminaries

In this section, the definition of LPFS and the operational laws of LPFSs, as well as the definitions of BM and PBM, are briefly reviewed.

2.1. Linguistic Pythagorean Fuzzy Sets (LPFSs). Garg [19] combined the theory of PFSs with the concept of LTSs and put forward a novel qualitative assessment model called the linguistic Pythagorean fuzzy sets (LPFSs) as follows.

Definition 1 (see [19]). Let Y be a finite reference set and let $S = \{s_\varepsilon \mid \varepsilon \in [0, \tau]\}$ with a positive integer τ be a continuous linguistic term set (LTS) [33]; then a LPFS on the finite set Y is expressed in a mathematical form as

$$L = \{(y, s_m(y), s_n(y)) \mid y \in Y\} \tag{1}$$

where $s_m(y)$ and $s_n(y)$ are two functions, which are responsible for returning the MD and the NMD of the element y being the member of the LPFS L. Each two-tuple $(s_m(y), s_n(y))$ in the LPFS is named as a LPFN or LPFV and it can be expressed in a simplified form as $\alpha = (s_m, s_n)$ satisfying that $0 \le m \le \tau$, $0 \le n \le \tau$, and $0 \le m^2 + n^2 \le \tau^2$. Let $s_\pi(y) = s_{\sqrt{\tau^2 - m^2 - n^2}}$; then $s_\pi(y)$ can be referred to as the hesitance degree (HD) of the element y being the member of the LPFS L.

Garg [19] also presented the score value and the accuracy value for LPFNs and then developed a method for comparing two LPFNs.

Definition 2 (see [19]). There exist a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0, \tau]\}$ and a LPFN $\alpha = (s_m, s_n)$ with $s_m, s_n \in S$; then the score value of the LPFN α is computed as

$$D(\alpha) = s_{\sqrt{(\tau^2 + m^2 - n^2)/2}} \tag{2}$$

and the accuracy value is calculated as

$$J(\alpha) = s_{\sqrt{m^2 + n^2}} \tag{3}$$

The method for comparing two LPFNs α and β are described as follows:

(i) If $D(\alpha) > D(\beta)$, then $\alpha \succ \beta$, which means α is preferred to β.

(ii) If $D(\alpha) = D(\beta)$, then the accuracy values of the LPFNs α and β should be compared.

(a) If $J(\alpha) > J(\beta)$, then $\alpha \succ \beta$.

(b) If $J(\alpha) = J(\beta)$, then $\alpha = \beta$.

Based on the t-conorm and t-norm, Garg gave several operational rules to compute two LPFNs as follows.

Definition 3 (see [19]). Assume a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0, \tau]\}$ with a positive integer τ and two LPFNs $\alpha_1 = (s_{m_1}, s_{n_1})$ and $\alpha_2 = (s_{m_2}, s_{n_2})$ with $s_{m_1}, s_{n_1}, s_{m_2}, s_{n_2} \in S$; then

(i) $\alpha_1 \oplus \alpha_2 = (s_{\tau\sqrt{m_1^2/\tau^2+m_2^2/\tau^2-m_1^2m_2^2/\tau^4}}, s_{\tau(n_1n_2/\tau^2)})$;

(ii) $\alpha_1 \otimes \alpha_2 = (s_{\tau(m_1m_2/\tau^2)}, s_{\tau\sqrt{n_1^2/\tau^2+n_2^2/\tau^2-n_1^2n_2^2/\tau^4}})$;

(iii) $\lambda\alpha_1 = (s_{\tau\sqrt{1-(1-m_1^2/\tau^2)^\lambda}}, s_{\tau(n_1/\tau)^\lambda})$;

(iv) $\alpha_1^\lambda = (s_{\tau(m_1/\tau)^\lambda}, s_{\tau\sqrt{1-(1-n_1^2/\tau^2)^\lambda}})$.

2.2. The BM Operator and PBM Operator. Aggregation operators are an important tool introduced to fuse a set of input values into a single one. They are expressed in the form of mathematical functions as follows.

Definition 4 (see [34]). An aggregation operator is a nondecreasing function $f : [0,1]^n \longrightarrow [0,1]$ satisfying that $f(0, 0, \ldots, 0) = 0$ and $f(1, 1, \ldots, 1) = 1$.

The BM operator, initially put forward by Bonferroni [35], is a valuable aggregation function that can capture the interrelationship between each attribute value and the other ones. Its mathematical expression is defined as follows.

Definition 5 (see [36]). Given a set of nonnegative input values $\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, as well as two real numbers $p, q \geq 0$, the mathematical expression of the BM operator is described as

$$BM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = \left(\frac{1}{K(K-1)} \sum_{\substack{u,v=1 \\ u\neq v}}^{K} \alpha_u^p \alpha_v^q\right)^{1/(p+q)} \tag{4}$$

Its geometric version is also presented as follows.

Definition 6 (see [37]). Given a set of nonnegative input values $\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, as well as two real numbers $p, q \geq 0$, the mathematical expression of the geometric BM (GBM) operator is described as

$$GBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = \frac{1}{p+q}\left(\prod_{\substack{u,v=1 \\ u\neq v}}^{K} (p\alpha_u + q\alpha_v)\right)^{1/K(K-1)} \tag{5}$$

To capture the interrelationship that the attribute values in the same set are related and the ones in the different sets are not related, the PBM operator is put forward in [30] to extend the BM operator as follows.

Definition 7 (see [30]). Given two nonnegative real numbers $p, q \geq 0$, as well as a set of nonnegative input values $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different groups $G_1, G_2, \ldots, G_g$, with $\bigcup_{t=1}^{g} G_t = O$, the mathematical expression of the PBM operator is described as

$$PBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = \frac{1}{g}\left(\sum_{t=1}^{g}\left(\frac{1}{|G_t|} \cdot \sum_{\alpha_u\in G_t} \alpha_u^p \left(\frac{1}{|G_t|-1} \sum_{\substack{\alpha_v\in G_t \\ \alpha_v\neq\alpha_u}} \alpha_v^q\right)\right)^{1/(p+q)}\right) \tag{6}$$

where $|G_t|$ is the number of elements in the group G_t and g denotes the number of groups $\sum_{t=1}^{g} |G_t| = K$.

Its geometric version is given as follows.

Definition 8 (see [38]). Given two nonnegative real numbers $p, q \geq 0$, as well as a set of nonnegative input values $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different groups $G_1, G_2, \ldots, G_g$, with $\bigcup_{t=1}^{g} G_t = O$, the mathematical expression of the PGBM operator is described as

$$PGBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = \left(\prod_{t=1}^{g}\left(\frac{1}{p+q}\left(\prod_{\substack{\alpha_u,\alpha_v\in G_t \\ \alpha_u\neq\alpha_v}} (p\alpha_u + q\alpha_v)\right)^{1/|G_t|(|G_t|-1)}\right)\right)^{1/g} \tag{7}$$

where $|G_t|$ is the number of elements in the group G_t and g denotes the number of groups and $\sum_{t=1}^{g} |G_t| = K$.

3. Interactional Operational Rules for LPFNs

In this section, the drawbacks of the operational rules proposed by Garg [19] for LPFNs are analyzed and then some novel interactional operational rules are developed for the computations of LPFNs. Finally, their properties are discussed.

The operational rules presented in Definition 3 do not work well for all the LPFNs, especially when the MD or NMD of one of LPFNs is s_0. Assume that there are two LPFNs, $\alpha_1 = (s_{m_1}, s_0)$ and $\alpha_2 = (s_{m_2}, s_{n_2})$. Based on the additive operation in Definition 3, the aggregated NMD of $\alpha_1 \oplus \alpha_2$ is always 0 regardless of the value of the NMD of α_2. Obviously, this operational result is unreasonable, which also happens in the situation that the multiplicative operation is performed to compute two LPFNs, of which the MD is s_0. Moreover, the aggregation operators based on the operational rules may also generate unreasonable results. For example, if the LPFWA operator presented in [19] is used to aggregate the above two LPFNs, the NMD obtained from $LPFWA(\alpha_1, \alpha_2)$ is s_0. In a word, the value of NMD in the LPFN α_2 does not influence the value of NMD of operational results and aggregation results.

To overcome the above drawback, in this section, several novel interactional operational rules are put forward to compute the LPFNs.

Definition 9. Suppose that a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0,\tau]\}$ with a positive integer τ and two LPFNs $\alpha_1 = (s_{m_1}, s_{n_1})$ and $\alpha_2 = (s_{m_2}, s_{n_2})$ with $s_{m_1}, s_{n_1}, s_{m_2}, s_{n_2} \in S$; then

$$\begin{aligned}
&\text{(i)}\quad \alpha_1 \oplus \alpha_2 \\
&\quad = \left(s_{\tau\sqrt{1-\prod_{i=1}^{2}(1-m_i^2/\tau^2)}}, s_{\tau\sqrt{\prod_{i=1}^{2}(1-m_i^2/\tau^2)-\prod_{i=1}^{2}(1-m_i^2/\tau^2-n_i^2/\tau^2)}}\right); \\
&\text{(ii)}\quad \alpha_1 \otimes \alpha_2 \\
&\quad = \left(s_{\tau\sqrt{\prod_{i=1}^{2}(1-n_i^2/\tau^2)-\prod_{i=1}^{2}(1-m_i^2/\tau^2-n_i^2/\tau^2)}}, s_{\tau\sqrt{1-\prod_{i=1}^{2}(1-n_i^2/\tau^2)}}\right); \\
&\text{(iii)}\quad \lambda\alpha_1 = \left(s_{\tau\sqrt{1-(1-{m_1}^2/\tau^2)^\lambda}}, s_{\tau\sqrt{(1-m_1^2/\tau^2)^\lambda-(1-m_1^2/\tau^2-n_1^2/\tau^2)^\lambda}}\right); \\
&\text{(iv)}\quad \alpha_1^\lambda = \left(s_{\tau\sqrt{(1-n_1^2/\tau^2)^\lambda-(1-m_1^2/\tau^2-n_1^2/\tau^2)^\lambda}}, s_{\tau\sqrt{1-(1-n_1^2/\tau^2)^\lambda}}\right).
\end{aligned} \tag{8}$$

Example 10. Let a LTS $S = \{s_\varepsilon \mid \varepsilon \in [0,8]\}$, as well as two LPFNs $\alpha_1 = (s_6, s_0)$ and $\alpha_2 = (s_5, s_2)$; if the additive operation in Definition 3 is used, then $\alpha_1 \oplus \alpha_2 = (s_{6.85}, 0)$. If the additive operation in Definition 9 is used, then $\alpha_1 \oplus \alpha_2 = (s_{6.85}, s_{1.32})$. Obviously, the operational result derived from the operational rules in Definition 9 is more reasonable.

Some theorems can be derived from Definition 9 as follows.

Theorem 11. *Assume that a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0,\tau]\}$ with a positive integer τ, as well as two LPFNs $\alpha_1 = (s_{m_1}, s_{n_1})$ and $\alpha_2 = (s_{m_2}, s_{n_2})$ with $s_{m_1}, s_{n_1}, s_{m_2}, s_{n_2} \in S$; then $\alpha_1 \oplus \alpha_2$ is also a LPFN.*

Proof. (i) $\tau\sqrt{1-\prod_{i=1}^{2}(1-m_i^2/\tau^2)}$, $\tau\sqrt{\prod_{i=1}^{2}(1-m_i^2/\tau^2)-\prod_{i=1}^{2}(1-m_i^2/\tau^2-n_i^2/\tau^2)} \in [0,\tau]$ should be proven.

According to Definition 9, $\alpha_1 \oplus \alpha_2 = (s_{\tau\sqrt{1-\prod_{i=1}^{2}(1-m_i^2/\tau^2)}}, s_{\tau\sqrt{\prod_{i=1}^{2}(1-m_i^2/\tau^2)-\prod_{i=1}^{2}(1-m_i^2/\tau^2-n_i^2/\tau^2)}})$.

Since $0 \le m_i \le \tau$, then $\tau\sqrt{1-(1-0^2/\tau^2)^2} \le \tau\sqrt{1-\prod_{i=1}^{2}(1-m_i^2/\tau^2)} \le \tau\sqrt{1-(1-\tau^2/\tau^2)^2} \Longrightarrow 0 \le \tau\sqrt{1-\prod_{i=1}^{2}(1-m_i^2/\tau^2)} \le \tau$.

Let $f = \tau\sqrt{\prod_{i=1}^{2}(1-m_i^2/\tau^2)-\prod_{i=1}^{2}(1-m_i^2/\tau^2-n_i^2/\tau^2)}$ and $u = \prod_{i=1}^{2}(1-{m_i}^2/\tau^2)-\prod_{i=1}^{2}(1-{m_i}^2/\tau^2-{n_i}^2/\tau^2)$; then

$$\begin{aligned}
\frac{df}{dm_1} &= \frac{\tau}{2}\cdot u^{-1/2}\cdot\frac{du}{dm_1} = \frac{\tau}{2}\cdot u^{-1/2}\cdot\left[\left(1-\frac{{m_2}^2}{\tau^2}\right)\right. \\
&\quad \left.\cdot\left(-\frac{2m_1}{\tau^2}\right)-\left(1-\frac{{m_2}^2}{\tau^2}-\frac{{n_2}^2}{\tau^2}\right)\cdot\left(-\frac{2m_1}{\tau^2}\right)\right] \\
&= -\frac{m_1{n_2}^2}{\tau^3}u^{-1/2} \le 0 \\
\frac{df}{dm_2} &= \frac{\tau}{2}\cdot u^{-1/2}\cdot\frac{du}{dm_2} = \frac{\tau}{2}\cdot u^{-1/2}\cdot\left[\left(1-\frac{{m_1}^2}{\tau^2}\right)\right. \\
&\quad \left.\cdot\left(-\frac{2m_2}{\tau^2}\right)-\left(1-\frac{{m_1}^2}{\tau^2}-\frac{{n_1}^2}{\tau^2}\right)\cdot\left(-\frac{2m_2}{\tau^2}\right)\right] \\
&= -\frac{m_2{n_1}^2}{\tau^3}u^{-1/2} \le 0 \\
\frac{df}{dn_1} &= \frac{\tau}{2}\cdot u^{-1/2}\cdot\frac{du}{dn_1} = \frac{\tau}{2}\cdot u^{-1/2} \\
&\quad \cdot\left[-\left(1-\frac{{m_2}^2}{\tau^2}-\frac{{n_2}^2}{\tau^2}\right)\cdot\left(-\frac{2n_1}{\tau^2}\right)\right] \\
&= \frac{n_1\left(\tau^2-{m_2}^2-{n_2}^2\right)}{\tau^3}u^{-1/2} \ge 0 \\
\frac{df}{dn_2} &= \frac{\tau}{2}\cdot u^{-1/2}\cdot\frac{du}{dn_2} = \frac{\tau}{2}\cdot u^{-1/2} \\
&\quad \cdot\left[-\left(1-\frac{{m_1}^2}{\tau^2}-\frac{{n_1}^2}{\tau^2}\right)\cdot\left(-\frac{2n_2}{\tau^2}\right)\right] \\
&= \frac{n_2\left(\tau^2-{m_1}^2-{n_1}^2\right)}{\tau^3}u^{-1/2} \ge 0
\end{aligned} \tag{9}$$

Thus, the value of f decreases as m_1 or m_2 increases, while it increases as n_1 or n_2 increases.

