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Research Article

Finite-Time Synchronization and Synchronization Dynamics Analysis for Two Classes of Markovian Switching Multiweighted Complex Networks from Synchronization Control Rule Viewpoint

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This paper is mainly concerned with how nonlinear coupled one impacts synchronization dynamics of a class of nonlinear coupled Markovian switching multiweighted complex networks (NCMSMWCNs). Firstly, sufficient conditions of finite-time synchronization for a class of NCMSMWCNs and a class of linear coupled Markovian switching multiweighted complex networks (LCMSMWCNs) are investigated. Secondly, based on the derived results, how nonlinear coupled one affects synchronization dynamics of the NCMSMWCNs is analyzed from synchronization control rule. Thirdly, in order to further explore how nonlinear coupled one affects synchronization dynamics of the NCMSMWCNs, synchronization dynamics relationship of the NCMSMWCNs and the LCMSMWCNs is built. Furthermore, this relationship can also show how linear coupled one affects synchronization dynamics of the LCMSMWCNs. At last, numerical examples are provided to demonstrate the effectiveness of the obtained theory.

1. Introduction

In recent years, synchronization of nonlinear coupled complex networks has gained considerable attention because synchronization is one of the most important collective behaviors of complex networks [1–8]. Furthermore, nonlinear coupled one was important factor impacting synchronization dynamics of nonlinear coupled complex networks. In [1], Zhang et al. considered outer synchronization for a class of nonlinear coupled drive-response networks. In [2], exponential quasi-synchronization for a class of nonlinear coupled drive-response memristive neural networks was considered. In [3], a class of nonlinear inner coupling drive-response complex networks was proposed and sufficient conditions of its generalized matrix projective outer synchronization were given. By analyzing [1–8], it is worth noting that, except [4], there was no topic on how nonlinear coupled one affects synchronization dynamics of nonlinear coupled complex networks. The literature [4] adopted simulation method to discuss how nonlinear coupled one impacts synchronization dynamics of the addressed nonlinear coupled complex networks. It is pity that there was no theoretical analysis in [4]. The literatures [1–3, 5–8] mainly emphasized how to obtain sufficient conditions of the considered nonlinear coupled complex networks. Therefore, it is necessary and significant to further develop the above issue from theoretical aspect. Until now, to the best of our knowledge, there are a few literatures to mention the issue. In addition, linear coupled one can also affect synchronization dynamics of linear coupled complex networks [7]. Notice that although a lot of results for synchronization problems of linear coupled complex networks have been obtained [8–10], how linear coupled one impacts synchronization dynamics of linear coupled complex networks still needs to be further researched. For instance,

in [8], Ali et al. studied extended dissipative synchronization for a class of linear coupled complex networks with additive time-varying delay and discrete-time information. The literature [9] was concerned with asymptotical synchronization of a class of linear coupled complex networks with actuator faults and unknown coupling weights via adaptive control schemes. From [8–10], it is seen that although some feasible synchronization theories were derived, the relationship of linear coupled one and synchronization dynamics was still not noticed.

As a matter of fact, because the connections of many real networks have different weights, such as email networks and public traffic networks, these networks can be modeled by multiweighted complex networks [11–19]. Therefore, recently, some works for synchronization problems of multiweighted complex networks began to develop [11–16]. For example, in [11], the authors studied synchronization of complex networks with multiweights and its application in public traffic network. Qiu et al. [12] investigated synchronization and H_∞ synchronization of multiweighted complex delayed dynamical networks with fixed and switching topologies. Furthermore, although the literatures [17–19] gave several classes of multiweighted complex networks, they addressed some passivity problems, instead of synchronization issue. From [11–19], it is not difficult to find that until now there are still a few literatures to discuss synchronization of multiweighted complex networks. Besides these, it should be noted that, in many engineering areas, there sometimes exist abrupt variations which are often caused by random failures or repairs of the components in systems. In this case, Markovian switching can model this phenomenon [20]. This caused that Markovian switching systems have gained widespread attention [21–27]. Simultaneously, due to processing speeds, finite information transmission, and random perturbations which are often from environment elements, it is inevitable that time-delays and stochastic noises often happen [28]. Thus, many useful results for synchronization of Markovian switching systems with time-delays and stochastic noises were derived [21, 29, 30].

In fact, from the viewpoint of system dynamics, if a system can achieve stability state from initial state, there should exist convergence time t^*. That is to say, if a system satisfies stability conditions, it can get to stability state within finite-time t^*. How to get t^*? In this situation, finite-time problems of many systems have been attracting increasing interest [31–42]. For example, Liu et al. [31] investigated finite/fixed-time synchronization of complex networks with stochastic disturbances. The literature [32] was concerned with finite-time consensus of multiple nonholonomic chained-form systems based on recursive distributed observer. Liu et al. [33] studied finite-time synchronization switched coupled neural networks with discontinuous or continuous activations. In [34], finite-time robust passive control for a class of switched reaction-diffusion stochastic complex dynamical networks with coupling delays and impulsive control was considered. According to the obtained results of [31–42], it is observed that in some effective finite-time t^* estimation approaches for synchronization, passivity, and consensus, the proposed systems were derived. Unfortunately, to our knowledge, few researchers focused attention on finite-time synchronization of linear or nonlinear coupled Markovian switching multiweighted complex networks.

Inspired by the above discussion, this paper investigates sufficient condition of finite-time synchronization for a class of NCMSMWCNs. Based on the derived results and Lyapunov stability theory, how nonlinear coupled one affects synchronization dynamics of the NCMSMWCNs is analyzed from synchronization control rule viewpoint. Besides this, sufficient condition of finite-time synchronization for a class of LCMSMWCNs is proposed. Moreover, by comparing synchronization dynamics of the NCMSMWCNs and the LCMSMWCNs, how nonlinear coupled one impacts synchronization dynamics of the NCMSMWCNs is further explored. Numerical simulation examples illustrate the effectiveness of the derived results. The novelties and contributions of the paper are highlighted as follows:

(1) Based on the derived sufficient condition of finite-time synchronization of the NCMSMWCNs, how nonlinear coupled one impacts synchronization dynamics of the NCMSMWCNs is discussed.

(2) By building sufficient condition relationship of finite-time synchronization for two classes of the considered complex networks, synchronization control rule relationship of the NCMSMWCNs and the LCMSMWCNs is presented.

(3) How nonlinear coupled one and linear coupled one affect synchronization dynamics of the NCMSMWCNs and the LCMSMWCNs is analyzed from synchronization control rule relationship aspect.

(4) Compared with the existing results about synchronization problems of nonlinear coupled or linear coupled complex networks such as [1–16], the work in this paper can further extend the analysis ideas of synchronization problems.

The rest of this paper is organized as follows. Section 2 gives model and preliminaries. In Section 3, sufficient conditions and sufficient condition relationship of finite-time synchronization of the NCMSMWCNs and the LCMSMWCNs are derived. Furthermore, synchronization dynamics of the addressed systems are analyzed. Sections 4 and 5 provide simulation results and the conclusions, respectively. At last, acknowledgments and conflicts of interest are given.

2. Problem Formulation and Preliminaries

Consider two classes of Markovian switching multiweighted complex networks as follows:

$$\begin{aligned} d\dot{\tilde{z}}_{\tilde{i}}(t) = \Bigg[& \tilde{f}\left(\tilde{z}_{\tilde{i}}(t)\right) + \tilde{B}\tilde{z}_{\tilde{i}}(t) \\ & + \sum_{\tilde{k}=1}^{m}\sum_{\tilde{j}=1}^{N}\tilde{c}_{\tilde{k}}\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t))\,\Gamma_{\tilde{k}}\tilde{g}\left(\tilde{z}_{\tilde{j}}(t), \tilde{z}_{\tilde{j}}(t-\tilde{\tau}(t))\right) \\ & + \tilde{u}_{\tilde{i}}(t, \tilde{r}(t)) \Bigg] dt + \tilde{\sigma}_{\tilde{i}}(\tilde{z}(t), \tilde{z}(t-\tilde{\tau}(t)), t, \\ & \tilde{r}(t))\, d\varpi_i(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

$$d\dot{\tilde{z}}_{\tilde{i}}(t) = \Bigg[\tilde{f}\left(\tilde{z}_{\tilde{i}}(t)\right) + \tilde{B}\tilde{z}_{\tilde{i}}(t) + \sum_{\tilde{k}=1}^{m}\sum_{\tilde{j}=1}^{N} \tilde{c}_{\tilde{k}} \tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t))\, \Gamma_{\tilde{k}} \left(\tilde{z}_{\tilde{j}}(t) + \tilde{z}_{\tilde{j}}(t-\tilde{\tau}(t))\right) + \tilde{u}_{\tilde{i}}(t,\tilde{r}(t)) \Bigg] dt + \tilde{\sigma}_{\tilde{i}}(\tilde{z}(t), \tilde{z}(t-\tilde{\tau}(t)), t, \tilde{r}(t))\, d\tilde{\omega}_i(t), \quad i = 1,2,\ldots,N, \tag{2}$$

where $\tilde{f}(\cdot) : \mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ is the activity function of $\tilde{i}$th node, $\tilde{z}_{\tilde{i}}(t) = (\tilde{z}_{\tilde{i}1}(t), \tilde{z}_{\tilde{i}2}(t), \ldots, \tilde{z}_{\tilde{i}n}(t))^T$, $\tilde{B} \in \mathfrak{R}^{n\times n}$, $\tilde{c}_{\tilde{k}} > 0$ is coupling strength, $\Gamma_{\tilde{k}} \in \mathfrak{R}^{n\times n}$, which satisfies $\Gamma_{\tilde{k}} = \mathrm{diag}(\gamma_{\tilde{k}1}, \gamma_{\tilde{k}2}, \ldots, \gamma_{\tilde{k}n}) > 0$, is inner coupling matrix, $\tilde{g}(\cdot,\cdot) : \mathfrak{R}^n \times \mathfrak{R}^n \longrightarrow \mathfrak{R}^n$ is nonlinear coupling function, and $\tilde{D}_{\tilde{k}}(\tilde{r}(t)) = (\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t)))_{N\times N}$ stands for the $\tilde{k}$th outer-coupling weight matrix. If there exists a connection from node j to node i ($i \neq j$), $\tilde{D}_{\tilde{k}}(\tilde{r}(t)) \neq 0$, otherwise, $\tilde{D}_{\tilde{k}}(\tilde{r}(t)) = 0$. $\tilde{\tau}(t)$ is coupling time-varying delay, $\tilde{u}_{\tilde{i}}(t,\tilde{r}(t)$ is the controller of networks (1) and (2), $\tilde{\sigma}_{\tilde{i}}(\cdot,\cdot,\cdot,\cdot)$ is the noise intensity, and $\tilde{r}(t)$ is right-continuous Markov chain with known transition rates and is independent of the Brownian motion $\tilde{\omega}_i(t)$. In order to make notation simplicity, let $\tilde{D}_{\tilde{k}}(\tilde{r}(t)) = \tilde{D}_{\tilde{k}}^{\tilde{r}}$, $\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t)) = \tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k},\tilde{r}}$, $\tilde{u}_{\tilde{i}}(t,\tilde{r}(t)) = \tilde{u}_{\tilde{i}}(t,\tilde{r})$, and $\tilde{r}(t) = \tilde{r}$.

Remark 1. Comparing with models (1) and (2), it is seen that there is no difference except coupled functions. The coupled functions of models (1) and (2) are nonlinear function $\tilde{g}(\tilde{z}_{\tilde{j}}(t), \tilde{z}_{\tilde{j}}(t-\tilde{\tau}(t)))$ and linear coupled function $\tilde{z}_{\tilde{j}}(t) + \tilde{z}_{\tilde{j}}(t-\tilde{\tau}(t))$, respectively. Why are the two similar models considered? The main reason is related to the motivations of this paper. From the analysis of introduction, it is known that one of the motivations is under the same synchronization control rule, and comparing with synchronization dynamics of models (1) and (2), it is derived how nonlinear coupled function and linear coupled function affect synchronization dynamics of the addressed system. The detailed analysis is in Remark 15. Besides this, in models (1) and (2), diffusive coupling condition $\tilde{d}_{\tilde{i}\tilde{i}}^{\tilde{k}}(\tilde{r}(t)) = -\sum_{\tilde{j}=1,\tilde{j}\neq\tilde{i}}^{N} \tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t))$ is removed. Thus, the obtained results are more general. Actually, the scheme has been adopted in [19]. It is necessary to emphasize that although some classes of multiweighted complex networks have been proposed [11–19], these models are liner coupled multiweighted complex networks. Furthermore, in these models, there is no Markovian switching. Therefore, models (1) and (2) are two classes of new multiweighted complex networks.

From the networks (1) and (2), we get

$$d\tilde{s}(t) = \tilde{f}(\tilde{s}(t))\, dt + \tilde{B}\tilde{s}(t)\, dt + \tilde{\sigma}(\tilde{s}(t), \tilde{s}(t-\tilde{\tau}(t)), t, \tilde{r})\, d\tilde{\omega}(t), \tag{3}$$

where $\tilde{s}(t)$ is the synchronization state of networks (1) and (2).

The controller $\tilde{u}_{\tilde{i}}(t,\tilde{r})$ of networks (1) and (2) is defined by

$$\tilde{u}_{\tilde{i}}(t,\tilde{r}) = -\sum_{\tilde{k}=1}^{m} \theta_{\tilde{i}}^{\tilde{k},\tilde{r}} \omega_{\tilde{i}}(t) - \frac{1}{\alpha^{\tilde{r}}} \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \kappa_{\tilde{i}}^{\tilde{k},\tilde{r}} \tilde{Q}^{(\tilde{\gamma}-1)/2} \Gamma_{\tilde{k}}^{(\tilde{\gamma}+1)/2} \cdot \mathrm{sign}\left(\omega_{\tilde{i}}(t)\right) \left|\omega_{\tilde{i}}(t)\right|^{\tilde{\gamma}} - \mu \tilde{Q}^{-1} \sum_{\tilde{k}=1}^{m} \kappa_{\tilde{i}}^{\tilde{k},\tilde{r}} \left(\frac{\tilde{c}_{\tilde{k}}}{1-\tilde{\lambda}} \cdot \int_{t-\tilde{\tau}(t)}^{t} \omega_{\tilde{i}}^{T}(s)\, \tilde{F}_{\tilde{i}}^{\tilde{k}} \omega_{\tilde{i}}(s)\, ds \right)^{(\tilde{\gamma}+1)/2} \frac{\omega_{\tilde{i}}(t)}{\left\|\omega_{\tilde{i}}(t)\right\|^2}, \tag{4}$$

where $\theta_{\tilde{i}}^{\tilde{k},\tilde{r}} > 0$, $\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}} > 0$, $\alpha^{\tilde{r}} > 0$, $0 < \tilde{\gamma} < 1$, $\tilde{\gamma} \in \mathfrak{R}$, $\tilde{F}_{\tilde{i}}^{\tilde{k}} \in \mathfrak{R}^{n\times n} > 0$, $\omega_{\tilde{i}}(t) = (\omega_{\tilde{i}1}(t), \omega_{\tilde{i}2}(t), \ldots, \omega_{\tilde{i}n}(t))^T$ and $\mathrm{sign}(\omega_{\tilde{i}}(t))$ is the sign function, $\mu = \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \min_{\tilde{k}=1,2,\ldots,m}\{\lambda_{min}(\Gamma_{\tilde{k}}^{(1+\tilde{\gamma})/2})\}$.

