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Research Article

Finite Switchboard State Machines Based on Cubic Sets

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We study the cubic finite switchboard state machines (cubic fssms) intensively. The notions of (resp., good, strong) subsystem of cubic fssm are introduced. We prove that the Cartesian composition of two (resp., good, strong) subsystems of cubic fssms is (resp., good, strong) subsystem of cubic fssm. Similarly, the product of two (resp., good, strong) subsystems of cubic fssms is (resp. good, strong) subsystem of cubic fssm. We define the homomorphism between two of cubic fssms and proved some related results. Many examples on each case are provided.

1. Introduction

The idea of fuzzy set was first introduced by Zadeh [1]. He defined fuzzy sets which have been applied in daily real life as traffic monitoring systems, artificial intelligence, medicine, air conditions, washing machine, and computer science.

This paper is an extesion of cubic finite state machines which have been done by Abughazalah and Yaqoob [2]. Abdullah, Naz and Pedrycz [3] defined cubic finite state machines. Cartesian composition of automata was introduced first in [4]. Malik, Mordeson, and Sen [5–7] provided the concepts of both the product and the "Cartesian composition" of ffsm and they studied subsystem and strong subsystem of fuzzy finite state machines. Ebas, Hamzh, etc. [8] studied the algebraic properties of finite switchboard state machines and studied their properties. Constructions of product of finite switchboard state machines were given by Kumbhojkar and Chaudhari [9]. Mahmood and Khan [10] studied the notion of interval neutrosphic finite switchboard state machines. Hussain and Shabbir [11] introduced the concept of soft finite state machines and applied soft set theory to finite state machines.

The authors in [12–14] introduced intuitionistic fuzzy finite state machines and intuitionistic fuzzy finite switchboard state machines. Finite state machines in terms of bipolar fuzzy set were studied in [15] and some types of subsystem and strong subsystem in bipolar fuzzy set were introduced in [16]. Properties of bipolar fuzzy finite switchboard state machines and the concepts of bipolar submachine, bipolar connected, and bipolar retrievable are investigated in [17]. Jun et al. [18] studied the behaviour of cubic p–ideal (resp., a–ideal, q–ideal) of BCI-algebras. The concepts of (closed) cubic soft ideals in BCK/BCI- algebras are introduced and the relations between them are discussed in [19]. Liu et al. [20] studied the product of "Mealy- type fuzzy finite state machines".

2. Preliminaries

Definition 1 (see [1]). "A map $\lambda : X \longrightarrow [0,1]$ is called a fuzzy set of X."

Definition 2 (see [1]). "An interval valued fuzzy set (briefly, IVF-set) $\widetilde{\delta}_A$ on X is defined as

$$\widetilde{\delta}_A = \{\langle x, [\delta_A^-(x), \delta_A^+(x)]\rangle : x \in X\}, \tag{1}$$

where $\delta_A^-(x) \leq \delta_A^+(x)$, for all $x \in X$. Then the ordinary fuzzy sets $\delta_A^- : X \longrightarrow [0,1]$ and $\delta_A^+ : X \longrightarrow [0,1]$ are called a

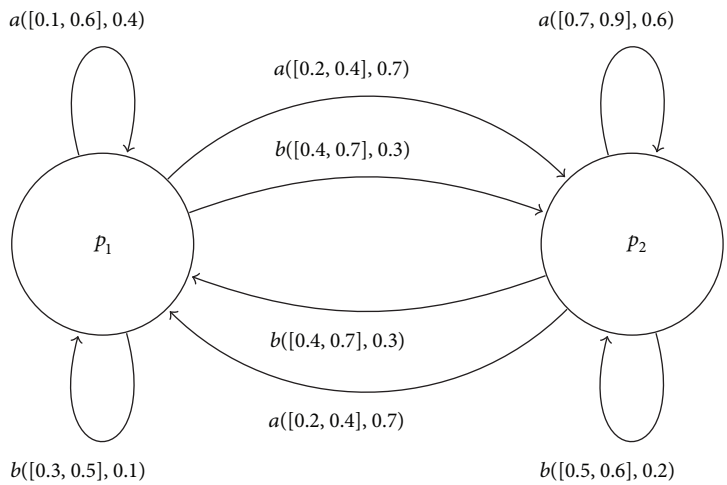

Figure 1: A switching cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$.

lower fuzzy set and an upper fuzzy set of $\widetilde{\delta}$, respectively. Let $\widetilde{\delta}_A(x) = [\delta_A^-(x), \delta_A^+(x)]$. Then

$$A = \left\{ \left\langle x, \widetilde{\delta}_A(x) \right\rangle : x \in X \right\}, \tag{2}$$

where $\widetilde{\delta}_A : X \longrightarrow D[0,1]$."

Definition 3 (see [15]). "Let X be a non-empty set. By a cubic set in X we mean a structure

$$\varphi = \left\{ \left\langle x, \widetilde{\delta}_\varphi(x), \omega_\varphi(x) \right\rangle \mid x \in X \right\} \tag{3}$$

in which $\widetilde{\delta}_\varphi$ is an IVF set in X and ω_φ is a fuzzy set in X."

Definition 4 (see [3]). "A cubic finite state machine (cubic fsm, shortly) is a triple $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$, where $\mathcal{P}$ and X are finite non-empty sets, called the set of states and the set of input symbols, respectively, and $\varphi = \langle \widetilde{\delta}_\varphi, \omega_\varphi \rangle$ is a cubic set in $\mathcal{P} \times X \times \mathcal{P}$."

"Let X^* denote the set of all words of elements of X of finite length. Let ξ denote the empty word in X^* and $|\xi|$ denote the length of ξ for every $\xi \in X^*$. See [3]."

Definition 5 (see [3]). "Let $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ be a cubic fsm. Define a cubic set $\varphi^* = \langle \widetilde{\delta}_\varphi^*, \omega_\varphi^* \rangle$ in $\mathcal{P} \times X^* \times \mathcal{P}$ by

$$\begin{aligned} \widetilde{\delta}_\varphi^*(p_1, \xi, p_2) &= \begin{cases} [1,1] & \text{if } p_1 = p_2 \\ [0,0] & \text{if } p_1 \neq p_2, \end{cases} \\ \omega_\varphi^*(p_1, \xi, p_2) &= \begin{cases} 0 & \text{if } p_1 = p_2 \\ 1 & \text{if } p_1 \neq p_2 \end{cases} \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\widetilde{\delta}_\varphi^*(p_1, xa, p_2) \\ &\quad = \bigvee_{p \in \mathcal{P}} \left[r \min \left\{ \widetilde{\delta}_\varphi^*(p_1, x, p), \widetilde{\delta}_\varphi(p, a, p_2) \right\} \right] \\ &\omega_\varphi^*(p_1, xa, p_2) \\ &\quad = \bigwedge_{p \in \mathcal{P}} \left[\max \left\{ \omega_\varphi^*(p_1, x, p), \omega_\varphi(p, a, p_2) \right\} \right] \end{aligned} \tag{5}$$

for all $p_1, p_2 \in \mathcal{P}$, $x \in X^*$ and $a \in X$."

Lemma 6 (see [3]). *"Let $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ be a cubic fsm. Then*

$$\begin{aligned} &\widetilde{\delta}_\varphi^*(p_1, xy, p_2) \\ &\quad = \bigvee_{p \in \mathcal{P}} \left[r \min \left\{ \widetilde{\delta}_\varphi^*(p_1, x, p), \widetilde{\delta}_\varphi^*(p, y, p_2) \right\} \right] \\ &\omega_\varphi^*(p_1, xy, p_2) \\ &\quad = \bigwedge_{p \in \mathcal{P}} \left[\max \left\{ \omega_\varphi^*(p_1, x, p), \omega_\varphi(p, y, p_2) \right\} \right] \end{aligned} \tag{6}$$

for all $p_1, p_2 \in \mathcal{P}$ and $x, y \in X^*$."