Since $0 \le n_i \le \tau$, then

$$\begin{aligned}
&\tau\sqrt{\left(1-\frac{\tau^2}{\tau^2}\right)^2-\left(1-\frac{\tau^2}{\tau^2}-\frac{0^2}{\tau^2}\right)^2} \le f \\
&\quad = \tau\sqrt{\prod_{i=1}^{2}\left(1-\frac{m_i^2}{\tau^2}\right)-\prod_{i=1}^{2}\left(1-\frac{m_i^2}{\tau^2}-\frac{n_i^2}{\tau^2}\right)} \\
&\quad \le \tau\sqrt{\left(1-\frac{0^2}{\tau^2}\right)^2-\left(1-\frac{0^2}{\tau^2}-\frac{\tau^2}{\tau^2}\right)^2} \Longrightarrow 0 \le f \\
&\quad = \tau\sqrt{\prod_{i=1}^{2}\left(1-\frac{m_i^2}{\tau^2}\right)-\prod_{i=1}^{2}\left(1-\frac{m_i^2}{\tau^2}-\frac{n_i^2}{\tau^2}\right)} \le \tau
\end{aligned} \tag{10}$$

(ii) $(\tau\sqrt{1-\prod_{i=1}^{2}(1-m_i^2/\tau^2)})^2 + (\tau\sqrt{\prod_{i=1}^{2}(1-m_i^2/\tau^2)-\prod_{i=1}^{2}(1-m_i^2/\tau^2-n_i^2/\tau^2)})^2 \le \tau^2$ should be proven.

$$\begin{aligned}
&\left(\tau\sqrt{1-\prod_{i=1}^{2}\left(1-\frac{m_i^2}{\tau^2}\right)}\right)^2 \\
&\quad + \left(\tau\sqrt{\prod_{i=1}^{2}\left(1-\frac{m_i^2}{\tau^2}\right)-\prod_{i=1}^{2}\left(1-\frac{m_i^2}{\tau^2}-\frac{n_i^2}{\tau^2}\right)}\right)^2 \\
&\quad = \tau^2-\prod_{i=1}^{2}\left(\tau^2-m_i^2-n_i^2\right) \le \tau^2
\end{aligned} \tag{11}$$

which completes the proof of Theorem 11. □

Theorem 12. *Assume that a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0,\tau]\}$ with a positive integer τ, as well as two LPFNs $\alpha_1 = (s_{m_1}, s_{n_1})$ and $\alpha_2 = (s_{m_2}, s_{n_2})$ with $s_{m_1}, s_{n_1}, s_{m_2}, s_{n_2} \in S$; then $\alpha_1 \otimes \alpha_2$ is also a LPFN.*

Proof. The proof is similar to that of Theorem 11. □

Theorem 13. *Given a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0,\tau]\}$ with a positive integer τ, three real numbers $\lambda, \lambda_1, \lambda_2$, as well as any three LPFNs $\alpha = (s_m, s_n)$, $\alpha_1 = (s_{m_1}, s_{n_1})$, and $\alpha_2 = (s_{m_2}, s_{n_2})$ with $s_m, s_n, s_{m_1}, s_{n_1}, s_{m_2}, s_{n_2} \in S$, one has*

(i) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;

(ii) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$;

(iii) $(\lambda_1 + \lambda_2)\alpha = \lambda_1\alpha \oplus \lambda_2\alpha$;

(iv) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;

(v) $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda$;

(vi) $\alpha^{\lambda_1+\lambda_2} = \alpha^{\lambda_1} \otimes \alpha^{\lambda_2}$.

Proof. Let $\phi_1 = 1 - m_1^2/\tau^2$, $\varphi_1 = 1 - m_1^2/\tau^2 - n_1^2/\tau^2$, $\phi_2 = 1 - m_2^2/\tau^2$, $\varphi_2 = 1 - m_2^2/\tau^2 - n_2^2/\tau^2$, $\phi = 1 - m^2/\tau^2$, $\varphi = 1 - m^2/\tau^2 - n^2/\tau^2$, $\gamma_1 = 1 - n_1^2/\tau^2$, $\gamma_2 = 1 - n_2^2/\tau^2$, and $\gamma = 1 - n^2/\tau^2$.

$$\text{(i)}\ \alpha_1 \oplus \alpha_2 = \left(s_{\tau\sqrt{1-\prod_{i=1}^2(1-m_i^2/\tau^2)}}, s_{\tau\sqrt{\prod_{i=1}^2(1-m_i^2/\tau^2)-\prod_{i=1}^2(1-m_i^2/\tau^2-n_i^2/\tau^2)}}\right) = \left(s_{\tau\sqrt{1-\phi_1\phi_2}}, s_{\tau\sqrt{\phi_1\phi_2-\varphi_1\varphi_2}}\right) = \alpha_2 \oplus \alpha_1;$$

$$\text{(ii)}\ \lambda(\alpha_1 \oplus \alpha_2) = \lambda\left(s_{\tau\sqrt{1-\phi_1\phi_2}}, s_{\tau\sqrt{\phi_1\phi_2-\varphi_1\varphi_2}}\right) = \left(s_{\tau\sqrt{1-(1-\tau^2(1-\phi_1\phi_2)/\tau^2)^\lambda}}, s_{\tau\sqrt{(1-\tau^2(1-\phi_1\phi_2)/\tau^2)^\lambda-(1-\tau^2(1-\phi_1\phi_2)/\tau^2-\tau^2(\phi_1\phi_2-\varphi_1\varphi_2)/\tau^2)^\lambda}}\right) = \left(s_{\tau\sqrt{1-(\phi_1\phi_2)^\lambda}}, s_{\tau\sqrt{(\phi_1\phi_2)^\lambda-(\varphi_1\varphi_2)^\lambda}}\right) \tag{12}$$

Since $\lambda\alpha_1 = (s_{\tau\sqrt{1-\phi_1^\lambda}}, s_{\tau\sqrt{\phi_1^\lambda-\varphi_1^\lambda}})$ and $\lambda\alpha_2 = (s_{\tau\sqrt{1-\phi_2^\lambda}}, s_{\tau\sqrt{\phi_2^\lambda-\varphi_2^\lambda}})$, then

$$\lambda\alpha_1 \oplus \lambda\alpha_2 = \left(s_{\tau\sqrt{1-\prod_{i=1}^2(1-(\tau\sqrt{1-\phi_i^\lambda})^2/\tau^2)}}, s_{\tau\sqrt{\prod_{i=1}^2(1-(\tau\sqrt{1-\phi_i^\lambda})^2/\tau^2)-\prod_{i=1}^2(1-(\tau\sqrt{1-\phi_i^\lambda})^2/\tau^2-(\tau\sqrt{\phi_i^\lambda-\varphi_i^\lambda})^2/\tau^2)}}\right) = \left(s_{\tau\sqrt{1-\prod_{i=1}^2\phi_i^\lambda}}, s_{\tau\sqrt{\prod_{i=1}^2\phi_i^\lambda-\prod_{i=1}^2\varphi_i^\lambda}}\right) = \lambda(\alpha_1 \oplus \alpha_2) \tag{13}$$

$$\text{(iii)}\ (\lambda_1 + \lambda_2)\alpha = \left(s_{\tau\sqrt{1-(1-m^2/\tau^2)^{\lambda_1+\lambda_2}}}, s_{\tau\sqrt{(1-m^2/\tau^2)^{\lambda_1+\lambda_2}-(1-m^2/\tau^2-n^2/\tau^2)^{\lambda_1+\lambda_2}}}\right) = \left(s_{\tau\sqrt{1-\phi^{\lambda_1+\lambda_2}}}, s_{\tau\sqrt{\phi^{\lambda_1+\lambda_2}-\varphi^{\lambda_1+\lambda_2}}}\right)$$

Since $\lambda_1\alpha = (s_{\tau\sqrt{1-\phi^{\lambda_1}}}, s_{\tau\sqrt{\phi^{\lambda_1}-\varphi^{\lambda_1}}})$ and $\lambda_2\alpha = (s_{\tau\sqrt{1-\phi^{\lambda_2}}}, s_{\tau\sqrt{\phi^{\lambda_2}-\varphi^{\lambda_2}}})$, then

$$\lambda_1\alpha \oplus \lambda_2\alpha = \left(s_{\tau\sqrt{1-(1-(\tau\sqrt{1-\phi^{\lambda_1}})^2/\tau^2)(1-(\tau\sqrt{1-\phi^{\lambda_2}})^2/\tau^2)}}, s_{\tau\sqrt{(1-(\tau\sqrt{1-\phi^{\lambda_1}})^2/\tau^2)(1-(\tau\sqrt{1-\phi^{\lambda_2}})^2/\tau^2)-(1-(\tau\sqrt{1-\phi^{\lambda_1}})^2/\tau^2-(\tau\sqrt{\phi^{\lambda_1}-\varphi^{\lambda_1}})^2/\tau^2)(1-(\tau\sqrt{1-\phi^{\lambda_2}})^2/\tau^2-(\tau\sqrt{\phi^{\lambda_2}-\varphi^{\lambda_2}})^2/\tau^2)}}\right) = \left(s_{\tau\sqrt{1-\phi^{\lambda_1+\lambda_2}}}, s_{\tau\sqrt{\phi^{\lambda_1+\lambda_2}-\varphi^{\lambda_1+\lambda_2}}}\right) = (\lambda_1 + \lambda_2)\alpha \tag{14}$$

$$\text{(iv)}\ \alpha_1 \otimes \alpha_2 = \left(s_{\tau\sqrt{\gamma_1\gamma_2-\varphi_1\varphi_2}}, s_{\tau\sqrt{1-\gamma_1\gamma_2}}\right) = \alpha_2 \otimes \alpha_1$$

$$\text{(v)}\ (\alpha_1 \otimes \alpha_2)^\lambda = \left(s_{\tau\sqrt{(1-(\tau\sqrt{1-\gamma_1\gamma_2})^2/\tau^2)^\lambda-(1-(\tau\sqrt{\gamma_1\gamma_2-\varphi_1\varphi_2})^2/\tau^2-(\tau\sqrt{1-\gamma_1\gamma_2})^2/\tau^2)^\lambda}}, s_{\tau\sqrt{1-(1-(\tau\sqrt{1-\gamma_1\gamma_2})^2/\tau^2)^\lambda}}\right) = \left(s_{\tau\sqrt{(\gamma_1\gamma_2)^\lambda-(\varphi_1\varphi_2)^\lambda}}, s_{\tau\sqrt{1-(\gamma_1\gamma_2)^\lambda}}\right)$$

Since $\alpha_1^\lambda = (s_{\tau\sqrt{\gamma_1^\lambda-\varphi_1^\lambda}}, s_{\tau\sqrt{1-\gamma_1^\lambda}})$ and $\alpha_2^\lambda = (s_{\tau\sqrt{\gamma_2^\lambda-\varphi_2^\lambda}}, s_{\tau\sqrt{1-\gamma_2^\lambda}})$, then

$$\alpha_1 \otimes \alpha_2 = \left(s_{\tau\sqrt{\prod_{i=1}^2(1-(\tau\sqrt{1-\gamma_i^\lambda})^2/\tau^2)-\prod_{i=1}^2(1-(\tau\sqrt{\gamma_i^\lambda-\varphi_i^\lambda})^2/\tau^2-(\tau\sqrt{1-\gamma_i^\lambda})^2/\tau^2)}}, s_{\tau\sqrt{1-\prod_{i=1}^2(1-(\tau\sqrt{1-\gamma_i^\lambda})^2/\tau^2)}}\right) = \left(s_{\tau\sqrt{\prod_{i=1}^2\gamma_i^\lambda-\prod_{i=1}^2\varphi_i^\lambda}}, s_{\tau\sqrt{1-\prod_{i=1}^2\gamma_i^\lambda}}\right)$$

$$= (\alpha_1 \otimes \alpha_2)^{\lambda} \tag{15}$$

which completes the proof of Theorem 13. □

4. Linguistic Pythagorean Fuzzy PBM Operators Considering the Interactional Operational Rules

In this section, the PBM operator and the novel interactional operational rules are combined to design the LPFIPBM operator, the LPFWIPBM operator, the LPFIPGBM operator, and the LPFWIPGBM operator for fusing the LPFNs. Then their special cases and properties are also discussed.

4.1. The LPFIPBM Operator and the LPFWIPBM Operator

Definition 14. Given a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0,\tau]\}$ with a positive integer τ, two real numbers $p, q \geq 0$, and a set of LPFNs $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different subsets $G_1, G_2, \ldots, G_g$, where $\alpha_k = (s_{m_k}, s_{n_k})(k = 1, 2, \ldots, K; s_{m_k}, s_{n_k} \in S)$ and $\bigcup_{t=1}^{g} G_t = O$, the mathematical expression of the linguistic Pythagorean fuzzy interaction partitioned Bonferroni mean (LPFIPBM) aggregation operator is defined as

$$LPFIPBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = \frac{1}{g}\left(\bigoplus_{t=1}^{g}\left(\frac{1}{|G_t|}\bigoplus_{\alpha_u \in G_t}\left(\alpha_u^p \otimes \left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}} \alpha_v^q\right)\right)\right)^{1/(p+q)}\right) \tag{16}$$

where $|G_t|$ is the number of elements in the subset G_t and g denotes the number of subsets and $\sum_{t=1}^{g} |G_t| = K$.

