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Research Article

Extended Heronian Mean Based on Hesitant Fuzzy Linguistic Information for Multiple Attribute Group Decision-Making

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This paper investigates the hesitant fuzzy linguistic multiple attribute group decision-making (MAGDM) problem with the heterogeneous relationship among the attribute variables that cannot be solved by most existing decision-making methods. To address this problem, a new operator is introduced based on partitioning attribute variables into different sets according to their interrelationship. This operator is called the extended Heronian mean (EHM) operator. To obtain each expert's comprehensive values of the alternatives in the hesitant fuzzy linguistic MAGDM problem, we investigate the EHM operator under a hesitant fuzzy linguistic environment and propose the hesitant fuzzy linguistic EHM operator and the hesitant fuzzy linguistic linear support degree weighted EHM operator. In addition, the axiom definition of a linguistic type similarity measure of hesitant fuzzy linguistic term sets is proposed. The weight of the experts can be determined based on this type similarity measure. Finally, a practical case is presented to demonstrate the steps of our method, and a comparison analysis illustrates its feasibility and effectiveness.

1. Introduction

In such a complicated and changeable economic and social environment, it is difficult for a single expert to fully understand and master the decision-making problem. Therefore, it is necessary that experts from different disciplines take part in the decision-making process, which evolves the process into multiple attribute group decision-making (MAGDM). MAGDM has been applied in many fields, such as academic assessments of higher education institutions [1], financial risk evaluation [2, 3], investment objective selection [4], and green supply chain management [5]. The decision-maker may use qualitative values instead of quantitative values to express his/her assessment information due to characteristics of decision-making objects, information integrality cannot be guaranteed, and human knowledge is limited. For example, when evaluating the degree of economic activity in one location, people may use linguistic terms such as "low," "medium," and "high."

A reasonable selection of the means of information representation is essential to solving linguistic MAGDM problems. In early research, fuzzy linguistic term sets (FLTS) were commonly used to represent decision-making information and were described and analyzed by Zadeh [6]. Based on FLTS, many other types of information representation models have been proposed in recent decades, such as the 2-tuple fuzzy linguistic representation model [7], the linguistic intuitionistic fuzzy set [8, 9], the uncertain linguistic fuzzy soft set [10], the interval neutrosophic uncertain linguistic model [11], the probabilistic linguistic term set [12], and picture 2-tuple linguistic set [13]. However, when experts vacillate among several linguistic terms, it is quite difficult for them to model such situations by the abovementioned representation models. Therefore, Rodriguez et al. [14] presented hesitant fuzzy linguistic term set (HFLTSs) which can reflect the hesitant psychology feature of the decision-maker. Hesitant fuzzy linguistic decision-making has received much attention by many scholars and has produced a large amount of research results [15–17].

Similarity measure is a very important research topic in the field of linguistic decision-making. It has been widely used in many fields, such as semantic translation, cluster analysis, and machine learning. For example, Chen et al. [18] introduced a new similarity measure formula for interval linguistic terms based on the likelihood of the comparison between two intervals. For the trapezoid fuzzy linguistic variables, Xu [19] put forward the similarity measure calculation method. Based on Xu's work, Liao et al. [20] further proposed the axioms of similarity measures for HFLTSs and then used them to sort alternatives in multiple attribute decision-making (MADM). To enrich the hesitant fuzzy linguistic similarity measure calculation, Hesamian et al. [21] proposed Gower-Legendre and Tversky similarity measures, Liao et al. [22] introduced cosine similarity measures, Gou et al. [23] developed some cross-entropy measures, Farhadinia et al. [24] defined some entropy measures, and Song et al. [25] proposed vector similarity measures.

Information aggregation is also an important component of linguistic decision-making. An operator is a primary tool for linguistic information aggregation, which is widely accepted and used in practical decision-making. In earlier research, linguistic aggregation operators were made based on the assumption that the input arguments are independent of each other, such as the linguistic geometric averaging operator [26], the linguistic hybrid geometric operator [27], and the uncertain linguistic ordered weighted averaging operator [28]. However, in recent years, scholars have gradually realized the significance of considering the associated relationship in information aggregation [29, 30]. The Heronian mean (HM) operator and Bonferroni mean (BM) operator can capture the interrelationship of the aggregation variables through the crossover operation of aggregation variables. Wei et al. and Tian et al. investigated a weighted BM operator under uncertain linguistic fuzzy environment [31] and simplified neutrosophic linguistic environment [32], respectively. Nie et al. proposed a Pythagorean fuzzy partitioned normalized weighted Bonferroni mean [33]. Liu et al. [34] analyzed the structure principle of the BM operator and HM operator and then pointed out that the HM operator can avoid the disadvantage of redundant information and make the information fusion more efficient. Li et al. [35] developed some 2-tuple linguistic HM aggregation operators and studied their properties. Liu et al. [36] proposed some new HM operators, based on the HM operator and the geometric HM operator, to solve intuitionistic uncertain linguistic MAGDM problems. Yu et al. [37] defined some of the reducible weighted linguistic hesitant fuzzy HM operators and then discussed their special cases and desirable properties.

Analyzing the existing research literature on linguistic similarity measures and linguistic aggregation operators shows that the following problems can be found:

(1) Most of the existing studies on the similarity measures of linguistic fuzzy sets use a crisp number to measure the degree of similarity. Whether this numerical value measurement is appropriate for linguistic input arguments is worth discussing because it conflicts with the motivation for using linguistic fuzzy set in decision-making that is close to the cognitive and understanding structure of people [38]. For example, consider a medical and health institution that wants to develop an online medical diagnosis system (OMDS) to provide the results of identified illnesses in an easily understandable way. The OMDS may suggest "similar," "moderately similar," and "high degree of similarity" as description forms. Thus, it is necessary to carry out research on the linguistic similarity measures of HFLTS to enrich and develop the theories and means of hesitant fuzzy linguistic decision-making.

(2) Although existing HM operators consider the interrelationship of the argument variables, they are built on the assumption that a relationship exists between any two input arguments. However, in some real-world problems, this assumption is not always true. For example, consider an international company selecting a suitable department manager according to the following attributes: C_1: Educational background; C_2: Work experience; C_3: Professional knowledge; C_4: Working skill; C_5: Age; and C_6: Health. Based on their interrelationship, the attributes can be divided into two classes: $S_1 = \{C_1, C_2, C_3, C_4\}$ and $S_2 = \{C_5, C_6\}$. It is easy to see that the elements in the same set are connected, but there is no relationship between the elements from different sets. Therefore, if we use the existing HM operators, they would generate disturbing information in the process of aggregation which would affect the reliability of decision-making.

In accordance with the above analysis, this paper begins by providing the axiom definition of linguistic similarity measures for HFLTS and then proposes a new operator under the inspiration of previous studies. This new operator is called extended Heronian mean (EHM). Next, we investigate the EHM operator in a hesitant fuzzy environment and propose some operators to infuse HFLTSs. Based on these operators and linguistic similarity measures, a new method for solving the hesitant fuzzy linguistic MAGDM problem is designed. Weights that are completely unknown or represented by a crisp numbers (linguistic terms) situation are taken into consideration in the decision-making process. This makes the new method more practical. The framework of this article is as follows:

Section 2 reviews some concepts of HFLTSs and proposes linguistic similarity measures for HFLTS and EHM operators. Section 3 introduces the HFLEHM and HFLLSDWEHM operators and investigates their properties. Section 4 designs a group decision-making method under hesitant fuzzy linguistic environments. Section 5 provides a practical example to illustrate the method and discuss the effect of parameters on decision-making results. Section 6 illustrates the feasibility and effectiveness of the proposed operators and method through comparison and analysis. Lastly, Section 7 summarizes this study.

2. Preliminaries

2.1. Hesitant Fuzzy Linguistic Term Set. To build a linguistic decision model, a set of evaluation scales can be built as $S = \{s_\alpha \mid \alpha = 0, 1, \ldots, t\}$, where s_α denotes a linguistic measurement level of evaluation objectives, and t is an even number which always takes the values of 4, 6, and 8 [26]. This requires the following properties to be satisfied: (1) $s_\alpha < s_\beta$ *iff* $\alpha < \beta$ and (2) $N(s_\alpha) = s_{t-\alpha}$. To make the linguistic decision model have an obvious logicality and system, Xu extended the above discrete evaluation scales set to a continuous set $\overline{S} = \{s_\alpha \mid \alpha \in [0,q],\ q > t\}$ and defined the following operational laws [39]:

Definition 1. Let $s_\alpha, s_\beta \in \overline{S}, \lambda_1, \lambda_2, \lambda > 0$; then, the following are given.

(1) $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta}$; (2) $\lambda s_\alpha = s_{\lambda\alpha}$

(3) $(\lambda_1 + \lambda_2)s_\alpha = \lambda_1 s_\alpha \oplus \lambda_2 s_\alpha$; (4) $\lambda(s_\alpha \oplus s_\beta) = \lambda s_\alpha \oplus \lambda s_\beta$

With the decision environment becoming more and more complex and uncertain, this kind of linguistic term sets is difficult to satisfy with the request of decision modeling. Because of this, Rodriguez et al. [14] proposed the HFLTS, and then Liao et al. [20] further put forward its mathematical form.

Definition 2 (see [20]). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, and let $S = \{s_\alpha \mid \alpha = 0, 1, \ldots, t\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS) on X is characterized by the following:

$$H_S = \{\langle x_i, h_S(x_i)\rangle \mid x_i \in X,\ i = 1, 2, \ldots, n\} \tag{1}$$

where $h_S(x_i) = \{s_{\eta_l}(x_i) \mid s_{\eta_l}(x_i) \in S,\ l = 1, 2, \ldots, L\}$ is a subset of S, and L denotes the number of linguistic terms in $h_S(x_i)$. For convenience, we call $h_S(x_i)$ a hesitant fuzzy linguistic element (HFLE), denoted by h_S.

Definition 3 (see [20]). Let $S = \{s_\alpha \mid \alpha = 0, 1, \ldots, t\}$ be a linguistic term set, and let $h_S = \{s_{\eta_l} \mid s_{\eta_l} \in S,\ l = 1, 2, \ldots, L\}$ be a HFLE. $\mu(h_S) = (1/L)\sum_{i=1}^{L} \eta_i$ is called the score of h_S, and $v(h_S) = (1/L)\sqrt{\sum_{i\neq k}^{L}(\eta_i - \eta_k)^2}$ is called the deviation degree of h_S. Then, the comparison law between two HFLEs ($h_S^1 = \{s_{\eta_l^1} \mid s_{\eta_l^1} \in S,\ l = 1, 2, \ldots, L\}$ and $h_S^2 = \{s_{\eta_l^2} \mid s_{\eta_l^2} \in S,\ l = 1, 2, \ldots, L\}$) can be defined as follows:

(a) If $\mu(h_S^1) > \mu(h_S^2)$, then $h_S^1 > h_S^2$

(b) If $\mu(h_S^1) = \mu(h_S^2)$, then

(i) if $v(h_S^1) = v(h_S^2)$, then $h_S^1 = h_S^2$

(ii) if $v(h_S^1) > v(h_S^2)$, then $h_S^1 < h_S^2$

There are also other comparison rules of HFLEs in the literature. For example, Liu et al. proposed a comparison rule between HFLEs based on a linguistic scale function [40]. Wei et al. introduced a novel comparison rule between HFLEs, which takes into account the average linguistic term and the hesitant degree [41]. Motivated by the possibility degree of intervals, Lee et al. introduced a likelihood based comparison rule between HFLEs [42]. In consideration of both the hesitant degrees and the unbalanced linguistic terms in evaluations, Liao et al. introduced a new comparison rule between HFLEs based on the psychological characteristics of experts [43]. On the basis of pairwise comparison of each linguistic term in the two HELEs, Huang et al. offered a comparison rule between HELEs [44]. For details, please refer to the survey papers [45]. However, the comparison rule proposed by Liao et al. [20], as shown in Definition 3, is simple and convenient and is thus widely used in decision-making.