3. Cubic Finite Switchboard State Machines

Definition 7. A cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ is said to be switching if it satisfies

$$\begin{aligned} \widetilde{\delta}_\varphi(p_1, x, p_2) &= \widetilde{\delta}_\varphi(p_2, x, p_1), \\ \omega_\varphi(p_1, x, p_2) &= \omega_\varphi(p_2, x, p_1) \end{aligned} \tag{7}$$

for all $p_1, p_2 \in \mathcal{P}$ and $x \in X$.

Example 8. Let $\mathcal{P} = \{p_1, p_2\}$ and $X = \{a, b\}$. Let $\varphi = \langle \widetilde{\delta}_\varphi, \omega_\varphi \rangle$ be a cubic subset in $\mathcal{P} \times X \times \mathcal{P}$. Then the cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ is switching, as shown in Figure 1.

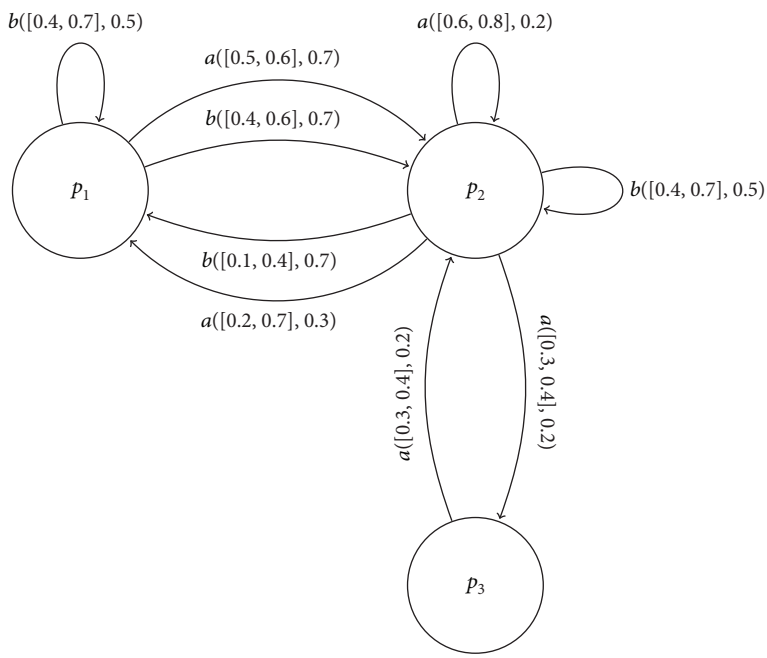

Figure 2: A commutative cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$.

Definition 9. A cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ is said to be commutative if it satisfies

$$\begin{aligned} \widetilde{\delta}_{\varphi}(p_1, xy, p_2) &= \widetilde{\delta}_{\varphi}(p_1, yx, p_2), \\ \omega_{\varphi}(p_1, xy, p_2) &= \omega_{\varphi}(p_1, yx, p_2) \end{aligned} \tag{8}$$

for all $p_1, p_2 \in \mathcal{P}$ and $x, y \in X$.

Example 10. Let $\mathcal{P} = \{p_1, p_2, p_3\}$ and $X = \{a, b\}$. Let $\varphi = \langle \widetilde{\delta}_{\varphi}, \omega_{\varphi} \rangle$ be a cubic subset in $\mathcal{P} \times X \times \mathcal{P}$. Then the cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ is commutative, as shown in Figure 2.

The cubic fsm shown in Figure 1 is switching but not commutative because

$$\begin{aligned} [0.2, 0.6] &= \widetilde{\delta}_{\varphi}(p_1, ab, p_2) \neq \widetilde{\delta}_{\varphi}(p_1, ba, p_2) \\ &= [0.4, 0.7], \\ 0.4 &= \omega_{\varphi}(p_1, ab, p_2) \neq \omega_{\varphi}(p_1, ba, p_2) = 0.6. \end{aligned} \tag{9}$$

Also, the cubic fsm shown in Figure 2 is commutative but not switching because

$$\begin{aligned} \widetilde{\delta}_{\varphi}(p_1, a, p_2) &\neq \widetilde{\delta}_{\varphi}(p_2, a, p_1), \\ \omega_{\varphi}(p_1, a, p_2) &\neq \omega_{\varphi}(p_2, a, p_1). \end{aligned} \tag{10}$$

Proposition 11. *If $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ is a commutative cubic fsm, then*

$$\begin{aligned} \widetilde{\delta}^*_{\varphi}(p_1, xy, p_2) &= \widetilde{\delta}^*_{\varphi}(p_1, yx, p_2), \\ \omega^*_{\varphi}(p_1, xy, p_2) &= \omega^*_{\varphi}(p_1, yx, p_2), \end{aligned} \tag{11}$$

for all $p_1, p_2 \in \mathcal{P}$, $y \in X$ and $x \in X^*$.

Proof. Let $p_1, p_2 \in \mathcal{P}$, $y \in X$, and $x \in X^*$. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \xi$; hence

$$\begin{aligned} \widetilde{\delta}^*_{\varphi}(p_1, xy, p_2) &= \widetilde{\delta}^*_{\varphi}(p_1, \xi y, p_2) = \widetilde{\delta}^*_{\varphi}(p_1, y, p_2) \\ &= \widetilde{\delta}^*_{\varphi}(p_1, y\xi, p_2) = \widetilde{\delta}^*_{\varphi}(p_1, yx, p_2), \end{aligned} \tag{12}$$

and

$$\begin{aligned} \omega^*_{\varphi}(p_1, xy, p_2) &= \omega^*_{\varphi}(p_1, \xi y, p_2) = \omega^*_{\varphi}(p_1, y, p_2) \\ &= \omega^*_{\varphi}(p_1, y\xi, p_2) = \omega^*_{\varphi}(p_1, yx, p_2). \end{aligned} \tag{13}$$

Therefore the result is true for $n = 0$. Suppose that the result is true for $|u| = n - 1$. That is, for all $u \in X^*$ with $|u| = n - 1$, $n > 0$. Let $d \in X$ be such that $x = ud$. Then

$$\begin{aligned} \widetilde{\delta}^*_{\varphi}(p_1, xy, p_2) &= \widetilde{\delta}^*_{\varphi}(p_1, udy, p_2) \\ &= \bigvee_{p \in \mathcal{P}} \left[r \min \left\{ \widetilde{\delta}^*_{\varphi}(p_1, u, p), \widetilde{\delta}^*_{\varphi}(p, dy, p_2) \right\} \right] \\ &= \bigvee_{p \in \mathcal{P}} \left[r \min \left\{ \widetilde{\delta}^*_{\varphi}(p_1, u, p), \widetilde{\delta}^*_{\varphi}(p, yd, p_2) \right\} \right] \\ &= \widetilde{\delta}^*_{\varphi}(p_1, uyd, p_2) \\ &= \bigvee_{p \in \mathcal{P}} \left[r \min \left\{ \widetilde{\delta}^*_{\varphi}(p_1, uy, p), \widetilde{\delta}^*_{\varphi}(p, d, p_2) \right\} \right] \\ &= \bigvee_{p \in \mathcal{P}} \left[r \min \left\{ \widetilde{\delta}^*_{\varphi}(p_1, yu, p), \widetilde{\delta}^*_{\varphi}(p, d, p_2) \right\} \right] \\ &= \widetilde{\delta}^*_{\varphi}(p_1, yud, p_2) = \widetilde{\delta}^*_{\varphi}(p_1, yx, p_2), \end{aligned} \tag{14}$$

and

$$\begin{aligned}\omega_\varphi^*(p_1, xy, p_2) &= \omega_\varphi^*(p_1, udy, p_2)\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p_1, u, p), \omega_\varphi^*(p, dy, p_2)\right\}\right]\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p_1, u, p), \omega_\varphi^*(p, yd, p_2)\right\}\right]\\ &= \omega_\varphi^*(p_1, uyd, p_2)\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p_1, uy, p), \omega_\varphi^*(p, d, p_2)\right\}\right]\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p_1, yu, p), \omega_\varphi^*(p, d, p_2)\right\}\right]\\ &= \omega_\varphi^*(p_1, yud, p_2) = \omega_\varphi^*(p_1, yx, p_2).\end{aligned} \tag{15}$$

Hence the result is true for $|u| = n$. This completes the proof. □

Definition 12. A cubic fsm $\mathscr{T} = \langle\mathscr{P}, X, \varphi\rangle$ is called a cubic finite switchboard state machine (cubic fssm, shortly) if it is switching and commutative.