According to the novel interactional operational rules developed for LPFNs in Definition 9, several theorems can be derived from (16) as follows.

Theorem 15. *Given a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0,\tau]\}$ with a positive integer τ, two real numbers $p, q \geq 0$, and a set of LPFNs $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different subsets $G_1, G_2, \ldots, G_g$, where $\alpha_k = (s_{m_k}, s_{n_k})(k = 1, 2, \ldots, K; s_{m_k}, s_{n_k} \in S)$ and $\bigcup_{t=1}^{g} G_t = O$, the aggregated result obtained from (16) is still a LPFN, which is expressed as*

$$\begin{aligned} &LPFIPBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) \\ &= \Bigg(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}))^{1/g}}}, \\ &s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}))^{1/g}-(\prod_{t=1}^{g}((\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)})^{1/g}}}\Bigg) \end{aligned} \tag{17}$$

where $\varepsilon_u = m_u/\tau, \gamma_u = n_u/\tau, \varepsilon_v = m_v/\tau, \gamma_v = n_v/\tau$, $1-\gamma_u^2 = \phi_u, 1-\gamma_v^2 = \phi_v$, $1-(\varepsilon_u^2+\gamma_u^2) = \varphi_u, 1-(\varepsilon_v^2+\gamma_v^2) = \varphi_v$, and $\rho = (\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}}(1-\phi_v^q+\varphi_v^q))^{1/(|G_t|-1)}, \eta = (\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}}\varphi_v^q)^{1/(|G_t|-1)}$.

Proof. According to Definition 9, we have

$$\begin{aligned} \alpha_u^p &= \left(s_{\tau\sqrt{(1-n_u^2/\tau^2)^p-(1-(n_u^2/\tau^2+m_u^2/\tau^2))^p}}, s_{\tau\sqrt{1-(1-n_u^2/\tau^2)^p}}\right), \\ \alpha_v^q &= \left(s_{\tau\sqrt{(1-n_v^2/\tau^2)^q-(1-(n_v^2/\tau^2+m_v^2/\tau^2))^q}}, s_{\tau\sqrt{1-(1-n_v^2/\tau^2)^q}}\right). \end{aligned} \tag{18}$$

Let $\varepsilon_u = m_u/\tau, \gamma_u = n_u/\tau, \varepsilon_v = m_v/\tau, \gamma_v = n_v/\tau$; then

$$\begin{aligned} \alpha_u^p &= \left(s_{\tau\sqrt{(1-\gamma_u^2)^p-(1-(\varepsilon_u^2+\gamma_u^2))^p}}, s_{\tau\sqrt{1-(1-\gamma_u^2)^p}}\right), \\ \alpha_v^q &= \left(s_{\tau\sqrt{(1-\gamma_v^2)^q-(1-(\varepsilon_v^2+\gamma_v^2))^q}}, s_{\tau\sqrt{1-(1-\gamma_u^2)^q}}\right). \end{aligned} \tag{19}$$

Let $1-\gamma_u^2 = \phi_u, 1-\gamma_v^2 = \phi_v$ and $1-(\varepsilon_u^2+\gamma_u^2) = \varphi_u, 1-(\varepsilon_v^2+\gamma_v^2) = \varphi_v$; then

$$\begin{aligned} \alpha_u^p &= \left(s_{\tau\sqrt{\phi_u^p-\varphi_u^p}}, s_{\tau\sqrt{1-\phi_u^p}}\right), \\ \alpha_v^q &= \left(s_{\tau\sqrt{\phi_v^q-\varphi_v^q}}, s_{\tau\sqrt{1-\phi_v^q}}\right), \\ \bigoplus_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}} \alpha_v^q &= \left(s_{\tau\sqrt{1-\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}}(1-\phi_v^q+\varphi_v^q)}}, s_{\tau\sqrt{\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}}(1-\phi_v^q+\varphi_v^q)-\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}}\varphi_v^q}}\right) \end{aligned} \tag{20}$$

According to Definition 9, we have

$$\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q=\left(s_{\tau\sqrt{1-(1-(\tau\sqrt{1-\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q)})^2/\tau^2)^{1/(|G_t|-1)}}},\right.$$
$$\left.s_{\tau\sqrt{(1-(\tau\sqrt{1-\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q)})^2/\tau^2)^{1/(|G_t|-1)}-(1-(\tau\sqrt{1-\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q)})^2/\tau^2-(\tau\sqrt{\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q)-\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\varphi_v^q})^2/\tau^2)^{1/(|G_t|-1)}}}\right)$$
$$=\left(s_{\tau\sqrt{1-(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q))^{1/(|G_t|-1)}}},s_{\tau\sqrt{(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q))^{1/(|G_t|-1)}-(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\varphi_v^q)^{1/(|G_t|-1)}}}\right) \tag{21}$$

and

$$\alpha_u^p\otimes\left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q\right)=\left(s_{\tau\sqrt{\phi_u^p(1-(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q))^{1/(|G_t|-1)}+(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\varphi_v^q)^{1/(|G_t|-1)})-\varphi_u^p(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\varphi_v^q)^{1/(|G_t|-1)}}},\right.$$
$$\left.s_{\tau\sqrt{1-\phi_u^p(1-(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q))^{1/(|G_t|-1)}+(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\varphi_v^q)^{1/(|G_t|-1)})}}\right) \tag{22}$$

Let $\rho=(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v^q+\varphi_v^q))^{1/(|G_t|-1)},\eta=(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\varphi_v^q)^{1/(|G_t|-1)}$; then the above equation can be transformed into

$$\alpha_u^p\otimes\left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q\right)=\left(s_{\tau\sqrt{\phi_u^p(1-\rho+\eta)-\varphi_u^p\eta}},s_{\tau\sqrt{1-\phi_u^p(1-\rho+\eta)}}\right) \tag{23}$$

Thus,

$$\bigoplus_{\alpha_u\in G_t}\left(\alpha_u^p\otimes\left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q\right)\right)$$
$$=\left(s_{\tau\sqrt{1-\prod_{\alpha_u\in G_t}(1-(\tau\sqrt{\phi_u^p(1-\rho+\eta)-\varphi_u^p\eta})^2/\tau^2)}},s_{\tau\sqrt{\prod_{\alpha_u\in G_t}(1-(\tau\sqrt{\phi_u^p(1-\rho+\eta)-\varphi_u^p\eta})^2/\tau^2)-\prod_{\alpha_u\in G_t}(1-(\tau\sqrt{\phi_u^p(1-\rho+\eta)-\varphi_u^p\eta})^2/\tau^2-(\tau\sqrt{1-\phi_u^p(1-\rho+\eta)})^2/\tau^2)}}\right)$$
$$=\left(s_{\tau\sqrt{1-\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta)}},s_{\tau\sqrt{\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta)-\prod_{\alpha_u\in G_t}(\varphi_u^p\eta)}}\right) \tag{24}$$

and

$$\frac{1}{|G_t|}\bigoplus_{\alpha_u\in G_t}\left(\alpha_u^p\otimes\left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q\right)\right)=\left(s_{\tau\sqrt{1-(1-(\tau\sqrt{1-\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta)})^2/\tau^2)^{1/|G_t|}}},\right.$$
$$\left.s_{\tau\sqrt{(1-(\tau\sqrt{1-\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta)})^2/\tau^2)^{1/|G_t|}-(1-(\tau\sqrt{1-\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta)})^2/\tau^2-(\tau\sqrt{\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta)-\prod_{\alpha_u\in G_t}(\varphi_u^p\eta)})^2/\tau^2)^{1/|G_t|}}}\right)$$
$$=\left(s_{\tau\sqrt{1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}}},\ s_{\tau\sqrt{(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}-(\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|}}}\right) \tag{25}$$

Then, we have

$$\left(\frac{1}{|G_t|}\bigoplus_{\alpha_u\in G_t}\left(\alpha_u^p\otimes\left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q\right)\right)\right)^{1/(p+q)}$$
$$=\left(s_{\tau\sqrt{(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}-((\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}}},\right.$$
$$\left.s_{\tau\sqrt{1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}}}\right) \tag{26}$$

and

$$\bigoplus_{t=1}^{g}\left(\frac{1}{|G_t|}\bigoplus_{\alpha_u\in G_t}\left(\alpha_u^p\otimes\left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q\right)\right)\right)^{1/(p+q)}$$
$$=\left(s_{\tau\sqrt{1-\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)})}},\right.$$
$$\left.s_{\tau\sqrt{\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)})-\prod_{t=1}^{g}((\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}}}\right) \tag{27}$$

Thus,

$$\frac{1}{g}\left(\bigoplus_{t=1}^{g}\left(\frac{1}{|G_t|}\bigoplus_{\alpha_u\in G_t}\left(\alpha_u^p\otimes\left(\frac{1}{|G_t|-1}\bigoplus_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\alpha_v^q\right)\right)\right)^{1/(p+q)}\right)$$
$$=\left(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}))^{1/g}}},\right.$$
$$\left.s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}))^{1/g}-(\prod_{t=1}^{g}((\prod_{\alpha_u\in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)})^{1/g}}}\right) \tag{28}$$

which completes the proof of Theorem 15. □

We continue to discuss the properties of the LPFIPBM operator as follows.

Theorem 16 (idempotency). *Given a continuous LTS $S=\{s_\varepsilon \mid \varepsilon\in[0,\tau]\}$ with a positive integer τ, two real numbers $p,q\geq 0$, and a set of LPFNs $O=\{\alpha_1,\alpha_2,\ldots,\alpha_K\}$, which is divided into g different subsets $G_1,G_2,\ldots,G_g$, where $\alpha_k=(s_{m_k},s_{n_k})(k=1,2,\ldots,K;s_{m_k},s_{n_k}\in S)$ and $\bigcup_{t=1}^{g}G_t=O$, if all α_k are equal for any k, namely, $\alpha_k=\alpha=(s_m,s_n)$, then*

$$LPFIPBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_K)=\alpha=(s_m,s_n) \tag{29}$$

Proof. Since $m_u=m_v=m$ and $n_u=n_v=n$, then we have

$$\begin{aligned}\frac{m_u}{\tau}&=\frac{m_v}{\tau}=\varepsilon_u=\varepsilon_v=\varepsilon,\\ \frac{n_u}{\tau}&=\frac{n_v}{\tau}=\gamma_u=\gamma_v=\gamma,\\ 1-\gamma_u^2&=1-\gamma_v^2=\phi_u=\phi_v=\phi,\\ 1-\left(\varepsilon_u^2+\gamma_u^2\right)&=1-\left(\varepsilon_v^2+\gamma_v^2\right)=\varphi_u=\varphi_v=\varphi,\end{aligned} \tag{30}$$

and thus

$$\rho = \left(\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}} \left(1 - \phi_v^q + \varphi_v^q\right) \right)^{1/(|G_t|-1)} = \left(1 - \phi^q + \varphi^q\right),$$

$$\eta = \left(\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}} \varphi_v^q \right)^{1/(|G_t|-1)} = \varphi^q \tag{31}$$

Bring the above equation into (17); then we have

$$\begin{aligned} &LPFIPBM^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_K\right) \\ &= \Bigg(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi^p(1-(1-\phi^q+\varphi^q)+\varphi^q)+\varphi^p\varphi^q))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi^p\varphi^q))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi^p\varphi^q))^{1/|G_t|})^{1/(p+q)}))^{1/g}}}, \\ &s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi^p(1-(1-\phi^q+\varphi^q)+\varphi^q)+\varphi^p\varphi^q))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi^p\varphi^q))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi^p\varphi^q))^{1/|G_t|})^{1/(p+q)}))^{1/g}-(\prod_{t=1}^{g}((\prod_{\alpha_u \in G_t}(\varphi^p\varphi^q))^{1/|G_t|})^{1/(p+q)})^{1/g}}} \Bigg) \\ &= \left(s_{\tau\sqrt{\phi-\varphi}}, s_{\tau\sqrt{1-\phi}}\right) = \left(s_{\tau\sqrt{1-n^2/\tau^2-(1-m^2/\tau^2-n^2/\tau^2)}}, s_{\tau\sqrt{1-(1-n^2/\tau^2)}}\right) = \left(s_m, s_n\right) = \alpha \end{aligned} \tag{32}$$

which finishes the proof of idempotency of the LPFIPBM operator. □

Theorem 17 (commutativity). *Given a continuous LTS $S = \{s_\varepsilon \mid 0 \leq \varepsilon \leq \tau\}$ with a positive integer τ, two real numbers $p, q \geq 0$, and a set of LPFNs $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different subsets $G_1, G_2, \ldots, G_g$, where $\alpha_k = (s_{m_k}, s_{n_k})(k = 1, 2, \ldots, K; s_{m_k}, s_{n_k} \in S)$ and $\bigcup_{t=1}^{g} G_t = O$, if a set of LPFNs $G_t' = \{\alpha_{t1}', \alpha_{t1}', \ldots, \alpha_{t1}'\}$ is any permutation of $G_t = \{\alpha_{t1}, \alpha_{t1}, \ldots, \alpha_{t1}\}$ for each t, then*

$$\begin{aligned} &LPFIPBM^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_K\right) \\ &= LPFIPBM^{p,q}\left(\alpha_1', \alpha_2', \ldots, \alpha_K'\right). \end{aligned} \tag{33}$$

Proof. According to Theorem 15, we have

$$\begin{aligned} &LPFIPBM^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_K\right) \\ &= \Bigg(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}))^{1/g}}}, \\ &s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi_u^p(1-\rho+\eta)+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)}))^{1/g}-(\prod_{t=1}^{g}((\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|})^{1/(p+q)})^{1/g}}} \Bigg) \end{aligned} \tag{34}$$

and

$$\begin{aligned} &LPFIPBM^{p,q}\left(\alpha_1', \alpha_2', \ldots, \alpha_K'\right) \\ &= \Bigg(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi_u'^p(1-\rho'+\eta')+\varphi_u'^p\eta'))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi_u'^p\eta'))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi_u'^p\eta'))^{1/|G_t|})^{1/(p+q)}))^{1/g}}}, \\ &s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u \in G_t}(1-\phi_u'^p(1-\rho'+\eta')+\varphi_u^p\eta))^{1/|G_t|}+(\prod_{\alpha_u \in G_t}(\varphi_u'^p\eta'))^{1/|G_t|})^{1/(p+q)}+((\prod_{\alpha_u \in G_t}(\varphi_u'^p\eta'))^{1/|G_t|})^{1/(p+q)}))^{1/g}-(\prod_{t=1}^{g}((\prod_{\alpha_u \in G_t}(\varphi_u'^p\eta'))^{1/|G_t|})^{1/(p+q)})^{1/g}}} \Bigg) \end{aligned} \tag{35}$$

For each t, we have $(\prod_{\alpha_u \in G_t}(1 - \phi_u'^p(1 - \rho' + \eta') + \varphi_u'^p\eta'))^{1/|G_t|} = (\prod_{\alpha_u \in G_t}(1 - \phi_u^p(1 - \rho + \eta) + \varphi_u^p\eta))^{1/|G_t|}$ and $(\prod_{\alpha_u \in G_t}(\varphi_u'^p\eta'))^{1/|G_t|} = (\prod_{\alpha_u \in G_t}(\varphi_u^p\eta))^{1/|G_t|}$.