2.2. A New Similarity Measure for HFLTSs. The similarity measure is an important tool for studying and applying linguistic fuzzy sets. Liao et al. proposed the numeric axiom definition of similarity measures of HFLTSs [22]. Based on this, we propose a linguistic axiom definition as follows:

Definition 4. Let $S = \{s_\alpha \mid \alpha = 0, 1, \ldots, t\}$ and $S^* = \{s_\alpha^* \mid \alpha = 0, 1, \ldots, t\}$ be two linguistic term sets, and H_S^1 and H_S^2 are two HFLTSs on S; then the linguistic similarity measure between H_S^1 and H_S^2 is defined as $\rho_l(H_S^1, H_S^2)$, which satisfies the following:

(1) $s_0^* \leq \rho_l(H_S^1, H_S^2) \leq s_t^*$; (2) $\rho_l(H_S^1, H_S^2) = s_t^*$ iff $H_s^1 = H_s^2$; (3) $\rho_l(H_S^1, H_S^2) = \rho_l(H_S^2, H_S^1)$.

Remark 5. s_α^* denotes a linguistic similarity measure level and s_t^* and s_0^* represent the maximum and minimum linguistic similarity measure levels, respectively. Moreover, S^* must have the same number of elements as S.

Definition 6. Let $S = \{s_\alpha \mid \alpha = 0, 1, \ldots, t\}$ and $S^* = \{s_\alpha^* \mid \alpha = 0, 1, \ldots, t\}$ be two linguistic term sets, $H_S^1 = \{h_S^{11}, h_S^{12}, \ldots h_S^{1k}\}$ and $H_S^2 = \{h_S^{21}, h_S^{22}, \ldots h_S^{2k}\}$ are two HFLTSs on S, $h_S^{1i} = \{s_{\alpha_1^{1i}}, s_{\alpha_2^{1i}}, \ldots, s_{\alpha_l^{1i}}\}$, and $h_S^{2i} = \{s_{\beta_1^{2i}}, s_{\beta_2^{2i}}, \ldots, s_{\beta_l^{2i}}\}$, $i = 0, 1, \ldots, k$. Then $\rho_l(H_S^1, H_S^2) = S^*_{(1/lk)\sum_{i=1}^{k}\sum_{j=1}^{l}(t-|\alpha_j^{1i}-\beta_j^{2i}|)}$ is called the linguistic similarity measure between H_S^1 and H_S^2.

Remark 7. If $k = 1$, then the linguistic similarity measure between two HFLEs, h_S^{11} and h_S^{21}, is defined as $\rho_l(h_S^{11}, h_S^{21}) = S^*_{(1/l)\sum_{j=1}^{l}(t-|\alpha_j^{11}-\beta_j^{21}|)}$.

Example 8. Let $h_S^1 = \{s_2, s_3, s_5\}$, $h_S^2 = \{s_2, s_4, s_6\}$, and $h_S^3 = \{s_3, s_4, s_6\}$ be three HFLEs on $S = \{s_\alpha \mid \alpha = 0, 1, \ldots, 6\}$. S^* can be built as $\{s_0^*$: extremely far, s_1^* : far, s_2^* : slightly far, s_3^* : medium, s_4^* : slightly close, s_5^* : close, and s_6^* : extremely close$\}$. According to the definition of linguistic similarity measure of HFLTSs, we obtain the following:

$$\begin{aligned} \rho_l\left(h_S^1, h_S^2\right) &= S^*_{(1/3)\{(6-|2-2|)+(6-|3-4|)+(6-|5-6|)\}} = S^*_{16/3}, \\ \rho_l\left(h_S^1, h_S^3\right) &= S^*_{(1/3)\{(6-|2-3|)+(6-|3-4|)+(6-|5-6|)\}} = S^*_5. \end{aligned} \tag{2}$$

This means that the similarity degree between h_S^1 and h_S^2 is higher than close, and the similarity degree between h_S^1 and h_S^3 is close. Therefore, $\rho_l(h_S^1, h_S^3) < \rho_l(h_S^1, h_S^2)$.

Utilizing the similarity measure defined by Hesamian et al. [21], we can then obtain the following:

$$\begin{aligned}\rho_2^1\left(h_S^1, h_S^2\right) &= 1 - \frac{\left|h_S^1 \triangle h_S^2\right|}{7} = 1 - \frac{4}{7} = \frac{3}{7},\\ \rho_2^1\left(h_S^1, h_S^3\right) &= 1 - \frac{\left|h_S^1 \triangle h_S^3\right|}{7} = 1 - \frac{4}{7} = \frac{3}{7},\end{aligned} \tag{3}$$

which means that $\rho_2^1(h_S^1, h_S^2) = \rho_2^1(h_S^1, h_S^3)$.

Utilizing the similarity measure defined by Liao et al. based on the Hamming distance [20], we can then obtain the following:

$$\begin{aligned}S\left(h_S^1, h_S^2\right) &= 1 - \left(\frac{1}{3}\left(\frac{|2-2|}{7} + \frac{|3-4|}{7} + \frac{|5-6|}{7}\right)\right)\\ &= \frac{19}{21},\\ S\left(h_S^1, h_S^3\right) &= 1 - \left(\frac{1}{3}\left(\frac{|2-3|}{7} + \frac{|3-4|}{7} + \frac{|5-6|}{7}\right)\right)\\ &= \frac{18}{21},\end{aligned} \tag{4}$$

which means that $\rho_l(h_S^1, h_S^3) < \rho_l(h_S^1, h_S^2)$.

From the above calculation results, we can see that the method proposed in this paper and the method given by Liao et al. have a high degree of discrimination. However, the representation of similarity degree of the two methods is quite different. One is a quantitative representation and the other is a qualitative representation. Users should reasonably choose according to the application environment.

2.3. Extended Heronian Mean Operator

Definition 9 (see [34]). Let $p \geq 0$, $q \geq 0$, $p + q \neq 0$ and a_i $(i = 1, 2, \ldots, n)$ be nonnegative real numbers. Then

$$\mathrm{HM}^{p,q}\left(a_1, a_2, \ldots, a_n\right) = \left(\frac{2}{n(n+1)} \sum_{i=1, j=i}^{n} a_i^p a_j^q\right)^{1/(p+q)} \tag{5}$$

is called Heronian mean (HM).

The HM operator is based on the assumption that a correlation exists between any two input arguments. However, in some real-world applications the above assumption does not hold because a partial correlation may exist between input arguments. Hence, an EHM is introduced to provide more precise aggregate information.

Definition 10. Let $p \geq 0$, $q \geq 0$, $p + q \neq 0$ and a_i $(i = 1, 2, \ldots, n)$ be nonnegative real numbers. Then, the extended Heronian mean (EHM) operator is defined as follows:

$$\begin{aligned}&\mathrm{EHM}^{p,q}\left(a_1, a_2, \ldots, a_n\right)\\ &\quad = \left(\frac{1}{\sum_{i=1}^{n} I\left(A_i\right)} \sum_{i=1}^{n} a_i^p \sum_{a_{i_j} \in A_i} a_{i_j}^q\right)^{1/(p+q)}\end{aligned} \tag{6}$$

where set A_i consists of some elements of $\{a_i, a_{i+1}, \ldots, a_n\}$ which have a correlation with a_i, and $I(A_i)$ represents the cardinality of A_i.

Remark 11. If $I(A_i) = 1$ $(i = 1, 2, \ldots, n)$, that is to say, all input arguments are independent, then the EHM operator reduces to the following:

$$\begin{aligned}&\mathrm{EHM}^{p,q}\left(a_1, a_2, \ldots, a_n\right)\\ &\quad = \left(\frac{1}{\sum_{i=1}^{n} I\left(A_i\right)} \sum_{i=1}^{n} a_i^p \sum_{a_{i_j} \in A_i} a_{i_j}^q\right)^{1/(p+q)}\\ &\quad = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^p a_i^q\right)^{1/(p+q)} = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^r\right)^{1/r}, \quad r = p + q\end{aligned} \tag{7}$$

which is called generalized arithmetic averaging (GAA) operator.

Remark 12. If $I(A_i) = n - i + 1$ $(i = 1, 2, \ldots, n)$, that is to say, all input arguments are related to each other, then EHM operator reduces to the HM operator.

$$\begin{aligned}&\mathrm{EHM}^{p,q}\left(a_1, a_2, \ldots, a_n\right)\\ &\quad = \left(\frac{1}{\sum_{i=1}^{n} I\left(A_i\right)} \sum_{i=1}^{n} a_i^p \sum_{a_{i_j} \in A_i} a_{i_j}^q\right)^{1/(p+q)}\\ &\quad = \left(\frac{2}{n(n+1)} \sum_{i=1, j=i}^{n} a_i^p a_j^q\right)^{1/(p+q)}\end{aligned} \tag{8}$$

Obviously, the EHM operator has the following properties:

(1) If $a_i = a$ $(i = 1, 2, \ldots, n)$, then $\mathrm{EHM}^{p,q}(a, a, \ldots, a) = a$

(2) If $a_i \geq d_i$ $(i = 1, 2, \ldots, n)$, then $\mathrm{EHM}^{p,q}(a_1, a_2, \ldots, a_n) \leq \mathrm{EHM}^{p,q}(d_1, d_2, \ldots, d_n)$

Remark 13. From properties (1) and (2), we can get

$$\min_{1 \leq i \leq n}\left\{a_i\right\} \leq \mathrm{EHM}^{p,q}\left(a_1, a_2, \ldots, a_n\right) \leq \max_{1 \leq i \leq n}\left\{a_i\right\}. \tag{9}$$

In the following section, we further extend the EHM operator to the hesitant fuzzy linguistic environment and discuss its relevant properties.

3. Hesitant Fuzzy Linguistic Extended Heronian Means

3.1. HFLEHM Operator

Definition 14. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}}, \ldots, s_{\alpha_m^{(i)}}\}$ $(i = 1, 2, \ldots, n)$ be a collection of HFLEs, for any $p, q \geq 0$,

$$\begin{aligned}&\mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n)\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \cdot \bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\}\end{aligned} \tag{10}$$

We call $\mathrm{HFLEHM}^{p,q}$ a hesitant fuzzy linguistic extended Heronian mean (HFLEHM) operator, where set B_k^i consists of some elements of $\{\alpha_k^{(i)}, \alpha_k^{(i+1)}, \ldots, \alpha_k^{(n)}\}$ which have correlation with $\alpha_k^{(i)}$, and $I(B_k^i)$ represents the cardinality of B_k^i.

Remark 15. If $I(B_k^i) = 1$ $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m)$, that is to say, all h_i $(i = 1, 2, \ldots, n)$ are independent, then HFLEHM operator reduces to

$$\begin{aligned}&\mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n)\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \cdot \bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\}\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{1}{n}\bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p \otimes \left(s_{\alpha_k^{(i)}}\right)^q\right)^{1/(p+q)}\right\} = \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{1}{n} \cdot \bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^{p+q}\right)^{1/(p+q)}\right\}\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{1}{n}\bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^{r}\right)^{1/r}\right\},\\&\qquad r = p + q,\end{aligned} \tag{11}$$

which is called hesitant fuzzy linguistic generalized arithmetic averaging (HFLGAA) operator.