Proposition 13. *If $\mathscr{T} = \langle\mathscr{P}, X, \varphi\rangle$ is a cubic fssm, then*

$$\begin{aligned}\widetilde{\delta}_\varphi^*(p_1, x, p_2) &= \widetilde{\delta}_\varphi^*(p_2, x, p_1),\\ \omega_\varphi^*(p_1, x, p_2) &= \omega_\varphi^*(p_2, x, p_1)\end{aligned} \tag{16}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X^*$.

Proof. Let $p_1, p_2 \in \mathscr{P}$ and $x \in X^*$. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \xi$, hence

$$\begin{aligned}\widetilde{\delta}_\varphi^*(p_1, x, p_2) &= \widetilde{\delta}_\varphi^*(p_1, \xi, p_2) = \widetilde{\delta}_\varphi^*(p_2, \xi, p_1)\\ &= \widetilde{\delta}_\varphi^*(p_2, x, p_1),\end{aligned} \tag{17}$$

and

$$\begin{aligned}\omega_\varphi^*(p_1, x, p_2) &= \omega_\varphi^*(p_1, \xi, p_2) = \omega_\varphi^*(p_2, \xi, p_1)\\ &= \omega_\varphi^*(p_2, x, p_1).\end{aligned} \tag{18}$$

Therefore the result is true for $n = 0$. Assume that the result is true for $|u| = n - 1$. That is, for all $u \in X^*$ with $|u| = n - 1$, $n > 0$, we have

$$\begin{aligned}\widetilde{\delta}_\varphi^*(p_1, x, p_2) &= \widetilde{\delta}_\varphi^*(p_2, x, p_1),\\ \omega_\varphi^*(p_1, x, p_2) &= \omega_\varphi^*(p_2, x, p_1).\end{aligned} \tag{19}$$

Let $a \in X$ and $u \in X^*$ be such that $x = ua$. Then

$$\begin{aligned}\widetilde{\delta}_\varphi^*(p_1, x, p_2) &= \widetilde{\delta}_\varphi^*(p_1, ua, p_2)\\ &= \bigvee_{p\in\mathscr{P}}\left[r\min\left\{\widetilde{\delta}_\varphi^*(p_1, u, p), \widetilde{\delta}_\varphi(p, a, p_2)\right\}\right]\\ &= \bigvee_{p\in\mathscr{P}}\left[r\min\left\{\widetilde{\delta}_\varphi^*(p, u, p_1), \widetilde{\delta}_\varphi(p_2, a, p)\right\}\right]\\ &= \bigvee_{p\in\mathscr{P}}\left[r\min\left\{\widetilde{\delta}_\varphi^*(p, u, p_1), \widetilde{\delta}_\varphi^*(p_2, a, p)\right\}\right]\\ &= \bigvee_{p\in\mathscr{P}}\left[r\min\left\{\widetilde{\delta}_\varphi^*(p_2, a, p), \widetilde{\delta}_\varphi^*(p, u, p_1)\right\}\right]\\ &= \widetilde{\delta}_\varphi^*(p_2, au, p_1) = \widetilde{\delta}_\varphi^*(p_2, ua, p_1)\\ &= \widetilde{\delta}_\varphi^*(p_2, x, p_1),\end{aligned} \tag{20}$$

and

$$\begin{aligned}\omega_\varphi^*(p_1, x, p_2) &= \omega_\varphi^*(p_1, ua, p_2)\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p_1, u, p), \omega_\varphi(p, a, p_2)\right\}\right]\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p, u, p_1), \omega_\varphi(p_2, a, p)\right\}\right]\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p, u, p_1), \omega_\varphi^*(p_2, a, p)\right\}\right]\\ &= \bigwedge_{p\in\mathscr{P}}\left[\max\left\{\omega_\varphi^*(p_2, a, p), \omega_\varphi^*(p, u, p_1)\right\}\right]\\ &= \omega_\varphi^*(p_2, au, p_1) = \omega_\varphi^*(p_2, ua, p_1)\\ &= \omega_\varphi^*(p_2, x, p_1).\end{aligned} \tag{21}$$

This shows that the result is true for $|u| = n$. This completes the proof. □

Proposition 14. *If $\mathscr{T} = \langle\mathscr{P}, X, \varphi\rangle$ is a cubic fssm, then*

$$\begin{aligned}\widetilde{\delta}_\varphi^*(p_1, xy, p_2) &= \widetilde{\delta}_\varphi^*(p_1, yx, p_2),\\ \omega_\varphi^*(p_1, xy, p_2) &= \omega_\varphi^*(p_1, yx, p_2),\end{aligned} \tag{22}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x, y \in X^*$.

Proof. Proof easily follows from Propositions 11 and 13. □

Definition 15. Consider a cubic fssm $\mathscr{T} = \langle\mathscr{P}, X, \varphi\rangle$ and a cubic subset $\widehat{\varphi} = \langle\widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}}\rangle$ of $\mathscr{P}$. Then $\langle\mathscr{P}, \widehat{\varphi}, X, \varphi\rangle$ is said to be a subsystem of $\mathscr{T}$ if and only if

$$\begin{aligned}\widetilde{\delta}_{\widehat{\varphi}}(p_2) &\geq r\min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}_\varphi(p_1, x, p_2)\right\},\\ \omega_{\widehat{\varphi}}(p_2) &\leq \max\left\{\omega_{\widehat{\varphi}}(p_1), \omega_\varphi(p_1, x, p_2)\right\},\end{aligned} \tag{23}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

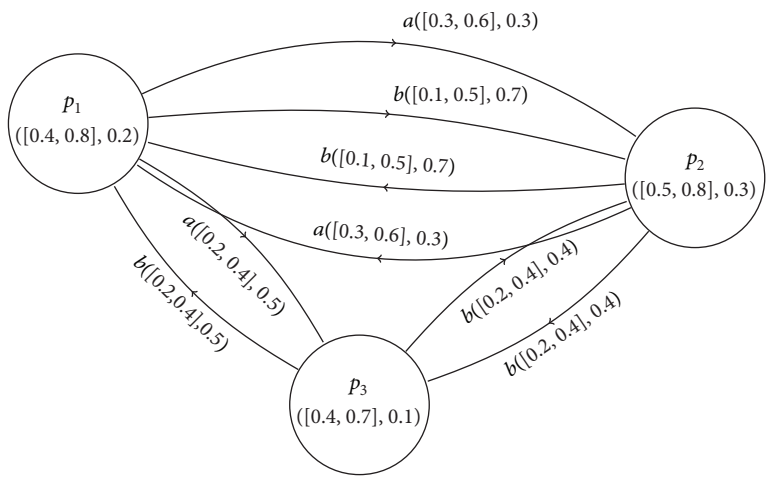

FIGURE 3: A commutative cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$.

If $\langle \mathcal{P}, \widehat{\varphi}, X, \varphi \rangle$ is a subsystem of $\mathcal{T}$, then we write simply $\widehat{\varphi}$ for $\langle \mathcal{P}, \widehat{\varphi}, X, \varphi \rangle$.

Example 16. Let $\mathcal{P} = \{p_1, p_2, p_3\}$ and $X = \{a, b\}$. Let $\varphi = \langle \widetilde{\delta}_{\varphi}, \omega_{\varphi} \rangle$ and $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ be a two cubic subsets in $\mathcal{P} \times X \times \mathcal{P}$ and $\mathcal{P}$, respectively. Then subsystem $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ of the cubic fsm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ is shown in Figure 3.

Theorem 17. *Consider a cubic fssm $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ and a cubic subset $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ of $\mathcal{P}$. Then $\langle \mathcal{P}, \widehat{\varphi}, X, \varphi \rangle$ is subsystem of $\mathcal{T}$ if and only if*

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi}}(p_2) &\succeq r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}^*_{\varphi}(p_1, x, p_2)\right\}, \\ \omega_{\widehat{\varphi}}(p_2) &\leq \max\left\{\omega_{\widehat{\varphi}}(p_1), \omega^*_{\varphi}(p_1, x, p_2)\right\}. \end{aligned} \tag{24}$$

for all $p_1, p_2 \in \mathcal{P}$ and $x \in X^*$.