Here, let $\omega_i(t) = \tilde{z}_{\tilde{i}}(t) - \tilde{s}(t)$, $F(\omega_{\tilde{i}}(t)) = \tilde{f}(\tilde{z}_{\tilde{i}}(t)) - \tilde{f}(\tilde{s}(t))$, $\tilde{G}(\omega_{\tilde{j}}(t), \omega_{\tilde{j}}(t-\tilde{\tau}(t))) = \tilde{g}(\tilde{z}_{\tilde{j}}(t), \tilde{z}_{\tilde{j}}(t-\tilde{\tau}(t))) - \tilde{g}(\tilde{s}(t), \tilde{s}(t-\tilde{\tau}(t)))$, $\hat{\sigma}_{\tilde{i}}(\omega(t), \omega(t-\tilde{\tau}(t)), t, \tilde{r}) = \tilde{\sigma}_{\tilde{i}}(\omega(t), \omega(t-\tilde{\tau}(t)), t, \tilde{r}) - \tilde{\sigma}(\tilde{s}(t), \tilde{s}(t-\tilde{\tau}(t)), t, \tilde{r})$. Subtracting (3) from (1) and (2), we can obtain the following error systems (5) and (6) of networks (1) and (2):

$$d\omega_{\tilde{i}}(t) = \Bigg[F\left(\omega_{\tilde{i}}(t)\right) + \tilde{B}\omega_{\tilde{i}}(t) + \sum_{\tilde{k}=1}^{m}\sum_{\tilde{j}=1}^{N} \tilde{c}_{\tilde{k}} \tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k},\tilde{r}} \Gamma_{\tilde{k}} \tilde{G}\left(\omega_{\tilde{j}}(t), \omega_{\tilde{j}}(t-\tilde{\tau}(t))\right) + \tilde{u}_{\tilde{i}}(t,\tilde{r}) \Bigg] dt + \hat{\sigma}_{\tilde{i}}(\omega(t), \omega(t-\tilde{\tau}(t)), t, \tilde{r})\, d\tilde{\omega}_i(t), \tag{5}$$

$$d\omega_{\tilde{i}}(t) = \Bigg[F\left(\omega_{\tilde{i}}(t)\right) + \tilde{B}\omega_{\tilde{i}}(t) + \sum_{\tilde{k}=1}^{m}\sum_{\tilde{j}=1}^{N} \tilde{c}_{\tilde{k}} \tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k},\tilde{r}} \Gamma_{\tilde{k}} \left(\omega_{\tilde{i}}(t) + \omega_{\tilde{i}}(t-\tilde{\tau}(t))\right) + \tilde{u}_{\tilde{i}}(t,\tilde{r}) \Bigg] dt + \hat{\sigma}_{\tilde{i}}(\omega(t), \omega(t-\tilde{\tau}(t)), t, \tilde{r})\, d\tilde{\omega}_i(t). \tag{6}$$

Remark 2. Although there is the *sign* function in controller (4), it cannot cause the phenomenon of chattering. The reason is that according to [43, 44], if the switching control term $\mathrm{sign}(\omega_{\tilde{i}}(t))|\omega_{\tilde{i}}(t)|^{\tilde{\gamma}}$ in controller (4) is softened to be a smooth signal, the phenomenon of chattering of networks (1) and (2) cannot happen. Assume that $\omega_{\tilde{i}}(t)$ satisfies $\|\lim_{\triangle t\longrightarrow 0}(\omega_{\tilde{i}}(t+\triangle t) - \omega_{\tilde{i}}(t))\| = \xi_{\tilde{i}}$, where $\xi_{\tilde{i}} > 0$. Then, there is $\|\dot{\omega}_{\tilde{i}}(t)\| = \|\lim_{\triangle t\longrightarrow 0}((\omega_{\tilde{i}}(t+\triangle t) - \omega_{\tilde{i}}(t))/\triangle t)\| = \|\lim_{\triangle t\longrightarrow 0}(\xi_{\tilde{i}}/\triangle t)\| \longrightarrow +\infty$. Combining the error systems (5) and (6), it is derived that $\|\omega_{\tilde{i}}(t)\| \longrightarrow +\infty$. This shows that the above assumption

is wrong. In order to make $\|\dot{\omega}_{\tilde{i}}(t)\|$ bounded, there must be $\|\lim_{\triangle t \longrightarrow 0}(\omega_{\tilde{i}}(t+\triangle t)-\omega_{\tilde{i}}(t))\| \longrightarrow 0$. This shows that $\omega_{\tilde{i}}(t)$ is a smooth function. This causes $\operatorname{sign}(\omega_{\tilde{i}}(t))|\omega_{\tilde{i}}(t)|^{\tilde{\gamma}}$ to be also a smooth function. The analysis is completed.

In order to obtain the derived and analysis results, the definition of finite-time synchronization of networks (1) and (2), some assumptions, and lemmas are needed.

Definition 3. If $\exists\, t^*$ such that

$$\lim_{t \longrightarrow t^*} \mathbb{E}\left\|\tilde{z}_{\tilde{i}}(t)-\tilde{s}(t)\right\| = 0, \tag{7}$$

and $\|\tilde{z}_{\tilde{i}}(t)-\tilde{s}(t)\| = 0$ for $t > t^*$, the network (1) or (2) is said to achieve synchronization within finite-time t^*, where $t^* > 0$, $\tilde{i} \in \{1, 2, \ldots, N\}$.

Assumption 4. The functions $\tilde{f}(\cdot)$ and $\tilde{g}(\cdot,\cdot)$ satisfy the Lipschitz conditions and $\tilde{g}(0,0) = 0$. That means there exist constants $\tilde{L}, \tilde{L}_1 > 0$ and $\tilde{L}_2 > 0$ such that

$$\begin{aligned} &\left\|\tilde{f}(x)-\tilde{f}(y)\right\| \le \tilde{L}\|x-y\|, \quad \forall x, y, \in \mathfrak{R}^n, \\ &\|\tilde{g}(x_1,y_1)-\tilde{g}(x_2,y_2)\| \\ &\quad \le \tilde{L}_1\|x_1-x_2\| + \tilde{L}_2\|y_1-y_2\|, \\ &\qquad\qquad \forall x_1, x_2, y_1, y_2 \in \mathfrak{R}^n. \end{aligned} \tag{8}$$

Remark 5. In [1–7], some synchronization problems for some classes of nonlinear coupled complex networks have been proposed. Furthermore, it is observed that some methods can be used to deal with nonlinear coupled one [1–7]. According to these methods, it is derived that these methods satisfy the following two properties. The first is that these methods are linearization techniques. That is to say, by using these methods, nonlinear coupled function will become linear function. For example, in [5–7], nonlinear coupling function of the addressed networks is $g(x(t))$. In order to obtain the derived results, one assumed that $g(x(t))$ satisfied $\|g_i(x_1(t))-g_i(x_2(t))\| \le L_i\|x_1(t)-x_2(t)\| + \tilde{\theta}_i(t)$, $|(g_i(x_1(t))-\alpha x_1(t))-(g_i(x_1(t))-\alpha x_2(t))| \le \varepsilon|x_1(t)-x_2(t)|$, $\|g(x_j(t))-g(s(t))\| \le \gamma_j\|x_j(t)-s(t)\|$, respectively, where $L_i > 0$, $\tilde{\theta}_i(t) \ge 0$, $\alpha \ge 0$, $\varepsilon \ge 0$, $\gamma_j > 0$, L_i, $\tilde{\theta}_i(t)$, α, ε and γ_j were constants. It is clear that $L_i\|x_1(t)-x_2(t)\| + \tilde{\theta}_i(t)$, $\varepsilon|x_1(t)-x_2(t)|$, and $\gamma_j\|x_j(t)-s(t)\|$ are linear function. The second is that nonlinearity of nonlinear coupling function is more serious, and linearizing parameters will become larger. For instance, in [1], nonlinear coupled function $f(x)$ satisfied $\|f(x)-f(y)\| \le \alpha\|x-y\|$, where $\alpha > 0$. From $\|f(x)-f(y)\| \le \alpha\|x-y\|$, one has $\|f(x)-f(y)\|/\|x-y\| \le \alpha$. This shows that if $\|x-y\|$ is fixed, $\|f(x)-f(y)\|$ is larger and α is also larger. That means if nonlinearity of $f(x)$ is increased, α is also increased.

Remark 6. From Assumption 8, it is easily seen that the method of dealing with nonlinear coupling function $\tilde{g}(x,y)$ is a linearization method. In order to build the relationship between nonlinearity of $\tilde{g}(x,y)$ and linearizing parameters $\tilde{L}_1$ and $\tilde{L}_2$, we make a rule as the following steps. (I) Assume that $\tilde{g}_{(1)}(x,y)$, $\tilde{g}_{(2)}(x,y)$ satisfy Assumption 8, and nonlinearity of $\tilde{g}_{(2)}(x,y)$ is more serious than that of $\tilde{g}_{(1)}(x,y)$. (II) According to Assumption 4, there are $\|\tilde{g}_{(1)}(x,y)\| \le \tilde{L}_1^{(1)}\|x\| + \tilde{L}_2^{(1)}\|y\|$ and $\|\tilde{g}_{(2)}(x,y)\| \le \tilde{L}_1^{(2)}\|x\| + \tilde{L}_2^{(2)}\|y\|$. (III) Fixing x and y, there must be $\|\tilde{g}_{(1)}(x,y)\| < \|\tilde{g}_{(2)}(x,y)\|$. Thus, let $\tilde{L}_1^{(1)}\|x\| + \tilde{L}_2^{(1)}\|y\| < \tilde{L}_1^{(2)}\|x\| + \tilde{L}_2^{(2)}\|y\|$ hold, where $\tilde{L}_1^{(1)} < \tilde{L}_1^{(2)}$ and $\tilde{L}_2^{(1)} < \tilde{L}_2^{(2)}$. By using the rule, it can be obtained that if linearizing parameters $\tilde{L}_1$ and $\tilde{L}_2$ become larger, nonlinearity of $\tilde{g}(x,y)$ will become more serious. This shows that under the principle, nonlinearity of $\tilde{g}(x,y)$ can be analyzed by $\tilde{L}_1$ and $\tilde{L}_2$.

Assumption 7. There exist nonnegative constants $\iota^{(1)}_{(\tilde{r})ij}$ and $\iota^{(2)}_{(\tilde{r})ij}$ $(\tilde{i}, \tilde{j} = 1, 2, \ldots, N, \tilde{r} \in \mathfrak{T})$, such that

$$\begin{aligned} &trace(\hat{\sigma}_{\tilde{i}}^T(\omega(t), \omega(t-\tilde{\tau}(t)), t, \tilde{r}) \\ &\quad \cdot \hat{\sigma}_{\tilde{i}}(\omega(t), \omega(t-\tilde{\tau}(t)), t, \tilde{r}) \le \sum_{\tilde{j}=1}^{N} \iota^{(1)}_{(\tilde{r})\tilde{i}\tilde{j}} \omega_{\tilde{i}}^T(t)\,\omega_{\tilde{i}}(t) \\ &\quad + \sum_{\tilde{j}=1}^{N} \iota^{(2)}_{(\tilde{r})\tilde{i}\tilde{j}} \omega_{\tilde{i}}^T(t-\tilde{\tau}(t))\,\omega_{\tilde{i}}(t-\tilde{\tau}(t)). \end{aligned} \tag{9}$$

Assumption 8. Coupling time-vary delay $\tilde{\tau}(t)$ in networks (1) and (2) satisfies $0 \le \tilde{\tau}(t) \le \tilde{\tau}_M$ and $0 \le \dot{\tilde{\tau}}(t) \le \tilde{\lambda} < 1$.

Lemma 9 (Wang and Xiao [45]). *Let $\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_n \ge 0$ and $0 < \tilde{p} \le 1$, then*

$$\left(\sum_{\tilde{i}=1}^{n} \tilde{\nu}_{\tilde{i}}\right)^{\tilde{p}} \le \sum_{\tilde{i}=1}^{n} \tilde{\nu}_{\tilde{i}}^{\tilde{p}}. \tag{10}$$

Lemma 10 (Boyd et al. [46]). *For $\forall x, y \in \mathfrak{R}^n$ and $\varphi \in \mathfrak{R}^{n\times n} > 0$, then*

$$2x^T y \le x^T \varphi^{-1} x + y^T \varphi y. \tag{11}$$

Lemma 11 (Bhat et al. [47]). *Let $V(t)$ be a Lyapunov function. If $V(t) : [0,\infty) \longrightarrow [0,\infty)$ is differentiable (the derivative of $V(t)$ at 0 is, in fact, its right derivative) and*

$$\frac{dV(t)}{dt} \le -\vartheta V^{\zeta}(t), \tag{12}$$

where $\vartheta > 0$ and $0 < \zeta < 1$, $V(t)$ will reach zero at finite-time $t^* \le V^{1-\zeta}(t)/(\vartheta(1-\zeta))$ and $V(t) = 0$ for all $t \ge t^*$.

Remark 12. From Lemma 11, it is observed that if $t \ge t^*$, there is $V(t) = 0$. Because $V(t)$ is Lyapunov function with respect to a system, this causes that if $V(t) = 0$, the system must achieve stability state within finite-time t^*. Finite-time synchronization, which is as a special case of stability, can be processed by the above Lemma 11. According to Lemma 11, if $dV(t)/dt \le -\vartheta V^{\zeta}(t)$, there is $t^* \le V^{1-\zeta}(t)/(\vartheta(1-\zeta))$. Therefore, in order to get t^*, it is needed to prove that

Lyapunov function $V(t)$ of the system satisfies $dV(t)/dt \leq -\vartheta V^{\zeta}(t)$. It should be pointed out that finite-time algorithm of Lemma 11 is a feasible finite-time estimation approach. Besides this, in $d\dot{\tilde{z}}_{\tilde{i}}(t)$ of networks (1)-(2) and $d\omega_{\tilde{i}}(t)$ of the error systems (5)-(6) there exists the notation d. Actually, d of $d\dot{\tilde{z}}_{\tilde{i}}(t)$ and $d\omega_{\tilde{i}}(t)$ stands for small change in any quantity over a small subsequent time interval dt, and $\varpi_i(t)$ of systems (1), (2), (5), and (6) represents a bounded vector-form Brownian motion process [48]. Therefore, systems (1), (2), (5), and (6) are two classes of stochastic differential equations with Markovian switching in nature. Because Definition 3 is built on systems (1) and (2), $\mathbb{E}$ in Definition 3 represents expectation. For instance, $\mathbb{E}[x]$ means the expectation of the random variable x.

3. Main Results

In this section, sufficient conditions of finite-time synchronization of networks (1) and (2) are derived. Based on sufficient condition of network (1), how nonlinear coupled one $\tilde{g}(\cdot,\cdot)$ affects synchronization dynamics of network (1) is analyzed. Besides this, according to the derived results, synchronization dynamics relationships of the networks (1) and (2) are built. By using synchronization dynamics relationship and synchronization control rule viewpoint, how nonlinear coupled one $\tilde{g}(\cdot,\cdot)$ and linear coupled one $\tilde{z}_{\tilde{j}}(t)+\tilde{z}_{\tilde{j}}(t-\tilde{\tau}(t))$ impact synchronization dynamics of networks (1) and (2) is further explored.

A. *Sufficient Conditions of Finite-Time Synchronization of the Networks (1), (2) and Nonlinear Coupled One $\tilde{g}(\cdot,\cdot)$ Effect Analysis on Synchronization Dynamics of the Network (1).*

Theorem 13. *Network (1) with controller (4) can achieve finite-time synchronization within t^* if Assumptions 4–8 and the following conditions (1)-(3) hold:*

(1) If $\tilde{p} \neq \tilde{r}$, $\tilde{\eta}_{\tilde{p}}\alpha^{\tilde{p}}/(1-\tilde{\lambda}) - \tilde{\beta}_{\tilde{r}} \leq 0$, otherwise, if $\tilde{p} = \tilde{r}$, $\tilde{\eta}_{\tilde{p}}\alpha^{\tilde{p}}/(1-\tilde{\lambda}) - \tilde{\beta}_{\tilde{r}} \geq 0$, where $\tilde{r}, \tilde{p} \in \mathfrak{J}$, $\tilde{\eta}_{\tilde{r}} \geq \alpha^{\tilde{r}} > 0$, $\tilde{\beta}_{\tilde{r}} > 0$, $0 \leq \tilde{\lambda} < 1$.

(2) The following inequalities are satisfied:

$$\begin{aligned}\tilde{\Xi}^{(1)}_{\tilde{r}} = \Bigg[& \left(\tilde{L}^2 \|\varphi_{(1)}\| + 2\tilde{L}^2_{(1)} \|\varphi_{(2)\tilde{r}}\| \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}}\right) I_N \otimes I_n \\ & + \sum_{\tilde{p}=1}^{v} \frac{\delta_{\tilde{r}\tilde{p}}\tilde{\eta}_{\tilde{p}}}{\tilde{\eta}_{\tilde{r}}} I_N \otimes \tilde{Q} + \frac{\alpha^{\tilde{r}}}{1-\tilde{\lambda}} \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \tilde{F}^{\tilde{k}} + 2 I_N \otimes \tilde{Q}\tilde{B} \\ & + I_N \otimes \tilde{Q}\varphi^{-1}_{(1)}\tilde{Q}^T \\ & + \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \left(\tilde{D}^{\tilde{r}}_{\tilde{k}} \otimes \tilde{Q}\Gamma_{\tilde{k}}\right) \varphi^{-1}_{(2)\tilde{r}} \left(\tilde{D}^{\tilde{r}}_{\tilde{k}} \otimes \tilde{Q}\Gamma_{\tilde{k}}\right)^T \\ & + \tilde{q}_{max}\left(\Lambda^{(1)}_{\tilde{r}} \otimes I_n\right) - 2\left(\sum_{\tilde{k}=1}^{m} \Theta^{\tilde{k}}_{\tilde{r}} \otimes \tilde{Q}\right)\Bigg] \leq 0,\end{aligned} \tag{13}$$

$$\begin{aligned}\tilde{\Xi}^{(2)}_{\tilde{r}} = \Bigg[& 2\tilde{L}^2_{(2)} \|\varphi_{(2)\tilde{r}}\| \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \left(I_N \otimes I_n\right) \\ & + \tilde{q}_{max}\left(\Lambda^{(2)}_{\tilde{r}} \otimes I_n\right) - \alpha^{\tilde{r}} \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \tilde{F}^{\tilde{k}}\Bigg] \leq 0,\end{aligned} \tag{14}$$

where $\varphi_{(1)} \in \mathfrak{R}^{n\times n} > 0$, $\varphi_{(2)\tilde{r}} \in \mathfrak{R}^{Nn\times Nn} > 0$, $\Theta^{\tilde{k}}_{\tilde{r}} = \mathrm{diag}(\theta^{\tilde{k},\tilde{r}}_1, \theta^{\tilde{k},\tilde{r}}_2, \ldots, \theta^{\tilde{k},\tilde{r}}_N)$, $\tilde{\eta}_{\tilde{r}} \geq 1$, $\tilde{Q} = \mathrm{diag}(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n) > 0$, $\tilde{F}^{\tilde{k}} = \mathrm{diag}(\tilde{F}^{\tilde{k}}_1, \tilde{F}^{\tilde{k}}_2, \ldots, \tilde{F}^{\tilde{k}}_N) \in \mathfrak{R}^{Nn\times Nn} > 0$, $\tilde{\iota}^{(1)}_{(\tilde{r})\tilde{i}} = \sum_{\tilde{j}=1}^{N} \tilde{\iota}^{(1)}_{(\tilde{r})\tilde{i}\tilde{j}}$, $\tilde{\iota}^{(2)}_{(\tilde{r})\tilde{i}} = \sum_{\tilde{j}=1}^{N} \tilde{\iota}^{(2)}_{(\tilde{r})\tilde{i}\tilde{j}}$, $\tilde{q}_{max} = \lambda_{max}(\tilde{Q})$, $\Lambda^{(1)}_{\tilde{r}} = \mathrm{diag}(\tilde{\iota}^{(1)}_{(\tilde{r})1}, \tilde{\iota}^{(1)}_{(\tilde{r})2}, \ldots, \tilde{\iota}^{(1)}_{(\tilde{r})N})$, $\Lambda^{(2)}_{\tilde{r}} = \mathrm{diag}(\tilde{\iota}^{(2)}_{(\tilde{r})1}, \tilde{\iota}^{(2)}_{(\tilde{r})2}, \ldots, \tilde{\iota}^{(2)}_{(\tilde{r})N})$.