Remark 16. If $I(B_k^i) = n - i + 1$ $(i = 1, 2, \ldots, n)$, that is to say, all h_i $(i = 1, 2, \ldots, n)$ are related to each other, then HFLEHM operator reduces to HFLHM operator.

$$\begin{aligned}&\mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n)\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \cdot \bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\}\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{j=i}^{n}\left(s_{\alpha_k^{(j)}}\right)^q\right)^{1/(p+q)}\right\} = \mathrm{HFLHM}^{p,q}(h_1, h_2, \ldots,\\&\quad h_n).\end{aligned} \tag{12}$$

Remark 17. If $I(B_k^i) = n - i + 1$ $(i = 1, 2, \ldots, n)$ and $q \longrightarrow 0$, then HFLEHM operator reduces to

$$\begin{aligned}&\lim_{q\longrightarrow 0}\mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n)\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\lim_{q\longrightarrow 0}\left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)}\bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\}\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\lim_{q\longrightarrow 0}\left(\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{j=1}^{n}\left(s_{\alpha_k^{(j)}}\right)^q\right)^{1/(p+q)}\right\}\\&= \bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots, s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\frac{2\bigoplus_{i=1}^{n}(n-i+1)\left(s_{\alpha_k^{(i)}}\right)^p}{nn(n+1)}\right)^{1/p}\right\}\\&= \bigcup_{1\le k\le m}\left\{\left(\bigoplus_{i=1}^{n}\frac{n-i+1}{n(n+1)/2}\left(s_{\alpha_k^{(i)}}\right)^p\right)^{1/p}\right\}\\&= \bigcup_{1\le k\le m}\left\{\left(\bigoplus_{i=1}^{n} w_i\left(s_{\alpha_k^{(i)}}\right)^p\right)^{1/p}\right\}\end{aligned} \tag{13}$$

which is called hesitant fuzzy linguistic generalized linear decrease weighted averaging (HFLGLDWA) operator, where $w_i = (n-i+1)/(n(n+1)/2)$ $(i = 1, 2, \ldots, n)$ meet the following conditions: (1) $w_i \geq 0$; (2) $\sum w_i = 1$; (3) $w_1 \geq w_2 \geq \cdots \geq w_n$.

Remark 18. If $I(B_k^i) = n - i + 1$ $(i = 1, 2, \ldots, n)$ and $p \longrightarrow 0$, then HFLEHM operator reduces to

$$
\begin{aligned}
&\lim_{p\longrightarrow 0}\mathrm{HFLEHM}^{p,q}\left(h_1,h_2,\ldots,h_n\right)\\
&=\bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1,s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}}\left\{\lim_{p\longrightarrow 0}\left(\frac{1}{\sum_{i=1}^{n}I\left(B_k^i\right)}\bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p\otimes\bigoplus_{\alpha_k^{(i_j)}\in h_{i_j},h_{i_j}\in B_k^i}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\}\\
&=\bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1,s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}}\left\{\lim_{p\longrightarrow 0}\left(\frac{2}{n\left(n+1\right)}\bigoplus_{i=1}^{n}\left(s_{\alpha_k^{(i)}}\right)^p\otimes\bigoplus_{j=i}^{n}\left(s_{\alpha_k^{(j)}}\right)^q\right)^{1/(p+q)}\right\}\\
&=\bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1,s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}}\left\{\left(\frac{2\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(s_{\alpha_k^{(j)}}\right)^q}{n\left(n+1\right)}\right)^{1/q}\right\}\\
&=\bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1,s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}}\left\{\left(\bigoplus_{j=1}^{n}\frac{j}{n\left(n+1\right)/2}\left(s_{\alpha_k^{(j)}}\right)^q\right)^{1/q}\right\}\\
&=\bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1,s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}}\left\{\left(\bigoplus_{j=1}^{n}w_j\left(s_{\alpha_k^{(j)}}\right)^q\right)^{1/q}\right\}
\end{aligned}
\tag{14}
$$

which is called hesitant fuzzy linguistic generalized linear increase weighted averaging (HFLGLIWA) operator, where $w_j = j/(n(n+1)/2)$ $(j = 1,2,\ldots,n)$ meet the following conditions: (1) $w_j \ge 0$; (2) $\sum w_j = 1$; (3) $w_1 \le w_2 \le \cdots \le w_n$.

Property 19. Let h_i $(i = 1,2,\ldots,n)$ be a collection of HFLEs; if $h_1 = h_2 = \cdots = h_n = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_m}\} = h$, then

$$\mathrm{HFLEHM}^{p,q}\left(h_1,h_2,\ldots,h_n\right)=h. \tag{15}$$

Property 20. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}}, \ldots, s_{\alpha_m^{(i)}}\}$ and $h_i' = \{s_{\beta_1^{(i)}}, s_{\beta_2^{(i)}}, \ldots, s_{\beta_m^{(i)}}\}$ $(i = 1,2,\ldots,n)$ be hesitant fuzzy linguistic evaluation values of same attributes, and $s_{\alpha_k^{(i)}} \le s_{\beta_k^{(i)}}$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots,m$; then $\mathrm{HFLEHM}^{p,q}(h_1,h_2,\ldots,h_n) \le \mathrm{HFLEHM}^{p,q}(h_1',h_2',\ldots,h_n')$.

Property 21. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}}, \ldots, s_{\alpha_m^{(i)}}\}$ $(i = 1,2,\ldots,n)$ be a collection of HFLEs, $\alpha_k^+ = \max_i\{\alpha_k^{(i)}\}$, $\alpha_k^- = \min_i\{\alpha_k^{(i)}\}$, $h^+ = \{\alpha_1^+, \alpha_2^+, \ldots, \alpha_m^+\}$, and $h^- = \{\alpha_1^-, \alpha_2^-, \ldots, \alpha_m^-\}$; then $h^- \le \mathrm{HFLEHM}^{p,q}(h_1,h_2,\ldots,h_n) \le h^+$.

3.2. HFLWEHM Operator. It should be noted that the HFLEHM operator considers the aggregate variables to have the same importance. In real decision-making it is difficult to satisfy this condition. To reflect the importance of the aggregate variables in the aggregation operator, two weighted forms of hesitant fuzzy linguistic EHM operators are introduced in this subsection.

Definition 22. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}}, \ldots, s_{\alpha_m^{(i)}}\}$ $(i = 1,2,\ldots,n)$ be a collection of HFLEs, with the weight w_i $(i = 1,2,\ldots,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. For any $p, q \ge 0$,

$$
\begin{aligned}
&\mathrm{HFLWEHM}^{p,q}\left(h_1,h_2,\ldots,h_n\right)\\
&=\bigcup_{\substack{1\le k\le m\\ s_{\alpha_k^{(1)}}\in h_1,s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}}\left\{\left(\bigoplus_{i=1}^{n}\lambda_i\left(s_{\alpha_k^{(i)}}\right)^p\right.\right.\\
&\left.\left.\otimes\bigoplus_{\alpha_k^{(i_j)}\in h_{i_j},h_{i_j}\in B_k^i}\lambda_{i_j}'\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\}
\end{aligned}
\tag{16}
$$

which is called the hesitant fuzzy linguistic weighted extended Heronian mean (HFLEHM) operator, where $\lambda_i = w_i I(B_k^i)/\sum_{i=1}^{n} w_i I(B_k^i)$ and $\lambda_{i_j}' = w_{i_j}/\sum w_{i_j}$.

It is worth noting that if the importance of the aggregate variables is given by the form of HFLEs, for example, $\widehat{S} = \{\widehat{s}_\alpha \mid \alpha = 0,1,\ldots,6\}$ = {very unimportant, unimportant, slightly unimportant, medium, slightly important, important, very important}, $w_i = \widehat{h}_i$ $(i = 1,2,\dots,n)$ being a collection of HFLEs on $\widehat{S}$, then $w_i = \mu(\widehat{h}_i)/\sum_{j=1}^{n}\mu(\widehat{h}_j)$, $\lambda_i = w_i I(B_k^i)/\sum_{i=1}^{n} w_i I(B_k^i)$, and $\lambda'_{i_j} = w_{i_j}/\sum w_{i_j}$.

Remark 23. If $I(B_k^i) = 1$ $(i = 1,2,\dots,n;\ k = 1,2,\dots,m)$, that is to say, all h_i $(i = 1,2,\dots,n)$ are independent, then HFLWEHM operator reduces to

$$\begin{aligned} &\mathrm{HFLWEHM}^{p,q}(h_1,h_2,\dots,h_n) \\ &= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\dots,s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\bigoplus_{i=1}^{n}\lambda_i\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i}\lambda'_{i_j}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\} \\ &= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\dots,s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\bigoplus_{i=1}^{n} w_i\left(s_{\alpha_k^{(i)}}\right)^{p+q}\right)^{1/(p+q)}\right\} \\ &= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\dots,s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\bigoplus_{i=1}^{n} w_i\left(s_{\alpha_k^{(i)}}\right)^{r}\right)^{1/r}\right\}, \\ &\qquad r = p+q \end{aligned} \tag{17}$$

which is called hesitant fuzzy linguistic generalized weighted arithmetic averaging (HFLGWAA) operator.

Remark 24. If $q = 0$ $(i = 1,2,\dots,n)$, then

$$\begin{aligned} &\mathrm{HFLWEHM}^{p,q}(h_1,h_2,\dots,h_n) \\ &= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\dots,s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\bigoplus_{i=1}^{n}\lambda_i\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i}\lambda'_{i_j}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\} \\ &= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\dots,s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\bigoplus_{i=1}^{n}\lambda_i\left(s_{\alpha_k^{(i)}}\right)^{p}\right)^{1/p}\right\}. \end{aligned} \tag{18}$$

Property 25. If $w_i = (1/n)$ $(i = 1,2,\dots,n)$, then

$$\begin{aligned} &\mathrm{HFLWEHM}^{p,q}(h_1,h_2,\dots,h_n) \\ &\quad = \mathrm{HFLEHM}^{p,q}(h_1,h_2,\dots,h_n). \end{aligned} \tag{19}$$

Property 26. Let h_i $(i = 1,2,\dots,n)$ be a collection of HFLEs with the weight w_i $(i = 1,2,\dots,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. If $h_1 = h_2 = \cdots = h_n = \{s_{\alpha_1}, s_{\alpha_2},\dots,s_{\alpha_m}\} = h$, then

$$\mathrm{HFLWEHM}^{p,q}(h_1,h_2,\dots,h_n) = h. \tag{20}$$

The proofs of Properties 27 and 28 are similar to those of Properties 20 and 21; therefore we omit the details.

Property 27. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}},\dots,s_{\alpha_m^{(i)}}\}$ and $h'_i = \{s_{\beta_1^{(i)}}, s_{\beta_2^{(i)}},\dots,s_{\beta_m^{(i)}}\}$ $(i = 1,2,\dots,n)$ be two collections of HFLEs, with the same weight w_i $(i = 1,2,\dots,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. If $s_{\alpha_k^{(i)}} \le s_{\beta_k^{(i)}}$, $i = 1,2,\dots,n$, $k = 1,2,\dots,m$, then $\mathrm{HFLWEHM}^{p,q}(h_1,h_2,\dots,h_n) \le \mathrm{HFLWEHM}^{p,q}(h'_1,h'_2,\dots,h'_n)$.