Thus, $LPFIPBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = LPFIPBM^{p,q}(\alpha_1', \alpha_2', \ldots, \alpha_K')$, which finishes the proof of Theorem 17. □

By adjusting the values of the parameters $p, q \geq 0$, respectively, some special equations can be derived from the LPFIPBM operator as follows:

(i) When $q \longrightarrow 0$, $\rho = (\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}}(1 - \phi_v^q + \varphi_v^q))^{1/(|G_t|-1)} = 1$ and $\eta = (\prod_{\substack{\alpha_v \in G_t \\ \alpha_v \neq \alpha_u}} \varphi_v^q)^{1/(|G_t|-1)} = 1$. Thus,

$$LPFIPBM^{p,0}\left(\alpha_1,\alpha_2,\ldots,\alpha_K\right)=\left(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p+\varphi_u^p))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p))^{1/|G_t|})^{1/p}+((\prod_{\alpha_u\in G_t}(\varphi_u^p))^{1/|G_t|})^{1/p}))^{1/g}}},\right.$$
$$\left.s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u^p+\varphi_u^p))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u^p))^{1/|G_t|})^{1/p}+((\prod_{\alpha_u\in G_t}(\varphi_u^p))^{1/|G_t|})^{1/p}))^{1/g}-(\prod_{t=1}^{g}((\prod_{\alpha_u\in G_t}(\varphi_u^p))^{1/|G_t|})^{1/p})^{1/g}}}\right) \tag{36}$$

(ii) When $p=1$ and $q\longrightarrow 0$, the above equation can be transformed into

$$LPFIPBM^{1,0}\left(\alpha_1,\alpha_2,\ldots,\alpha_K\right)=\left(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(\prod_{\alpha_u\in G_t}(1-\phi_u+\varphi_u))^{1/|G_t|})^{1/g}}},\right.$$
$$\left.s_{\tau\sqrt{(\prod_{t=1}^{g}(\prod_{\alpha_u\in G_t}(1-\phi_u+\varphi_u))^{1/|G_t|})^{1/g}-(\prod_{t=1}^{g}(\prod_{\alpha_u\in G_t}\varphi_u)^{1/|G_t|})^{1/g}}}\right) \tag{37}$$

(iii) When $p\longrightarrow 0$, we have

$$LPFIPBM^{0,q}\left(\alpha_1,\alpha_2,\ldots,\alpha_K\right)=\left(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-\rho+\eta)^{1/q}+\eta^{1/q}))^{1/g}}},\right.$$
$$\left.s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-\rho+\eta)^{1/q}+\eta^{1/q}))^{1/g}-(\prod_{t=1}^{g}\eta^{1/q})^{1/g}}}\right) \tag{38}$$

(iv) When $p=1$ and $q=1$, we have $\rho=(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}(1-\phi_v+\varphi_v))^{1/(|G_t|-1)}$ and $\eta=(\prod_{\substack{\alpha_v\in G_t\\ \alpha_v\neq\alpha_u}}\varphi_v)^{1/(|G_t|-1)}$. Thus,

$$LPFIPBM^{1,1}\left(\alpha_1,\alpha_2,\ldots,\alpha_K\right)=\left(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u(1-\rho+\eta)+\varphi_u\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u\eta))^{1/|G_t|})^{1/2}+((\prod_{\alpha_u\in G_t}(\varphi_u\eta))^{1/|G_t|})^{1/2}))^{1/g}}},\right.$$
$$\left.s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\alpha_u\in G_t}(1-\phi_u(1-\rho+\eta)+\varphi_u\eta))^{1/|G_t|}+(\prod_{\alpha_u\in G_t}(\varphi_u\eta))^{1/|G_t|})^{1/2}+((\prod_{\alpha_u\in G_t}(\varphi_u\eta))^{1/|G_t|})^{1/2}))^{1/g}-(\prod_{t=1}^{g}((\prod_{\alpha_u\in G_t}(\varphi_u\eta))^{1/|G_t|})^{1/2})^{1/g}}}\right) \tag{39}$$

Although the LPFIPBM operator could capture the interrelationship between each attribute value and the other ones in each independent subset, it is not capable of considering the weights of attributes and the decision-makers. To deal with the drawback of the LPFIPBM operator, the LPFWIPBM operator is devised as follows.

Definition 18. Given a continuous LTS $S=\{s_\varepsilon \mid \varepsilon\in[0,\tau]\}$ with a positive integer τ, two real numbers $p,q\geq 0$, and a set of LPFNs $O=\{\alpha_1,\alpha_2,\ldots,\alpha_K\}$, which is divided into g different subsets $G_1,G_2,\ldots,G_g$, where $\alpha_k=(s_{m_k},s_{n_k})(k=1,2,\ldots,K;s_{m_k},s_{n_k}\in S)$ and $\bigcup_{t=1}^{g}G_t=O$, the mathematical expression of the linguistic Pythagorean fuzzy weighted interaction partitioned Bonferroni mean (LPFWIPBM) aggregation operator is defined as

$$LPFWIPBM^{p,q}\left(\alpha_1,\alpha_2,\ldots,\alpha_K\right)=\frac{1}{g}\left(\bigoplus_{t=1}^{g}\left(\frac{1}{|G_t|\left(|G_t|-1\right)}\bigoplus_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}\left(\left(\omega_u\alpha_u\right)^p\otimes\left(\omega_v\alpha_v\right)^q\right)\right)^{1/(p+q)}\right) \tag{40}$$

where $|G_t|$ is the number of LPFNs in the subset G_t, g denotes the number of subsets and $\sum_{t=1}^{g}|G_t|=K$, $\omega=(\omega_1,\omega_2,\ldots,\omega_K)^T$ denotes the weight vector of the set of LPFNs O with $\omega_i\in[0,1]$, $i=1,2,\ldots,K$, and $\sum_{i=1}^{K}\omega_i=1$.

According to the novel interactional operational rules devised for LPFNs in Definition 9, a theorem can be derived from (40) as follows.

Theorem 19. *Given a continuous LTS $S=\{s_\varepsilon \mid \varepsilon\in[0,\tau]\}$ with a positive integer τ, two real numbers $p,q\geq 0$, a set of LPFNs $O=\{\alpha_1,\alpha_2,\ldots,\alpha_K\}$, which is partitioned into g different subsets $G_1,G_2,\ldots,G_g$, where $\alpha_k=(s_{m_k},s_{n_k})(k=1,2,\ldots,K;s_{m_k},s_{n_k}\in S)$ and $\bigcup_{t=1}^{g}G_t=O$, and a weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_K)^T$ of LPFNs in O where $\omega_i\in[0,1]$, $i=1,2,\ldots,K$, and $\sum_{i=1}^{K}\omega_i=1$, the aggregated result of (40) is still a LPFN, which is expressed as*

$$LPFWIPBM^{p,q}\left(\alpha_1,\alpha_2,\ldots,\alpha_K\right)=\left(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}+\eta^{1/(|G_t|(|G_t|-1)(p+q))}))^{1/g}}},\right.$$
$$\left.s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}+\eta^{1/(|G_t|(|G_t|-1)(p+q))}))^{1/g}-(\prod_{t=1}^{g}(\eta^{1/(|G_t|(|G_t|-1)(p+q))}))^{1/g}}}\right) \tag{41}$$

where $\varepsilon_u = m_u/\tau, \gamma_u = n_u/\tau, \varepsilon_v = m_v/\tau, \gamma_v = n_v/\tau, (1-\varepsilon_u^2)^{\omega_u} = \phi_u, (1-\varepsilon_v^2)^{\omega_v} = \phi_v, (1-\varepsilon_u^2-\gamma_u^2)^{\omega_u} = \varphi_u, (1-\varepsilon_v^2-\gamma_v^2)^{\omega_v} = \varphi_v, \prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(1-(1-\phi_u+\varphi_u)^p(1-\phi_v+\varphi_v)^q+\varphi_u^p\varphi_v^q) = \rho$, *and* $\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(\varphi_u^p\varphi_v^q) = \eta$.

Proof. According to Definition 9, we have

$$\begin{aligned}
&\omega_u\alpha_u\\
&= \left(s_{\tau\sqrt{1-(1-m_u{}^2/\tau^2)^{\omega_u}}}, s_{\tau\sqrt{(1-m_u^2/\tau^2)^{\omega_u}-(1-m_u^2/\tau^2-n_u^2/\tau^2)^{\omega_u}}}\right),\\
&\omega_v\alpha_v\\
&= \left(s_{\tau\sqrt{1-(1-m_v{}^2/\tau^2)^{\omega_v}}}, s_{\tau\sqrt{(1-m_v^2/\tau^2)^{\omega_v}-(1-m_v^2/\tau^2-n_v^2/\tau^2)^{\omega_v}}}\right)
\end{aligned} \tag{42}$$

Let $\varepsilon_u = m_u/\tau, \gamma_u = n_u/\tau, \varepsilon_v = m_v/\tau, \gamma_v = n_v/\tau$; then the above equations can be transformed as

$$\begin{aligned}
\omega_u\alpha_u &= \left(s_{\tau\sqrt{1-(1-\varepsilon_u^2)^{\omega_u}}}, s_{\tau\sqrt{(1-\varepsilon_u^2)^{\omega_u}-(1-\varepsilon_u^2-\gamma_u^2)^{\omega_u}}}\right),\\
\omega_v\alpha_v &= \left(s_{\tau\sqrt{1-(1-\varepsilon_v^2)^{\omega_v}}}, s_{\tau\sqrt{(1-\varepsilon_v^2)^{\omega_v}-(1-\varepsilon_v^2-\gamma_v^2)^{\omega_v}}}\right)
\end{aligned} \tag{43}$$

Thus,

$$\begin{aligned}
(\omega_u\alpha_u)^p = \Bigg(&s_{\tau\sqrt{(1-(1-\varepsilon_u^2)^{\omega_u}+(1-\varepsilon_u^2-\gamma_u^2)^{\omega_u})^p-((1-\varepsilon_u^2-\gamma_u^2)^{\omega_u})^p}},\\
&s_{\tau\sqrt{1-(1-(1-\varepsilon_u^2)^{\omega_u}+(1-\varepsilon_u^2-\gamma_u^2)^{\omega_u})^p}}\Bigg)
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
(\omega_v\alpha_v)^q = \Bigg(&s_{\tau\sqrt{(1-(1-\varepsilon_v^2)^{\omega_v}+(1-\varepsilon_v^2-\gamma_v^2)^{\omega_v})^q-((1-\varepsilon_v^2-\gamma_v^2)^{\omega_v})^q}},\\
&s_{\tau\sqrt{1-(1-(1-\varepsilon_v^2)^{\omega_v}+(1-\varepsilon_v^2-\gamma_v^2)^{\omega_v})^q}}\Bigg)
\end{aligned} \tag{45}$$

Let $(1-\varepsilon_u^2)^{\omega_u} = \phi_u, (1-\varepsilon_v^2)^{\omega_v} = \phi_v$ and $(1-\varepsilon_u^2-\gamma_u^2)^{\omega_u} = \varphi_u, (1-\varepsilon_v^2-\gamma_v^2)^{\omega_v} = \varphi_v$; then we have

$$\begin{aligned}
(\omega_u\alpha_u)^p &= \left(s_{\tau\sqrt{(1-\phi_u+\varphi_u)^p-\varphi_u^p}}, s_{\tau\sqrt{1-(1-\phi_u+\varphi_u)^p}}\right),\\
(\omega_v\alpha_v)^q &= \left(s_{\tau\sqrt{(1-\phi_v+\varphi_v)^q-\varphi_v^q}}, s_{\tau\sqrt{1-(1-\phi_v+\varphi_v)^q}}\right)\\
(\omega_u\alpha_u)^p\otimes(\omega_v\alpha_v)^q &= \Bigg(s_{\tau\sqrt{(1-\phi_u+\varphi_u)^p(1-\phi_v+\varphi_v)^q-\varphi_u^p\varphi_v^q}},\\
&\quad s_{\tau\sqrt{1-(1-\phi_u+\varphi_u)^p(1-\phi_v+\varphi_v)^q}}\Bigg)
\end{aligned} \tag{46}$$

Thus,

$$\begin{aligned}
&\bigoplus_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}\left((\omega_u\alpha_u)^p\otimes(\omega_v\alpha_v)^q\right)\\
&= \Bigg(s_{\tau\sqrt{1-\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(1-(1-\phi_u+\varphi_u)^p(1-\phi_v+\varphi_v)^q+\varphi_u^p\varphi_v^q)}},\\
&\quad s_{\tau\sqrt{\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(1-(1-\phi_u+\varphi_u)^p(1-\phi_v+\varphi_v)^q+\varphi_u^p\varphi_v^q)-\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(\varphi_u^p\varphi_v^q)}}\Bigg)
\end{aligned} \tag{47}$$