(3) t^* is estimated by $t^* \leq \tilde{\tau}_M + (\alpha_{max})^{(1+\tilde{\gamma})/2}[V(\omega(0), 0, \tilde{r}(0))]^{(1-\tilde{\gamma})/2}/\kappa\mu(1-\tilde{\gamma})$, where $\alpha_{max} = \max_{\tilde{r}\in\mathfrak{J}}\{\alpha^{\tilde{r}}\}$, $\kappa_{\tilde{k}} = \min^{\tilde{r}=1,2,\ldots,v}_{\tilde{i}=1,2,\ldots,N}\{\kappa^{\tilde{k},\tilde{r}}_{\tilde{i}}\}$, $\kappa = \min_{\tilde{k}=1,2\ldots,m}\{\kappa_{\tilde{k}}\}$, $\mu = \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} * \min_{\tilde{k}=1,2,\ldots,m}\{\lambda_{min}(\Gamma^{(1+\tilde{\gamma})/2}_{\tilde{k}})\}$, $0 < \tilde{\gamma} < 1$, $V(\omega(0), 0, \tilde{r}(0)) = \tilde{\eta}_{\tilde{r}(0)} \sum_{\tilde{i}=1}^{N} \omega^T_{\tilde{i}}(0)\tilde{Q}\omega_{\tilde{i}}(0)$.

Proof. Construct a Lyapunov function candidate as

$$\begin{aligned}V(\omega(t), t, \tilde{r}) = \tilde{\eta}_{\tilde{r}} \Bigg[& \sum_{\tilde{i}=1}^{N} \omega^T_{\tilde{i}}(t)\tilde{Q}\omega_{\tilde{i}}(t) \\ & + \frac{\alpha^{\tilde{r}}}{1-\tilde{\lambda}} \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \int_{t-\tilde{\tau}(t)}^{t} \omega^T(s)\tilde{F}^{\tilde{k}}\omega(s)\,ds\Bigg],\end{aligned} \tag{15}$$

where $\tilde{\eta}_{\tilde{r}} \geq \alpha^{\tilde{r}} > 0$, $\tilde{\eta}_{\tilde{r}} \geq 1$, $\tilde{Q} = \mathrm{diag}(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n) > 0$, $\tilde{F}^{\tilde{k}} = \mathrm{diag}(\tilde{F}^{\tilde{k}}_1, \tilde{F}^{\tilde{k}}_2, \ldots, \tilde{F}^{\tilde{k}}_N) \in \mathfrak{R}^{Nn\times Nn} > 0$.

Computing $\mathscr{L}V(\omega(t), t, \tilde{r})$ along the trajectory of error system (5), we can obtain

$$\begin{aligned}\mathscr{L}V(\omega(t), t, \tilde{r}) = {} & V_t(\omega(t), t, \tilde{r}) + V_\omega(\omega(t), t, \tilde{r}) \\ & \cdot \Bigg[F(\omega_{\tilde{i}}(t)) + \tilde{B}\omega_{\tilde{i}}(t) \\ & + \sum_{\tilde{k}=1}^{m}\sum_{\tilde{j}=1}^{N} \tilde{c}_{\tilde{k}} d^{\tilde{k}}_{\tilde{i}\tilde{j}} \Gamma_{\tilde{k}} \tilde{G}\left(\omega_{\tilde{j}}(t), \omega_{\tilde{j}}(t-\tilde{\tau}(t))\right) - \sum_{\tilde{k}=1}^{m} \theta^{\tilde{k},\tilde{r}}_{\tilde{i}} \omega_{\tilde{i}}(t) \\ & - \frac{1}{\alpha^{\tilde{r}}} \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \kappa^{\tilde{k},\tilde{r}}_{\tilde{i}} \tilde{Q}^{(\tilde{\gamma}-1)/2} \Gamma^{(\tilde{\gamma}+1)/2}_{\tilde{k}} \mathrm{sign}(\omega_i(t)) |\omega_i(t)|^{\tilde{\gamma}} \\ & - \mu\tilde{Q}^{-1} \sum_{\tilde{k}=1}^{m} \kappa^{\tilde{k},\tilde{r}}_{\tilde{i}} \left(\frac{\tilde{c}_{\tilde{k}}}{1-\tilde{\lambda}} \int_{t-\tilde{\tau}(t)}^{t} \omega^T_i(s)\tilde{F}^{\tilde{k}}_{\tilde{i}}\omega_i(s)\,ds\right)^{(\tilde{\gamma}+1)/2} \\ & \cdot \frac{\omega_{\tilde{i}}(t)}{\|\omega_{\tilde{i}}(t)\|^2}\Bigg] + \sum_{\tilde{p}=1}^{v} \delta_{\tilde{r}\tilde{p}} V(\omega(t), t, \tilde{p}) \\ & + \tilde{\eta}_{\tilde{r}} \sum_{\tilde{i}=1}^{N} trace\left[\hat{\sigma}^T_{\tilde{i}}(\omega(t), \omega(t-\tilde{\tau}(t)), t, \tilde{r})\right.\end{aligned}$$

$$\cdot \widetilde{Q}\widehat{\sigma}_{\tilde{i}}(\omega(t),\omega(t-\tilde{\tau}(t)),t,\tilde{r})\Big] = \frac{\tilde{\eta}_{\tilde{r}}\alpha^{\tilde{r}}}{1-\tilde{\lambda}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\Big[\omega^T(t)$$
$$\cdot \widetilde{F}^{\tilde{k}}\omega(t) - \left(1-\dot{\tilde{\tau}}(t)\right)\omega^T(t-\tilde{\tau}(t))\widetilde{F}^{\tilde{k}}\omega(t-\tilde{\tau}(t))\Big]$$
$$+ 2\tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\Bigg[F(\omega_{\tilde{i}}(t)) + \widetilde{B}\omega_{\tilde{i}}(t)$$
$$+ \sum_{\tilde{k}=1}^{m}\sum_{\tilde{j}=1}^{N}\tilde{c}_{\tilde{k}}\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}\Gamma_{\tilde{k}}\widetilde{G}\left(\tilde{\omega}_{\tilde{j}}(t),\tilde{\omega}_{\tilde{j}}(t-\tilde{\tau}(t))\right) - \sum_{\tilde{k}=1}^{m}\theta_{\tilde{i}}^{\tilde{k},\tilde{r}}\omega_{\tilde{i}}(t)$$
$$- \frac{1}{\alpha^{\tilde{r}}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}}\widetilde{Q}^{(\tilde{\gamma}-1)/2}\Gamma_{\tilde{k}}^{(\tilde{\gamma}+1)/2}\operatorname{sign}(\omega_i(t))\left|\omega_i(t)\right|^{\tilde{\gamma}}$$
$$- \mu\widetilde{Q}^{-1}\sum_{\tilde{k}=1}^{m}\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}}\left(\frac{\tilde{c}_{\tilde{k}}}{1-\tilde{\lambda}}\int_{t-\tilde{\tau}(t)}^{t}\omega_i^T(s)\widetilde{F}_{\tilde{i}}^{\tilde{k}}\omega_i(s)\,ds\right)^{(\tilde{\gamma}+1)/2}$$
$$\cdot \frac{\omega_{\tilde{i}}(t)}{\left\|\omega_{\tilde{i}}(t)\right\|^2}\Bigg] + \sum_{\tilde{p}=1}^{v}\delta_{\tilde{r}\tilde{p}}V(\omega(t),t,\tilde{p})$$
$$+ \tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}trace\Big[\widehat{\sigma}_{\tilde{i}}^T(\omega(t),\omega(t-\tilde{\tau}(t)),t,\tilde{r})$$
$$\cdot \widetilde{Q}\widehat{\sigma}_{\tilde{i}}(\omega(t),\omega(t-\tilde{\tau}(t)),t,\tilde{r})\Big]. \tag{16}$$

From Assumption 4 and Lemma 10, we have

$$2\tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}F(\omega_{\tilde{i}}(t)) \le \tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\Big[\omega_{\tilde{i}}^T(t)\widetilde{Q}\varphi_{(1)}^{-1}\widetilde{Q}^T\omega_{\tilde{i}}(t)$$
$$+ F^T(\omega_{\tilde{i}}(t))\varphi_{(1)}F(\omega_{\tilde{i}}(t))\Big] \le \tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\Big[\omega_{\tilde{i}}^T(t)$$
$$\cdot \widetilde{Q}\varphi_{(1)}^{-1}\widetilde{Q}^T\omega_{\tilde{i}}(t) + \tilde{L}^2\left\|\varphi_{(1)}\right\|\omega_{\tilde{i}}^T(t)\omega_{\tilde{i}}(t)\Big], \tag{17}$$

$$2\tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\sum_{\tilde{k}=1}^{m}\sum_{\tilde{j}=1}^{N}\tilde{c}_{\tilde{k}}\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k},\tilde{r}}\Gamma_{\tilde{k}}\widetilde{G}\left(\omega_{\tilde{j}}(t),\omega_{\tilde{j}}(t-\tilde{\tau}(t))\right)$$
$$= 2\tilde{\eta}_{\tilde{r}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\omega^T(t)\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}}\otimes\widetilde{Q}\Gamma_{\tilde{k}}\right)\widetilde{G}(\omega(t),\omega(t-\tilde{\tau}(t)))$$
$$\le \tilde{\eta}_{\tilde{r}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\Big[\omega^T(t)\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}}\otimes\widetilde{Q}\Gamma_{\tilde{k}}\right)\varphi_{(2)\tilde{r}}^{-1}\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}}\otimes\widetilde{Q}\Gamma_{\tilde{k}}\right)^T$$
$$\cdot \omega(t) + \widetilde{G}^T(\omega(t),\omega(t-\tilde{\tau}(t)))$$
$$\cdot \varphi_{(2)\tilde{r}}\widetilde{G}(\omega(t),\omega(t-\tilde{\tau}(t)))\Big] \le \tilde{\eta}_{\tilde{r}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\Big[\omega^T(t)$$
$$\cdot \left(\widetilde{D}_{\tilde{k}}^{\tilde{r}}\otimes\widetilde{Q}\Gamma_{\tilde{k}}\right)\varphi_{(2)\tilde{r}}^{-1}\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}}\otimes\widetilde{Q}\Gamma_{\tilde{k}}\right)^T\omega(t) + 2\tilde{L}_{(1)}^2\left\|\varphi_{(2)\tilde{r}}\right\|$$
$$\cdot \omega^T(t)\omega(t) + 2\tilde{L}_{(2)}^2\left\|\varphi_{(2)\tilde{r}}\right\|\omega^T(t-\tilde{\tau}(t))$$
$$\cdot \omega(t-\tilde{\tau}(t))\Big]. \tag{18}$$

$$-2\tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\sum_{\tilde{k}=1}^{m}\theta_{\tilde{i}}^{\tilde{k},\tilde{r}}\omega_{\tilde{i}}(t) = -2\tilde{\eta}_{\tilde{r}}\omega^T(t)\left(\sum_{\tilde{k}=1}^{m}\Theta_{\tilde{r}}^{\tilde{k}}\otimes\widetilde{Q}\right)$$
$$\cdot \omega(t), \tag{19}$$

where $\Theta_{\tilde{r}}^{\tilde{k}} = \operatorname{diag}(\theta_1^{\tilde{k},\tilde{r}},\theta_2^{\tilde{k},\tilde{r}},\ldots,\theta_N^{\tilde{k},\tilde{r}})$.

By Lemma 9, we get

$$-2\tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\frac{1}{\alpha^{\tilde{r}}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}}\widetilde{Q}^{(\tilde{\gamma}-1)/2}\Gamma_{\tilde{k}}^{(\tilde{\gamma}+1)/2}\operatorname{sign}(\omega_{\tilde{i}}(t))\left|\omega_{\tilde{i}}(t)\right|^{\tilde{\gamma}}$$
$$= -\frac{2\tilde{\eta}_{\tilde{r}}}{\alpha^{\tilde{r}}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\sum_{\tilde{i}=1}^{N}\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}}\omega_{\tilde{i}}^T(t)\widetilde{Q}^{(1+\tilde{\gamma})/2}\Gamma_{\tilde{k}}^{(\tilde{\gamma}+1)/2}\operatorname{sign}(\omega_{\tilde{i}}(t))$$
$$\cdot \left|\omega_{\tilde{i}}(t)\right|^{\tilde{\gamma}} \le -2\kappa\mu\left(\frac{\tilde{\eta}_{\tilde{r}}}{\alpha^{\tilde{r}}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\omega_{\tilde{i}}(t)\right)^{(1+\tilde{\gamma})/2}, \tag{20}$$

$$-2\tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\mu\widetilde{Q}^{-1}\sum_{\tilde{k}=1}^{m}\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}}\Bigg(\frac{\tilde{c}_{\tilde{k}}}{1-\tilde{\lambda}}$$
$$\cdot \int_{t-\tilde{\tau}(t)}^{t}\omega_{\tilde{i}}^T(s)\widetilde{F}_{\tilde{i}}^{\tilde{k}}\omega_{\tilde{i}}(s)\,ds\Bigg)^{(1+\tilde{\gamma})/2}\frac{\omega_{\tilde{i}}(t)}{\left\|\omega_{\tilde{i}}(t)\right\|^2}$$
$$= -2\tilde{\eta}_{\tilde{r}}\mu\sum_{\tilde{k}=1}^{m}\sum_{\tilde{i}=1}^{N}\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}}\left(\frac{\tilde{c}_{\tilde{k}}}{1-\tilde{\lambda}}\int_{t-\tilde{\tau}(t)}^{t}\omega_{\tilde{i}}^T(s)\widetilde{F}_{\tilde{i}}^{\tilde{k}}\omega_{\tilde{i}}(s)\,ds\right)^{(1+\tilde{\gamma})/2}$$
$$\le -2\tilde{\eta}_{\tilde{r}}\mu\kappa\sum_{\tilde{k}=1}^{m}\left[\sum_{\tilde{i}=1}^{N}\left(\frac{\tilde{c}_{\tilde{k}}}{1-\tilde{\lambda}}\int_{t-\tilde{\tau}(t)}^{t}\omega_{\tilde{i}}^T(s)\widetilde{F}_{\tilde{i}}^{\tilde{k}}\omega_{\tilde{i}}(s)\,ds\right)^{(1+\tilde{\gamma})/2}\right]$$
$$\le -2\tilde{\eta}_{\tilde{r}}\mu\kappa\sum_{\tilde{k}=1}^{m}\left(\frac{\tilde{c}_{\tilde{k}}}{1-\tilde{\lambda}}\int_{t-\tilde{\tau}(t)}^{t}\omega^T(s)\widetilde{F}^{\tilde{k}}\omega(s)\,ds\right)^{(1+\tilde{\gamma})/2}$$
$$\le -2\kappa\mu\left(\frac{\tilde{\eta}_{\tilde{r}}}{1-\tilde{\lambda}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\int_{t-\tilde{\tau}(t)}^{t}\omega^T(s)\widetilde{F}^{\tilde{k}}\omega(s)\,ds\right)^{(1+\tilde{\gamma})/2}, \tag{21}$$

where $\kappa_{\tilde{k}} = \min_{\tilde{i}=1,2,\ldots,N}^{\tilde{r}=1,2,\ldots,v}\{\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}}\}$, $\kappa = \min_{\tilde{k}=1,2\ldots,m}\{\kappa_{\tilde{k}}\}$, $\mu = \sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}} * \min_{\tilde{k}=1,2,\ldots,m}\{\lambda_{min}(\Gamma_{\tilde{k}}^{(1+\tilde{\gamma})/2})\}$.