Property 28. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}},\dots,s_{\alpha_m^{(i)}}\}$ $(i = 1,2,\dots,n)$ be a collection of HFLEs with the weight $w_i (i = 1,2,\dots,n)$, $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. $\alpha_k^+ = \max_i\{\alpha_k^{(i)}\}$, $\alpha_k^- = \min_i\{\alpha_k^{(i)}\}$, $\alpha^+ = \{\alpha_1^+,\alpha_2^+,\dots,\alpha_m^+\}$, and $\alpha^- = \{\alpha_1^-,\alpha_2^-,\dots,\alpha_m^-\}$; then $\alpha^- \le \mathrm{HFLWEHM}^{p,q}(\alpha_1,\alpha_2,\dots,\alpha_n) \le \alpha^+$.

Due to the complexity of the decision-making system, people may encounter a situation wherein the importance of the aggregate variables is unknown. Inspired by the work of Xu and Yager [46], we introduce a new operator to deal with this kind of problem in decision-making.

Definition 29. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}},\dots,s_{\alpha_m^{(i)}}\}$ $(i = 1,2,\dots,n)$ be a collection of HFLEs. For any $p,q > 0$,

$$\begin{aligned} &\mathrm{HFLLSDWEHM}^{p,q}(h_1,h_2,\dots,h_n) \\ &= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\dots,s_{\alpha_k^{(n)}}\in h_n}} \left\{\left(\bigoplus_{i=1}^{n} w_i\left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i} w'_{i_j}\left(s_{\alpha_k^{(i_j)}}\right)^q\right)^{1/(p+q)}\right\} \end{aligned} \tag{21}$$

which is called the hesitant fuzzy linguistic linear support degree weighted extended Heronian mean (HFLLSDWEHM) operator, where $w_i = g(B_i/TV) - g(B_{i-1}/TV)$ is the support degree of h_i in $\{h_1, h_2, \ldots, h_n\}$, $T(h_i) = \sum_{j=1, j\neq i}^{n} \sup(h_i, h_j)$, $TV = \sum_{i=1}^{n}(1 + T(h_i))$, $B_i = \sum_{j=1}^{i}(1 + T(h_j))$, and $g(x)$ has the following properties. (1) $g(0) = 0$, $g(1) = 1$; (2) $\forall x, y \in [0,1]$, $g(x) \geq g(y)$, if $x > y$. Similarly, $w'_{i_j} = g(B'_i/TV') - g(B'_{i-1}/TV')$ is the support degree of h_{i_j} in B^i_k.

Property 30. Let h_i $(i = 1, 2, \ldots, n)$ be a collection of HFLEs. If $h_1 = h_2 = \cdots = h_n = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_m}\} = h$, then $\text{HFLLSDWEHM}^{p,q}(h_1, h_2, \ldots, h_n) = h$.

Property 31. Let $h_i = \{s_{\alpha_1^{(i)}}, s_{\alpha_2^{(i)}}, \ldots, s_{\alpha_m^{(i)}}\}$ $(i = 1, 2, \ldots, n)$ be a collection of HFLEs, $\alpha_k^+ = \max_i\{\alpha_k^{(i)}\}$, $\alpha_k^- = \min_i\{\alpha_k^{(i)}\}$, $\alpha^+ = \{\alpha_1^+, \alpha_2^+, \ldots, \alpha_m^+\}$, and $\alpha^- = \{\alpha_1^-, \alpha_2^-, \ldots, \alpha_m^-\}$; then

$$\alpha^- \leq \text{HFLLSDWEHM}^{p,q}\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \leq \alpha^+. \tag{22}$$

We should emphasize that, unlike the HFLWEHM operator, the HFLLSDWEHM operator does not have the monotonicity property.

4. An Approach to MAGDM with Hesitant Fuzzy Linguistic Information

The critical components of solving MAGDM problems include attribute weights, expert weights, and attribute evaluation values. In this section, we utilize the proposed operators to solve MAGDM problems in which attribute weights are known or unknown, and the evaluation values are expressed by HFLTSs.

For a MAGDM problem, assume that $D = \{d_1, d_2, \ldots, d_m\}$ are decision experts with the weight vectors $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ and $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set of n alternatives, and $G = \{g_1, g_2, \ldots, g_l\}$ is a set of attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_l)^T$. Set $G_c = \{g_{c_1}, g_{c_2}, \ldots, g_{c_d}\}$ is constructed using the elements of $\{g_c, g_{c+1}, \ldots, g_n\}$ which have a correlation with g_c and $\diamond(G_c) = \{c_1, c_1, \ldots, c_d\}$. Suppose that the decision experts $d_k \in D$ provide the hesitant fuzzy linguistic information to evaluate the characteristics of the alternatives $x_i \in X$ under attribute $g_j \in G$, denoted by a HFLE $h_{ij}^{(k)}$; then the hesitant fuzzy linguistic decision matrix $H^{(k)} = (h_{ij}^{(k)})_{n\times l}$ is constructed.

An approach to multiple attribute group decision-making problems is provided in the following steps:

Step 1 (normalize the evaluation information matrices). For benefit attributes, higher evaluation values indicate the better alternative. However, for cost attributes, smaller evaluation values indicate the better alternative. All attributes of the alternative should have the same judgment standard in decision-making [47]. Therefore, decision matrix $H^{(k)} = (h_{ij}^{(k)})_{n\times l}$ is transformed into the normalized decision matrix $\overline{H}^{(k)} = (\overline{h}_{ij}^{(k)})_{n\times l}$, where $\overline{h}_{ij}^{(k)} = h_{ij}^{(k)}$, for benefit criteria g_j; $\overline{h}_{ij}^{(k)} = N(h_{ij}^{(k)})$, for cost criteria g_j, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, l$; and $k = 1, 2, \ldots, m$.

Step 2. Calculate each expert's comprehensive values of the alternatives.

Case 1. If the weighting vectors of the attributes are known, then the HFLWEHM operator is used to calculate the d_k $(k = 1, 2, \ldots, m)$ comprehensive evaluation value $h_i^{(k)}$ of x_i $(i = 1, 2, \cdots, n)$.

$$\begin{aligned} h_i^{(k)} &= \text{HFLWEHM}^{p,q}\left(\overline{h}_{i1}^{(k)}, \overline{h}_{i2}^{(k)} \ldots, \overline{h}_{il}^{(k)}\right) \\ &= \bigcup_{\substack{1\leq t\leq r \\ s_{\alpha_t^{(kj1)}}\in\overline{h}_{i1}^{(k)}, s_{\alpha_t^{(kj2)}}\in\overline{h}_{i2}^{(k)}, \ldots, s_{\alpha_t^{(kjl)}}\in\overline{h}_{il}^{(k)}}} \left\{\left(\bigoplus_{s=1}^{l}\lambda_s\left(s_{\alpha_t^{(k_{is})}}\right)^p \otimes \bigoplus_{s_{\alpha_t^{(k_{j_s})}}\in\overline{h}_{j_s}^{(k)}, j_s\in\diamond(G_s)} \lambda'_{j_s}\left(s_{\alpha_t^{(k_{j_s})}}\right)^q\right)^{1/(p+q)}\right\} \end{aligned} \tag{23}$$

where $\lambda_s = w_s I(G_s)/\sum_{i=1}^{n} w_i I(G_s)$ and $\lambda'_{j_s} = w_{j_s}/\sum w_{j_s}$.

Case 2. If the weighting vectors of the attributes are unknown, then the HFLLSDWEHM operator is used to calculate d_k $(k = 1, 2, \ldots, m)$ comprehensive evaluation value $h_i^{(k)}$ of x_i $(i = 1, 2, \ldots, n)$.

$$\begin{aligned} h_i^{(k)} &= \text{HFLLSDWEHM}^{p,q}\left(\overline{h}_{i1}^{(k)}, \overline{h}_{i2}^{(k)} \ldots, \overline{h}_{il}^{(k)}\right) \\ &= \bigcup_{\substack{1\leq t\leq r \\ s_{\alpha_t^{(ki1)}}\in\overline{h}_{i1}^{(k)}, s_{\alpha_t^{(ki2)}}\in\overline{h}_{i2}^{(k)}, \ldots, s_{\alpha_t^{(kil)}}\in\overline{h}_{il}^{(k)}}} \left\{\left(\bigoplus_{s=1}^{l} w_s^k\left(s_{\alpha_t^{(k_{is})}}\right)^p \otimes \bigoplus_{s_{\alpha_t^{(k_{j_s})}}\in\overline{h}_{j_s}^{(k)}, j_s\in\diamond(G_s)} \widetilde{w}_{j_s}^{(k)}\left(s_{\alpha_t^{(k_{j_s})}}\right)^q\right)^{1/(p+q)}\right\} \end{aligned} \tag{24}$$

where $w_s^k = g(B_s^k/TV) - g(B_{s-1}^k/TV)$ is the support degree of $\overline{h}_s^{(k)}$ in $\{\overline{h}_s^{(k)}, \overline{h}_{s+1}^{(k)}, \ldots, \overline{h}_l^{(k)}\}$, $\overline{h}_s^{(k)}$ denotes the s column of $\overline{H}^{(k)}$, $T(\overline{h}_s^{(k)}) = \sum_{b=1,b\neq s}^{l} \sup(\overline{h}_s^{(k)}, \overline{h}_b^{(k)})$, $TV = \sum_{s=1}^{l}(1 + T(\overline{h}_s^{(k)}))$, and $B_j^k = \sum_{s=1}^{j}(1 + T(\overline{h}_s^{(k)}))$. Similarly, $\widetilde{w}_{j_s}^{(k)} = g(\widetilde{B}_{j_s}^k/TV') - g(\widetilde{B}_{j-1_s}^k/TV')$ is the support degree of $\overline{h}_{j_s}^{(k)}$ in $\{\overline{h}_{j_s}^{(k)}, \overline{h}_{j+1_s}^{(k)}, \ldots, \overline{h}_{d_s}^{(k)}\}$, and $\overline{h}_{j_s}^{(k)}$ denotes the j_s column of $\overline{H}^{(k)}$. In this paper, without loss of generality, we let $\sup(\overline{h}_s^{(k)}, \overline{h}_b^{(k)}) = \mu(\rho_l(\overline{h}_s^{(k)}, \overline{h}_b^{(k)}))/t$.

Step 3. Calculate the group comprehensive evaluation value.

Case (i). If the weighting vectors of the expert are known, then the HFLGWAA operator (in this paper we let $r = 1$) is used to calculate the group comprehensive evaluation value h_i of each alternative x_i $(i = 1, 2, \ldots, n)$.

$$h_i = \text{HFLGWAA}\left(h_i^{(1)}, h_i^{(2)} \ldots, h_i^{(m)}\right) = \bigcup_{\substack{1\le k\le r \\ s_{\alpha_k^{(1)}}\in h_i^{(1)}, s_{\alpha_k^{(2)}}\in h_i^{(2)}, \ldots, s_{\alpha_k^{(m)}}\in h_i^{(m)}}} \left\{\bigoplus_{i=1}^{m} \omega_i s_{\alpha_k^{(i)}}\right\} \quad (25)$$

Case (ii). If the weighting vectors of the expert are unknown, then we first calculate the linguistic similarity lsm_{ef} between any two experts d_e and d_f $(e, f = 1, 2, \ldots, m)$.

$$lsm_{ef} = \sum_{i=1}^{n} \rho_l\left(h_i^{(d_e)}, h_i^{(d_f)}\right) \quad (26)$$

These linguistic similarity values are summarized as follows:

$$LSM = \begin{pmatrix} lsm_{11} & lsm_{12} & \cdots & lsm_{1m} \\ lsm_{21} & lsm_{22} & \cdots & lsm_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ lsm_{m1} & lsm_{m2} & \cdots & lsm_{mm} \end{pmatrix} \quad (27)$$

Next, the average similarity of each expert d_e $(e = 1, 2, \ldots, m)$ in the group is calculated as $lsm_e = (1/(l-1))\bigoplus_{f=1,f\neq e}^{l} lsm_{ef}$, and the weight of each expert d_e is obtained as $\omega_e = \mu(lsm_e)/\sum_{e=1}^{m}\mu(lsm_e)$. Finally, the group comprehensive evaluation values can be obtained using (25).