Let $\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(1-(1-\phi_u+\varphi_u)^p(1-\phi_v+\varphi_v)^q+\varphi_u^p\varphi_v^q) = \rho$ and $\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(\varphi_u^p\varphi_v^q) = \eta$; then the above equation can be transformed into

$$\bigoplus_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}\left((\omega_u\alpha_u)^p\otimes(\omega_v\alpha_v)^q\right) = \left(s_{\tau\sqrt{1-\rho}}, s_{\tau\sqrt{\rho-\eta}}\right) \tag{48}$$

and we can have

$$\begin{aligned}
&\frac{1}{|G_t|(|G_t|-1)}\bigoplus_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}\left((\omega_u\alpha_u)^p\otimes(\omega_v\alpha_v)^q\right)\\
&\quad= \left(s_{\tau\sqrt{1-\rho^{1/(|G_t|(|G_t|-1))}}}, s_{\tau\sqrt{\rho^{1/(|G_t|(|G_t|-1))}-\eta^{1/(|G_t|(|G_t|-1))}}}\right).
\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}
&\left(\frac{1}{|G_t|(|G_t|-1)}\bigoplus_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}\left((\omega_u\alpha_u)^p\otimes(\omega_v\alpha_v)^q\right)\right)^{1/(p+q)}\\
&\quad= \Bigg(s_{\tau\sqrt{(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}-\eta^{1/(|G_t|(|G_t|-1)(p+q))}}},\\
&\quad s_{\tau\sqrt{1-(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}}}\Bigg)
\end{aligned} \tag{50}$$

and

$$\overset{g}{\underset{t=1}{\oplus}}\left(\frac{1}{|G_t|(|G_t|-1)}\underset{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}{\oplus}\left((\omega_u\alpha_u)^p\otimes(\omega_v\alpha_v)^q\right)\right)^{1/(p+q)}=\left(s_{\tau\sqrt{1-\prod_{t=1}^{g}(1-(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}+\eta^{1/(|G_t|(|G_t|-1)(p+q))})}},\right.$$
$$\left.s_{\tau\sqrt{\prod_{t=1}^{g}(1-(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}+\eta^{1/(|G_t|(|G_t|-1)(p+q))})-\prod_{t=1}^{g}(\eta^{1/(|G_t|(|G_t|-1)(p+q))})}}\right) \tag{51}$$

Finally, we can have

$$\frac{1}{g}\left(\overset{g}{\underset{t=1}{\oplus}}\left(\frac{1}{|G_t|(|G_t|-1)}\underset{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}{\oplus}\left((\omega_u\alpha_u)^p\otimes(\omega_v\alpha_v)^q\right)\right)^{1/(p+q)}\right)$$
$$=\left(s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}+\eta^{1/(|G_t|(|G_t|-1)(p+q))}))^{1/g}}},\right.$$
$$\left.s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-\rho^{1/(|G_t|(|G_t|-1))}+\eta^{1/(|G_t|(|G_t|-1))})^{1/(p+q)}+\eta^{1/(|G_t|(|G_t|-1)(p+q))}))^{1/g}-(\prod_{t=1}^{g}(\eta^{1/(|G_t|(|G_t|-1)(p+q))}))^{1/g}}}\right) \tag{52}$$

which finishes the proof of Theorem 19. □

4.2. The LPFIPGBM Operator and the LPFWIPGBM Operator. The aggregation operators are mainly divided into two categories: the arithmetic operators and the geometric operators [39]. The aggregated results derived from the former ones are easily influenced by the extreme values, while the aggregated results derived from the latter ones could take the balance among all the values into consideration. Thus, the geometric operators perform better than the arithmetic operators, especially for the social economy data. The study presented in [40] reported that the aggregated results of the arithmetic operators are higher than those of the geometric ones when the same data set is fused. Hence, if the decision-makers are optimistic, then the arithmetic operators are used. If they are pessimistic, then the geometric operators can be selected. To enhance the theory of aggregation operators, the PGBM operator and novel interactional operational rules are combined to devise the LPFIPGBM and LPFWIPGBM operators for LPFNs.

Definition 20. Given a continuous LTS $S=\{s_\varepsilon \mid \varepsilon\in[0,\tau]\}$ with a positive integer τ, two real numbers $p,q\geq 0$, and a set of LPFNs $O=\{\alpha_1,\alpha_2,\ldots,\alpha_K\}$, which is divided into g different subsets $G_1,G_2,\ldots,G_g$, where $\alpha_k=(s_{m_k},s_{n_k})(k=1,2,\ldots,K;s_{m_k},s_{n_k}\in S)$ and $\bigcup_{t=1}^{g}G_t=O$, the mathematical expression of the linguistic Pythagorean fuzzy interaction partitioned geometric Bonferroni mean (LPFIPGBM) operator is defined as

$$LPFIPGBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_K)$$
$$=\left(\overset{g}{\underset{t=1}{\otimes}}\left(\frac{1}{p+q}\left(\underset{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}{\otimes}(p\alpha_u\oplus q\alpha_v)\right)^{1/|G_t|(|G_t|-1)}\right)\right)^{1/g} \tag{53}$$

where $|G_t|$ is the number of elements in the subset G_t and g denotes the number of subsets and $\sum_{t=1}^{g}|G_t|=K$.

According to the novel interactional operational rules developed for LPFNs in Definition 9, several theorems can be derived from (53) as follows.

Theorem 21. *Given a continuous LTS $S=\{s_\varepsilon \mid \varepsilon\in[0,\tau]\}$ with a positive integer τ, two real numbers $p,q\geq 0$, and a set of LPFNs $O=\{\alpha_1,\alpha_2,\ldots,\alpha_K\}$, which is divided into g different subsets $G_1,G_2,\ldots,G_g$, where $\alpha_k=(s_{m_k},s_{n_k})(k=1,2,\ldots,K;s_{m_k},s_{n_k}\in S)$ and $\bigcup_{t=1}^{g}G_t=O$, the aggregated result obtained from (53) is still a LPFN, which is expressed as*

$$LPFIPGBM^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_K)$$
$$=\left(s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-\rho+\eta)^{1/(p+q)}+\eta^{1/(p+q)}))^{1/g}-(\prod_{t=1}^{g}(\eta^{1/(p+q)}))^{1/g}}},\right. \tag{54}$$
$$\left.s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-\rho+\eta)^{1/(p+q)}+\eta^{1/(p+q)}))^{1/g}}}\right)$$

where $\varepsilon_u=m_u/\tau$, $\gamma_u=n_u/\tau$, $\varepsilon_v=m_v/\tau$, $\gamma_v=n_v/\tau$, $1-\varepsilon_u^2=\phi_u$, $1-\varepsilon_v^2=\phi_v$, $1-\varepsilon_u^2-\gamma_u^2=\varphi_u$, $1-\varepsilon_v^2-\gamma_v^2=\varphi_v$, $\rho=(\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(1-\phi_u^p\phi_v^q+\varphi_u^p\varphi_v^q))^{1/|G_t|(|G_t|-1)}$, and $\eta=(\prod_{\substack{\alpha_u,\alpha_v\in G_t\\ \alpha_u\neq\alpha_v}}(\varphi_u^p\varphi_v^q))^{1/|G_t|(|G_t|-1)}$.

Proof. The proof process is similar to that of Theorem 15, which is omitted here. □

Then, some properties of the LPFIPGBM operator are given as follows.

Theorem 22 (idempotency). *Given a continuous LTS $S=\{s_\varepsilon \mid \varepsilon\in[0,\tau]\}$ with a positive integer τ, two real numbers $p,q\geq 0$,*

and a set of LPFNs $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different subsets $G_1, G_2, \ldots, G_g$, where $\alpha_k = (s_{m_k}, s_{n_k})(k = 1, 2, \ldots, K; s_{m_k}, s_{n_k} \in S)$ and $\bigcup_{t=1}^{g} G_t = O$, if all α_k are equal for any k, namely, $\alpha_k = \alpha = (s_m, s_n)$, then

$$LPFIPGBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = \alpha = (s_m, s_n). \tag{55}$$

Theorem 23 (commutativity). *Given a continuous LTS $S = \{s_\varepsilon \mid 0 \le \varepsilon \le \tau\}$ with a positive integer τ, two real numbers $p, q \ge 0$, and a set of LPFNs $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different subsets $G_1, G_2, \ldots, G_g$, where $\alpha_k = (s_{m_k}, s_{n_k})(k = 1, 2, \ldots, K; s_{m_k}, s_{n_k} \in S)$ and $\bigcup_{t=1}^{g} G_t = O$, if a set of LPFNs $G_t' = \{\alpha_{t1}', \alpha_{t1}', \ldots, \alpha_{t1}'\}$ is any permutation of $G_t = \{\alpha_{t1}, \alpha_{t1}, \ldots, \alpha_{t1}\}$ for each t, then*

$$\begin{aligned} &LPFIPGBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) \\ &\quad = LPFIPGBM^{p,q}(\alpha_1', \alpha_2', \ldots, \alpha_K'). \end{aligned} \tag{56}$$

Proof. The proof processes of Theorems 22 and 23 are similar to those of Theorems 16 and 17, which are omitted here. □

By adjusting the values of the parameters $p, q \ge 0$, respectively, some special equations can be derived from the LPFIPGBM operator as follows:

(i) When $p = 1$ and $q \longrightarrow 0$, $\rho = (\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (1 - \phi_u + \varphi_u))^{1/|G_t|(|G_t|-1)}$ and $\eta = (\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} \varphi_u)^{1/|G_t|(|G_t|-1)}$. Thus,

$$\begin{aligned} &LPFIPGBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) \\ &= \left(s_{\tau\sqrt{(\prod_{t=1}^{g}(\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (1-\phi_u+\varphi_u))^{1/|G_t|(|G_t|-1)})^{1/g} - (\prod_{t=1}^{g}(\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} \varphi_u)^{1/|G_t|(|G_t|-1)})^{1/g}}}, s_{\tau\sqrt{1-(\prod_{t=1}^{g}(\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (1-\phi_u+\varphi_u))^{1/|G_t|(|G_t|-1)})^{1/g}}} \right) \end{aligned} \tag{57}$$

(ii) When $p = 1$ and $q = 1$, $\rho = (\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (1 - \phi_u\phi_v + \varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)}$, and $\eta = (\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)}$. Thus,

$$\begin{aligned} &LPFIPGBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) \\ &= \left(s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-(\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (1-\phi_u\phi_v+\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)}+(\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)})^{1/2}+((\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)})^{1/2}))^{1/g}-(\prod_{t=1}^{g}(((\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)})^{1/2}))^{1/g}}}, \\ &\quad s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-(\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (1-\phi_u\phi_v+\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)}+(\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)})^{1/2}+((\prod_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (\varphi_u\varphi_v))^{1/|G_t|(|G_t|-1)})^{1/2}))^{1/g}}} \right). \end{aligned} \tag{58}$$

Similarly, the LPFIPGBM operator can capture the interrelationship between each attribute value and the other ones in each independent subset; it is not capable of taking the weights of attributes and decision-makers into account. To enhance the capability of the LPFIPGBM operator, the LPFWIPGBM operator is devised as follows.

Definition 24. Given a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0, \tau]\}$ with a positive integer τ, two real numbers $p, q \ge 0$, and a set of LPFNs $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is divided into g different subsets $G_1, G_2, \ldots, G_g$, where $\alpha_k = (s_{m_k}, s_{n_k})(k = 1, 2, \ldots, K; s_{m_k}, s_{n_k} \in S)$ and $\bigcup_{t=1}^{g} G_t = O$, the mathematical expression of the linguistic Pythagorean fuzzy weighted interaction partitioned geometric Bonferroni mean (LPFWIPGBM) aggregation operator is defined as

$$\begin{aligned} &LPFWIPGBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) \\ &= \left(\bigotimes_{t=1}^{g} \left(\frac{1}{p+q} \left(\bigotimes_{\substack{\alpha_u, \alpha_v \in G_t \\ \alpha_u \ne \alpha_v}} (p\alpha_u^{\omega_u} \oplus q\alpha_v^{\omega_v}) \right)^{1/|G_t|(|G_t-1|)} \right) \right)^{1/g} \end{aligned} \tag{59}$$

where $|G_t|$ is the number of LPFNs in the subset G_t, g denotes the number of subsets, and $\sum_{t=1}^{g} |G_t| = K$, $\omega = (\omega_1, \omega_2, \ldots, \omega_K)^T$ denotes the weight vector of the set of LPFNs O with $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, K$, and $\sum_{i=1}^{K} \omega_i = 1$.

According to the novel interactional operational rules devised for LPFNs in Definition 9, a theorem can be derived from (59) as follows.

Theorem 25. *Given a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0, \tau]\}$ with a positive integer τ, two real numbers $p, q \ge 0$, a set*

of LPFNs $O = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, which is partitioned into g different subsets $G_1, G_2, \ldots, G_g$, where $\alpha_k = (s_{m_k}, s_{n_k})(k = 1, 2, \ldots, K; s_{m_k}, s_{n_k} \in S)$ and $\bigcup_{t=1}^{g} G_t = O$, and a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_K)^T$ of LPFNs in O where $\omega_i \in [0, 1]$, $i = 1, 2, \ldots, K$, and $\sum_{i=1}^{K} \omega_i = 1$, the aggregated result of (59) is still a LPFN, which is expressed as

$$LPFWIPGBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_K) = \left(s_{\tau\sqrt{(\prod_{t=1}^{g}(1-(1-\rho^{1/|G_t|(|G_t-1|)}+\eta^{1/|G_t|(|G_t-1|)})^{1/p+q}+(\eta^{1/|G_t|(|G_t-1|)})^{1/p+q}))^{1/g}-(\prod_{t=1}^{g}(\eta^{1/|G_t|(|G_t-1|)})^{1/p+q})^{1/g}}}, \right.$$
$$\left. s_{\tau\sqrt{1-(\prod_{t=1}^{g}(1-(1-\rho^{1/|G_t|(|G_t-1|)}+\eta^{1/|G_t|(|G_t-1|)})^{1/p+q}+(\eta^{1/|G_t|(|G_t-1|)})^{1/p+q}))^{1/g}}} \right) \tag{60}$$

where $\varepsilon_u = m_u/\tau, \gamma_u = n_u/\tau$, $\varepsilon_v = m_v/\tau$, $\gamma_v = n_v/\tau$, $(1-\gamma_u^2)^{\omega_u} = \phi_u$, $(1-\gamma_v^2)^{\omega_v} = \phi_v$, $(1-\varepsilon_u^2-\gamma_u^2)^{\omega_u} = \varphi_u$, $(1-\varepsilon_v^2-\gamma_v^2)^{\omega_v} = \varphi_v$, $\prod_{\substack{\alpha_u,\alpha_v \in G_t \\ \alpha_u \neq \alpha_v}} (1-(1-\phi_u+\varphi_u)^p(1-\phi_v+\varphi_v)^q+\varphi_u^p\varphi_v^q) = \rho$, and $\prod_{\substack{\alpha_u,\alpha_v \in G_t \\ \alpha_u \neq \alpha_v}} (\varphi_u^p\varphi_v^q) = \eta$.