Proof. Suppose $\widehat{\varphi}$ is a subsystem of $\mathcal{T}$. Let $p_1, p_2 \in \mathcal{P}$ and $x \in X^*$. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \xi$. Now, if $p_1 = p_2$ then

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi}}(p_2) &= r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}^*_{\varphi}(p_1, x, p_2)\right\}, \\ \omega_{\widehat{\varphi}}(p_2) &= \max\left\{\omega_{\widehat{\varphi}}(p_1), \omega^*_{\varphi}(p_1, x, p_2)\right\}. \end{aligned} \tag{25}$$

Now if $p_1 \neq p_2$, then

$$r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}^*_{\varphi}(p_1, x, p_2)\right\} = [0, 0] \preceq \widetilde{\delta}_{\widehat{\varphi}}(p_2), \tag{26}$$

and

$$\max\left\{\omega_{\widehat{\varphi}}(p_1), \omega^*_{\varphi}(p_1, x, p_2)\right\} = 1 \geq \omega_{\widehat{\varphi}}(p_2). \tag{27}$$

Thus the result is true for $n = 0$. Suppose the result is true for all $y \in X^*$ such that $|y| = n - 1$, $n > 0$. Let $x = ya$, $y \in X^*$, $a \in X$ and $|y| = n - 1$. Then

$$\begin{aligned} r \min&\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}^*_{\varphi}(p_1, x, p_2)\right\} = r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1),\right. \\ &\left.\widetilde{\delta}^*_{\varphi}(p_1, ya, p_2)\right\} = r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1),\right. \\ &\left.\bigvee_{p \in \mathcal{P}} \left\{r \min\left\{\widetilde{\delta}^*_{\varphi}(p_1, y, p), \widetilde{\delta}_{\varphi}(p, a, p_2)\right\}\right\}\right\} = \bigvee_{p \in \mathcal{P}} r \\ &\cdot \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}^*_{\varphi}(p_1, y, p), \widetilde{\delta}_{\varphi}(p, a, p_2)\right\} \preceq \bigvee_{p \in \mathcal{P}} r \\ &\cdot \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p), \widetilde{\delta}_{\varphi}(p, a, p_2)\right\} \preceq \widetilde{\delta}_{\widehat{\varphi}}(p_2), \end{aligned} \tag{28}$$

and

$$\begin{aligned} \max&\left\{\omega_{\widehat{\varphi}}(p_1), \omega^*_{\varphi}(p_1, x, p_2)\right\} = \max\left\{\omega_{\widehat{\varphi}}(p_1),\right. \\ &\left.\omega^*_{\varphi}(p_1, ya, p_2)\right\} = \max\left\{\omega_{\widehat{\varphi}}(p_1),\right. \\ &\left.\bigwedge_{p \in \mathcal{P}} \left\{\max\left\{\omega^*_{\varphi}(p_1, y, p), \omega_{\varphi}(p, a, p_2)\right\}\right\}\right\} \\ &= \bigwedge_{p \in \mathcal{P}} \max\left\{\omega_{\widehat{\varphi}}(p_1), \omega^*_{\varphi}(p_1, y, p), \omega_{\varphi}(p, a, p_2)\right\} \\ &\geq \bigwedge_{p \in \mathcal{P}} \max\left\{\omega_{\widehat{\varphi}}(p), \omega_{\varphi}(p, a, p_2)\right\} \geq \omega_{\widehat{\varphi}}(p_2). \end{aligned} \tag{29}$$

The converse is obvious. □

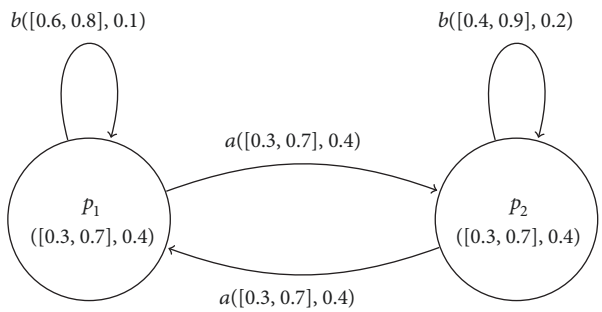

Figure 4: Good subsystem $\widehat{\varphi_g}$ of cubic fssm $\mathscr{T}$.

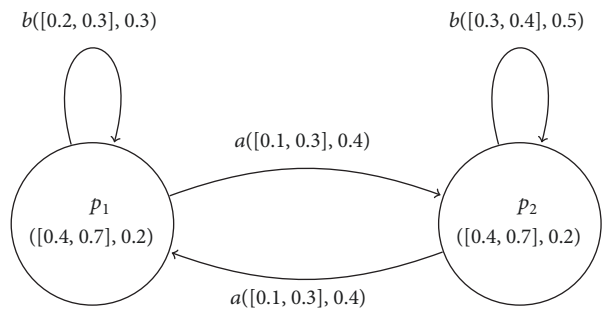

Figure 5: Strong subsystem $\widehat{\varphi_s}$ of cubic fssm $\mathscr{T}$.

Definition 18. Consider a cubic fssm $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and a cubic subset $\widehat{\varphi_g} = \langle \widetilde{\delta}_{\widehat{\varphi_g}}, \omega_{\widehat{\varphi_g}} \rangle$ of $\mathscr{P}$. Then $\langle \mathscr{P}, \widehat{\varphi_g}, X, \varphi \rangle$ is said to be a good subsystem of $\mathscr{T}$ if and only if

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi_g}}(p_2) &= r \min\left\{\widetilde{\delta}_{\widehat{\varphi_g}}(p_1), \widetilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, \\ \omega_{\widehat{\varphi_g}}(p_2) &= \max\left\{\omega_{\widehat{\varphi_g}}(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{30}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

Example 19. The good subsystem $\langle \mathscr{P}, \widehat{\varphi_g}, X, \varphi \rangle$ of a cubic fssm $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ is shown in Figure 4.

Theorem 20. *Consider a cubic fssm $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and a cubic subset $\widehat{\varphi_g} = \langle \widetilde{\delta}_{\widehat{\varphi_g}}, \omega_{\widehat{\varphi_g}} \rangle$ of $\mathscr{P}$. Then $\langle \mathscr{P}, \widehat{\varphi_g}, X, \varphi \rangle$ is good subsystem of $\mathscr{T}$ if and only if*

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi_g}}(p_2) &= r \min\left\{\widetilde{\delta}_{\widehat{\varphi_g}}(p_1), \widetilde{\delta}^*_{\varphi}(p_1, x, p_2)\right\}, \\ \omega_{\widehat{\varphi_g}}(p_2) &= \max\left\{\omega_{\widehat{\varphi_g}}(p_1), \omega^*_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{31}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X^*$.

Proof. The proof is similar to the proof of Theorem 17. □

Definition 21. Consider a cubic fssm $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and a cubic subset $\widehat{\varphi_s} = \langle \widetilde{\delta}_{\widehat{\varphi_s}}, \omega_{\widehat{\varphi_s}} \rangle$ of $\mathscr{P}$. Then $\langle \mathscr{P}, \widehat{\varphi_s}, X, \varphi \rangle$ is said to be a strong subsystem of $\mathscr{T}$ if and only if

$$\begin{aligned} \widetilde{\delta}_{\varphi}(p_1, x, p_2) &\succ [0, 0], \\ \omega_{\varphi}(p_1, x, p_2) &< 1, \end{aligned} \tag{32}$$

then

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi_s}}(p_2) &\succeq \widetilde{\delta}_{\widehat{\varphi_s}}(p_1), \\ \omega_{\widehat{\varphi_s}}(p_2) &\leq \omega_{\widehat{\varphi_s}}(p_1), \end{aligned} \tag{33}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

Example 22. The strong subsystem $\langle \mathscr{P}, \widehat{\varphi_s}, X, \varphi \rangle$ of a cubic fssm $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ is shown in Figure 5.

Remark 23. A strong subsystem may be or may be not a good subsystem.