Step 4. Rank x_i $(i = 1, 2, \ldots, n)$ in ascending order according to h_i $(i = 1, 2, \ldots, n)$ by the ranking method defined in Definition 3.

The flow chart of the proposed MAGDM method is shown in Figure 1.

5. A Numerical Example

With the improvement of people's health awareness, environmental protection, and energy conservation, an increasing number of "green" products are produced. The corresponding industrial structure also shows a trend toward green development, and this has initiated the concept of a "green supply chain." The construction of a green supply chain has become the primary challenge and trend for enterprises to provide green products and develop sustainability in a society. The crucial factor for implementing successful green supply chain management lies in the correct choice of suppliers, especially in terms of which meet enterprise requirements and have sustainable development strategies. Therefore, we consider the green supplier selection in the hesitant fuzzy linguistic environment as follows.

As we know, electric bicycles, featuring convenience, and energy conservation are widely promoted and utilized in China. It can be concluded from data released by the National Bureau of Statistics of China that the cumulative number of products in China since 2017 has exceeded 60 million. However, problems with batteries are a key bottleneck that limits the development of electric bicycles. Good and poor batteries are intermingled, so their elimination rate is high. Unfortunately, if they are not properly disposed of, the batteries also emit heavy metals, including lead and acid, causing serious damage to the environment. Therefore, it is of particular importance for electric bicycle manufacturers to locate green battery suppliers. Take, for example, that an electric bicycle enterprise has established five possible battery sources x_i $(i = 1, 2, \ldots, 5)$ and five assessment attributes: (a) g_1 environmental damage from production materials; (b) g_2 product price; (c) g_3 resource consumption; (d) g_4 service level; and (e) enterprise reputation. The linguistic term set $S = \{s_\alpha \mid \alpha = 0, 1, \ldots, 8\}$ = {extremely low, very low, low, slightly low, average, slightly high, high, very high, extremely high} is used to evaluate the five alternatives and $\widehat{S} = \{s_0$: extremely unimportant, s_1: very unimportant, s_2: unimportant, s_3: slightly unimportant, s_4: average, s_5: slightly important, s_6: important, s_7: very important, s_8: extremely important} is used to evaluate the attribute weights. To achieve an objective and logical decision, the company invited three experts d_i $(i = 1, 2, 3)$ to evaluate these five feasible alternatives. Attribute evaluation values provided by the experts are represented by HFLTSs, as presented in Tables 1–3. The weight vectors $w^{(i)}$ of attributes by the experts d_i $(i = 1, 2, 3)$ are shown in Table 4.

Table 1: Expert information matrix $H^{(1)}$ [16].

g_1	g_2	g_3	g_4	g_5
$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_3\}$	$\{s_5, s_6, s_7\}$	$\{s_7, s_8\}$
$\{s_1, s_2\}$	$\{s_3, s_4, s_5\}$	$\{s_2, s_3, s_4\}$	$\{s_6, s_7\}$	$\{s_6, s_7\}$
$\{s_1, s_2, s_3\}$	$\{s_2, s_3, s_4\}$	$\{s_1, s_2, s_3\}$	$\{s_6\}$	$\{s_6\}$
$\{s_3, s_4\}$	$\{s_1, s_2, s_3\}$	$\{s_2, s_3, s_4\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$
$\{s_3, s_4, s_5\}$	$\{s_2, s_3\}$	$\{s_3, s_4\}$	$\{s_6, s_7\}$	$\{s_5, s_6\}$

5.1. Using the Proposed Method.

Step 1 (normalize the evaluation information matrices). It is easy to see that g_1-g_3 are cost criteria and g_4, g_5 are benefit

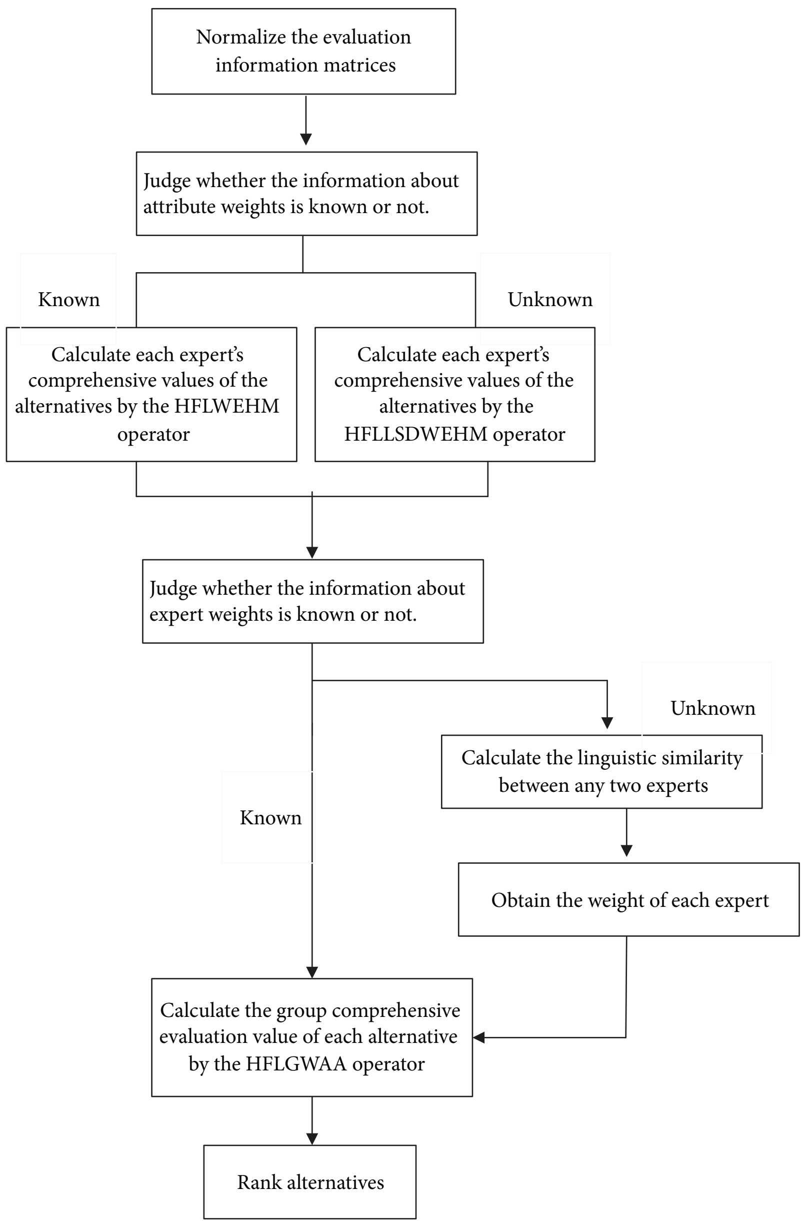

Figure 1: The flow chart of the proposed MAGDM method.

Table 2: Expert information matrix $H^{(2)}$ [15].

g_1	g_2	g_3	g_4	g_5
$\{s_0, s_1, s_2\}$	$\{s_2, s_3, s_4\}$	$\{s_2, s_3\}$	$\{s_4, s_5, s_6\}$	$\{s_6, s_7\}$
$\{s_1\}$	$\{s_3, s_4\}$	$\{s_2, s_3\}$	$\{s_5, s_6\}$	$\{s_6\}$
$\{s_1, s_2\}$	$\{s_0, s_2, s_3\}$	$\{s_1, s_3, s_4\}$	$\{s_6\}$	$\{s_6, s_7\}$
$\{s_2, s_3, s_4\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_5, s_6, s_7\}$	$\{s_4, s_5, s_6\}$
$\{s_3, s_4\}$	$\{s_1, s_2\}$	$\{s_3, s_4\}$	$\{s_5, s_6\}$	$\{s_6\}$

Table 3: Expert information matrix $H^{(3)}$ [16].

g_1	g_2	g_3	g_4	g_5
$\{s_2, s_3, s_4\}$	$\{s_2, s_3\}$	$\{s_3, s_4\}$	$\{s_5, s_6, s_7\}$	$\{s_5, s_6, s_7\}$
$\{s_1, s_3\}$	$\{s_4, s_5\}$	$\{s_2, s_3\}$	$\{s_6\}$	$\{s_5, s_6\}$
$\{s_2, s_3, s_4\}$	$\{s_3, s_4, s_5\}$	$\{s_1, s_2, s_3\}$	$\{s_5, s_6\}$	$\{s_5, s_6\}$
$\{s_3, s_4\}$	$\{s_2, s_3, s_4\}$	$\{s_1, s_2, s_3\}$	$\{s_2, s_3, s_4\}$	$\{s_4, s_5\}$
$\{s_3, s_5\}$	$\{s_5, s_6, s_7\}$	$\{s_2, s_3\}$	$\{s_6, s_7, s_8\}$	$\{s_4, s_5, s_6\}$

criteria. Therefore, the experts' information matrices need to be normalized, as shown in Tables 5–7.

Step 2 (calculate each expert's comprehensive values of the alternatives). Based on the interconnectedness of the attribute variables, we obtain $G_1 = \{g_1, g_2, g_3\}$, $G_2 = \{g_2, g_3\}$, $G_3 = \{g_3\}$, $G_4 = \{g_4, g_5\}$, and $G_5 = \{g_5\}$. Hence, by using the HFLWEHM operator ($p = q = 1$), the individual comprehensive values $h_i^{(k)}$ ($i = 1, 2, \ldots, 5$; $k = 1, 2, 3$) are obtained, as shown in Table 8.