Proof. The proof process of this theorem is similar to that of Theorem 19, which is omitted here. □

5. Novel MAGDM Method Using the LPFWIPBM or LPFWIPGBM Operator

In the MAGDM problems, there exists a set of alternatives $A = \{A_1, A_2, \ldots, A_h\}$ with a set of attributes $B = \{B_1, B_2, \ldots, B_w\}$, which can be split into two categories, namely, the benefit attributes and the cost attributes, respectively. A group of decision-makers $M = \{M_1, M_2, \ldots, M_r\}$ are invited to select the linguistic terms from a continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0, \tau]\}$ with a positive integer τ to construct the LPFNs for modeling the preference information. Suppose that $L^l = (\alpha_{ij}^l)_{h\times w}$ is a preference information matrix that is given by the decision-maker M_l, where $\alpha_{ij}^l = (s_{m_{ij}}^l, s_{n_{ij}}^l)$ is a LPFN modeling the preference information given by the decision-maker M_l when the alternative A_i with respect to the attribute B_j is assessed. Suppose that $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_r)^T$ represents the weight vector of the group of decision-makers M, where κ_k is the weight of the decision-maker M_k satisfying $0 \le \kappa_k \le 1$ and $\sum_{k=1}^{r} \kappa_k = 1$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_w)^T$ express the weight vector of the set of attributes B, where ω_t denotes the weight of the attribute B_t satisfying $0 \le \omega_t \le 1$ and $\sum_{t=1}^{w} \omega_t = 1$.

For the above MAGDM problems, a novel MAGDM method exploiting the LPFWIPBM operator or the LPFWIPGBM operator is currently devised as follows.

Step 1. Normalize the preference information matrix $L^l = (\alpha_{ij}^l)_{h\times w}$ that is provided by each decision-maker. If there exist cost attributes, then it is necessary to transform their values into the benefit-type ones. The normalized preference information matrix denoted as $Z^l = (z_{ij}^l)_{h\times w}$ $(l = 1, 2, \ldots, r)$ is derived according to the following rule:

$$z_{ij}^l = \left(s_{m_{ij}}^l, s_{n_{ij}}^l\right) = \begin{cases} \left(s_{m_{ij}}^l, s_{n_{ij}}^l\right) & \textit{for benefit-type attribute } B_j \\ \left(s_{n_{ij}}^l, s_{m_{ij}}^l\right) & \textit{for cost-type attribute } B_j \end{cases} \tag{61}$$

Step 2. Exploit the LPFWIPBM operator

$$z_i^l = \left(s_{m_i}^l, s_{n_i}^l\right) = LPFWIPBM\left(z_{i1}^l, z_{i2}^l, \ldots, z_{iw}^l\right) \tag{62}$$

or the LPFWIPGBM operator

$$z_i^l = \left(s_{m_i}^l, s_{n_i}^l\right) = LPFWIPGBM\left(z_{i1}^l, z_{i2}^l, \ldots, z_{iw}^l\right) \tag{63}$$

to compute the fused value of all the attribute values of each alternative A_i which are given by the decision-maker M_l.

Step 3. Exploit the LPFWIPBM operator

$$z_i = \left(s_{m_i}, s_{n_i}\right) = LPFWIPBM\left(z_i^1, z_i^2, \ldots, z_i^r\right) \tag{64}$$

or the LPFWIPGBM operator

$$z_i = \left(s_{m_i}, s_{n_i}\right) = LPFWIPGBM\left(z_i^1, z_i^2, \ldots, z_i^r\right) \tag{65}$$

to compute the fused value of the fused attribute value of each alternative A_i.

Step 4. Use Definition 2 to compute the score value and the accuracy value of all the fused value z_i as

$$D(z_i) = s_{\sqrt{(\tau^2+m_i^2-n_i^2)/2}} \tag{66}$$

and

$$J(z_i) = s_{\sqrt{m_i^2+n_i^2}} \tag{67}$$

Step 5. All the alternatives $\{A_1, A_2, \ldots, A_h\}$ are ranked based on their score value and accuracy value, and the optimal one(s) is selected.

Step 6. End.

6. Illustrative Example and Comparison Analysis

In this section, a practical example is presented to illustrate the detailed implementation process of the proposed MAGDM method, and then it is compared with the existing study.

Table 1: The linguistic Pythagorean fuzzy preference information matrix L^1 provided by the decision-maker M_1.

	B_1	B_2	B_3	B_4
A_1	(s_1, s_6)	(s_3, s_7)	(s_6, s_2)	(s_7, s_2)
A_2	(s_4, s_3)	(s_5, s_4)	(s_1, s_3)	(s_1, s_5)
A_3	(s_7, s_3)	(s_7, s_1)	(s_2, s_6)	(s_1, s_6)
A_4	(s_3, s_7)	(s_4, s_3)	(s_2, s_6)	(s_6, s_4)

Table 2: The linguistic Pythagorean fuzzy preference information matrix L^2 provided by the decision-maker M_2.

	B_1	B_2	B_3	B_4
A_1	(s_1, s_7)	(s_2, s_6)	(s_5, s_2)	(s_7, s_2)
A_2	(s_4, s_3)	(s_7, s_1)	(s_2, s_6)	(s_1, s_5)
A_3	(s_7, s_3)	(s_6, s_1)	(s_2, s_6)	(s_2, s_5)
A_4	(s_7, s_1)	(s_3, s_5)	(s_2, s_3)	(s_7, s_2)

Table 3: The linguistic Pythagorean fuzzy preference information matrix L^3 provided by the decision-maker M_3.

	B_1	B_2	B_3	B_4
A_1	(s_2, s_7)	(s_1, s_6)	(s_7, s_3)	(s_6, s_2)
A_2	(s_6, s_1)	(s_4, s_1)	(s_2, s_3)	(s_3, s_5)
A_3	(s_6, s_2)	(s_7, s_2)	(s_1, s_7)	(s_4, s_5)
A_4	(s_5, s_2)	(s_2, s_3)	(s_6, s_1)	(s_4, s_6)

6.1. Illustrative Example. In this subsection, we introduce a practical case concerning the selection of SSD productions to show the implementation process of the proposed MAGDM method.

Example 26. A company plans to purchase a batch of SSDs to upgrade their computer systems. Currently, there are four kinds of SSD productions $A = \{A_1, A_2, A_3, A_4\}$ available to be chosen. Three decision-makers $M = \{M_1, M_2, M_3\}$ are invited to assess these SSD productions considering four attributes, which are the basic requirements B_1, physical characteristics B_2, brand B_3, and overall performance B_4. Naturally, the former two attributes are benefit-type, while the latter two attributes are cost-type. Therefore, these four attributes are split into two groups, $G_1 = \{B_1, B_2\}$ and $G_2 = \{B_3, B_4\}$. The weight vector of the attributes is $\omega = (0.25, 0.25, 0.20, 0.30)^T$ and the weight vector of the decision-makers is $\kappa = (0.40, 0.25, 0.35)^T$. Based on the continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0, 8]\}$, the decision-makers utilize the LPFNs to evaluate each SSD production A_i with respect to each attribute B_j and then construct the preference information matrices $L^l = (\alpha_{ij}^l)_{4\times 4}$ $(l = 1, 2, 3)$ that are presented in Tables 1–3.

(1) Rank all the SSD productions using the LPFWIPBM operator

Step 1. Here, there exist cost-type attributes $G_2 = \{B_3, B_4\}$. Hence, (61) is used to normalize the linguistic Pythagorean fuzzy preference information matrices shown in Tables 1–3. The normalized ones are listed in Tables 4–6.

Step 2. Let $p = 1, q = 1$; we utilize the LPFWIPBM operator in (62) to compute the fused value of all the attribute values of each alternative A_i which are given by the decision-maker M_l as

$$\begin{aligned}
z_1^1 &= (s_{1.1383}, s_{4.3667}), \\
z_2^1 &= (s_{2.3993}, s_{1.6564}), \\
z_3^1 &= (s_{3.9857}, s_{1.8209}), \\
z_4^1 &= (s_{2.3005}, s_{3.7101}) \\
z_1^2 &= (s_{0.9214}, s_{4.0755}), \\
z_2^2 &= (s_{3.2940}, s_{1.2182}), \\
z_3^2 &= (s_{3.5949}, s_{1.8665}), \\
z_4^2 &= (s_{2.5429}, s_{3.0728}) \\
z_1^3 &= (s_{1.0700}, s_{4.2451}), \\
z_2^3 &= (s_{2.6405}, s_{1.2113}), \\
z_3^3 &= (s_{3.7648}, s_{1.8823}), \\
z_4^3 &= (s_{2.4596}, s_{2.5731})
\end{aligned} \tag{68}$$

Step 3. Use the LPFWIPBM operator in (64) to compute the fused value of the fused attribute value of each alternative A_i as

$$z_1 = (s_{0.6175}, s_{2.6090}),$$

Table 4: The normalized linguistic Pythagorean fuzzy preference information matrix Z^1 provided by the decision-maker M_1.

	B_1	B_2	B_3	B_4
A_1	(s_1, s_6)	(s_3, s_7)	(s_2, s_6)	(s_2, s_7)
A_2	(s_4, s_3)	(s_5, s_4)	(s_3, s_1)	(s_5, s_1)
A_3	(s_7, s_3)	(s_7, s_1)	(s_6, s_2)	(s_6, s_1)
A_4	(s_3, s_7)	(s_4, s_3)	(s_6, s_2)	(s_4, s_6)

Table 5: The normalized linguistic Pythagorean fuzzy preference information matrix Z^2 provided by the decision-maker M_2.

	B_1	B_2	B_3	B_4
A_1	(s_1, s_7)	(s_2, s_6)	(s_2, s_5)	(s_2, s_7)
A_2	(s_4, s_3)	(s_7, s_1)	(s_6, s_2)	(s_5, s_1)
A_3	(s_7, s_3)	(s_6, s_1)	(s_6, s_2)	(s_5, s_2)
A_4	(s_7, s_1)	(s_3, s_5)	(s_3, s_2)	(s_2, s_7)

Table 6: The normalized linguistic Pythagorean fuzzy preference information matrix Z^3 provided by the decision-maker M_3.

	B_1	B_2	B_3	B_4
A_1	(s_2, s_7)	(s_1, s_6)	(s_3, s_7)	(s_2, s_6)
A_2	(s_6, s_1)	(s_4, s_1)	(s_3, s_2)	(s_5, s_3)
A_3	(s_6, s_2)	(s_7, s_2)	(s_7, s_1)	(s_5, s_4)
A_4	(s_5, s_2)	(s_2, s_3)	(s_1, s_6)	(s_6, s_4)

$$\begin{aligned} z_2 &= (s_{1.6129}, s_{0.8502}), \\ z_3 &= (s_{2.2977}, s_{1.1792}), \\ z_4 &= (s_{1.4199}, s_{1.9703}) \end{aligned} \tag{69}$$

Step 4. Use (66) to compute the score value of all the fused value z_i as

$$\begin{aligned} D(z_1) &= s_{5.3654}, \\ D(z_2) &= s_{5.7393}, \\ D(z_3) &= s_{5.8262}, \\ D(z_4) &= s_{5.5738} \end{aligned} \tag{70}$$

Step 5. According to their score values, these SSD productions can be ranked as

$$A_3 \succ A_2 \succ A_4 \succ A_1 \tag{71}$$

Thus, the optimal one is A_3.

Step 6. End.

(2) Rank all the SSD productions using the LPFWIPGBM operator

Step 1. Here, there exist cost-type attributes $G_2 = \{B_3, B_4\}$. Hence, (61) is utilized to normalize the linguistic Pythagorean fuzzy preference information matrices shown in Tables 1–3. The normalized ones are listed in Tables 4–6.

Step 2. Let $p = 1, q = 1$; we utilize the LPFWIPGBM operator in (63) to compute the fused value of all the attribute values of each alternative A_i which are given by the decision-maker M_l as

$$\begin{aligned} z_1^1 &= (s_{1.9546}, s_{4.0674}), \\ z_2^1 &= (s_{2.5722}, s_{1.3727}), \\ z_3^1 &= (s_{4.2678}, s_{0.9934}), \\ z_4^1 &= (s_{2.9957}, s_{3.1754}) \\ z_1^2 &= (s_{1.3856}, s_{3.9419}), \\ z_2^2 &= (s_{3.3852}, s_{0.9351}), \\ z_3^2 &= (s_{3.8904}, s_{1.1276}), \\ z_4^2 &= (s_{2.7264}, s_{2.9112}) \\ z_1^3 &= (s_{1.8911}, s_{3.9484}), \\ z_2^3 &= (s_{2.7107}, s_{1.0449}), \\ z_3^3 &= (s_{3.9817}, s_{1.3648}), \\ z_4^3 &= (s_{2.8544}, s_{2.1267}) \end{aligned} \tag{72}$$

Step 3. Utilize the LPFWIPGBM operator in (65) to compute the fused value of the fused attribute value of each alternative A_i as

$$z_1 = (s_{1.1601}, s_{2.4170}),$$

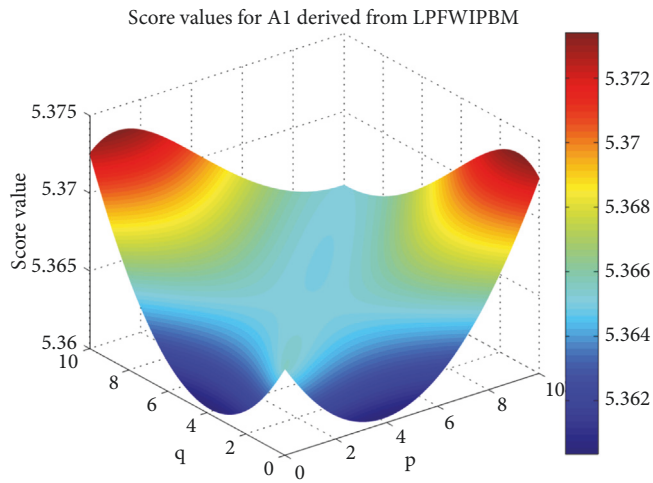

Figure 2: Score values of alternative A1 derived from LPFWIPBM when $p, q \in (0, 10)$.