Theorem 24. *Consider a cubic fssm $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and a cubic subset $\widehat{\varphi_s} = \langle \widetilde{\delta}_{\widehat{\varphi_s}}, \omega_{\widehat{\varphi_s}} \rangle$ of $\mathscr{P}$. Then $\langle \mathscr{P}, \widehat{\varphi_s}, X, \varphi \rangle$ is strong subsystem of $\mathscr{T}$ if and only if*

$$\begin{aligned} \widetilde{\delta}^*_{\varphi}(p_1, x, p_2) &\succ [0, 0], \\ \omega^*_{\varphi}(p_1, x, p_2) &< 1, \end{aligned} \tag{34}$$

then

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi_s}}(p_2) &\succeq \widetilde{\delta}_{\widehat{\varphi_s}}(p_1), \\ \omega_{\widehat{\varphi_s}}(p_2) &\leq \omega_{\widehat{\varphi_s}}(p_1), \end{aligned} \tag{35}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X^*$.

Proof. Suppose that $\widehat{\varphi_s}$ is a strong subsystem. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \xi$. Now, if $p_1 = p_2$ then $\widetilde{\delta}^*_{\varphi}(p_2, \xi, p_2) = [1, 1]$ and $\omega^*_{\varphi}(p_2, \xi, p_2) = 0$. Thus $\widetilde{\delta}_{\widehat{\varphi_s}}(p_2) = \widetilde{\delta}_{\widehat{\varphi_s}}(p_2)$ and $\omega_{\widehat{\varphi_s}}(p_2) = \omega_{\widehat{\varphi_s}}(p_2)$. Now if $p_1 \neq p_2$, then $\widetilde{\delta}^*_{\varphi}(p_1, \xi, p_2) = [0, 0]$ and $\omega^*_{\varphi}(p_1, \xi, p_2) = 1$. Thus the result is true for $n = 0$. Suppose the result is true for all $y \in X^*$ such that $|y| = n - 1$, $n > 0$. Let $x = ya$, $y \in X^*$, $a \in X$ and $|y| = n - 1$. Suppose that $\widetilde{\delta}^*_{\varphi}(p_1, x, p_2) \succ [0, 0]$. Then

$$\begin{aligned} \widetilde{\delta}^*_{\varphi}(p_1, x, p_2) &\succ [0, 0] \\ \widetilde{\delta}^*_{\varphi}(p_1, ya, p_2) &\succ [0, 0] \\ \bigvee_{p \in \mathscr{P}} \left[r \min\left\{\widetilde{\delta}^*_{\varphi}(p_1, y, p), \widetilde{\delta}^*_{\varphi}(p, a, p_2)\right\}\right] &\succ [0, 0]. \end{aligned} \tag{36}$$

Thus there exists $p \in \mathscr{P}$ such that $\widetilde{\delta}^*_{\varphi}(p_1, y, p) \succ [0, 0]$ and $\widetilde{\delta}^*_{\varphi}(p, a, p_2) \succ [0, 0]$. Hence $\widetilde{\delta}_{\widehat{\varphi_s}}(p_2) \succeq \widetilde{\delta}_{\widehat{\varphi_s}}(p)$ and $\widetilde{\delta}_{\widehat{\varphi_s}}(p) \succeq \widetilde{\delta}_{\widehat{\varphi_s}}(p_1)$. Thus $\widetilde{\delta}_{\widehat{\varphi_s}}(p_2) \succeq \widetilde{\delta}_{\widehat{\varphi_s}}(p_1)$. Now suppose that $\omega^*_{\varphi}(p_1, x, p_2) < 1$. Then

$$\begin{aligned} \omega^*_{\varphi}(p_1, x, p_2) &< 1 \\ \omega^*_{\varphi}(p_1, ya, p_2) &< 1 \\ \bigwedge_{p \in \mathscr{P}} \left[\max\left\{\omega^*_{\varphi}(p_1, y, p), \omega^*_{\varphi}(p, a, p_2)\right\}\right] &< 1. \end{aligned} \tag{37}$$

Thus there exists $p \in \mathscr{P}$ such that $\omega_{\varphi}^{*}(p_1, y, p) < 1$ and $\omega_{\varphi}^{*}(p, a, p_2) < 1$. Hence $\omega_{\widehat{\varphi_s}}(p_2) \leq \omega_{\widehat{\varphi_s}}(p)$ and $\omega_{\widehat{\varphi_s}}(p) \leq \omega_{\widehat{\varphi_s}}(p_1)$. Thus $\omega_{\widehat{\varphi_s}}(p_2) \leq \omega_{\widehat{\varphi_s}}(p_1)$. The converse is obvious. □

Theorem 25. *Let $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ be a cubic fssm. Let $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$ be two subsystems (resp., good subsystems, strong subsystems) of $\mathscr{T}$. Then the following conditions hold.*

(i) $\widehat{\varphi_1} \wedge \widehat{\varphi_2}$ is a subsystem (resp., good subsystem, strong subsystem) of $\mathscr{T}$.

(ii) $\widehat{\varphi_1} \vee \widehat{\varphi_2}$ is a subsystem (resp., good subsystem, strong subsystem) of $\mathscr{T}$.

Proof. Suppose that $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$ be two subsystems of $\mathscr{T}$. Then

$$\begin{aligned} \tilde{\delta}_{\widehat{\varphi_1}}(p_2) &\geq r \min\left\{\tilde{\delta}_{\widehat{\varphi_1}}(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, \\ \omega_{\widehat{\varphi_1}}(p_2) &\leq \max\left\{\omega_{\widehat{\varphi_1}}(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{38}$$

also

$$\begin{aligned} \tilde{\delta}_{\widehat{\varphi_2}}(p_2) &\geq r \min\left\{\tilde{\delta}_{\widehat{\varphi_2}}(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, \\ \omega_{\widehat{\varphi_2}}(p_2) &\leq \max\left\{\omega_{\widehat{\varphi_2}}(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{39}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

(i) Now to prove that $\widehat{\varphi_1} \wedge \widehat{\varphi_2}$ is a subsystem of $\mathscr{T}$. It is enough to prove that

$$\begin{aligned} &\left(\tilde{\delta}_{\widehat{\varphi_1}} \wedge \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_2) \\ &\quad \geq r \min\left\{\left(\tilde{\delta}_{\widehat{\varphi_1}} \wedge \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{40}$$

and

$$\begin{aligned} &\left(\omega_{\widehat{\varphi_1}} \wedge \omega_{\widehat{\varphi_2}}\right)(p_2) \\ &\quad \leq \max\left\{\left(\omega_{\widehat{\varphi_1}} \wedge \omega_{\widehat{\varphi_2}}\right)(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}. \end{aligned} \tag{41}$$

Now

$$\begin{aligned} &\left(\tilde{\delta}_{\widehat{\varphi_1}} \wedge \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_2) = r \min\left\{\tilde{\delta}_{\widehat{\varphi_1}}(p_2), \tilde{\delta}_{\widehat{\varphi_2}}(p_2)\right\} \geq r \\ &\quad \cdot \min\left\{r \min\left\{\tilde{\delta}_{\widehat{\varphi_1}}(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, r\right. \\ &\quad \cdot \left.\min\left\{\tilde{\delta}_{\widehat{\varphi_2}}(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}\right\} = r \min\left\{r\right. \\ &\quad \cdot \min\left\{\tilde{\delta}_{\widehat{\varphi_1}}(p_1), \tilde{\delta}_{\widehat{\varphi_2}}(p_1)\right\}, \tilde{\delta}_{\varphi}(p_1, x, p_2)\Big\} = r \\ &\quad \cdot \min\left\{\left(\tilde{\delta}_{\widehat{\varphi_1}} \wedge \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{42}$$

and

$$\begin{aligned} &\left(\omega_{\widehat{\varphi_1}} \wedge \omega_{\widehat{\varphi_2}}\right)(p_2) = \max\left\{\omega_{\widehat{\varphi_1}}(p_2), \omega_{\widehat{\varphi_2}}(p_2)\right\} \\ &\quad \leq \max\left\{\max\left\{\omega_{\widehat{\varphi_1}}(p_1), \omega_{\varphi}(p_1, x, p_2)\right\},\right. \\ &\quad \left.\max\left\{\omega_{\widehat{\varphi_2}}(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}\right\} \\ &\quad = \max\left\{\max\left\{\omega_{\widehat{\varphi_1}}(p_1), \omega_{\widehat{\varphi_2}}(p_1)\right\}, \omega_{\varphi}(p_1, x, p_2)\right\} \\ &\quad = \max\left\{\left(\omega_{\widehat{\varphi_1}} \wedge \omega_{\widehat{\varphi_2}}\right)(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}. \end{aligned} \tag{43}$$

Hence this shows that $\widehat{\varphi_1} \wedge \widehat{\varphi_2}$ is a subsystem of $\mathscr{T}$.