Table 4: Attribute weights information matrix [16].

	g_1	g_2	g_3	g_4	g_5
$w^{(1)}$	$\{s_1\}$	$\{s_2, s_3\}$	$\{s_6, s_7\}$	$\{s_5, s_6\}$	$\{s_5\}$
$w^{(2)}$	$\{s_1\}$	$\{s_3\}$	$\{s_6\}$	$\{s_6, s_7\}$	$\{s_6\}$
$w^{(3)}$	$\{s_1, s_2\}$	$\{s_3, s_4\}$	$\{s_7, s_8\}$	$\{s_7\}$	$\{s_7\}$

Table 5: Normalize information matrix $\overline{H}^{(1)}$.

g_1	g_2	g_3	g_4	g_5
$\{s_6, s_7\}$	$\{s_5, s_6\}$	$\{s_5\}$	$\{s_5, s_6, s_7\}$	$\{s_7, s_8\}$
$\{s_6, s_7\}$	$\{s_3, s_4, s_5\}$	$\{s_4, s_5, s_6\}$	$\{s_6, s_7\}$	$\{s_6, s_7\}$
$\{s_5, s_6, s_7\}$	$\{s_4, s_5, s_6\}$	$\{s_5, s_6, s_7\}$	$\{s_6\}$	$\{s_6\}$
$\{s_4, s_5\}$	$\{s_5, s_6, s_7\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$
$\{s_3, s_4, s_5\}$	$\{s_5, s_6\}$	$\{s_4, s_5\}$	$\{s_6, s_7\}$	$\{s_5, s_6\}$

Table 6: Normalize information matrix $\overline{H}^{(2)}$.

g_1	g_2	g_3	g_4	g_5
$\{s_6, s_7, s_8\}$	$\{s_4, s_5, s_6\}$	$\{s_5, s_6\}$	$\{s_4, s_5, s_6\}$	$\{s_6, s_7\}$
$\{s_7\}$	$\{s_4, s_5\}$	$\{s_5, s_6\}$	$\{s_5, s_6\}$	$\{s_6\}$
$\{s_6, s_7\}$	$\{s_5, s_6, s_8\}$	$\{s_4, s_5, s_7\}$	$\{s_6\}$	$\{s_6, s_7\}$
$\{s_4, s_5, s_6\}$	$\{s_6, s_7\}$	$\{s_5, s_6\}$	$\{s_5, s_6, s_7\}$	$\{s_4, s_5, s_6\}$
$\{s_4, s_5\}$	$\{s_6, s_7\}$	$\{s_4, s_5\}$	$\{s_5, s_6\}$	$\{s_6\}$

Table 7: Normalize information matrix $\overline{H}^{(3)}$.

g_1	g_2	g_3	g_4	g_5
$\{s_4, s_5, s_6\}$	$\{s_5, s_6\}$	$\{s_4, s_5\}$	$\{s_5, s_6, s_7\}$	$\{s_5, s_6, s_7\}$
$\{s_5, s_7\}$	$\{s_3, s_4\}$	$\{s_5, s_6\}$	$\{s_6\}$	$\{s_5, s_6\}$
$\{s_4, s_5, s_6\}$	$\{s_3, s_4, s_5\}$	$\{s_5, s_6, s_7\}$	$\{s_5, s_6\}$	$\{s_5, s_6\}$
$\{s_4, s_5\}$	$\{s_4, s_5, s_6\}$	$\{s_5, s_6, s_7\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$
$\{s_3, s_5\}$	$\{s_5, s_6, s_7\}$	$\{s_5, s_6\}$	$\{s_6, s_7, s_8\}$	$\{s_4, s_5, s_6\}$

Step 3 (calculate the weight of the experts). We obtain the experts' similarity matrix *LSM* using the linguistic similarity measure formula (26).

$$LSM = \begin{pmatrix} \sim & lsm_{12} = s_{7.647} & lsm_{13} = s_{7.604} \\ lsm_{21} = s_{7.647} & \sim & lsm_{23} = s_{7.640} \\ lsm_{31} = s_{7.604} & lsm_{32} = s_{7.640} & \sim \end{pmatrix} \quad (28)$$

Next, the average similarity of each expert d_e $(e = 1, 2, 3)$ in the group is calculated as $d_1 = 7.625$, $d_2 = 7.643$, and $d_3 = 7.622$, and the weight of each expert ω_e $(e = 1, 2, 3)$ is given as $\omega_1 = 0.333$, $\omega_2 = 0.334$, and $\omega_3 = 0.333$.

Step 4 (calculate the group comprehensive evaluation value). Aggregate the individual comprehensive values $h_i^{(k)}$ $(i = 1, 2, \ldots, 5;\ k = 1, 2, 3)$ using Eq. (25) to obtain the group comprehensive values h_i $(i = 1, 2, \ldots, 5)$ of the alternatives x_i $(i = 1, 2, \ldots, 5)$, as $h_1 = \{s_{5.057}, s_{5.735}, s_{6.178}\}$; $h_2 = \{s_{5.137}, s_{5.593}, s_{5.945}\}$; $h_3 = \{s_{5.189}, s_{5.742}, s_{6.146}\}$; $h_4 = \{s_{4.421}, s_{5.251}, s_{5.819}\}$; and $h_5 = \{s_{5.004}, s_{5.579}, s_{6.04}\}$.

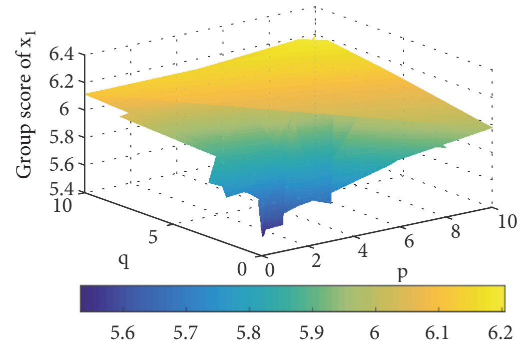

Figure 2: Group scores of the x_1.

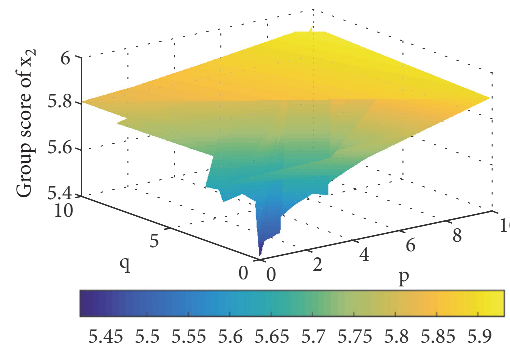

Figure 3: Group scores of the x_2.

Step 5 (rank all the alternatives). By calculating the score values of h_i $(i = 1, 2, \ldots, 5)$, we can obtain

$$\begin{aligned} \eta(h_1) &= 5.657; \\ \eta(h_2) &= 5.558; \\ \eta(h_3) &= 5.692; \\ \eta(h_4) &= 5.164; \\ \eta(h_5) &= 5.541 \end{aligned} \quad (29)$$

According to the binary relation of HFLETs described in Definition 3, the final ranking is $x_4 < x_5 < x_2 < x_1 < x_3$. Therefore, the optimal choice is x_3.

5.2. The Parametric Analysis of the HFLWEHM Operator. The above decision-making process is repeated multiple times by setting different parameter values (in Step 2) to analyze the influence of the parameters of the HFLWEHM operator on information fusion. From Figures 2–6, we can see that the variance of the score values of the group evaluation to the parameters of the HFLWEHM operator is irregular. This means that the HFLWEHM operator can bring about various decision-making possibilities. Through further analysis, we find that if $p = q = m$, the scores obtained by the HFLWEHM operator trend to increase along with the parameters, as shown in Figure 7. Therefore, the decision-maker's risk appetite can be reflected by the parameter m. If one takes an aggressive decision-making position, the decision-maker

Table 8: Individual comprehensive evaluation values H.

d_1	d_2	d_3
$h_1^{(1)} = \{s_{5.586}, s_{6.09}, s_{6.606}\}$	$h_1^{(2)} = \{s_{4.894}, s_{5.612}, s_{6.336}\}$	$h_1^{(3)} = \{s_{4.692}, s_{5.503}, s_{6.318}\}$
$h_2^{(1)} = \{s_{5.144}, s_{5.807}, s_{6.488}\}$	$h_2^{(2)} = \{s_{5.217}, s_{5.554}, s_{5.897}\}$	$h_2^{(3)} = \{s_{5.051}, s_{5.418}, s_{5.792}\}$
$h_3^{(1)} = \{s_{5.445}, s_{5.886}, s_{6.371}\}$	$h_3^{(2)} = \{s_{5.406}, s_{5.909}, s_{6.844}\}$	$h_3^{(3)} = \{s_{4.716}, s_{5.431}, s_{6.159}\}$
$h_4^{(1)} = \{s_{4.121}, s_{4.971}, s_{8.827}\}$	$h_4^{(2)} = \{s_{4.847}, s_{5.638}, s_{6.441}\}$	$h_4^{(3)} = \{s_{4.295}, s_{5.143}, s_{5.994}\}$
$h_5^{(1)} = \{s_{4.919}, s_{5.436}, s_{5.956}\}$	$h_5^{(2)} = \{s_{5.156}, s_{5.503}, s_{5.855}\}$	$h_5^{(3)} = \{s_{4.936}, s_{5.797}, s_{6.662}\}$

Table 9: Hesitant fuzzy linguistic information matrix L [4].

	g_1	g_2	g_3	g_4	g_5
x_1	$\{s_5, s_6\}$	$\{s_3, s_4\}$	$\{s_4, s_5\}$	$\{s_5, s_6, s_7\}$	$\{s_3, s_4, s_5\}$
x_2	$\{s_6, s_7\}$	$\{s_6, s_7\}$	$\{s_4, s_5, s_6\}$	$\{s_4, s_5\}$	$\{s_4, s_5, s_6\}$
x_3	$\{s_5, s_6, s_7\}$	$\{s_4, s_5, s_6\}$	$\{s_5, s_6, s_7\}$	$\{s_6\}$	$\{s_4, s_5\}$
x_4	$\{s_5, s_6\}$	$\{s_6, s_7\}$	$\{s_5, s_6\}$	$\{s_4, s_5, s_6\}$	$\{s_5, s_6, s_7\}$
w	0.2	0.3	0.2	0.2	0.1

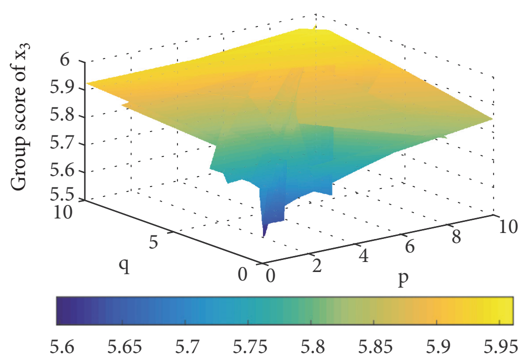

Figure 4: Group scores of the x_3.

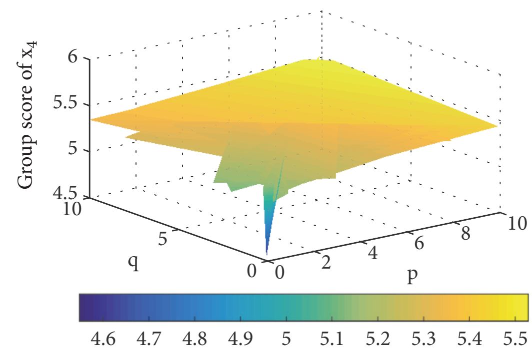

Figure 5: Group scores of the x_4.

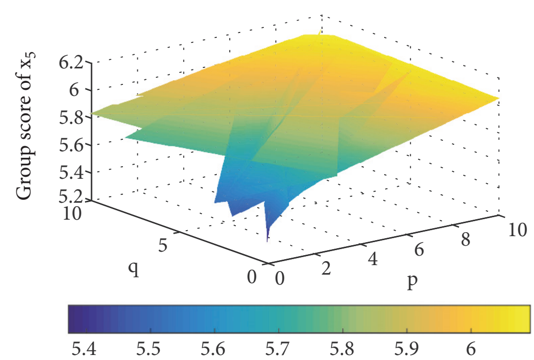

Figure 6: Group scores of the x_5.

can use the HFLWEHM operator with a larger value of m. However, if one takes defensive decision-making, the decision-maker can use the HFLWEHM operator with a smaller value of m.