According to $\sum_{\tilde{r},\tilde{p}\in\mathfrak{J}}\delta_{\tilde{r}\tilde{p}} = 0$, for $\forall\tilde{\beta}_{\tilde{r}} > 0(\tilde{r}\in\mathfrak{J})$, there is $\sum_{\tilde{r},\tilde{p}\in\mathfrak{J}}\delta_{\tilde{r}\tilde{p}}\tilde{\beta}_{\tilde{r}} = 0$. Therefore, we have

$$\sum_{\tilde{p}=1}^{v}\delta_{\tilde{r}\tilde{p}}V(\omega(t),t,\tilde{p}) = \sum_{\tilde{p}=1}^{v}\delta_{\tilde{r}\tilde{p}}\tilde{\eta}_{\tilde{p}}\Bigg[\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\omega_{\tilde{i}}(t)$$
$$+ \frac{\alpha^{\tilde{p}}}{1-\tilde{\lambda}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\int_{t-\tilde{\tau}(t)}^{t}\omega^T(s)\widetilde{F}^{\tilde{k}}\omega(s)\,ds\Bigg] = \sum_{\tilde{p}=1}^{v}\delta_{\tilde{r}\tilde{p}}\tilde{\eta}_{\tilde{p}}$$
$$\cdot \sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^T(t)\widetilde{Q}\omega_{\tilde{i}}(t) + \sum_{\tilde{p}=1}^{v}\delta_{\tilde{r}\tilde{p}}\left(\frac{\tilde{\eta}_{\tilde{p}}\alpha^{\tilde{p}}}{1-\tilde{\lambda}} - \tilde{\beta}_{\tilde{r}}\right)$$
$$\cdot \sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\int_{t-\tilde{\tau}(t)}^{t}\omega^T(s)\widetilde{F}^{\tilde{k}}\omega(s)\,ds. \tag{22}$$

From Assumption 7, we obtain

$$\begin{aligned}
&\tilde{\eta}_{\tilde{r}}\sum_{\tilde{i}=1}^{N} trace\left[\hat{\sigma}_{\tilde{i}}^{T}\left(\omega(t),\omega(t-\tilde{\tau}(t)),t,\tilde{r}\right)\right.\\
&\quad\left.\cdot\widetilde{Q}\hat{\sigma}_{\tilde{i}}\left(\omega(t),\omega(t-\tilde{\tau}(t)),t,\tilde{r}\right)\right]\\
&\quad\le\tilde{\eta}_{\tilde{r}}\tilde{q}_{\max}\sum_{\tilde{i}=1}^{N}\left[\tilde{\iota}_{(\tilde{r})\tilde{i}}^{(1)}\omega_{\tilde{i}}^{T}(t)\,\omega_{\tilde{i}}(t)+\tilde{\iota}_{(\tilde{r})\tilde{i}}^{(2)}\omega_{\tilde{i}}^{T}(t-\tilde{\tau}(t))\right.\\
&\quad\left.\cdot\,\omega_{\tilde{i}}(t-\tilde{\tau}(t))\right]=\tilde{\eta}_{\tilde{r}}\tilde{q}_{\max}\left[\omega^{T}(t)\left(\Lambda_{\tilde{r}}^{(1)}\otimes I_n\right)\omega(t)\right.\\
&\quad\left.+\,\omega^{T}(t-\tilde{\tau}(t))\left(\Lambda_{\tilde{r}}^{(2)}\otimes I_n\right)\omega(t-\tilde{\tau}(t))\right],
\end{aligned}\tag{23}$$

where $\tilde{\iota}_{(\tilde{r})\tilde{i}}^{(1)}=\sum_{\tilde{j}=1}^{N}\tilde{\iota}_{(\tilde{r})\tilde{i}\tilde{j}}^{(1)}$, $\tilde{\iota}_{(\tilde{r})\tilde{i}}^{(2)}=\sum_{\tilde{j}=1}^{N}\tilde{\iota}_{(\tilde{r})\tilde{i}\tilde{j}}^{(2)}$, $\tilde{q}_{\max}=\lambda_{\max}(\widetilde{Q})$, $\Lambda_{\tilde{r}}^{(1)}=\operatorname{diag}(\tilde{\iota}_{(\tilde{r})1}^{(1)},\tilde{\iota}_{(\tilde{r})2}^{(1)},\ldots,\tilde{\iota}_{(\tilde{r})N}^{(1)})$, $\Lambda_{\tilde{r}}^{(2)}=\operatorname{diag}(\tilde{\iota}_{(\tilde{r})1}^{(2)},\tilde{\iota}_{(\tilde{r})2}^{(2)},\ldots,\tilde{\iota}_{(\tilde{r})N}^{(2)})$.

Substituting inequalities (17)-(23) into (16), combining condition (1) of Theorem 13, and taking the expectation on both sides of (24), we can derive that

$$\begin{aligned}
&\mathbb{E}\left[\mathscr{L}V\left(\omega(t),t,\tilde{r}\right)\right]\le\mathbb{E}\left\{\tilde{\eta}_{\tilde{r}}\left[\omega^{T}(t)\,\tilde{\Xi}_{\tilde{r}}^{(1)}\omega(t)+\omega^{T}(t\right.\right.\\
&\quad\left.-\,\tilde{\tau}(t))\,\tilde{\Xi}_{\tilde{r}}^{(2)}\omega(t-\tilde{\tau}(t))\right]-2\kappa\mu\left[\left(\frac{\tilde{\eta}_{\tilde{r}}}{\alpha^{\tilde{r}}}\right.\right.\\
&\quad\left.\cdot\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^{T}(t)\,\widetilde{Q}\omega_{\tilde{i}}(t)\right)^{(1+\tilde{\gamma})/2}+\left(\frac{\tilde{\eta}_{\tilde{r}}}{1-\tilde{\lambda}}\right.\\
&\quad\left.\left.\left.\cdot\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\int_{t-\tilde{\tau}(t)}^{t}\omega^{T}(s)\,\tilde{F}^{\tilde{k}}\omega(s)\,ds\right)^{(1+\tilde{\gamma})/2}\right]\right\},
\end{aligned}\tag{24}$$

where

$$\begin{aligned}
\tilde{\Xi}_{\tilde{r}}^{(1)}&=\left(\tilde{L}^{2}\left\|\varphi_{(1)}\right\|+2\tilde{L}_{(1)}^{2}\left\|\varphi_{(2)\tilde{r}}\right\|\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\right)I_N\otimes I_n\\
&\quad+\sum_{\tilde{p}=1}^{v}\frac{\delta_{\tilde{r}\tilde{p}}\tilde{\eta}_{\tilde{p}}}{\tilde{\eta}_{\tilde{r}}}I_N\otimes\widetilde{Q}+\frac{\alpha^{\tilde{r}}}{1-\tilde{\lambda}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\tilde{F}^{\tilde{k}}+2I_N\\
&\quad\otimes\widetilde{Q}\widetilde{B}+I_N\otimes\widetilde{Q}\varphi_{(1)}^{-1}\widetilde{Q}^{T}\\
&\quad+\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}}\otimes\widetilde{Q}\Gamma_{\tilde{k}}\right)\varphi_{(2)\tilde{r}}^{-1}\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}}\otimes\widetilde{Q}\Gamma_{\tilde{k}}\right)^{T}\\
&\quad+\tilde{q}_{\max}\left(\Lambda_{\tilde{r}}^{(1)}\otimes I_n\right)-2\left(\sum_{\tilde{k}=1}^{m}\Theta_{\tilde{r}}^{\tilde{k}}\otimes\widetilde{Q}\right),
\end{aligned}\tag{25}$$

$$\begin{aligned}
\tilde{\Xi}_{\tilde{r}}^{(2)}&=2\tilde{L}_{(2)}^{2}\left\|\varphi_{(2)\tilde{r}}\right\|\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\left(I_N\otimes I_n\right)+\tilde{q}_{\max}\left(\Lambda_{\tilde{r}}^{(2)}\otimes I_n\right)\\
&\quad-\alpha^{\tilde{r}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\tilde{F}^{\tilde{k}}.
\end{aligned}\tag{26}$$

By the condition (2) of Theorem 13, and Lemma 9, we have

$$\begin{aligned}
&\mathbb{E}\left[\mathscr{L}V\left(\omega(t),t,\tilde{r}\right)\right]\\
&\quad\le\mathbb{E}\left\{-2\kappa\mu\left[\left(\frac{\tilde{\eta}_{\tilde{r}}}{\alpha^{\tilde{r}}}\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^{T}(t)\,\widetilde{Q}\omega_{\tilde{i}}(t)\right)^{(1+\tilde{\gamma})/2}\right.\right.\\
&\quad\left.\left.+\left(\frac{\tilde{\eta}_{\tilde{r}}}{1-\tilde{\lambda}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\int_{t-\tilde{\tau}(t)}^{t}\omega^{T}(s)\,\tilde{F}^{\tilde{k}}\omega(s)\,ds\right)^{(1+\tilde{\gamma})/2}\right]\right\}\\
&\quad\le\mathbb{E}\left\{-2\kappa\mu\left[\frac{\tilde{\eta}_{\tilde{r}}}{\alpha^{\tilde{r}}}\left(\sum_{\tilde{i}=1}^{N}\omega_{\tilde{i}}^{T}(t)\,\widetilde{Q}\omega_{\tilde{i}}(t)\right.\right.\right.\\
&\quad\left.\left.\left.+\frac{\alpha^{\tilde{r}}}{1-\tilde{\lambda}}\sum_{\tilde{k}=1}^{m}\tilde{c}_{\tilde{k}}\int_{t-\tilde{\tau}(t)}^{t}\omega^{T}(s)\,\tilde{F}^{\tilde{k}}\omega(s)\,ds\right)\right]^{(1+\tilde{\gamma})/2}\right\}\\
&\quad\le-\frac{2\kappa\mu}{\left(\alpha_{\max}\right)^{(1+\tilde{\gamma})/2}}\left(\mathbb{E}\left[V\left(\omega(t),t,\tilde{r}\right)\right]\right)^{(1+\tilde{\gamma})/2},
\end{aligned}\tag{27}$$

where $\alpha_{\max}=\max_{\tilde{r}\in\mathfrak{J}}\{\alpha^{\tilde{r}}\}$.

From Lemma 11, one gets

$$t^{*}\le\tilde{\tau}_{M}+\frac{\left(\alpha_{\max}\right)^{(1+\tilde{\gamma})/2}\left[V\left(\omega(0),0,\tilde{r}(0)\right)\right]^{(1-\tilde{\gamma})/2}}{\kappa\mu\left(1-\tilde{\gamma}\right)}.\tag{28}$$

That means if $t\ge t^{*}$, there is $V(\omega(t),t,\tilde{r})=0$. Combining equality (15), we obtain $\omega_{\tilde{i}}(t)=0$ if $t\ge t^{*}$. From Definition 3, it is seen that if $t\ge t^{*}$, there is $\mathbb{E}\|z_{\tilde{i}}(t)-\tilde{s}(t)\|=0$. Therefore, network (1) can achieve finite-time synchronization with the controller (4) within t^{*}. The proof is completed. □

Remark 14. In fact, if diffusive coupling condition $\tilde{d}_{\tilde{i}\tilde{i}}^{\tilde{k}}(\tilde{r}(t))=-\sum_{\tilde{j}=1,\tilde{j}\neq\tilde{i}}^{N}\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t))$ of network (1) is removed, Theorem 13 still exists. The reason is as follows. During the process of proving Theorem 13, in order to obtain inequality (18), Assumption 4 and Lemma 10 are used. From inequality (18), it is seen that diffusive coupling condition $\tilde{d}_{\tilde{i}\tilde{i}}^{\tilde{k}}(\tilde{r}(t))=-\sum_{\tilde{j}=1,\tilde{j}\neq\tilde{i}}^{N}\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t))$ of network (1) is not utilized. That means if coupled matrix $\widetilde{D}_{\tilde{k}}^{\tilde{r}}$ does not satisfy $\tilde{d}_{\tilde{i}\tilde{i}}^{\tilde{k}}(\tilde{r}(t))=-\sum_{\tilde{j}=1,\tilde{j}\neq\tilde{i}}^{N}\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t))$, inequality (18) still holds. This shows that Theorem 13 is not related to diffusive coupling condition $\tilde{d}_{\tilde{i}\tilde{i}}^{\tilde{k}}(\tilde{r}(t))=-\sum_{\tilde{j}=1,\tilde{j}\neq\tilde{i}}^{N}\tilde{d}_{\tilde{i}\tilde{j}}^{\tilde{k}}(\tilde{r}(t))$.

Remark 15. Combining Remark 6, Theorem 13, and Lyapunov stability theory, the effect of nonlinear coupled function $\tilde{g}(\cdot,\cdot)$ for synchronization dynamics of network (1) is

analyzed as follows. From the inequalities (18), (24), and (27), it is observed that if the same synchronization control rule makes inequalities (18), (24), and (27) hold, according to Remark 6, we can obtain that if nonlinearity of $\widetilde{g}(\cdot,\cdot)$ is increased, linearizing parameters $\widetilde{L}_1$ and $\widetilde{L}_2$ will also increase. This causes that $\mathbb{E}[\mathscr{L}V(\omega(t),t,\widetilde{r})] \leq 0$ also increases. That means with increasing nonlinearity of $\widetilde{g}(\cdot,\cdot)$, synchronization dynamics of network (1) becomes more and more poor. Actually, the conclusion is not entirely correct. The reason is as follows. From inequality (18), it is derived that under $t_0 \leq t < t^*$, where t_0 is the initial time and t^* is defined in Definition 3, if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$, $\omega^T(t) > 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$, there must be $2\widetilde{\eta}_{\widetilde{r}} \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \omega^T(t)(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}})\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$. According to Remark 6, if nonlinearity of $\widetilde{g}(\cdot,\cdot)$ becomes serious, $\|\widetilde{g}(\cdot,\cdot)\|$ is increased. This causes that $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t)))$ is also increased if $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$. Therefore, if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$, $\omega^T(t) > 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$, under Theorem 13 and the same synchronization rule, with increasing nonlinearity of $\widetilde{g}(\cdot,\cdot)$, synchronization dynamics of network (1) becomes poorer. By the above similar analysis, if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$, $\omega^T(t) < 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) < 0$, or if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$, $\omega^T(t) > 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) < 0$, or if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$, $\omega^T(t) < 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$, the quite same conclusion can be obtained. If $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$, $\omega^T(t) < 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$, or if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$, $\omega^T(t) > 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) < 0$, or if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$, $\omega^T(t) < 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$, or if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$, $\omega^T(t) > 0$ and $\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) < 0$, this leads to $2\widetilde{\eta}_{\widetilde{r}} \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \omega^T(t)(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}})\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) < 0$. Thus, it is obtained that if nonlinearity of $\widetilde{g}(\cdot,\cdot)$ is increased, under Theorem 13 and the same synchronization rule, synchronization dynamics of network (1) becomes better. If $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}$ satisfies $d_{\widetilde{i}\widetilde{i}}^{\widetilde{k}}(\widetilde{r}(t)) = -\sum_{\widetilde{j}=1,\widetilde{j}\neq\widetilde{i}}^{N} \widetilde{d}_{\widetilde{i}\widetilde{j}}^{\widetilde{k}}(\widetilde{r}(t))$, it is difficult to get the relationship between synchronization dynamics of network (1) and nonlinearity of nonlinear coupled function $\widetilde{g}(\cdot,\cdot)$. For example, let $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} = [-1,1;1,-1]$, then there is $\det(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}) = 0$. This means that if $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}$ satisfies $d_{\widetilde{i}\widetilde{i}}^{\widetilde{k}}(\widetilde{r}(t)) = -\sum_{\widetilde{j}=1,\widetilde{j}\neq\widetilde{i}}^{N} \widetilde{d}_{\widetilde{i}\widetilde{j}}^{\widetilde{k}}(\widetilde{r}(t))$, $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$ and $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$ must not hold. This causes that it is not derived how nonlinearity of nonlinear coupled function $\widetilde{g}(\cdot,\cdot)$ affects synchronization dynamics of network (1). Because of $\omega_i(t) = \widetilde{z}_{\widetilde{i}}(t) - \widetilde{s}(t)$ and $\widetilde{G}(\omega_{\widetilde{j}}(t),\omega_{\widetilde{j}}(t-\widetilde{\tau}(t))) = \widetilde{g}(\widetilde{z}_{\widetilde{j}}(t),\widetilde{z}_{\widetilde{j}}(t-\widetilde{\tau}(t))) - \widetilde{g}(\widetilde{s}(t),\widetilde{s}(t-\widetilde{\tau}(t)))$, combining the above analysis, synchronization dynamics of network (1) is related to the initial state $\widetilde{z}(0)$, synchronization state $\widetilde{s}(t)$, nonlinear coupled function $\widetilde{g}(\cdot,\cdot)$, and coupled matrix $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}$.