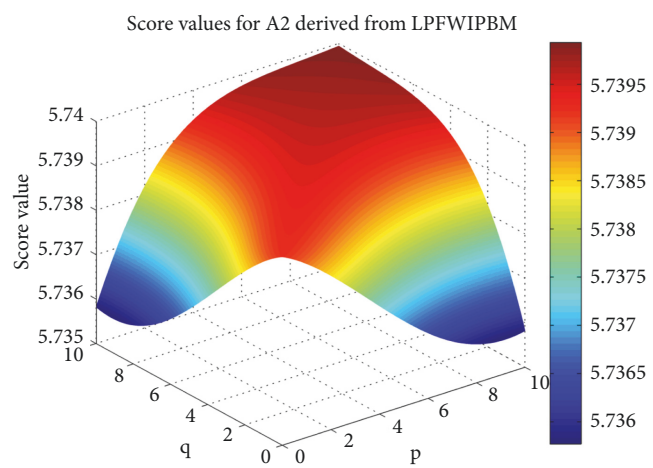

Figure 3: Score values of alternative A2 derived from LPFWIPBM when $p, q \in (0, 10)$.

$$z_2 = (s_{1.6939}, s_{0.6746}),$$
$$z_3 = (s_{2.4921}, s_{0.6776}),$$
$$z_4 = (s_{1.7840}, s_{1.6478}) \tag{73}$$

Step 4. Use (66) to compute the score value of the fused value z_i as

$$D(z_1) = s_{5.4545},$$
$$D(z_2) = s_{5.7626},$$
$$D(z_3) = s_{5.9056},$$
$$D(z_4) = s_{5.6775} \tag{74}$$

Step 5. According to their score values, these SSD productions can be ranked as

$$A_3 \succ A_2 \succ A_4 \succ A_1 \tag{75}$$

Thus, the optimal one is A_3.

Step 6. End.

From the above implementation process, it can be seen that both the LPFWIPBM and LPFWIPGBM operators get the same ranking result.

6.2. The Impact of the Parameters on the Decision Results. Figures 2–9 present the ranking results derived from the LPFWIPBM and LPFWIPGBM operators when the values of the parameters p, q change between 0 and 10.

As depicted in Figures 2–9, it can be seen that the ranking results derived from the LPFWIPBM and LPFWIPGBM operators remain unchanged as the values of the parameters p, q change and the optimal alternative is always A_3.

6.3. The Validation of Effectiveness of Our Studies. For verifying the effectiveness of our proposed LPFWIPBM and LPFWIPGBM operators, we replay the existing LPFWA and LPFWG operators, which are the only ones that are put forward to fuse the LPFNs to date, to deal with the linguistic Pythagorean fuzzy information in Example 26. For fair

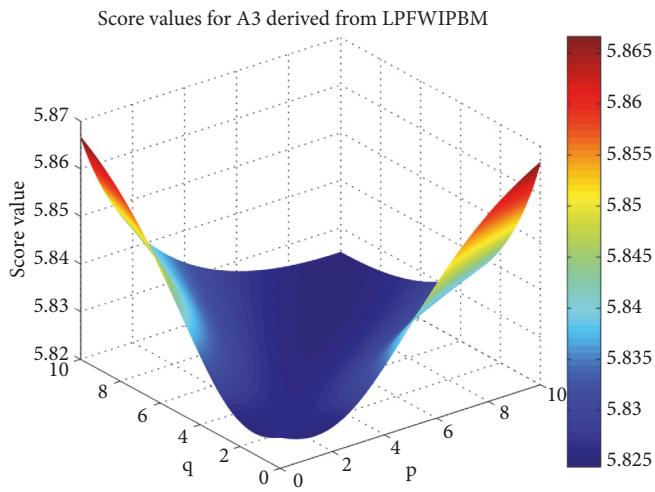

Figure 4: Score values of alternative A3 derived from LPFWIPBM when $p, q \in (0, 10)$.

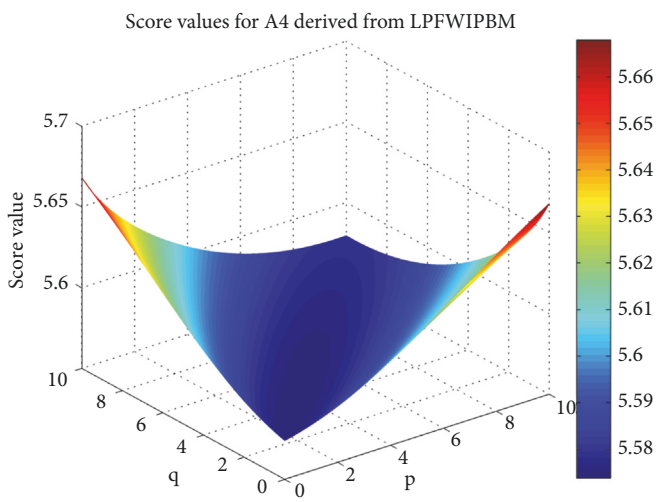

Figure 5: Score values of alternative A4 derived from LPFWIPBM when $p, q \in (0, 10)$.

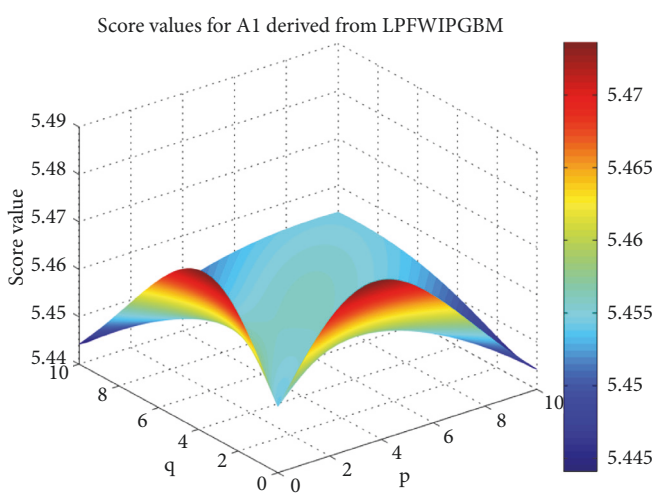

Figure 6: Score values of alternative A1 derived from LPFWIPGBM when $p, q \in (0, 10)$.

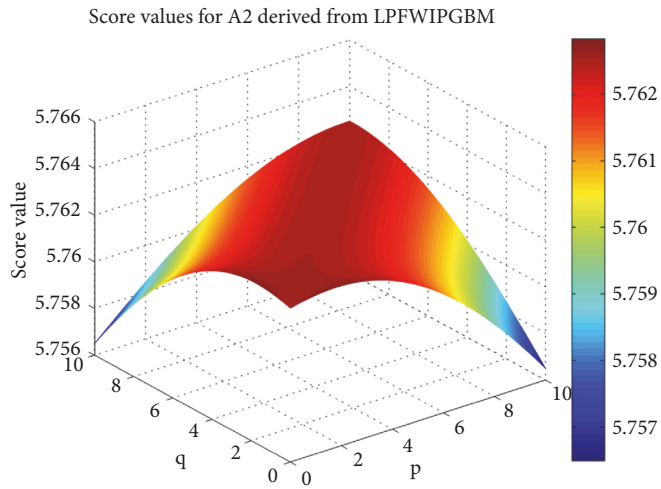

Figure 7: Score values of alternative A2 derived from LPFWIPGBM when $p, q \in (0, 10)$.

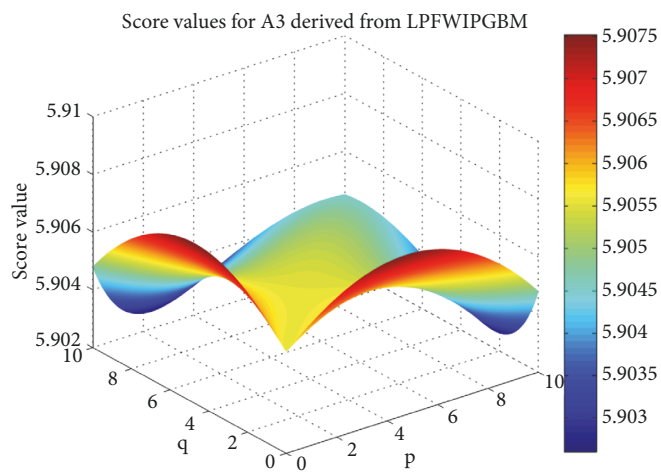

Figure 8: Score values of alternative A3 derived from LPFWIPGBM when $p, q \in (0, 10)$.

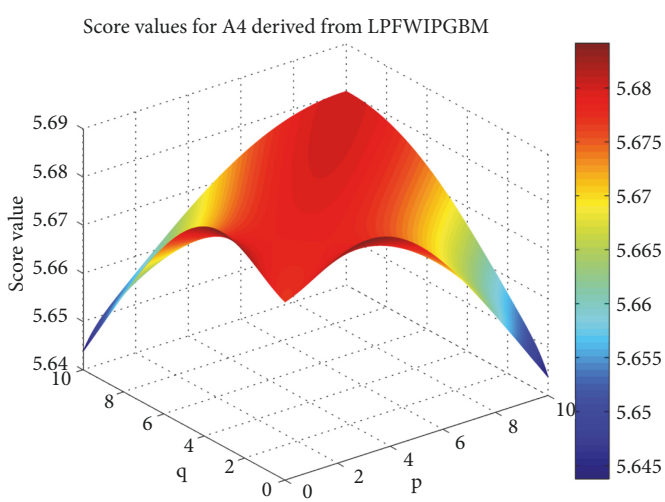

Figure 9: Score values of alternative A4 derived from LPFWIPGBM when $p, q \in (0, 10)$.

Table 7: The ranking results of our proposed operators and the operators in [19].

Methods	Score values	Ranking results
The LPFWA operator in [19]	$D(z_1) = s_{3.6553}, D(z_2) = s_{6.5676}$, $D(z_3) = s_{7.1377}, D(z_4) = s_{5.9919}$	$X_3 \succ X_2 \succ X_4 \succ X_1$
The LPFWG operator in [19]	$D(z_1) = s_{3.4561}, D(z_2) = s_{6.3172}$, $D(z_3) = s_{6.9701}, D(z_4) = s_{5.0140}$	$X_3 \succ X_2 \succ X_4 \succ X_1$
Our proposed LPFWIPBM operator	$D(z_1) = s_{5.3654}, D(z_2) = s_{5.7393}$, $D(z_3) = s_{5.8262}, D(z_4) = s_{5.5738}$	$X_3 \succ X_2 \succ X_4 \succ X_1$
Our proposed LPFWIPGBM operator	$D(z_1) = s_{5.4545}, D(z_2) = s_{5.7626}$, $D(z_3) = s_{5.9056}, D(z_4) = s_{5.6775}$	$X_3 \succ X_2 \succ X_4 \succ X_1$

Table 8: The linguistic Pythagorean fuzzy preference information matrix L provided by the decision-maker M.

	B_1	B_2	B_3	B_4
A_1	(s_4, s_3)	(s_6, s_4)	(s_0, s_3)	(s_1, s_7)
A_2	(s_6, s_2)	(s_1, s_0)	(s_2, s_5)	(s_3, s_7)
A_3	(s_5, s_2)	(s_4, s_2)	(s_1, s_6)	(s_0, s_5)
A_4	(s_4, s_1)	(s_5, s_3)	(s_0, s_2)	(s_5, s_6)

Table 9: The normalized linguistic Pythagorean fuzzy preference information matrix Z provided by the decision-maker M.

	B_1	B_2	B_3	B_4
A_1	(s_4, s_3)	(s_6, s_4)	(s_3, s_0)	(s_7, s_1)
A_2	(s_6, s_2)	(s_1, s_0)	(s_5, s_2)	(s_7, s_3)
A_3	(s_5, s_2)	(s_4, s_2)	(s_6, s_1)	(s_5, s_0)
A_4	(s_4, s_1)	(s_5, s_3)	(s_2, s_0)	(s_6, s_5)

comparison, the parameters p and q, the weight vectors of the attributes, and the decision-makers are set as $p = 1$ and $q = 1$, $\omega = (0.25, 0.25, 0.20, 0.30)^T$, and $\kappa = (0.40, 0.25, 0.35)^T$. Then the ranking results are listed in Table 7.

As shown in Table 7, the ranking results derived from the LPFWA and LPFWG operators in [19] are the same as those derived from our proposed LPFWIPBM and LPFWIPGBM operators. This comparison result can verify the effectiveness of our proposed LPFWIPBM and LPFWIPGBM operators.

6.4. The Validation of Advantages of Our Studies. In this subsection, we further validate the advantages of our proposed LPFWIPBM and LPFWIPGBM operators by introducing an extra example.

Example 27. Suppose that an investment corporation plans to expand the investment scale. There are four investment solutions $A = \{A_1, A_2, A_3, A_4\}$ available for selection, which are the real estate project, the biological medicine project, the children education project, and the IT project. Regarding these investment projects, four factors should be considered: the national policy B_1, the size of market B_2, the investment risk B_3, and the competitive power B_4. The weight vector of these factors is $\omega = (0.25, 0.25, 0.25, 0.25)^T$. The former two factors are benefit-type attributes, while the latter two ones are cost-type attributes. Hence, these four factors are divided into two groups, $G_1 = \{B_1, B_2\}$ and $G_2 = \{B_3, B_4\}$. Based on the continuous LTS $S = \{s_\varepsilon \mid \varepsilon \in [0, 8]\}$, the decision-maker utilizes the LPFNs to evaluate each project A_i with respect to each attribute B_j and then construct the preference information matrix $L = (\alpha_{ij})_{4\times4}$, which are presented in Table 8.

In Example 27, there exist two cost-type attributes $G_2 = \{B_3, B_4\}$. Hence, (61) is used to normalize the linguistic Pythagorean fuzzy preference information matrix shown in Table 8. The normalized one is listed in Table 9.