6. Comparative Analysis

6.1. Comparative Analysis of Operators. In this subsection, the EHMs are compared with other classic aggregation operators, including the hesitant fuzzy linguistic weighted averaging (HFLWA) operator [48], the hesitant fuzzy linguistic weighted Heronian mean (HFLWHM) operator [49], and the hesitant fuzzy linguistic weighted Bonferroni mean (HFLWBM) operator [17]. The decision-making matrix L is used based on the linguistic term set $S = \{s_\alpha \mid \alpha = 0, 1, 2, \ldots, 8\}$ to compare the above hesitant linguistic operators, as shown in Table 9.

From Table 10 we can see that the ranking results obtained by the proposed operators are different. This is mainly because they come from different viewpoints to aggregate information. Among the five hesitant linguistic operators, only the HFLWA operator considers argument variables that are independent of each other, which is a limitation of its application to decision-making. In addition,

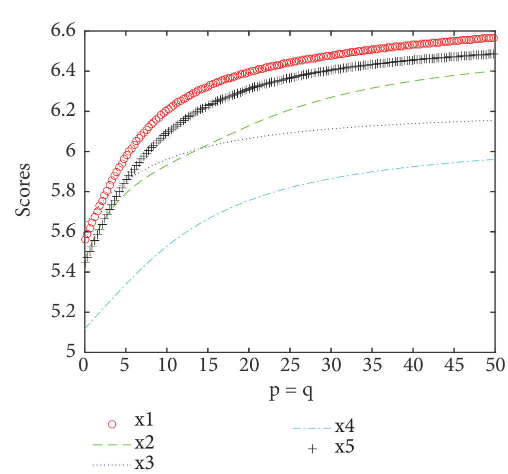

Figure 7: Group scores of the alternatives.

TABLE 10: Aggregation results by different operators ($p = q = 1$).

Operator	Aggregation values				Ranking
	x_1	x_2	x_3	x_4	
HFLWA	$\{s_4, s_{4.65}, s_{5.3}\}$	$\{s_{5.2}, s_{5.75}, s_{6.3}\}$	$\{s_{4.8}, s_{5.55}, s_{6.3}\}$	$\{s_{5.1}, s_{5.75}, s_{6.4}\}$	$x_1 < x_3 < x_2 < x_4$
HFLWHM	$\{s_{4.04}, s_{4.69}, s_{5.34}\}$	$\{s_{5.27}, s_{5.83}, s_{6.38}\}$	$\{s_{4.84}, s_{5.60}, s_{6.36}\}$	$\{s_{5.18}, s_{5.82}, s_{6.47}\}$	$x_1 < x_3 < x_4 < x_2$
HFLWBM	$\{s_{3.99}, s_{4.93}, s_{5.60}\}$	$\{s_{5.54}, s_{6.16}, s_{6.78}\}$	$\{s_{5.13}, s_{5.97}, s_{6.81}\}$	$\{s_{5.49}, s_{6.17}, s_{6.85}\}$	$x_1 < x_3 < x_2 < x_4$
HFLWEHM	$\{s_{4.00}, s_{4.65}, s_{5.30}\}$	$\{s_{5.26}, s_{5.82}, s_{6.38}\}$	$\{s_{4.79}, s_{5.56}, s_{6.35}\}$	$\{s_{5.18}, s_{5.79}, s_{6.41}\}$	$x_1 < x_3 < x_4 < x_2$
HFLLSDWEHM	$\{s_{3.88}, s_{4.58}, s_{5.28}\}$	$\{s_{5.20}, s_{5.69}, s_{6.21}\}$	$\{s_{4.75}, s_{5.48}, s_{6.23}\}$	$\{s_{5.05}, s_{5.74}, s_{6.44}\}$	$x_1 < x_3 < x_2 < x_4$

TABLE 11: Decision making results.

Method	Alternative	Value	Ranking
Xu et al.'s method (HFLOWD, $\lambda = 1$)	x_1	$d_1 = 0.098$	$x_4 < x_5 < x_2 < x_1 < x_3$
	x_2	$d_2 = 0.114$	
	x_3	$d_3 = 0.092$	
	x_4	$d_4 = 0.151$	
	x_5	$d_5 = 0.122$	
Rodríguez et al.'s Method (HFLWEHM, $p = q = 1$)	x_1	$NQD_1 = 0.935$	$x_4 < x_2 < x_5 < x_1 < x_3$
	x_2	$NQD_2 = 0.794 = 0.935$	
	x_3	$NQD_3 = 1$	
	x_4	$NQD_4 = 0.697$	
	x_5	$NQD_5 = 0.799$	

the HFLWA operator is a special form of the other operators. The HFLWHM and HFLWBM operators consider the associated relationship in information infusing. However, they are founded upon the assumption that every argument variable is associated with the rest of the argument variables. This assumption lacks accuracy and may impose an association on two variables that are totally unrelated. This would make the decision-making results unreliable. In view of this, we propose the HFLWEHM and HFLLSDWEHM operators to infuse the hesitant fuzzy linguistic information in which attribute weights are known or unknown.

6.2. Methods of Comparative Analysis. We dealt with the same illustrative example by utilizing two existing methods, including Rodríguez et al.'s method [50] and Xu et al.'s method [51], to explore the rationality and flexibility of the proposed hesitant fuzzy linguistic group decision-making method. The aggregation results are presented in Table 11.

We can see from Table 11 that the optimal choice of the three methods is the same when the operators' parameters have a fixed number of 1. This means that the proposed method in this paper is rational and feasible. Rodríguez et al. calculated the distance between the ideal choice and each alternative using the hesitant fuzzy linguistic fuzzy ordered weighted distance (HFLOWD) operator and then proposed a group decision-making method. However, this method does not consider the interrelationship of input arguments and cannot solve the problem in which the weight vector is represented by HFLTSs. This limits the application of this method. Xu et al.'s method is based on the envelope of HFLEs. It is pointed out in the literature [17, 52] that transforming HFLEs into an envelope may lead to information loss. For example, according to HFLEs envelope theory, $h^1 = \{s_1, s_3, s_5\}$ and $h^2 = \{s_1, s_5\}$ are equivalent, which does not actually correlate with the facts. The method proposed in this paper avoids those disadvantages.

7. Conclusion

Based on different perspectives on the Heronian mean, we proposed the extended Heronian mean (EHM) operator. Next, we introduced some new operators to infuse hesitant fuzzy linguistic decision-making information, including the HFLWEHM operator and the HFLLSDWEHM operator. New aggregation operators have some good properties, such as monotonicity, boundedness, and idempotency. In addition, by changing the parameters of the operators, some of their particular forms are investigated. In addition, the linguistic axiom definition and calculation formula of the similarity measures for HFLTSs have been given. Based on the above, we presented a new technique for hesitant fuzzy linguistic MAGDM and constructed a practical example study. The advantages and disadvantages of the proposed operators and the decision-making method were discussed using several comparisons.

The main contributions of this work are as follows:

(1) In an important expansion of the traditional measures theory, we proposed the linguistic axiom definition and calculation formula of similarity measures for HFLTSs.

(2) We introduced the HFLWEHM and HFLLSDWEHM operators, which have better characterization for the relation between input arguments compared with the traditional HM operator. Moreover, these extended HM operators can avoid

the crossover operation on unrelated input arguments during infusing.

(3) The group decision-making method proposed in this paper can deal with MAGDM in which the weight vector is unknown or can be expressed in numerical (or linguistic) form and therefore has more adaptability and flexibility. It should be emphasized that using the HFLLSDWEHM operator to aggregate information with unknown attribute weights can reduce the singular point in the decision-making results.

However, any kind of decision method has its own advantages and disadvantages. The decision-making method in this paper is slightly complicated and is suitable only for a hesitant fuzzy linguistic environment. In the future, we will try to simplify the model calculation and extend it to other fuzzy environments. In addition, we will also apply the proposed operators in this paper to data mining, fuzzy clustering, sustainability assessment, figure and pattern recognition, and so on.

Appendix

Proof of Property 19. Since $h_1 = h_2 = \cdots = h_n = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_m}\} = h$, we can get

$$\begin{aligned}
&\mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n) \\
&= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \cdot \bigoplus_{i=1}^{n} \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k^{(ij)}}\right)^q \right)^{1/(p+q)} \right\} \\
&= \bigcup_{1\le k\le m} \left\{ \left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \bigoplus_{i=1}^{n} \left(s_{\alpha_k}\right)^p \otimes \bigoplus_{s_{\alpha_k}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k}\right)^q \right)^{1/(p+q)} \right\} \\
&= \bigcup_{1\le k\le m} \left\{ \left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \cdot \bigoplus_{i=1}^{n} \left(s_{\alpha_k}\right)^{p+q} I\left(B_k^i\right) \right)^{1/(p+q)} \right\} \\
&= \bigcup_{1\le k\le m} \left\{ \left(\frac{\left(s_{\alpha_k}\right)^{p+q}}{\sum_{i=1}^{n} I(B_k^i)} \sum_{i=1}^{n} I\left(B_k^i\right) \right)^{1/(p+q)} \right\} \\
&= \bigcup_{1\le k\le m} \left\{ s_{\alpha_k} \right\} = h.
\end{aligned} \tag{A.1}$$

So, Property 19 is kept. □

Proof of Property 20. The elements of $\{h_1, h_2, \ldots, h_n\}$ and $\{h_1', h_2', \ldots, h_n'\}$ have the same correlation because they are evaluation values of the same attributes. Let set B_k^i represent the elements of $\{\alpha_k^{(i)}, \alpha_k^{(i+1)}, \ldots, \alpha_k^{(n)}\}$ which have a correlation with $\alpha_k^{(i)}$, and D_k^i represents the elements of $\{\beta_k^{(i)}, \beta_k^{(i+1)}, \ldots, \beta_k^{(n)}\}$ which have a correlation with $\beta_k^{(i)}$. Then, we obtain the following:

$$\begin{aligned}
&\bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k^{(ij)}}\right)^q \le \bigoplus_{\beta_k^{(ij)}\in h_{i_j}', h_{i_j}'\in D_k^i} \left(s_{\beta_k^{(ij)}}\right)^q \Longrightarrow \\
&\bigoplus_{i=1}^{n} \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k^{(ij)}}\right)^q \le \bigoplus_{i=1}^{n} \left(s_{\beta_k^{(i)}}\right)^p \\
&\otimes \bigoplus_{\beta_k^{(ij)}\in h_{i_j}', h_{i_j}'\in D_k^i} \left(s_{\beta_k^{(ij)}}\right)^q \Longrightarrow \\
&\left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \bigoplus_{i=1}^{n} \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k^{(ij)}}\right)^q \right)^{1/(p+q)} \\
&\le \left(\frac{1}{\sum_{i=1}^{n} I(D_k^i)} \bigoplus_{i=1}^{n} \left(s_{\beta_k^{(i)}}\right)^p \otimes \bigoplus_{\beta_k^{(ij)}\in h_{i_j}', h_{i_j}'\in D_k^i} \left(s_{\beta_k^{(ij)}}\right)^q \right)^{1/(p+q)}.
\end{aligned} \tag{A.2}$$

Therefore

$$\begin{aligned}
&\mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n) \\
&\quad \le \mathrm{HFLEHM}^{p,q}\left(h_1', h_2', \ldots, h_n'\right).
\end{aligned} \tag{A.3}$$

□

Proof of Property 21.

$$\begin{aligned}
&\mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n) \\
&= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\frac{1}{\sum_{i=1}^{n} I(B_k^i)} \cdot \bigoplus_{i=1}^{n} \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k^{(ij)}}\right)^q \right)^{1/(p+q)} \right\}
\end{aligned}$$

$$\leq \bigcup_{\substack{1\leq k\leq m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\frac{1}{\sum_{i=1}^{n} I\left(B_k^i\right)} \cdot \bigoplus_{i=1}^{n} \left(s_{\alpha_k^+}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k^+}\right)^q \right)^{1/(p+q)} \right\}$$

$$= \bigcup_{\substack{1\leq k\leq m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\frac{1}{\sum_{i=1}^{n} I\left(B_k^i\right)} \cdot \bigoplus_{i=1}^{n} I\left(B_k^i\right) \left(s_{\alpha_k^+}\right)^{p+q} \right)^{1/(p+q)} \right\}$$

$$= \bigcup_{1\leq k\leq m} \left\{ \left(\frac{\left(s_{\alpha_k^+}\right)^{p+q}}{\sum_{i=1}^{n} I\left(B_k^i\right)} \sum_{i=1}^{n} I\left(B_k^i\right) \right)^{1/(p+q)} \right\}$$

$$= \bigcup_{1\leq k\leq m} \left\{ s_{\alpha_k^+} \right\} = h^+ \tag{A.4}$$

Similarly, $h^- \leq \mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n)$.