(ii) Now to prove that $\widehat{\varphi_1} \vee \widehat{\varphi_2}$ is a subsystem of $\mathscr{T}$. It is enough to prove that

$$\begin{aligned} &\left(\tilde{\delta}_{\widehat{\varphi_1}} \vee \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_2) \\ &\quad \geq r \min\left\{\left(\tilde{\delta}_{\widehat{\varphi_1}} \vee \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\left(\omega_{\widehat{\varphi_1}} \vee \omega_{\widehat{\varphi_2}}\right)(p_2) \\ &\quad \leq \max\left\{\left(\omega_{\widehat{\varphi_1}} \vee \omega_{\widehat{\varphi_2}}\right)(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}. \end{aligned} \tag{45}$$

Now

$$\begin{aligned} &\left(\tilde{\delta}_{\widehat{\varphi_1}} \vee \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_2) = r \max\left\{(\tilde{\delta}_{\widehat{\varphi_1}}(p_2), \tilde{\delta}_{\widehat{\varphi_2}}(p_2)\right\} \geq r \\ &\quad \cdot \max\left\{r \min\left\{\tilde{\delta}_{\widehat{\varphi_1}}(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, r\right. \\ &\quad \cdot \left.\min\left\{\tilde{\delta}_{\widehat{\varphi_2}}(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}\right\} = r \min\left\{r\right. \\ &\quad \cdot \max\left\{\tilde{\delta}_{\widehat{\varphi_1}}(p_1), \tilde{\delta}_{\widehat{\varphi_2}}(p_1)\right\}, \tilde{\delta}_{\varphi}(p_1, x, p_2)\Big\} = r \\ &\quad \cdot \min\left\{\left(\tilde{\delta}_{\widehat{\varphi_1}} \vee \tilde{\delta}_{\widehat{\varphi_2}}\right)(p_1), \tilde{\delta}_{\varphi}(p_1, x, p_2)\right\}, \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\left(\omega_{\widehat{\varphi_1}} \vee \omega_{\widehat{\varphi_2}}\right)(p_2) = \min\left\{\omega_{\widehat{\varphi_1}}(p_2), \omega_{\widehat{\varphi_2}}(p_2)\right\} \\ &\quad \leq \min\left\{\max\left\{\omega_{\widehat{\varphi_1}}(p_1), \omega_{\varphi}(p_1, x, p_2)\right\},\right. \\ &\quad \left.\max\left\{\omega_{\widehat{\varphi_2}}(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}\right\} \\ &\quad = \max\left\{\min\left\{\omega_{\widehat{\varphi_1}}(p_1), \omega_{\widehat{\varphi_2}}(p_1)\right\}, \omega_{\varphi}(p_1, x, p_2)\right\} \\ &\quad = \max\left\{\left(\omega_{\widehat{\varphi_1}} \vee \omega_{\widehat{\varphi_2}}\right)(p_1), \omega_{\varphi}(p_1, x, p_2)\right\}. \end{aligned} \tag{47}$$

Hence this shows that $\widehat{\varphi_1} \vee \widehat{\varphi_2}$ is a subsystem of $\mathscr{T}$. The other cases can be seen in a similar way. □

4. Products of Subsystems of Cubic fssm

In [2], the authors constructed different types of products of subsystems of cubic fsms. Here in this section we will provide some results related to products of subsystems of cubic fssm.

Definition 26. Let $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$ be two cubic fssms and let $X \cap Y = \emptyset$. The Cartesian composition of $\mathscr{T}$ and $\mathscr{V}$ is denoted by

$$\mathscr{T} \circ \mathscr{V} = \langle \mathscr{P} \times \mathscr{Q}, X \cup Y, \varphi \circ \psi \rangle \tag{48}$$

and is defined as follows:

(i)

$$\begin{aligned} \left(\tilde{\delta}_{\varphi} \circ \tilde{\delta}_{\psi}\right)((p_1, q), a, (p_2, q)) &= \tilde{\delta}_{\varphi}(p_1, a, p_2) \\ \left(\omega_{\varphi} \circ \omega_{\psi}\right)((p_1, q), a, (p_2, q)) &= \omega_{\varphi}(p_1, a, p_2) \end{aligned} \tag{49}$$

for all $p_1, p_2 \in \mathscr{P}$, $q \in \mathscr{Q}$ and $a \in X \cup Y$.

(ii)

$$\left(\widetilde{\delta}_{\varphi} \times \widetilde{\delta}_{\psi}\right)((p,q_1),a,(p,q_2)) = \widetilde{\delta}_{\psi}(q_1,a,q_2)$$
$$\left(\omega_{\varphi} \times \omega_{\psi}\right)((p,q_1),a,(p,q_2)) = \omega_{\psi}(q_1,a,q_2) \tag{50}$$

for all $p \in \mathscr{P}$, $q_1, q_2 \in \mathscr{Q}$ and $a \in X \cup Y$.

Proposition 27. *The "Cartesian composition" of two cubic fssms is a cubic fssm.*

Proof. It is straightforward. □

Definition 28. Let $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ be the subsystems of cubic fssms $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, respectively, and let $X \cap Y = \emptyset$. The "Cartesian composition" of $\widehat{\varphi}$ and $\widehat{\psi}$ is denoted by

$$\widehat{\varphi} \circ \widehat{\psi} = \langle \mathscr{P} \times \mathscr{Q}, \widehat{\varphi} \circ \widehat{\psi}, X \cup Y, \varphi \circ \psi \rangle \tag{51}$$

and is defined as follows:

(i)

$$\left(\widetilde{\delta}_{\widehat{\varphi}} \circ \widetilde{\delta}_{\widehat{\psi}}\right)(p,q) = r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p), \widetilde{\delta}_{\widehat{\psi}}(q)\right\}$$
$$\left(\omega_{\widehat{\varphi}} \circ \omega_{\widehat{\psi}}\right)(p,q) = \max\left\{\omega_{\widehat{\varphi}}(p), \omega_{\widehat{\psi}}(q)\right\} \tag{52}$$

for all $(p,q) \in \mathscr{P} \times \mathscr{Q}$.

(ii)

$$\left(\widetilde{\delta}_{\varphi} \circ \widetilde{\delta}_{\psi}\right)((p_1,q),a,(p_2,q)) = \widetilde{\delta}_{\varphi}(p_1,a,p_2)$$
$$\left(\omega_{\varphi} \circ \omega_{\psi}\right)((p_1,q),a,(p_2,q)) = \omega_{\varphi}(p_1,a,p_2) \tag{53}$$

for all $p_1, p_2 \in \mathscr{P}$, $q \in \mathscr{Q}$ and $a \in X \cup Y$.

(iii)

$$\left(\widetilde{\delta}_{\varphi} \circ \widetilde{\delta}_{\psi}\right)((p,q_1),a,(p,q_2)) = \widetilde{\delta}_{\psi}(q_1,a,q_2)$$
$$\left(\omega_{\varphi} \circ \omega_{\psi}\right)((p,q_1),a,(p,q_2)) = \omega_{\psi}(q_1,a,q_2) \tag{54}$$

for all $p \in \mathscr{P}$, $q_1, q_2 \in \mathscr{Q}$ and $a \in X \cup Y$.