In the real applications, it usually happens that the MD or NMD of one of the aggregated LPFNs is s_0. To verify the superiority of the proposed interactional operational rules over the existing operational rules that are put forward by Garg [19] when the MD or NMD of one of the fused LPFNs is s_0, the LPFWA operator in [19] with novel interactional operational rules and the LPFWG operator in [19] with novel interactional operational rules are developed for comparison with the LPFWA operator in [19] and the LPFWG operator in [19]. To validate the superiority of the PBM operator over the existing averaging and geometric operators, our proposed LPFWIPBM and LPFWIPGBM operators are compared to the LPFWA operator in [19] with novel interactional operational rules and the LPFWG operator in [19] with novel interactional operational rules. As shown in Table 10, the LPFWA operator in [19] with novel operational rules and the LPFWA operator in [19] get different ranking results, and the LPFWG operator in [19] with novel operational rules and the LPFWG operator in [19] also obtain different ranking results. That is because the LPFWA and LPFWG operators in [19] utilize the operational rules, which do not consider the interactions

Table 10: The ranking results of our proposed operators and the operators in [19].

Method	Score value	Ranking results
The LPFWA operator in [19] with novel operational rules	$D(A_1) = s_{6.6182}$, $D(A_2) = s_{6.6330}$, $D(A_3) = s_{6.6355}$, $D(A_4) = s_{5.7542}$	$A_3 \succ A_2 \succ A_1 \succ A_4$
The LPFWG operator in [19] with novel operational rules	$D(A_1) = s_{6.6984}$, $D(A_2) = s_{6.9134}$, $D(A_3) = s_{6.6243}$, $D(A_4) = s_{6.5580}$	$A_2 \succ A_1 \succ A_3 \succ A_4$
The LPFWA operator in [19]	$D(A_1) = s_{6.9119}$, $D(A_2) = s_{6.9328}$, $D(A_3) = s_{6.7152}$, $D(A_4) = s_{6.5532}$	$A_2 \succ A_1 \succ A_3 \succ A_4$
The LPFWG operator in [19]	$D(A_1) = s_{6.3075}$, $D(A_2) = s_{6.0891}$, $D(A_3) = s_{6.5658}$, $D(A_4) = s_{5.9043}$	$A_3 \succ A_1 \succ A_2 \succ A_4$
Our proposed LPFWIPBM operator	$D(A_1) = s_{5.9453}$, $D(A_2) = s_{5.9648}$, $D(A_3) = s_{5.9623}$, $D(A_4) = s_{5.6419}$	$A_2 \succ A_3 \succ A_1 \succ A_4$
Our proposed LPFWIPGBM operator	$D(A_1) = s_{6.0780}$, $D(A_2) = s_{6.1478}$, $D(A_3) = s_{5.9807}$, $D(A_4) = s_{6.0086}$	$A_2 \succ A_1 \succ A_4 \succ A_3$

Table 11: The differences between our studies and existing studies for LPFNs.

Methods	Whether considering the interrelationships between input values	Whether considering the interactions between the MD and NMD	Whether considering the partition of input values
The LPFWA operator in [19]	No	No	No
The LPFWG operator in [19]	No	No	No
Our proposed LPFWIPBM operator	Yes	Yes	Yes
Our proposed LPFWIPGBM operator	Yes	Yes	Yes

between the MD and NMD from different LPFNs. Hence, they may result in unreasonable decision results, which has been discussed in Section 3. However, the LPFWA operator in [19] with novel interactional operational rules and the LPFWG operator in [19] with novel interactional operational rules utilize our proposed novel interactional operational rules, which can overcome the weakness of the operational rules in [19].

Although our proposed LPFWIPBM and LPFWIPGBM operators obtain the same optimal solution as the LPFWG operator in [19] with novel operational rules, their ranking results of the alternatives A_1, A_3, A_4 are different. Moreover, our proposed LPFWIPBM and LPFWIPGBM operators show the different ranking results from the LPFWA operator in [19] with novel interactional operational rules. The reason is that our proposed LPFWIPBM and LPFWIPGBM operators are capable of capturing the interrelationship that the attribute values in the same set are related and the ones in the different sets are not related, which exists in Example 27. However, the LPFWA operator in [19] with novel operational rules and the LPFWG operator in [19] with novel operational rules do not consider the interrelationship existing in Example 27.

The comparison results verify the superiority of our proposed operational rules and the LPFWIPBM and LPFWIPGBM operators.

As analyzed above, the differences between our studies and existing study reported for LPFNs could be summarized and presented in Table 11.

As presented in Table 11, it can be seen that our proposed LPFWIPBM operator and LPFWIPGBM operator adopt the novel interactional operational rules, which could consider the interactions between the MD and NMD from different LPFNs. Their PBM function can also capture the interrelationships between input values in the MAGDM problems that the input values in the same set are related and the ones in the different sets are not related. In a word, our proposed LPFWIPBM and LPFWIPGBM operators could deal with these complex MAGDM problems.

7. Conclusions

In this paper, we firstly devise some novel operational rules to overcome the drawbacks of the existing operational rules for LPFNs. Then, the novel operational rules and PBM/PGBM operators are combined to propose the LPFIPBM, LPFWIPBM, LPFIPGBM, and LPFWIPGBM operators for fusing LPFNs. At the same time, the properties of these operators are studied and some special cases are discussed. Based on the LPFWIPBM and LPFWIPGBM operators, a novel MAGDM method is developed to process the MAGDM problems with LPFNs. Finally, a practical example with respect to the selection of SSD (solid state drive) productions is presented to demonstrate the implementation process of the proposed MAGDM method. The effect of the parameters p and q on the aggregated results is also discussed and the comparison analysis between the proposed LPFWIPBM and LPFWIPGBM operators and existing study reported for LPFNs is conducted to verify the effectiveness and superiority of the proposed interactional operational rules as well as the LPFWIPBM and LPFWIPGBM operators.

In the future research, we plan to put forward some efficient operators to fuse the LPFNs or preference relations

[41, 42] in the MAGDM problems with heterogeneous relationship among attributes.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grants nos. 61872086 and 61502102, Natural Science Foundation of Fujian Province under Grant no. 2016J05149, and Distinguished Young Scientific Research Talents Plan in Universities of Fujian Province (2017).

References

[1] Q. Feng and X. Guo, "A novel approach to fuzzy soft set-based group decision-making," *Complexity*, vol. 2018, Article ID 2501489, p. 12, 2018.

[2] M. W. Lin, Z. S. Xu, Y. L. Zhai, and Z. Q. Yao, "Multi-attribute group decision-making under probabilistic uncertain linguistic environment," *Journal of the Operational Research Society*, vol. 69, no. 2, pp. 157–170, 2018.

[3] L. Zhang and F. Meng, "An approach to interval-valued hesitant fuzzy multiattribute group decision making based on the generalized Shapley-Choquet integral," *Complexity*, vol. 2018, Article ID 3941847, 19 pages, 2018.

[4] F. Meng, J. Tang, and C. Li, "Uncertain linguistic hesitant fuzzy sets and their application in multi-attribute decision making," *International Journal of Intelligent Systems*, vol. 33, no. 3, pp. 586–614, 2018.

[5] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.

[6] W. Zhou and Z. Xu, "Extended Intuitionistic Fuzzy Sets Based on the Hesitant Fuzzy Membership and their Application in Decision Making with Risk Preference," *International Journal of Intelligent Systems*, vol. 33, no. 2, pp. 417–443, 2018.

[7] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[8] P. Liu and S.-M. Chen, "Group Decision Making Based on Heronian Aggregation Operators of Intuitionistic Fuzzy Numbers," *IEEE Transactions on Cybernetics*, vol. 47, no. 9, pp. 2514–2530, 2017.

[9] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making," *International Journal of Intelligent Systems*, vol. 28, no. 5, pp. 436–452, 2013.

[10] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2014.

[11] P. Liu and J. Liu, "Some q-Rung Orthopai Fuzzy Bonferroni Mean Operators and Their Application to Multi-Attribute Group Decision Making," *International Journal of Intelligent Systems*, vol. 33, no. 2, pp. 315–347, 2018.

[12] P. Liu and P. Wang, "Multiple-Attribute Decision Making based on Archimedean Bonferroni Operators of q-Rung Orthopair Fuzzy Numbers," *IEEE Transactions on Fuzzy Systems*, 2018.

[13] H. Wang, Z. Xu, and X.-J. Zeng, "Hesitant fuzzy linguistic term sets for linguistic decision making: Current developments, issues and challenges," *Information Fusion*, vol. 43, pp. 1–12, 2018.

[14] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning I," *Information Sciences*, vol. 8, pp. 199–249, 1975.

[15] Z. Xu and H. Wang, "On the syntax and semantics of virtual linguistic terms for information fusion in decision making," *Information Fusion*, vol. 34, pp. 43–48, 2017.

[16] I. B. Turksen, "Type 2 representation and reasoning for CWW," *Fuzzy Sets and Systems*, vol. 127, no. 1, pp. 17–36, 2002.

[17] F. Herrera and L. Martínez, "A 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 6, pp. 746–752, 2000.

[18] R. M. Rodriguez, L. Martinez, and F. Herrera, "Hesitant fuzzy linguistic term sets for decision making," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 1, pp. 109–119, 2012.

[19] H. Garg, "Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process," *International Journal of Intelligent Systems*, 2018.

[20] H. M. Zhang, "Linguistic intuitionistic fuzzy sets and application in MAGDM," *Journal of Applied Mathematics*, vol. 2014, Article ID 432092, 11 pages, 2014.

[21] R. Sahin and G. D. Kucuk, "Group decision making with simplified neutrosophic ordered weighted distance operator," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 2, pp. 4795–4809, 2018.

[22] G. W. Wei, M. Lu, X. Y. Tang, and Y. Wei, "Pythagorean hesitant fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *International Journal of Intelligent Systems*, vol. 33, no. 6, pp. 1197–1233, 2018.

[23] Z. S. Xu, "Choquet integrals of weighted intuitionistic fuzzy information," *Information Sciences*, vol. 180, no. 5, pp. 726–736, 2010.

[24] X. Zhang, P. Liu, and Y. Wang, "Multiple attribute group decision making methods based on intuitionistic fuzzy frank power aggregation operators," *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, vol. 29, no. 5, pp. 2235–2246, 2015.

[25] F. Wang, S. Zeng, and C. Zhang, "A method based on intuitionistic fuzzy dependent aggregation operators for supplier selection," *Mathematical Problems in Engineering*, Article ID 481202, 2013.

[26] B. Zhu, Z. Xu, and M. Xia, "Hesitant fuzzy geometric Bonferroni means," *Information Sciences*, vol. 205, pp. 72–85, 2012.

[27] Y. He, Z. He, and H. Chen, "Extensions of atanassov's intuitionistic fuzzy interaction bonferroni means and their application to multiple-attribute decision making," *IEEE Transactions on Fuzzy Systems*, vol. 24, no. 3, pp. 558–573, 2016.

[28] P. Liu, S.-M. Chen, and J. Liu, "Multiple attribute group decision making based on intuitionistic fuzzy interaction partitioned Bonferroni mean operators," *Information Sciences*, vol. 411, pp. 98–121, 2017.

[29] D. Liang, A. P. Darko, and Z. Xu, "Interval-valued Pythagorean fuzzy extended Bonferroni mean for dealing with heterogenous

relationship among attributes," *International Journal of Intelligent Systems*, 2018.

[30] B. Dutta and D. Guha, "Partitioned Bonferroni mean based on linguistic 2-tuple for dealing with multi-attribute group decision making," *Applied Soft Computing*, vol. 37, pp. 166–179, 2015.

[31] F. Teng, Z. Liu, and P. Liu, "Some power Maclaurin symmetric mean aggregation operators based on Pythagorean fuzzy linguistic numbers and their application to group decision making," *International Journal of Intelligent Systems*, 2018.

[32] P. Liu, J. Liu, and J. M. Merigó, "Partitioned Heronian means based on linguistic intuitionistic fuzzy numbers for dealing with multi-attribute group decision making," *Applied Soft Computing*, vol. 62, pp. 395–422, 2018.

[33] Z. S. Xu, "A method based on linguistic aggregation operators for group decision making with linguistic preference relations," *Information Sciences*, vol. 166, no. 1–4, pp. 19–30, 2004.

[34] G. Beliakov, A. Pradera, and T. Calvo, *Aggregation functions: A guide for practitioners*, Springer, Berlin, Germany, 2007.

[35] C. Bonferroni, "Sulle medie multiple di potenze," *Bollettino della Unione Matematica Italiana*, vol. 5, pp. 267–270, 1950.

[36] D. Liang, Y. Zhang, Z. Xu, and A. P. Darko, "Pythagorean fuzzy Bonferroni mean aggregation operator and its accelerative calculating algorithm with the multithreading," *International Journal of Intelligent Systems*, vol. 33, no. 3, pp. 615–633, 2018.

[37] R. Zhang, J. Wang, X. Zhu, M. Xia, and M. Yu, "Some Generalized Pythagorean Fuzzy Bonferroni Mean Aggregation Operators with Their Application to Multiattribute Group Decision-Making," *Complexity*, vol. 2017, 16 pages, 2017.

[38] P. Liu and X. Liu, "Multiattribute Group Decision Making Methods Based on Linguistic Intuitionistic Fuzzy Power Bonferroni Mean Operators," *Complexity*, vol. 2017, Article ID 3571459, 15 pages, 2017.

[39] H. Gao, G. Wei, and Y. Huang, "Dual Hesitant Bipolar Fuzzy Hamacher Prioritized Aggregation Operators in Multiple Attribute Decision Making," *IEEE Access*, vol. 6, pp. 11508–11522, 2017.

[40] C. Fan, Y. Song, Q. Fu, L. Lei, and X. Wang, "New Operators for Aggregating Intuitionistic Fuzzy Information With Their Application in Decision Making," *IEEE Access*, vol. 6, pp. 27214–27238, 2018.

[41] F. Y. Meng, J. Tang, Q. X. An, and X. H. Chen, "Decision making with intuitionistic linguistic preference relations," in *International Transactions in Operational Research*, 2017.

[42] F. Meng, C. Tan, and X. Chen, "Multiplicative consistency analysis for interval fuzzy preference relations: A comparative study," *OMEGA - The International Journal of Management Science*, vol. 68, pp. 17–38, 2017.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
The Scientific World Journal
Journal of Probability and Statistics
International Journal of Mathematics and Mathematical Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of Optimization
International Journal of Engineering Mathematics
International Journal of Analysis
Journal of Complex Analysis
Advances in Numerical Analysis
Mathematical Problems in Engineering
International Journal of Differential Equations
Discrete Dynamics in Nature and Society
International Journal of Stochastic Analysis
Journal of Mathematics
Journal of Function Spaces
Abstract and Applied Analysis
Advances in Mathematical Physics