Theorem 16. *Network (2) with controller (4) can achieve finite-time synchronization within t^* if Assumptions 4–8 and the following conditions (1)-(3) hold:*

(1) If $\widetilde{p} \neq \widetilde{r}$, $\widetilde{\eta}_{\widetilde{p}}\alpha^{\widetilde{p}}/(1-\widetilde{\lambda}) - \widetilde{\beta}_{\widetilde{r}} \leq 0$, otherwise, if $\widetilde{p} = \widetilde{r}$, $\widetilde{\eta}_{\widetilde{p}}\alpha^{\widetilde{p}}/(1-\widetilde{\lambda}) - \widetilde{\beta}_{\widetilde{r}} \geq 0$, where $\widetilde{r},\widetilde{p} \in \mathfrak{J}$, $\widetilde{\eta}_{\widetilde{r}} \geq \alpha^{\widetilde{r}} > 0$, $\widetilde{\beta}_{\widetilde{r}} > 0$, $0 \leq \widetilde{\lambda} < 1$.

(2) The following inequalities are satisfied:

$$\begin{aligned}\widehat{\Xi}_{\widetilde{r}}^{(1)} = \Bigg[& \widetilde{L}^2 \|\varphi_{(1)}\| I_N \otimes I_n + \sum_{\widetilde{p}=1}^{v} \frac{\delta_{\widetilde{r}\widetilde{p}}\widetilde{\eta}_{\widetilde{p}}}{\widetilde{\eta}_{\widetilde{r}}} I_N \otimes \widetilde{Q} \\ & + \frac{\alpha^{\widetilde{r}}}{1-\widetilde{\lambda}} \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \widetilde{F}^{\widetilde{k}} + 2I_N \otimes \widetilde{Q}\widetilde{B} + I_N \otimes \widetilde{Q}\varphi_{(1)}^{-1}\widetilde{Q}^T \\ & + \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \left(2\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}}\right) \\ & + \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \left(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}}\right) \varphi_{(3)\widetilde{r}}^{-1} \left(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}}\right)^T \\ & + \widetilde{q}_{max}\left(\Lambda_{\widetilde{r}}^{(1)} \otimes I_n\right) - 2\left(\sum_{\widetilde{k}=1}^{m} \Theta_{\widetilde{r}}^{\widetilde{k}} \otimes \widetilde{Q}\right)\Bigg] \leq 0,\end{aligned} \tag{29}$$

$$\begin{aligned}\widehat{\Xi}_{\widetilde{r}}^{(2)} = & \left[\sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \varphi_{(3)\widetilde{r}} + \widetilde{q}_{max}\left(\Lambda_{\widetilde{r}}^{(2)} \otimes I_n\right) - \alpha^{\widetilde{r}} \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \widetilde{F}^{\widetilde{k}}\right] \\ & \leq 0,\end{aligned} \tag{30}$$

where $\varphi_{(1)} \in \mathfrak{R}^{n\times n} > 0$, $\varphi_{(3)\widetilde{r}} \in \mathfrak{R}^{Nn\times Nn} > 0$, $\Theta_{\widetilde{r}}^{\widetilde{k}} = \mathrm{diag}(\theta_1^{\widetilde{k},\widetilde{r}}, \theta_2^{\widetilde{k},\widetilde{r}}, \ldots, \theta_N^{\widetilde{k},\widetilde{r}})$, $\widetilde{\eta}_{\widetilde{r}} \geq 1$, $\widetilde{Q} = \mathrm{diag}(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_n) > 0$, $\widetilde{F}^{\widetilde{k}} = \mathrm{diag}(\widetilde{F}_1^{\widetilde{k}}, \widetilde{F}_2^{\widetilde{k}}, \ldots, \widetilde{F}_N^{\widetilde{k}}) \in R^{Nn\times Nn} > 0$, $\widetilde{t}_{(\widetilde{r})\widetilde{i}}^{(1)} = \sum_{\widetilde{j}=1}^{N} \widetilde{t}_{(\widetilde{r})\widetilde{i}\widetilde{j}}^{(1)}$, $\widetilde{t}_{(\widetilde{r})\widetilde{i}}^{(2)} = \sum_{\widetilde{j}=1}^{N} \widetilde{t}_{(\widetilde{r})\widetilde{i}\widetilde{j}}^{(2)}$, $\widetilde{q}_{max} = \lambda_{max}(\widetilde{Q})$, $\Lambda_{\widetilde{r}}^{(1)} = \mathrm{diag}(\widetilde{t}_{(\widetilde{r})1}^{(1)}, \widetilde{t}_{(\widetilde{r})2}^{(1)}, \ldots, \widetilde{t}_{(\widetilde{r})N}^{(1)})$, $\Lambda_{\widetilde{r}}^{(2)} = \mathrm{diag}(\widetilde{t}_{(\widetilde{r})1}^{(2)}, \widetilde{t}_{(\widetilde{r})2}^{(2)}, \ldots, \widetilde{t}_{(\widetilde{r})N}^{(2)})$.

(3) t^* is estimated by $t^* \leq \widetilde{\tau}_M + (\alpha_{max})^{(1+\widetilde{\gamma})/2}[V(\omega(0),0,\widetilde{r}(0))]^{(1-\widetilde{\gamma})/2}/\kappa\mu(1-\widetilde{\gamma})$, where $\alpha_{max} = \max_{\widetilde{r}\in\mathfrak{J}}\{\alpha^{\widetilde{r}}\}$, $\kappa_{\widetilde{k}} = \min_{\widetilde{i}=1,2,\ldots,N}^{\widetilde{r}=1,2,\ldots,\upsilon}\{\kappa_{\widetilde{i}}^{\widetilde{k},\widetilde{r}}\}$, $\kappa = \min_{\widetilde{k}=1,2\ldots,m}\{\kappa_{\widetilde{k}}\}$, $0 < \widetilde{\gamma} < 1$, $V(\omega(0),0,\widetilde{r}(0)) = \widetilde{\eta}_{\widetilde{r}(0)} \sum_{\widetilde{i}=1}^{N} \omega_{\widetilde{i}}^T(0)\widetilde{Q}\omega_{\widetilde{i}}(0)$, $\mu = \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} * \min_{\widetilde{k}=1,2,\ldots,m}\{\lambda_{min}(\Gamma_{\widetilde{k}}^{(1+\widetilde{\gamma})/2})\}$.

Proof. Construct a Lyapunov function candidate as

$$\begin{aligned}V(\omega(t),t,\widetilde{r}) = \widetilde{\eta}_{\widetilde{r}} \Bigg[& \sum_{\widetilde{i}=1}^{N} \omega_{\widetilde{i}}^T(t)\widetilde{Q}\omega_{\widetilde{i}}(t) \\ & + \frac{\alpha^{\widetilde{r}}}{1-\widetilde{\lambda}} \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \int_{t-\widetilde{\tau}(t)}^{t} \omega^T(s)\widetilde{F}^{\widetilde{k}}\omega(s)\,ds\Bigg].\end{aligned} \tag{31}$$

□

The next proof is similar to that of Theorem 13.

B. *Nonlinear Coupling Function and Linear Coupling Function Effect Analysis on Synchronization Dynamics of Networks (1) and (2) from Synchronization Control Rule Viewpoint.*

Corollary 17. *Under Theorem 13, network (1) with controller (4) must satisfy Theorem 16 if the following inequalities hold:*

$$\begin{aligned}\widetilde{\Xi}_{\widetilde{r}}^{(1)} = \sum_{\widetilde{k}=1}^{m} \widetilde{c}_{\widetilde{k}} \Bigg[& \left(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}}\right)\left(\varphi_{(2)\widetilde{r}}^{-1} - \varphi_{(3)\widetilde{r}}^{-1}\right)\left(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}}\right)^T \\ & + 2\left(\widetilde{L}_{(1)}^2 \|\varphi_{(2)\widetilde{r}}\| I_N \otimes I_n - \widetilde{D}_{\widetilde{k}}^{\widetilde{r}} \otimes \widetilde{Q}\Gamma_{\widetilde{k}}\right)\Bigg] > 0,\end{aligned} \tag{32}$$

$$\breve{\Xi}_{\tilde{r}}^{(2)} = \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \left(2\tilde{L}_{(2)}^{2} \left\|\varphi_{(2)\tilde{r}}\right\| I_N \otimes I_n - \varphi_{(3)\tilde{r}}\right) > 0. \tag{33}$$

Proof. From Theorems 13–16, one has

$$\begin{aligned}\breve{\Xi}_{\tilde{r}}^{(1)} &= \tilde{\Xi}_{\tilde{r}}^{(1)} - \hat{\Xi}_{\tilde{r}}^{(1)} \\ &= \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \left[\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}}\right)\left(\varphi_{(2)\tilde{r}}^{-1} - \varphi_{(3)\tilde{r}}^{-1}\right)\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}}\right)^{T}\right. \\ &\quad \left. + 2\left(\tilde{L}_{(1)}^{2} \left\|\varphi_{(2)\tilde{r}}\right\| I_N \otimes I_n - \widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}}\right)\right],\end{aligned} \tag{34}$$

$$\begin{aligned}\breve{\Xi}_{\tilde{r}}^{(2)} = \tilde{\Xi}_{\tilde{r}}^{(2)} - \hat{\Xi}_{\tilde{r}}^{(2)} &= \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \left(2\tilde{L}_{(2)}^{2} \left\|\varphi_{(2)\tilde{r}}\right\| I_N \otimes I_n \right. \\ &\quad \left. - \varphi_{(3)\tilde{r}}\right).\end{aligned} \tag{35}$$

If $\breve{\Xi}_{\tilde{r}}^{(1)} > 0$, $\breve{\Xi}_{\tilde{r}}^{(2)} > 0$, one gets $\tilde{\Xi}_{\tilde{r}}^{(1)} > \hat{\Xi}_{\tilde{r}}^{(1)}$ and $\tilde{\Xi}_{\tilde{r}}^{(2)} > \hat{\Xi}_{\tilde{r}}^{(2)}$. Under Theorem 13, there must be $\tilde{\Xi}_{\tilde{r}}^{(1)} \leq 0$ and $\tilde{\Xi}_{\tilde{r}}^{(2)} \leq 0$. Therefore, if $\breve{\Xi}_{\tilde{r}}^{(1)} > 0$, $\breve{\Xi}_{\tilde{r}}^{(2)} > 0$, $\tilde{\Xi}_{\tilde{r}}^{(1)} \leq 0$, and $\tilde{\Xi}_{\tilde{r}}^{(2)} \leq 0$, one obtains $\hat{\Xi}_{\tilde{r}}^{(1)} < 0$ and $\hat{\Xi}_{\tilde{r}}^{(2)} < 0$. That means under Theorem 13, if $\breve{\Xi}_{\tilde{r}}^{(1)} > 0$ and $\breve{\Xi}_{\tilde{r}}^{(2)} > 0$, network (1) with controller (4) must satisfy Theorem 16. This completes the proof. □

Corollary 18. *Under Corollary 17, synchronization control rule for network (1) must make network (2) achieve synchronization within finite-time t^*.*

Proof. From Corollary 17, there is $\breve{\Xi}_{\tilde{r}}^{(1)} = \tilde{\Xi}_{\tilde{r}}^{(1)} - \hat{\Xi}_{\tilde{r}}^{(1)} > 0$ and $\breve{\Xi}_{\tilde{r}}^{(2)} = \tilde{\Xi}_{\tilde{r}}^{(2)} - \hat{\Xi}_{\tilde{r}}^{(2)} > 0$, where $\tilde{\Xi}_{\tilde{r}}^{(1)} \leq 0$ and $\tilde{\Xi}_{\tilde{r}}^{(2)} \leq 0$. This causes $\hat{\Xi}_{\tilde{r}}^{(1)} < \tilde{\Xi}_{\tilde{r}}^{(1)} \leq 0$ and $\hat{\Xi}_{\tilde{r}}^{(2)} < \tilde{\Xi}_{\tilde{r}}^{(2)} \leq 0$. That means, under Corollary 17, synchronization control rule for network (1) must make network (2) achieve synchronization within finite-time t^*. This completes the proof. □

Remark 19. According to inequalities (32) and (33) of Corollary 17, it is derived $\tilde{L}_{(1)}^{2} I_N \otimes I_n > \widetilde{\Lambda}^{(1)}$ and $\tilde{L}_{(2)}^{2} I_N \otimes I_n > \widetilde{\Lambda}^{(2)}$, where $\widetilde{\Lambda}_{\tilde{r}}^{(1)} = (1/2\|\varphi_{(2)\tilde{r}}\|)[(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}})(\varphi_{(3)\tilde{r}}^{-1} - \varphi_{(2)\tilde{r}}^{-1})(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}})^{T} + 2\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}}]$, $\widetilde{\Lambda}_{\tilde{r}}^{(2)} = \varphi_{(3)\tilde{r}}/2\|\varphi_{(2)\tilde{r}}\|$, $\tilde{L}_{(1)} > 0$ and $\tilde{L}_{(2)} > 0$. That means if networks (1) and (2) satisfy Corollary 18, there must be $\tilde{L}_{(1)}^{2} I_N \otimes I_n > \widetilde{\Lambda}^{(1)}$ and $\tilde{L}_{(2)}^{2} I_N \otimes I_n > \widetilde{\Lambda}^{(2)}$. Under Corollary 18, although $\tilde{L}_{(1)}$ and $\tilde{L}_{(2)}$ satisfy $\tilde{L}_{(1)}^{2} I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(1)}$ and $\tilde{L}_{(2)}^{2} I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(2)}$, there must be $\mathbb{E}[\mathscr{L}V^{L}(\omega(t), t, \tilde{r})] < \mathbb{E}[\mathscr{L}V^{\tilde{g}}(\omega(t), t, \tilde{r})] \leq 0$, where $\mathbb{E}[\mathscr{L}V^{L}(\omega(t), t, \tilde{r})]$ and $\mathbb{E}[\mathscr{L}V^{\tilde{g}}(\omega(t), t, \tilde{r})]$ are with respect to networks (1) and (2); it is not derived that synchronization dynamics of network (2) is better than that of network (1). The reason is given in Remark 15. In the process of proving Theorems 13 and 16, there are $\hbar_{(1)}(\omega(t), \omega(t - \tilde{\tau}(t))) = 2\tilde{\eta}_{\tilde{r}} \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \omega^{T}(t)(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}})\widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t)))$ and $\hbar_{(2)}(\omega(t), \omega(t - \tilde{\tau}(t))) = 2\tilde{\eta}_{\tilde{r}} \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \omega^{T}(t)(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}})(\omega(t) + \omega(t - \tilde{\tau}(t)))$. By using similar analysis of Remark 15, it can be obtained that under Corollary 17, synchronization dynamics relationships of networks (1) and (2) are as follows: Case I: $\hbar_{(1)}(\cdot,\cdot) > \hbar_{(2)}(\cdot,\cdot)$. If $\widetilde{D}_{\tilde{k}}^{\tilde{r}} > 0$, $\omega^{T}(t) > 0$ and $\widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) > \omega(t) + \omega(t - \tilde{\tau}(t)) > 0$, or if $\widetilde{D}_{\tilde{k}}^{\tilde{r}} > 0$, $\omega^{T}(t) < 0$ and $\widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) < \omega(t) + \omega(t - \tilde{\tau}(t)) < 0$, or if $\widetilde{D}_{\tilde{k}}^{\tilde{r}} < 0$, $\omega^{T}(t) > 0$ and $\omega(t) + \omega(t - \tilde{\tau}(t)) > \widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) > 0$, or if $\widetilde{D}_{\tilde{k}}^{\tilde{r}} < 0$, $\omega^{T}(t) < 0$ and $\omega(t) + \omega(t - \tilde{\tau}(t)) < \widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) < 0$, synchronization dynamics of network (1) is poorer than that of network (2). Case II: $\hbar_{(1)}(\cdot,\cdot) < \hbar_{(2)}(\cdot,\cdot)$. If $\widetilde{D}_{\tilde{k}}^{\tilde{r}} > 0$, $\omega^{T}(t) > 0$ and $\omega(t) + \omega(t - \tilde{\tau}(t)) > \widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) > 0$, or if $\widetilde{D}_{\tilde{k}}^{\tilde{r}} > 0$, $\omega^{T}(t) < 0$ and $\omega(t) + \omega(t - \tilde{\tau}(t)) < \widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) < 0$, or if $\widetilde{D}_{\tilde{k}}^{\tilde{r}} < 0$, $\omega^{T}(t) > 0$ and $\widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) > \omega(t) + \omega(t - \tilde{\tau}(t)) > 0$, or if $\widetilde{D}_{\tilde{k}}^{\tilde{r}} < 0$, $\omega^{T}(t) < 0$ and $\widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) < \omega(t) + \omega(t - \tilde{\tau}(t)) < 0$, synchronization dynamics of network (1) is better than that of network (2).