Therefore, we obtain $h^- \leq \mathrm{HFLEHM}^{p,q}(h_1, h_2, \ldots, h_n) \leq h^+$. □

Proof of Property 25. Based on $w_i = (1/n)$ $(i = 1, 2, \ldots, n)$, we have

$$\begin{aligned} \lambda_i &= \frac{w_i I\left(B_k^i\right)}{\sum_{i=1}^{n} w_i I\left(B_k^i\right)} = \frac{I\left(B_k^i\right)}{\sum_{i=1}^{n} I\left(B_k^i\right)}, \\ \lambda'_{i_j} &= \frac{w_{i_j}}{\sum w_{i_j}} = \frac{1}{I\left(B_k^i\right)}. \end{aligned} \tag{A.5}$$

Thus

$$\begin{aligned} &\mathrm{HFLWEHM}^{p,q}\left(h_1, h_2, \ldots, h_n\right) \\ &= \bigcup_{\substack{1\leq k\leq m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\bigoplus_{i=1}^{n} \lambda_i \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \lambda'_{i_j} \left(s_{\alpha_k^{(ij)}}\right)^q \right)^{1/(p+q)} \right\} \\ &= \bigcup_{\substack{1\leq k\leq m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\bigoplus_{i=1}^{n} \frac{I\left(B_k^i\right)}{\sum_{i=1}^{n} I\left(B_k^i\right)} \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \frac{1}{I\left(B_k^i\right)} \left(s_{\alpha_k^{(ij)}}\right)^q \right)^{1/(p+q)} \right\} \\ &= \bigcup_{\substack{1\leq k\leq m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\frac{1}{\sum_{i=1}^{n} I\left(B_k^i\right)} \bigoplus_{i=1}^{n} \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \left(s_{\alpha_k^{(ij)}}\right)^q \right)^{1/(p+q)} \right\} \\ &= \mathrm{HFLEHM}^{p,q}\left(h_1, h_2, \ldots, h_n\right) \end{aligned} \tag{A.6}$$

which completes the proof. □

Proof of Property 26. Since $h_1 = h_2 = \cdots = h_n = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_m}\} = h$, we obtain

$$\begin{aligned} &\mathrm{HFLWEHM}^{p,q}\left(h_1, h_2, \ldots, h_n\right) \\ &= \bigcup_{\substack{1\leq k\leq m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\bigoplus_{i=1}^{n} \lambda_i \left(s_{\alpha_k^{(i)}}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \lambda'_{i_j} \left(s_{\alpha_k^{(ij)}}\right)^q \right)^{1/(p+q)} \right\} \\ &= \bigcup_{1\leq k\leq m} \left\{ \left(\bigoplus_{i=1}^{n} \lambda_i \left(s_{\alpha_k}\right)^p \otimes \bigoplus_{\alpha_k^{(ij)}\in h_{i_j}, h_{i_j}\in B_k^i} \lambda'_{i_j} \left(s_{\alpha_k}\right)^q \right)^{1/(p+q)} \right\} \\ &= \bigcup_{1\leq k\leq m} \left\{ \left(\left(s_{\alpha_k}\right)^{p+q} \sum_{i=1}^{n} \lambda_i \right. \right. \end{aligned}$$

$$\cdot \sum_{\alpha_k^{(i_j)} \in h_{i_j}, h_{i_j} \in B_k^i} \lambda'_{i_j} \Bigg) ^{1/(p+q)} \Bigg\} = \bigcup_{1\le k\le m} \Bigg\{ \Bigg(\left(s_{\alpha_k} \right)^{p+q}$$

$$\cdot \frac{\sum_{i=1}^n w_i I\left(B_k^i\right)}{\sum_{i=1}^n w_i I\left(B_k^i\right)} \cdot \frac{\sum w_{i_j}}{\sum w_{i_j}} \Bigg)^{1/(p+q)} \Bigg\}$$

$$= \bigcup_{1\le k\le m} \left\{ \left(\left(s_{\alpha_k} \right)^{p+q} \cdot \right)^{1/(p+q)} \right\} = \bigcup_{1\le k\le m} \left\{ s_{\alpha_k} \right\} = h, \tag{A.7}$$

which completes the proof. □

Proof of Property 30. According to $h_1 = h_2 = \cdots = h_n = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_m}\} = h$, we have

$$\mathrm{HFLLSDWEHM}^{p,q}(h_1, h_2, \ldots, h_n) = \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}} \in h_1, s_{\alpha_k^{(2)}} \in h_2, \ldots, s_{\alpha_k^{(n)}} \in h_n}} \left\{ \left(\bigoplus_{i=1}^n w_i \left(s_{\alpha_k^{(i)}} \right)^p \otimes \bigoplus_{\alpha_k^{(i_j)} \in h_{i_j}, h_{i_j} \in B_k^i} w'_{i_j} \left(s_{\alpha_k^{(i_j)}} \right)^q \right)^{1/(p+q)} \right\}$$

$$= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}} \in h_1, s_{\alpha_k^{(2)}} \in h_2, \ldots, s_{\alpha_k^{(n)}} \in h_n}} \left\{ \left(\bigoplus_{i=1}^n w_i \left(s_{\alpha_k} \right)^p \otimes \bigoplus_{\alpha_k^{(i_j)} \in h_{i_j}, h_{i_j} \in B_k^i} w'_{i_j} \left(s_{\alpha_k} \right)^q \right)^{1/(p+q)} \right\} = \bigcup_{1\le k\le m} \Bigg\{ \Bigg(\left(s_{\alpha_k} \right)^{p+q}$$

$$\cdot \sum_{i=1}^n \left(\left[g\left(\frac{B_i}{TV} \right) - g\left(\frac{B_{i-1}}{TV} \right) \right] \sum_{\alpha_k^{(i_j)} \in h_{i_j}, h_{i_j} \in B_k^i} \left[g\left(\frac{B'_i}{TV'} \right) - g\left(\frac{B'_{i-1}}{TV'} \right) \right] \right) \Bigg)^{1/(p+q)} \Bigg\} = \bigcup_{1\le k\le m} \Bigg\{ \Bigg(\left(s_{\alpha_k} \right)^{p+q} \tag{A.8}$$

$$\cdot \sum_{i=1}^n \left(\left[g\left(\frac{B_i}{TV} \right) - g\left(\frac{B_{i-1}}{TV} \right) \right] \left[g\left(\frac{\sum_{j=1}^l \left(1 + T\left(h_{i_j}\right)\right)}{\sum_{j=1}^l \left(1 + T\left(h_{i_j}\right)\right)} \right) - g\left(\frac{B'_0}{\sum_{j=1}^l \left(1 + T\left(h_{i_j}\right)\right)} \right) \right] \right) \Bigg)^{1/(p+q)} \Bigg\}$$

$$= \bigcup_{1\le k\le m} \left\{ \left(\left(s_{\alpha_k} \right)^{p+q} \sum_{i=1}^n \left(\left[g\left(\frac{B_i}{TV} \right) - g\left(\frac{B_{i-1}}{TV} \right) \right] \right) \right)^{1/(p+q)} \right\} = \bigcup_{1\le k\le m} \Bigg\{ \Bigg(\left(s_{\alpha_k} \right)^{p+q}$$

$$\cdot \sum_{i=1}^n \left(\left[g\left(\frac{\sum_{j=1}^n \left(1 + T\left(h_i\right)\right)}{\sum_{j=1}^n \left(1 + T\left(h_i\right)\right)} \right) - g\left(\frac{B'_0}{\sum_{j=1}^n \left(1 + T\left(h_i\right)\right)} \right) \right] \right) \Bigg)^{1/(p+q)} \Bigg\} = \bigcup_{1\le k\le m} \left\{ \left(\left(s_{\alpha_k} \right)^{p+q} \cdot \right)^{1/(p+q)} \right\} = \bigcup_{1\le k\le m} \left\{ s_{\alpha_k} \right\}$$

$$= h$$

which completes the proof. □

Proof of Property 31.

$$\mathrm{HFLLSDWEHM}^{p,q}(h_1, h_2, \ldots, h_n)$$

$$= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}} \in h_1, s_{\alpha_k^{(2)}} \in h_2, \ldots, s_{\alpha_k^{(n)}} \in h_n}} \left\{ \left(\bigoplus_{i=1}^n w_i \left(s_{\alpha_k^{(i)}} \right)^p \otimes \bigoplus_{\alpha_k^{(i_j)} \in h_{i_j}, h_{i_j} \in B_k^i} w'_{i_j} \left(s_{\alpha_k^{(i_j)}} \right)^q \right)^{1/(p+q)} \right\}$$

$$\le \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}} \in h_1, s_{\alpha_k^{(2)}} \in h_2, \ldots, s_{\alpha_k^{(n)}} \in h_n}} \left\{ \left(\bigoplus_{i=1}^n w_i \left(s_{\alpha_k^{+}} \right)^p \otimes \bigoplus_{\alpha_k^{(i_j)} \in h_{i_j}, h_{i_j} \in B_k^i} w'_{i_j} \left(s_{\alpha_k^{+}} \right)^q \right)^{1/(p+q)} \right\}$$

$$= \bigcup_{\substack{1\le k\le m \\ s_{\alpha_k^{(1)}}\in h_1, s_{\alpha_k^{(2)}}\in h_2,\ldots,s_{\alpha_k^{(n)}}\in h_n}} \left\{ \left(\left(s_{\alpha_k^+}\right)^{p+q} \sum_{i=1}^{n} \left(\left[g\left(\frac{B_i}{TV}\right) - g\left(\frac{B_{i-1}}{TV}\right) \right] \sum_{\alpha_k^{(i_j)}\in h_{i_j}, h_{i_j}\in B_k^i} \left[g\left(\frac{B_i'}{TV'}\right) - g\left(\frac{B_{i-1}'}{TV'}\right) \right] \right) \right)^{1/(p+q)} \right\}$$

$$= \bigcup_{1\le k\le m} \left\{ s_{\alpha_k^+} \right\}$$

$$= h^+ \text{ (From the proof of Property 30)} \tag{A.9}$$

Similarly, $h^- \le \mathrm{HFLLSDWEHM}^{p,q}(h_1, h_2, \ldots, h_n)$.

Therefore, we obtain $h^- \le \mathrm{HFLLSDWEHM}^{p,q}(h_1, h_2, \ldots, h_n) \le h^+$. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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