Proposition 29. *Let $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ be two subsystems (resp., good subsystems, strong subsystems) of cubic fssms $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, respectively. Then $\widehat{\varphi} \circ \widehat{\psi}$ is a subsystem (resp., good subsystem, strong subsystem) of cubic fssm $\mathscr{T} \circ \mathscr{V}$.*

Proof. It is straightforward. □

Theorem 30. *If $\widehat{\varphi} \circ \widehat{\psi}$ is a good subsystem of cubic fssm $\mathscr{T} \circ \mathscr{V}$, then at least $\widehat{\varphi}$ or $\widehat{\psi}$ must be good subsystem.*

Proof. Suppose that $\widehat{\varphi}$ and $\widehat{\psi}$ are not good subsystems. Then there exist $p_1, p_2 \in \mathscr{P}$, $q_1, q_2 \in \mathscr{Q}$, $x \in X$, and $y \in Y$ such that

$$\widetilde{\delta}_{\widehat{\varphi}}(p_2) \succ r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}_{\varphi}(p_1,x,p_2)\right\},$$
$$\widetilde{\delta}_{\widehat{\psi}}(q_2) \succ r \min\left\{\widetilde{\delta}_{\widehat{\psi}}(q_1), \widetilde{\delta}_{\psi}(q_1,y,q_2)\right\} \tag{55}$$

also

$$\omega_{\widehat{\varphi}}(p_2) < \max\left\{\omega_{\widehat{\varphi}}(p_1), \omega_{\varphi}(p_1,x,p_2)\right\},$$
$$\omega_{\widehat{\psi}}(q_2) < \max\left\{\omega_{\widehat{\psi}}(q_1), \omega_{\psi}(q_1,y,q_2)\right\}. \tag{56}$$

Now by the definition

$$\left(\widetilde{\delta}_{\varphi} \circ \widetilde{\delta}_{\psi}\right)((p,q_1),y,(p,q_2)) = \widetilde{\delta}_{\psi}(q_1,y,q_2) \tag{57}$$

and

$$\left(\omega_{\varphi} \circ \omega_{\psi}\right)((p,q_1),y,(p,q_2)) = \omega_{\psi}(q_1,y,q_2). \tag{58}$$

Now

$$\begin{aligned}
\left(\widetilde{\delta}_{\widehat{\varphi}} \circ \widetilde{\delta}_{\widehat{\psi}}\right)(p,q_2) &= r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p), \widetilde{\delta}_{\widehat{\psi}}(q_2)\right\} \succ r \\
&\cdot \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p), r \min\left\{\widetilde{\delta}_{\widehat{\psi}}(q_1), \widetilde{\delta}_{\psi}(q_1,y,q_2)\right\}\right. = r \\
&\cdot \min\left\{r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p), \widetilde{\delta}_{\widehat{\psi}}(q_1)\right\}, \widetilde{\delta}_{\psi}(q_1,y,q_2)\right\}\} \\
&= r \min\left\{\left(\widetilde{\delta}_{\widehat{\varphi}} \circ \widetilde{\delta}_{\widehat{\psi}}\right)(p,q_1), \widetilde{\delta}_{\psi}(q_1,y,q_2)\right\} = r \\
&\cdot \min\left\{\left(\widetilde{\delta}_{\widehat{\varphi}} \circ \widetilde{\delta}_{\widehat{\psi}}\right)(p,q_1), \left(\widetilde{\delta}_{\varphi} \circ \widetilde{\delta}_{\psi}\right)\right. \\
&\cdot \left.((p,q_1),y,(p,q_2))\right\},
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
\left(\omega_{\widehat{\varphi}} \circ \omega_{\widehat{\psi}}\right)(p,q_2) &= \max\left\{\omega_{\widehat{\varphi}}(p), \omega_{\widehat{\psi}}(q_2)\right\} \\
&< \max\left\{\omega_{\widehat{\varphi}}(p), \max\left\{\omega_{\widehat{\psi}}(q_1), \omega_{\psi}(q_1,y,q_2)\right\}\right. \\
&= \max\left\{\max\left\{\omega_{\widehat{\varphi}}(p), \omega_{\widehat{\psi}}(q_1)\right\}, \omega_{\psi}(q_1,y,q_2)\right\}\} \\
&= \max\left\{\left(\omega_{\widehat{\varphi}} \circ \omega_{\widehat{\psi}}\right)(p,q_1), \omega_{\psi}(q_1,y,q_2)\right\} \\
&= \max\left\{\left(\omega_{\widehat{\varphi}} \circ \omega_{\widehat{\psi}}\right)(p,q_1), \left(\omega_{\varphi} \circ \omega_{\psi}\right)\right. \\
&\cdot \left.((p,q_1),y,(p,q_2))\right\}.
\end{aligned} \tag{60}$$

This shows that $\widehat{\varphi} \circ \widehat{\psi}$ is not good subsystem, a contradiction. This completes the proof. □

The following examples show that if $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ is a good subsystem of $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ is not a good subsystem of $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, then $\widehat{\varphi} \circ \widehat{\psi}$ may or may not be a good subsystem.

Example 31. Consider $\widehat{\varphi}$ is a good subsystem and $\widehat{\psi}$ is not a good subsystem as shown in the Figures 6 and 7.

The Cartesian composition of $\widehat{\varphi}$ and $\widehat{\psi}$ is shown in Figure 8. It is clear from Figure 8 that $\widehat{\varphi} \circ \widehat{\psi}$ is a good subsystem.

Example 32. Consider $\widehat{\varphi}$ is a good subsystem and $\widehat{\psi}$ is not a good subsystem as shown in Figures 9 and 10.

The Cartesian composition of $\widehat{\varphi}$ and $\widehat{\psi}$ is shown in Figure 11. It is clear from Figure 11 that $\widehat{\varphi} \circ \widehat{\psi}$ is not a good subsystem

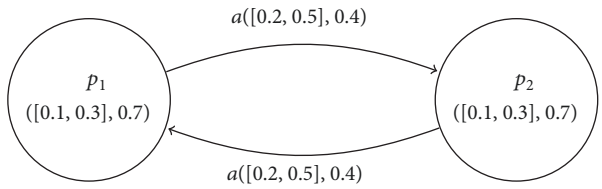

Figure 6: $\widehat{\varphi}$ is a good subsystem.

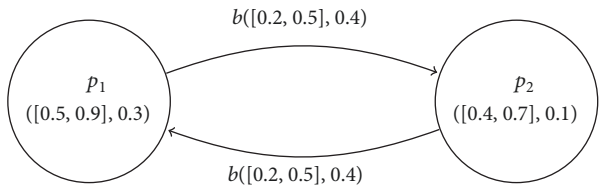

Figure 7: $\widehat{\psi}$ is not a good subsystem.

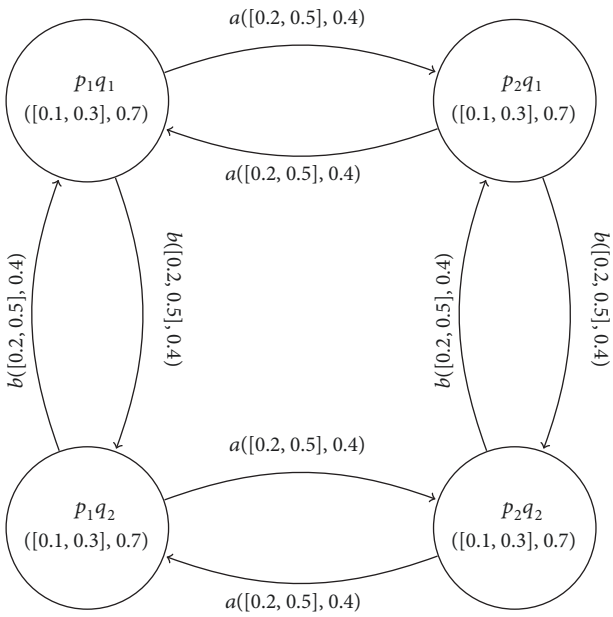

Figure 8: $\widehat{\varphi} \circ \widehat{\psi}$ is a good subsystem.