Corollary 20. *Under Theorem 16, network (2) with controller (4) must satisfy Theorem 13 if the following inequalities hold:*

$$\begin{aligned}\breve{\Xi}_{\tilde{r}}^{(1)} &= \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \left[\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}}\right)\left(\varphi_{(2)\tilde{r}}^{-1} - \varphi_{(3)\tilde{r}}^{-1}\right)\left(\widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}}\right)^{T}\right. \\ &\quad \left. + 2\left(\tilde{L}_{(1)}^{2} \left\|\varphi_{(2)\tilde{r}}\right\| I_N \otimes I_n - \widetilde{D}_{\tilde{k}}^{\tilde{r}} \otimes \widetilde{Q}\Gamma_{\tilde{k}}\right)\right] < 0,\end{aligned} \tag{36}$$

$$\breve{\Xi}_{\tilde{r}}^{(2)} = \sum_{\tilde{k}=1}^{m} \tilde{c}_{\tilde{k}} \left(2\tilde{L}_{(2)}^{2} \left\|\varphi_{(2)\tilde{r}}\right\| I_N \otimes I_n - \varphi_{(3)\tilde{r}}\right) < 0. \tag{37}$$

Proof. The proof is similar to that of Corollary 17. □

Corollary 21. *Under Corollary 20, synchronization control rule for network (2) must make network (1) get synchronization within finite-time t^*.*

Proof. The proof is similar to that of Corollary 18. □

Remark 22. Under Corollary 21, there are $0 < \tilde{L}_{(1)}^{2} I_N \otimes I_n < \widetilde{\Lambda}_{\tilde{r}}^{(1)}$, $0 < \tilde{L}_{(2)}^{2} I_N \otimes I_n < \widetilde{\Lambda}_{\tilde{r}}^{(2)}$ and synchronization dynamics relationship of networks (1) and (2) is similar to that of Remark 19. From the above, it is seen that under Corollaries 18 and 21, synchronization dynamics relationship of networks (1) and (2) is decided by the initial state $\tilde{z}_{\tilde{i}}(0)$, synchronization state $\tilde{s}(t)$, nonlinear coupled function $\tilde{g}(\cdot,\cdot)$, linear coupled function $\tilde{z}(t) + \tilde{z}(t - \tilde{\tau}(t))$, and coupled matrix $\widetilde{D}_{\tilde{k}}^{\tilde{r}}$. From synchronization dynamics relationship of networks (1) and (2), it is derived that how nonlinear coupled function $\tilde{g}(\cdot,\cdot)$ and linear coupled function $\tilde{z}(t) + \tilde{z}(t - \tilde{\tau}(t))$ impact synchronization dynamics of networks (1) and (2) is not only related to coupled function, but also connected with coupled matrix, the initial state, and synchronization state.

Remark 23. Compared with the recent results of synchronization problems for complex networks such as [8, 10, 16], the effectiveness of this paper can be reflected by synchronization dynamics analysis ideas of the addressed complex networks. From the analysis of Remarks 15–19, it is seen that the effect of nonlinear coupling function for synchronization dynamics is not only related to nonlinearity of nonlinear

coupling function, but also connected with coupled matrix, synchronization state, and the initial state of the considered system. Although some feasible synchronization results of complex networks were derived in [8, 10, 16], these papers mainly focused on how to obtain sufficient conditions of synchronization problems. Besides this, according to introduction of this paper, it is observed that the literature [4] only used simulation method to analyze how nonlinear coupling function impacts synchronization dynamics of the proposed complex networks. All these show that the work in this paper can extend the existed analysis ideas of synchronization problems for complex networks.

Remark 24. According the above analysis, it is not difficult to find that the results of Remarks 14–19 are closely related to the common synchronization control rule. This causes that although nonlinearity of nonlinear coupled function $\widetilde{g}(\cdot,\cdot)$ can affect synchronization dynamics of network (1), it is not reflected by synchronization finite-time t^*. The reason is that there is no function relationship between the estimation approach of synchronization finite-time t^* and nonlinearity of nonlinear coupled function $\widetilde{g}(\cdot,\cdot)$. This can be seen from condition (3) of Theorem 13. How to solve it? Besides this, the ideas of this paper can be extended to finite/fixed-time pinning synchronization of complex networks [31], nonsmooth finite-time synchronization of switched coupled neural networks [33], finite-time consensus of multiagent systems [35], tracking control uncertain interconnected nonlinear systems [49], distributed formation control of multiple quadrotor aircraft [50] and so on. All these are further works in the future.

4. Numerical Simulations

In order to testify the effectiveness of Remarks 15, 19, and 22, three examples are given in this section. Assume that networks (1) and (2) are composed of three coupled nodes, the initial conditions are $\widetilde{z}_1(0) = (1,2)^T$, $\widetilde{z}_2(0) = (3,4)^T$, $\widetilde{z}_3(0) = (5,6)^T$, and a Markovian switching with rate transition matrix of networks (1) and (2) is

$$\Pi = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}. \tag{38}$$

Define that synchronization total error is $\omega(t) = \sum_{i=1}^{3}\sum_{j=1}^{2}\omega_{ij}(t)$. Besides the above parameters, the other parameters of networks (1) and (2) are $\widetilde{f}(\widetilde{z}_{\widetilde{i}}(t)) = [0,0]^T$, $\widetilde{\sigma}_{\widetilde{i}}(\widetilde{z}(t),\widetilde{z}(t-\widetilde{\tau}(t)),t,\widetilde{r}(t)) = [0,0;0,0]$, $\widetilde{c}_1 = \widetilde{c}_2 = 1$, $\widetilde{\tau}(t) = 0.05$, $\widetilde{B} = \mathrm{diag}\{0,0\}$, $\Gamma_1 = \Gamma_2 = \mathrm{diag}\{1,1\}$.

Example 25. In order to test the analysis of Remark 15 and according to coupled matrix $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}$ and nonlinear coupled function $\widetilde{g}(\cdot,\cdot)$, four cases of simulation results are given as follows:

Case I. Coupled matrix $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$ of network (1), where $\widetilde{k},\widetilde{r} = 1,2$. Firstly, let

$$\widetilde{D}_1^1 = \widetilde{D}_2^1 = D^{(1)} = \begin{bmatrix} 1 & 0.3 & 0.5 \\ 0.3 & 1.2 & 0 \\ 0.5 & 0 & 1.2 \end{bmatrix},$$
$$\widetilde{D}_1^2 = \widetilde{D}_2^2 = D^{(2)} = \begin{bmatrix} 1.2 & 0.3 & 0.5 \\ 0.3 & 1.1 & 0.3 \\ 0.5 & 0.3 & 1 \end{bmatrix}, \tag{39}$$
$$\widetilde{g}_{(1)}\left(\widetilde{z}_{\widetilde{j}}(t),\widetilde{z}_{\widetilde{j}}(t-\widetilde{\tau}(t))\right) = \begin{bmatrix} \tanh\left(\widetilde{z}_{\widetilde{j}1}(t)\right) + \tanh\left(\widetilde{z}_{\widetilde{j}1}(t-\widetilde{\tau}(t))\right) \\ \tanh\left(\widetilde{z}_{\widetilde{j}2}(t)\right) + \tanh\left(\widetilde{z}_{\widetilde{j}2}(t-\widetilde{\tau}(t))\right) \end{bmatrix}.$$

By using Assumption 4, one can obtain $\widetilde{L} = 0$, $\widetilde{L}_{(1)}^{g_{(1)}} = \widetilde{L}_{(2)}^{g_{(1)}} = 1.2$. Let $\widetilde{Q} = I_2$, $\widetilde{F}_1 = \widetilde{F}_2 = I_6$. Combining Assumption 7 and inequality (23), one gets $\Lambda_{\widetilde{r}}^{(1)} = \Lambda_{\widetilde{r}}^{(2)} = 0$. From Assumption 8, there is $\widetilde{\lambda} = 0$. According to Lemma 10 and inequalities (17) and (18), one derives $\varphi_{(1)} = \mathrm{diag}\{1,1\}$ and $\varphi_{(2)} = I_6$. By conditions (1)-(2) of Theorem 13, one obtains $\widetilde{\eta}_{\widetilde{r}} = \alpha^{\widetilde{r}} = 4$, $\Theta_{\widetilde{r}}^{\widetilde{k}} = \mathrm{diag}\{8,8,8\}$ and $\widetilde{\beta}_{\widetilde{r}} = 16$. From initial condition of network (1), there is $V(\omega(0),0,\widetilde{r}(0)) = 364$. Let $\kappa_{\widetilde{i}}^{\widetilde{k},\widetilde{r}} = 15$ and $\widetilde{\gamma} = 0.5$, then using condition (3) of Theorem 13, one has $t_{\widetilde{g}_{(1)}}^* < 1.43$.

Secondly, nonlinear coupled function $\widetilde{g}(\widetilde{z}_{\widetilde{j}}(t),\widetilde{z}_{\widetilde{j}}(t-\widetilde{\tau}(t)))$ is chosen as follows:

$$\widetilde{g}_{(2)}\left(\widetilde{z}_{\widetilde{j}}(t),\widetilde{z}_{\widetilde{j}}(t-\widetilde{\tau}(t))\right) = \begin{bmatrix} 0.6 * \left(\tanh\left(0.6\widetilde{z}_{\widetilde{j}1}(t)\right) + \tanh\left(0.6\widetilde{z}_{\widetilde{j}1}(t-\widetilde{\tau}(t))\right)\right) \\ 0.6 * \left(\tanh\left(0.6\widetilde{z}_{\widetilde{j}2}(t)\right) + \tanh\left(0.6\widetilde{z}_{\widetilde{j}2}(t-\widetilde{\tau}(t))\right)\right) \end{bmatrix}, \tag{40}$$

$$\widetilde{g}_{(3)}\left(\widetilde{z}_{\widetilde{j}}(t),\widetilde{z}_{\widetilde{j}}(t-\widetilde{\tau}(t))\right) = \begin{bmatrix} 0.3 * \left(\tanh\left(0.3\widetilde{z}_{\widetilde{j}1}(t)\right) + \tanh\left(0.3\widetilde{z}_{\widetilde{j}1}(t-\widetilde{\tau}(t))\right)\right) \\ 0.3 * \left(\tanh\left(0.3\widetilde{z}_{\widetilde{j}2}(t)\right) + \tanh\left(0.3\widetilde{z}_{\widetilde{j}2}(t-\widetilde{\tau}(t))\right)\right) \end{bmatrix}. \tag{41}$$

From the above parameters, one has $\widetilde{\eta}_{\widetilde{r}} = \alpha^{\widetilde{r}} = 4$, $\Theta_{\widetilde{r}}^{\widetilde{k}} = \mathrm{diag}\{8,8,8\}$, $\kappa_{\widetilde{i}}^{\widetilde{k},\widetilde{r}} = 15$, and $\widetilde{\gamma} = 0.5$. By Assumption 4, one gets $\widetilde{L}_{(1)}^{\widetilde{g}_{(2)}} = \widetilde{L}_{(2)}^{\widetilde{g}_{(2)}} = 0.8$ and $\widetilde{L}_{(1)}^{\widetilde{g}_{(3)}} = \widetilde{L}_{(2)}^{\widetilde{g}_{(3)}} = 0.5$. Here, $\widetilde{L}_{(1)}^{\widetilde{g}_{(2)}}$, $\widetilde{L}_{(2)}^{\widetilde{g}_{(2)}}$, $\widetilde{L}_{(1)}^{\widetilde{g}_{(3)}}$, and $\widetilde{L}_{(2)}^{\widetilde{g}_{(3)}}$ are with respect to $\widetilde{g}_{(2)}(\cdot,\cdot)$ and $\widetilde{g}_{(3)}(\cdot,\cdot)$, respectively. Substituting $\widetilde{L}_{(1)}^{g_{(2)}}$, $\widetilde{L}_{(2)}^{g_{(2)}}$, $\widetilde{L}_{(1)}^{g_{(3)}}$, and $\widetilde{L}_{(1)}^{g_{(3)}}$ into inequalities (13) and (14) of condition (2) in Theorem 13, there are $\widetilde{\Xi}_{\widetilde{r}}^{(1),\widetilde{g}_{(2)}} < 0$, $\widetilde{\Xi}_{\widetilde{r}}^{(2),\widetilde{g}_{(2)}} < 0$, $\widetilde{\Xi}_{\widetilde{r}}^{(1),\widetilde{g}_{(3)}} < 0$ and $\widetilde{\Xi}_{\widetilde{r}}^{(2),\widetilde{g}_{(3)}} < 0$.

Case II: Coupled matrix $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$ of the network (1), where $\widetilde{k},\widetilde{r} = 1,2$. Let $\widetilde{D}_{\widetilde{k}}^1 = -D^{(1)}$ and $\widetilde{D}_{\widetilde{k}}^2 = -D^{(2)}$. The other parameters are the same as Case I.

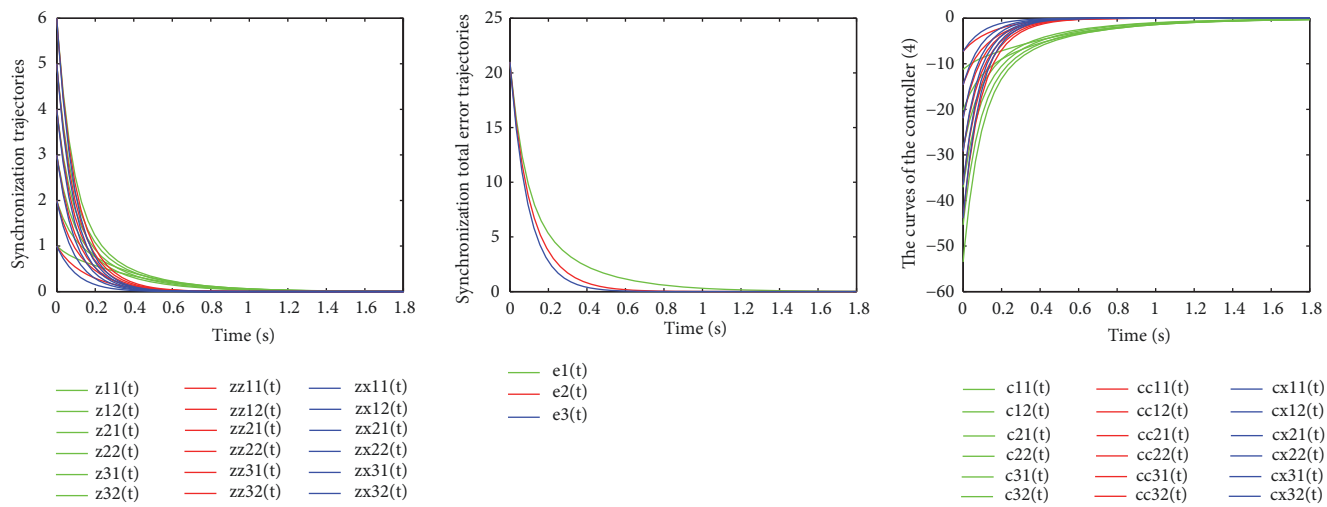

FIGURE 1: Synchronization trajectories, synchronization total error trajectories of network (1) with controller (4) and the curves of controller (4) for Case I of Example 25.

Case III: Let $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$, $\widetilde{D}_{\widetilde{k}}^{1} = -D^{(1)}$, $\widetilde{D}_{\widetilde{k}}^{2} = -D^{(2)}$ and $\widetilde{g}(\cdot,\cdot) = -\widetilde{g}_{(\iota)}(\cdot,\cdot)$, where $\widetilde{g}_{(\iota)}(\cdot,\cdot)$ is nonlinear coupled function of Case I and $\iota = 1,2,3$. The other parameters are the same as Case I.

Case IV: Assume that coupled matrix $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}$ satisfies diffusive coupling condition $\widetilde{d}_{\widetilde{i}\widetilde{i}}^{\widetilde{k}}(\widetilde{r}(t)) = -\sum_{\widetilde{j}=1,\widetilde{j}\neq\widetilde{i}}^{N} \widetilde{d}_{\widetilde{i}\widetilde{j}}^{\widetilde{k}}(\widetilde{r}(t))$ and let

$$\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} = \begin{bmatrix} -1 & 0.3 & 0.7 \\ 0.3 & -0.9 & 0.6 \\ 0.7 & 0.6 & -1.3 \end{bmatrix}, \tag{42}$$

where $\widetilde{k},\widetilde{r} = 1,2$. According to the Case I, we have $\widetilde{Q} = I_2$, $\widetilde{F}_1 = \widetilde{F}_2 = I_6$, $\widetilde{\eta}_{\widetilde{r}} = \alpha^{\widetilde{r}} = 4$, $\Theta_{\widetilde{r}}^{\widetilde{k}} = \mathrm{diag}\{8,8,8\}$, $\kappa_{\widetilde{i}}^{\widetilde{k},\widetilde{r}} = 15$ and $\widetilde{\gamma} = 0.5$. Nonlinear coupled function $\widetilde{g}_{(\iota)}(\cdot,\cdot)$ is the same as Case I, where $\iota = 1,2,3$. We can test that these parameters of Cases II, III, and IV satisfy Theorem 13, and finite-time $t_{\widetilde{g}_{(\iota)}}^{*} < 1.43$.

Remark 26. In Figures 1–4, green trajectories, red trajectories, and blue trajectories are with respect to $\widetilde{g}_{(1)}(\cdot,\cdot)$, $\widetilde{g}_{(2)}(\cdot,\cdot)$, and $\widetilde{g}_{(3)}(\cdot,\cdot)$, respectively. From $\widetilde{g}_{(1)}(\cdot,\cdot)$, $\widetilde{g}_{(2)}(\cdot,\cdot)$, and $\widetilde{g}_{(3)}(\cdot,\cdot)$, it is seen that nonlinearity of $\widetilde{g}_{(1)}(\cdot,\cdot)$ is more serious than that of $\widetilde{g}_{(2)}(\cdot,\cdot)$ and $\widetilde{g}_{(3)}(\cdot,\cdot)$, and nonlinearity of $\widetilde{g}_{(2)}(\cdot,\cdot)$ is more serious than that of $\widetilde{g}_{(3)}(\cdot,\cdot)$. Because of $\widetilde{f}(\widetilde{z}_{\widetilde{i}}(t)) = [0,0]^T$, $\widetilde{B} = \mathrm{diag}\{0,0\}$, $\widetilde{\sigma}_{\widetilde{i}}(\widetilde{z}(t),\widetilde{z}(t-\widetilde{\tau}(t)),t,\widetilde{r}(t)) = [0,0;0,0]$ and synchronization state $\widetilde{s}(t)$ which is given in equality (3), we can get $\widetilde{s}(t) = 0$. This causes $\omega_i(t) = \widetilde{z}_{\widetilde{i}}(t) - \widetilde{s}(t) > 0$, $\widetilde{G}(\omega_{\widetilde{j}}(t),\omega_{\widetilde{j}}(t-\widetilde{\tau}(t))) = \widetilde{g}_{(\iota)}(\widetilde{z}_{\widetilde{j}}(t),\widetilde{z}_{\widetilde{j}}(t-\widetilde{\tau}(t))) - \widetilde{g}(\widetilde{s}(t),\widetilde{s}(t-\widetilde{\tau}(t))) > 0$, where $\widetilde{g}_{(\iota)}(\cdot,\cdot)$ is nonlinear coupled function of Cases I and II. Due to $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} > 0$ of Case I and $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$ of Cases II-III, there are $2\widetilde{\eta}_{\widetilde{r}}\sum_{\widetilde{k}=1}^{m}\widetilde{c}_{\widetilde{k}}\omega^T(t)(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}\otimes\widetilde{Q}\Gamma_{\widetilde{k}})\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) > 0$ of Cases I, III, and $2\widetilde{\eta}_{\widetilde{r}}\sum_{\widetilde{k}=1}^{m}\widetilde{c}_{\widetilde{k}}\omega^T(t)(\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}\otimes\widetilde{Q}\Gamma_{\widetilde{k}})\widetilde{G}(\omega(t),\omega(t-\widetilde{\tau}(t))) < 0$ of Case II, where $\widetilde{Q} = I_2 > 0$, $\Gamma_{\widetilde{k}} = \mathrm{diag}\{1,1\} > 0$, $\widetilde{\eta}_{\widetilde{r}} = 4 > 0$, $\widetilde{c}_{\widetilde{k}} = 1$, $\widetilde{k},\widetilde{r} = 1,2$. According to the analysis of Remark 15, synchronization dynamics of network (1) will become poorer if nonlinearity of nonlinear coupled function $\widetilde{g}_{(\iota)}(\cdot,\cdot)$ is increased in Cases I and III, and with increasing nonlinearity of nonlinear coupled function $\widetilde{g}_{(\iota)}(\cdot,\cdot)$, synchronization dynamics of network (1) will become better. Figures 1–3 testify that the above analysis results are reasonable. In Case IV, it is observed that if coupled matrix $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}$ satisfies diffusive coupling condition, there is no relationship between nonlinearity of $\widetilde{g}_{(\iota)}(\cdot,\cdot)$ and synchronization dynamics of network (1) from Figure 4. Actually, the simulation result cannot be derived by theoretical aspect. The reason is given in Remark 15.

Remark 27. How to make inequality conditions (13) and (14) of Theorem 13 hold? From Theorem 13, it is seen that inequality conditions (13) and (14) are $\widetilde{\Xi}_{\widetilde{r}}^{(1)} \leq 0$ and $\widetilde{\Xi}_{\widetilde{r}}^{(2)} \leq 0$, respectively. Actually, in order to make $\widetilde{\Xi}_{\widetilde{r}}^{(1)} \leq 0$ and $\widetilde{\Xi}_{\widetilde{r}}^{(2)} \leq 0$, $\Theta_{\widetilde{r}}^{\widetilde{k}}$ and $\alpha^{\widetilde{r}}$ of the controller (4) can be designed. That is to say, $\Theta_{\widetilde{r}}^{\widetilde{k}}$ and $\alpha^{\widetilde{r}}$ of controller (4) can make $\widetilde{\Xi}_{\widetilde{r}}^{(1)} \leq 0$ and $\widetilde{\Xi}_{\widetilde{r}}^{(2)} \leq 0$ hold. Besides this, in $\widetilde{\Xi}_{\widetilde{r}}^{(1)} \leq 0$ and $\widetilde{\Xi}_{\widetilde{r}}^{(2)} \leq 0$, there are parameters $\widetilde{L}$, $\widetilde{L}_{(1)}$, $\widetilde{L}_{(2)}$, $\widetilde{\lambda}$, $\Lambda_{\widetilde{r}}^{(2)}$, $\Lambda_{\widetilde{r}}^{(1)}$, where $\Lambda_{\widetilde{r}}^{(1)} = \mathrm{diag}(\widetilde{\iota}_{(\widetilde{r})1}^{(1)},\widetilde{\iota}_{(\widetilde{r})2}^{(1)},\ldots,\widetilde{\iota}_{(\widetilde{r})N}^{(1)})$, $\Lambda_{\widetilde{r}}^{(2)} = \mathrm{diag}(\widetilde{\iota}_{(\widetilde{r})1}^{(2)},\widetilde{\iota}_{(\widetilde{r})2}^{(2)},\ldots,\widetilde{\iota}_{(\widetilde{r})N}^{(2)})$. From the process of proving Theorem 13, it is observed that the above parameters are built on Assumptions 4–8, respectively. That means in order to obtain these parameters, it is necessary to make Assumptions 4–8 hold. How to guarantee these Assumptions? This can be explained by Case I of Example 25. In Case I of Example 25, there are $\widetilde{L} = \widetilde{L}_{(1)}^{g_{(1)}} = \widetilde{L}_{(2)}^{g_{(1)}} = 1.2$, $\widetilde{f}(\widetilde{z}_{\widetilde{i}}(t)) = [0,0]^T$ and $\widetilde{g}(\cdot,\cdot)$ shown in equality (39). Substituting these parameters into Assumption 4, it is derived that Assumption 4 holds. By similar scheme, it is testified that Assumptions 7–8 also hold. From the above, it can be obtained that feasible parameters to guarantee Assumptions 4–8 are chosen.

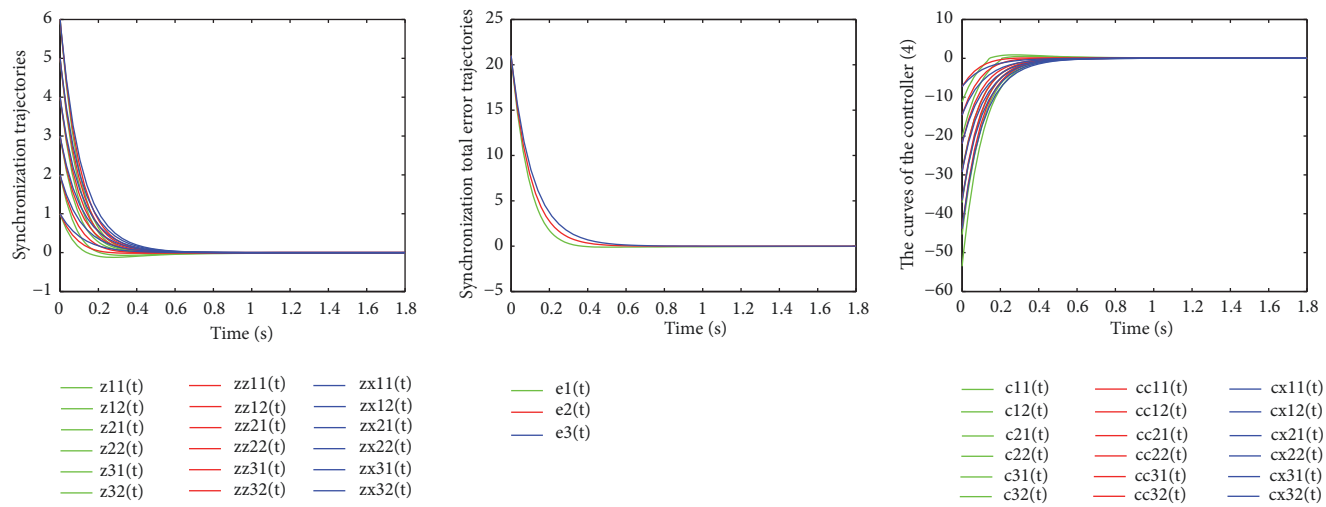

FIGURE 2: Synchronization trajectories, synchronization total error trajectories of network (1) with controller (4) and the curves of controller (4) for Case II of Example 25.

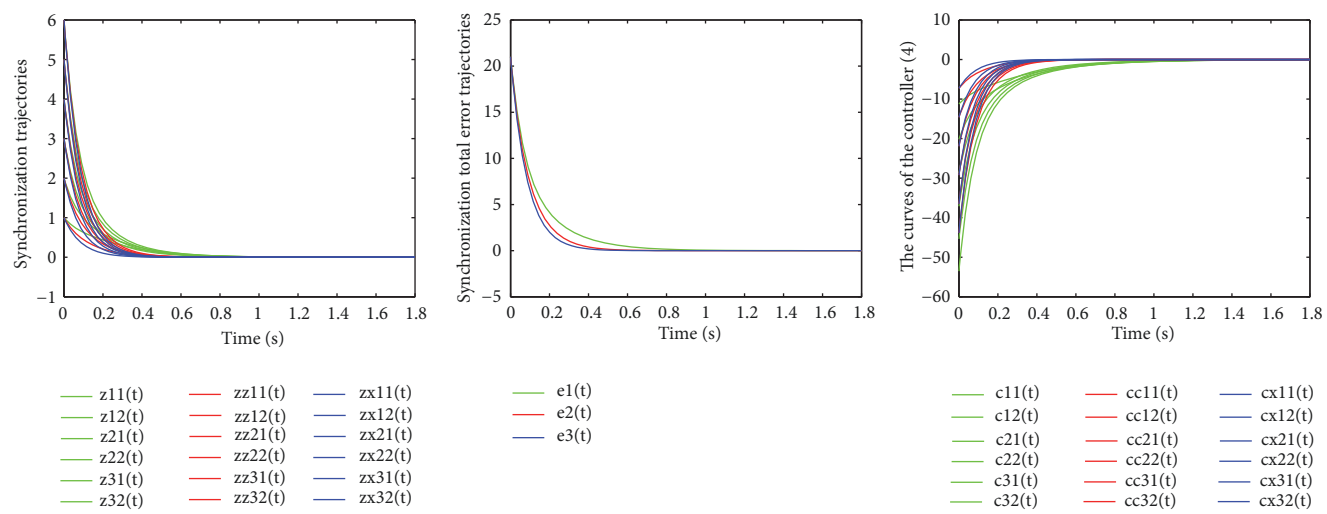

FIGURE 3: Synchronization trajectories, synchronization total error trajectories of network (1) with controller (4) and the curves of controller (4) for Case III of Example 25.

Example 28. In order to testify the analysis of Case I in Remark 19, this example is given. Firstly, let $\tilde{c}_1 = \tilde{c}_2 = 1.5$ and

$$\widetilde{D}_1^1 = \widetilde{D}_2^1 = D^{(1)} = \begin{bmatrix} 0.4 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0 \\ 0.1 & 0 & 0.3 \end{bmatrix},$$
$$\widetilde{D}_1^2 = \widetilde{D}_2^2 = D^{(2)} = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 \end{bmatrix}. \tag{43}$$

Secondly, according to Case I of Remark 19, there exist four subcases as follows:

Case I-1: $\widetilde{D}_{\tilde{k}}^{\tilde{r}} > 0$, $\omega^T(t) > 0$, and $\widetilde{G}(\omega(t), \omega(t-\tilde{\tau}(t))) > \omega(t) + \omega(t-\tilde{\tau}(t)) > 0$. Let

$$\tilde{g}\left(\tilde{z}_{\tilde{j}}(t), \tilde{z}_{\tilde{j}}(t-\tilde{\tau}(t))\right) = \begin{bmatrix} 1.2 * \left(\tilde{z}_{\tilde{j}1}(t) + \tilde{z}_{\tilde{j}1}(t-\tilde{\tau}(t))\right) \\ 1.2 * \left(\tilde{z}_{\tilde{j}2}(t) + \tilde{z}_{\tilde{j}2}(t-\tilde{\tau}(t))\right) \end{bmatrix}. \tag{44}$$

From Remark 26, there is $\tilde{s}(t) = 0$. Combining the initial condition $\tilde{z}_{\tilde{i}}(0) > 0$ and (44), where $\tilde{i} = 1, 2, 3$, it is obtained that the condition of Case I-1 is satisfied. From (44), there is $\widetilde{L}_{(1)} = \widetilde{L}_{(2)} = 1.2$. Let $\widetilde{Q} = I_2$, $\widetilde{F}_1 = \widetilde{F}_2 = I_6$. By using the similar steps of Example 25, one gets $\varphi_{(2)\tilde{r}} = \varphi_{(3)\tilde{r}} = I_6$, $\tilde{\eta}_{\tilde{r}} = \alpha^{\tilde{r}} = 4$ and $\Theta_{\tilde{r}}^{\tilde{k}} = \mathrm{diag}\{6.5, 6.5, 6.5\}$. Let $\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}} = 8$ and $\tilde{\gamma} = 0.5$, then $t^* < 2.71$. According to the above parameters, it

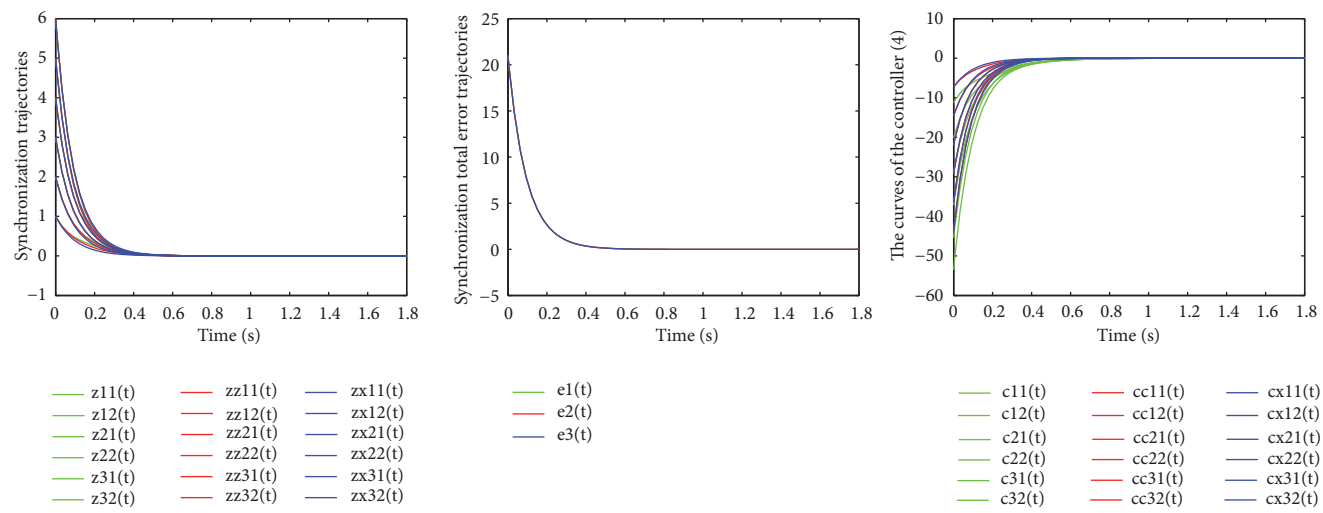

FIGURE 4: Synchronization trajectories, synchronization total error trajectories of network (1) with controller (4) and the curves of controller (4) for Case IV of Example 25.

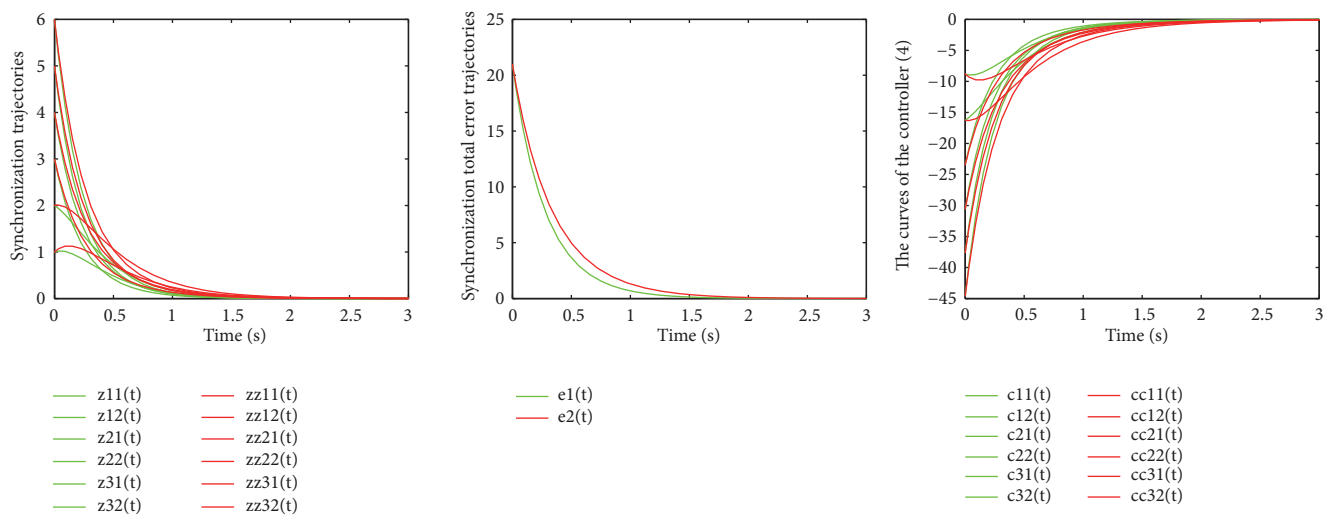

FIGURE 5: Synchronization trajectories, synchronization total error trajectories of networks (1) and (2) with controller (4) and the curves of controller (4) for Case I-1 of Example 28.

can be obtained that $\widetilde{L}_{(1)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(1)}$ and $\widetilde{L}_{(2)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(2)}$ hold.

Case I-2: $\widetilde{D}_{\tilde{k}}^{\tilde{r}} < 0$, $\omega^T(t) > 0$ and $\omega(t) + \omega(t - \tilde{\tau}(t)) > \widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) > 0$. Let $\widetilde{D}_1^1 = \widetilde{D}_2^1 = -D^{(1)}$, $\widetilde{D}_1^2 = \widetilde{D}_2^2 = -D^{(2)}$, and

$$\tilde{g}\left(\tilde{z}_{\tilde{j}}(t), \tilde{z}_{\tilde{j}}(t - \tilde{\tau}(t))\right) = \begin{bmatrix} 0.9 * \left(\tilde{z}_{\tilde{j}1}(t) + \tilde{z}_{\tilde{j}1}(t - \tilde{\tau}(t))\right) \\ 0.9 * \left(\tilde{z}_{\tilde{j}2}(t) + \tilde{z}_{\tilde{j}2}(t - \tilde{\tau}(t))\right) \end{bmatrix}. \quad (45)$$

It is easily tested that the condition of Case I-2, $\widetilde{L}_{(1)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(1)}$ and $\widetilde{L}_{(2)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(2)}$ hold. From $\tilde{g}(\cdot,\cdot)$ of Case I-2, there is $\widetilde{L}_{(1)} = \widetilde{L}_{(2)} = 0.9$. Similar to Case I-1, it is obtained that $\tilde{\eta}_{\tilde{r}} = \alpha^{\tilde{r}} = 3$ and $\Theta_{\tilde{r}}^{\tilde{k}} = \mathrm{diag}\{6, 6, 6\}$. Let $\kappa_{\tilde{i}}^{\tilde{k},\tilde{r}} = 8$ and $\tilde{\gamma} = 0.5$, then $t^* < 1.2$.

Case I-3: $\widetilde{D}_{\tilde{k}}^{\tilde{r}} > 0$, $\omega^T(t) < 0$, and $\widetilde{G}(\omega(t), \omega(t - \tilde{\tau}(t))) < \omega(t) + \omega(t - \tilde{\tau}(t)) < 0$. Let

$$\tilde{g}\left(\tilde{z}_{\tilde{j}}(t), \tilde{z}_{\tilde{j}}(t - \tilde{\tau}(t))\right) = \begin{bmatrix} 1.2 * \left(\tilde{z}_{\tilde{j}1}(t) + \tilde{z}_{\tilde{j}1}(t - \tilde{\tau}(t))\right) \\ 1.2 * \left(\tilde{z}_{\tilde{j}2}(t) + \tilde{z}_{\tilde{j}2}(t - \tilde{\tau}(t))\right) \end{bmatrix}. \quad (46)$$

In order to make $\omega^T(t) < 0$, the initial conditions are $\tilde{z}_1(0) = (-1, -2)^T$, $\tilde{z}_2(0) = (-3, -4)^T$, $\tilde{z}_3(0) = (-5, -6)^T$, and $\widetilde{D}_{\tilde{k}}^{\tilde{r}}$ is the same as Case I-1. It is obtained that the condition of Case I-3, $\widetilde{L}_{(1)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(1)}$ and $\widetilde{L}_{(2)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\tilde{r}}^{(2)}$ hold. According to $\tilde{g}(\cdot,\cdot)$ of Case I-3, let $\widetilde{L}_{(1)} = \widetilde{L}_{(2)} = 1.2$. By similar steps of Case

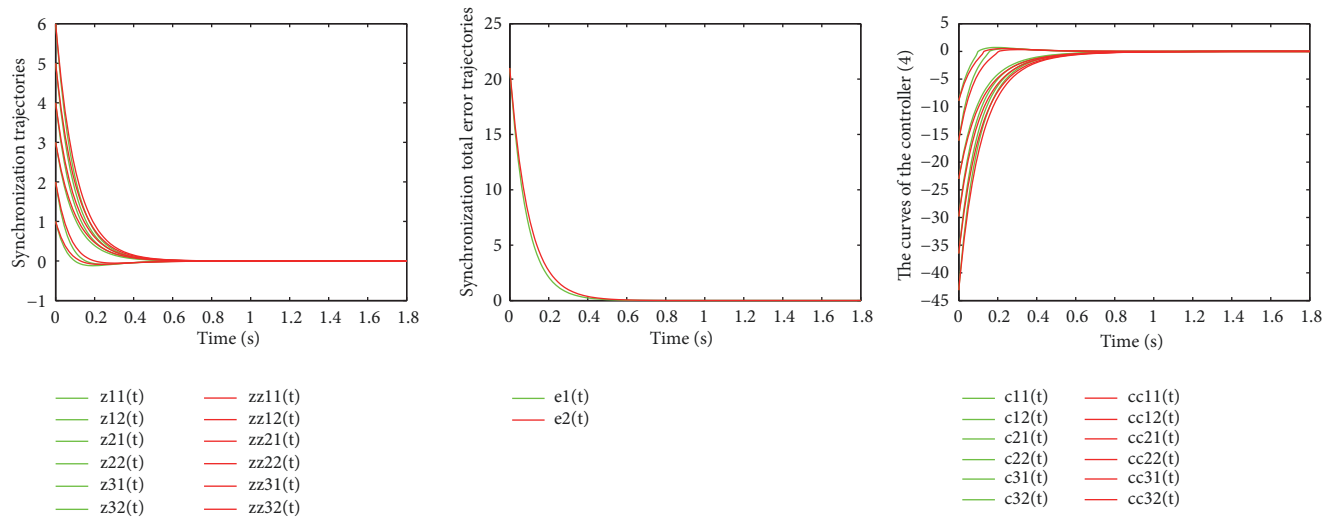

FIGURE 6: Synchronization trajectories, synchronization total error trajectories of networks (1) and (2) with controller (4) and the curves of controller (4) for Case I-2 of Example 28.

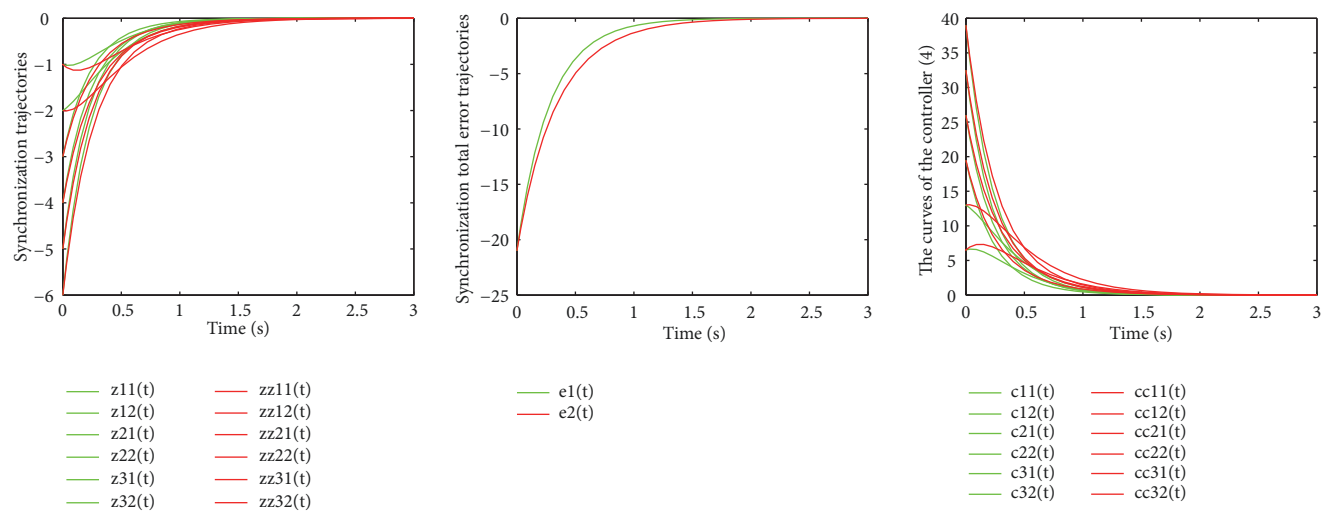

FIGURE 7: Synchronization trajectories, synchronization total error trajectories of networks (1) and (2) with controller (4) and the curves of controller (4) for Case I-3 of Example 28.

I-1, it is derived that $\widetilde{\eta}_{\widetilde{r}} = \alpha^{\widetilde{r}} = 4$ and $\Theta_{\widetilde{r}}^{\widetilde{k}} = \mathrm{diag}\{6.5, 6.5, 6.5\}$. Let $\kappa_{\widetilde{i}}^{\widetilde{k},\widetilde{r}} = 8$ and $\widetilde{\gamma} = 0.5$, then $t^* < 2.71$.

Case I-4: $\widetilde{D}_{\widetilde{k}}^{\widetilde{r}} < 0$, $\omega^T(t) < 0$ and $\omega(t) + \omega(t - \widetilde{\tau}(t)) < \widetilde{G}(\omega(t), \omega(t - \widetilde{\tau}(t))) < 0$. Let

$$\widetilde{g}\left(\widetilde{z}_{\widetilde{j}}(t), \widetilde{z}_{\widetilde{j}}(t - \widetilde{\tau}(t))\right) = \begin{bmatrix} 0.9 * \left(\widetilde{z}_{\widetilde{j}1}(t) + \widetilde{z}_{\widetilde{j}1}(t - \widetilde{\tau}(t))\right) \\ 0.9 * \left(\widetilde{z}_{\widetilde{j}2}(t) + \widetilde{z}_{\widetilde{j}2}(t - \widetilde{\tau}(t))\right) \end{bmatrix}. \quad (47)$$

$\widetilde{D}_{\widetilde{k}}^{\widetilde{r}}$ and the initial conditions are the same as Case I-2 and Case I-3, respectively. It is testified that the condition of Case I-4, $\widetilde{L}_{(1)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\widetilde{r}}^{(1)}$ and $\widetilde{L}_{(2)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\widetilde{r}}^{(2)}$ hold. Similar to Case I-2, there are $\widetilde{\eta}_{\widetilde{r}} = \alpha^{\widetilde{r}} = 3$ and $\Theta_{\widetilde{r}}^{\widetilde{k}} = \mathrm{diag}\{6, 6, 6\}$. Let $\kappa_{\widetilde{i}}^{\widetilde{k},\widetilde{r}} = 8$ and $\widetilde{\gamma} = 0.5$, then $t^* < 1.2$.

Remark 29. In Figures 5–8, red trajectories and green trajectories are with respect to $\widetilde{g}(\widetilde{z}_{\widetilde{j}}(t), \widetilde{z}_{\widetilde{j}}(t - \widetilde{\tau}(t)))$ and $\widetilde{z}_{\widetilde{j}}(t) + \widetilde{z}_{\widetilde{j}}(t - \widetilde{\tau}(t))$, respectively. Figures 5–8 show that in Cases I-1, 2, 3, and 4, if networks (1) and (2) satisfy Corollary 18, synchronization dynamics of network (1) is poorer than that of network (2). Besides this, simulation results of Figures 5–8 are built on Corollary 18. That means under Corollary 18, networks (1) and (2) must achieve synchronization within t^*. That is to say, synchronization dynamics relationship of networks (1) and (2) is based on Corollary 18. Therefore, in the above Cases I-1, 2, 3, and 4, $\widetilde{L}_{(1)}$ and $\widetilde{L}_{(2)}$, which are with respect to $\widetilde{g}(\cdot, \cdot)$, must satisfy $\widetilde{L}_{(1)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\widetilde{r}}^{(1)}$ and $\widetilde{L}_{(2)}^2 I_N \otimes I_n > \widetilde{\Lambda}_{\widetilde{r}}^{(2)}$. By using

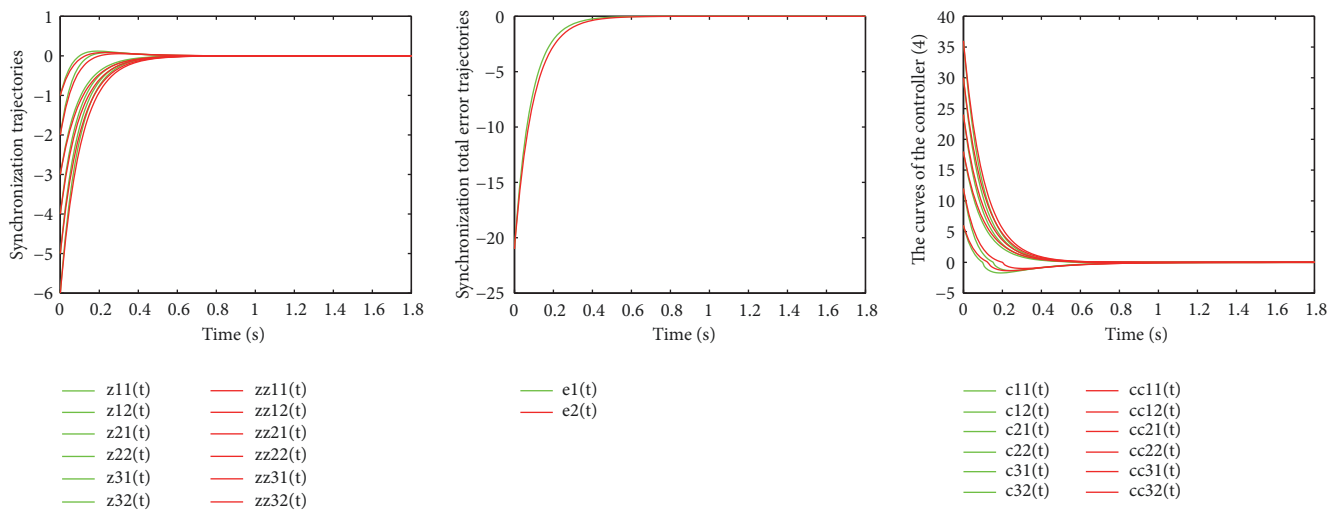

Figure 8: Synchronization trajectories, synchronization total error trajectories of networks (1) and (2) with controller (4) and the curves of controller (4) for Case I-3 of Example 28.

similar simulation scheme of Cases I-1, 2, 3, and 4, simulation results of Case II of Remark 19 and Cases I-II of Remark 22 can also be obtained. Therefore, simulation results of Case II of Remark 19 and Cases I-II of Remark 22 are not given.

5. Conclusions

In this paper, sufficient conditions of finite-time synchronization for a class of NCMSMWCNs and a class of LCMSMWCNs are studied. Based on the derived results, synchronization dynamics problems of the NCMSMWCNs and synchronization dynamics relationships of the NCMSMWCNs and the LCMSMWCNs are analyzed, respectively. Comparing synchronization dynamics between the NCMSMWCNs and the LCMSMWCNs, how nonlinear coupled one and linear coupled one affect synchronization dynamics of the NCMSMWCNs and the LCMSMWCNs is further explored. Numerical simulation results show the effectiveness of the derived theory.

Data Availability

No data were used to support this study. The reason was that this paper mainly concentrated on proving sufficient conditions of finite-time synchronization for the considered complex networks.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Xin Wang carried out the main part of this manuscript. Bin Yang corrected the main theorems and finished numerical simulation section. All authors read and approved the final manuscript.

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