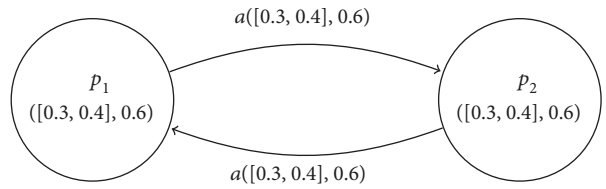

Figure 9: $\widehat{\varphi}$ is a good subsystem.

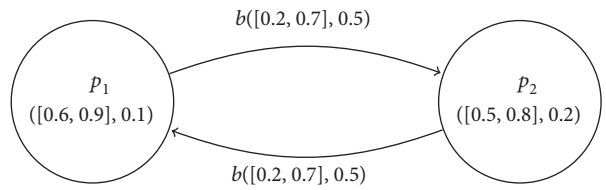

Figure 10: $\widehat{\psi}$ is not a good subsystem.

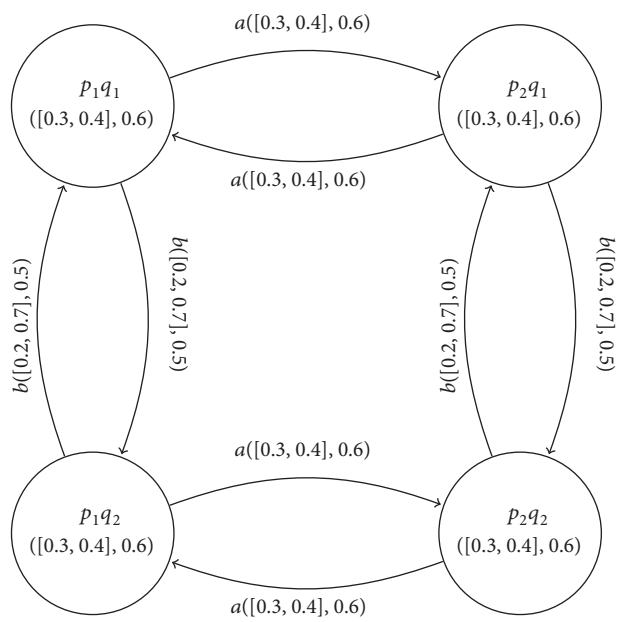

Figure 11: $\widehat{\varphi} \circ \widehat{\psi}$ is not a good subsystem.

Now we can state a new proposition without its proof.

Proposition 33. *Let $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ be a good subsystem of $\mathcal{T} = \langle \mathcal{P}, X, \varphi \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ be any subsystem of $\mathcal{V} = \langle \mathcal{Q}, Y, \psi \rangle$; then $\widehat{\varphi} \circ \widehat{\psi}$ is a good subsystem if and only if*

$$\begin{aligned} \widetilde{\delta}_{\varphi}(p_1, x, p_2) &= \widetilde{\delta}_{\psi}(q_1, y, q_2), \\ \omega_{\varphi}(p_1, x, p_2) &= \omega_{\psi}(q_1, y, q_2), \end{aligned} \tag{61}$$

also

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi}}(p_1) &\preceq \widetilde{\delta}_{\widehat{\psi}}(q_1), \\ \omega_{\widehat{\varphi}}(p_1) &\geq \omega_{\widehat{\psi}}(q_1), \end{aligned} \tag{62}$$

for all $p_1, p_2 \in \mathcal{P}$, $q_1, q_2 \in \mathcal{Q}$, $x \in X$ and $y \in Y$.

Theorem 34. *If $\widehat{\varphi} \circ \widehat{\psi}$ is a strong subsystem of cubic fssm $\mathcal{T} \circ \mathcal{V}$, then at least $\widehat{\varphi}$ or $\widehat{\psi}$ must be strong subsystem.*

Proof. Suppose that $\widehat{\varphi}$ and $\widehat{\psi}$ are not strong subsystems. Then there exist $p_1, p_2 \in \mathcal{P}$, $q_1, q_2 \in \mathcal{Q}$, $x \in X$ and $y \in Y$ such that

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi}}(p_2) &\prec \widetilde{\delta}_{\widehat{\varphi}}(p_1), \\ \widetilde{\delta}_{\widehat{\psi}}(q_2) &\prec \widetilde{\delta}_{\widehat{\psi}}(q_1) \end{aligned} \tag{63}$$

also

$$\begin{aligned} \omega_{\widehat{\varphi}}(p_2) &> \omega_{\widehat{\varphi}}(p_1), \\ \omega_{\widehat{\psi}}(q_2) &> \omega_{\widehat{\psi}}(q_1). \end{aligned} \tag{64}$$

We know that if $\widetilde{\delta}_{\psi}(q_1, y, q_2) \succ [0,0]$, then $(\widetilde{\delta}_{\varphi} \circ \widetilde{\delta}_{\psi})((p, q_1), y, (p, q_2)) \succ [0,0]$; also if $\omega_{\psi}(q_1, y, q_2) < 1$, then $(\omega_{\varphi} \circ \omega_{\psi})((p, q_1), y, (p, q_2)) < 1$. Now consider that

$$\begin{aligned}\left(\widetilde{\delta}_{\widehat{\varphi}} \circ \widetilde{\delta}_{\widehat{\psi}}\right)(p, q_2) &= r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p), \widetilde{\delta}_{\widehat{\psi}}(q_2)\right\} \\ &\prec r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p), \widetilde{\delta}_{\widehat{\psi}}(q_1)\right\} \\ &= \left(\widetilde{\delta}_{\widehat{\varphi}} \circ \widetilde{\delta}_{\widehat{\psi}}\right)(p, q_1)\end{aligned} \tag{65}$$

and

$$\begin{aligned}\left(\omega_{\widehat{\varphi}} \circ \omega_{\widehat{\psi}}\right)(p, q_2) &= \max\left\{\omega_{\widehat{\varphi}}(p), \omega_{\widehat{\psi}}(q_2)\right\} \\ &> \max\left\{\omega_{\widehat{\varphi}}(p), \omega_{\widehat{\psi}}(q_1)\right\} \\ &= \left(\omega_{\widehat{\varphi}} \circ \omega_{\widehat{\psi}}\right)(p, q_1).\end{aligned} \tag{66}$$

This shows that $\widehat{\varphi} \circ \widehat{\psi}$ is not a strong subsystem, a contradiction. This completes the proof. □

We can easily show with the help of an example that if $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ is a strong subsystem of $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ is not a strong subsystem of $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, then $\widehat{\varphi} \circ \widehat{\psi}$ may or may not be a strong subsystem.

Definition 35. Let $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$ be two cubic fssms and let $X \cap Y = \emptyset$. The direct product of $\mathscr{T}$ and $\mathscr{V}$ is denoted by

$$\mathscr{T} \times \mathscr{V} = \langle \mathscr{P} \times \mathscr{Q}, X \times Y, \varphi \times \psi \rangle \tag{67}$$

and is defined as follows:

(i)

$$\begin{aligned}\left(\widetilde{\delta}_{\varphi} \times \widetilde{\delta}_{\psi}\right)((p_1, q), x, (p_2, q)) &= \widetilde{\delta}_{\varphi}(p_1, x, p_2) \\ \left(\omega_{\varphi} \times \omega_{\psi}\right)((p_1, q), x, (p_2, q)) &= \omega_{\varphi}(p_1, x, p_2)\end{aligned} \tag{68}$$

for all $p_1, p_2 \in \mathscr{P}$, $q \in \mathscr{Q}$ and $x \in X$.

(ii)

$$\begin{aligned}\left(\widetilde{\delta}_{\varphi} \times \widetilde{\delta}_{\psi}\right)((p, q_1), y, (p, q_2)) &= \widetilde{\delta}_{\psi}(q_1, y, q_2) \\ \left(\omega_{\varphi} \times \omega_{\psi}\right)((p, q_1), y, (p, q_2)) &= \omega_{\psi}(q_1, y, q_2)\end{aligned} \tag{69}$$

for all $p \in \mathscr{P}$, $q_1, q_2 \in \mathscr{Q}$ and $y \in Y$.

(iii)

$$\begin{aligned}&\left(\widetilde{\delta}_{\varphi} \times \widetilde{\delta}_{\psi}\right)((p_1, q_1), (x, y), (p_2, q_2)) \\ &\quad = r \min\left\{\widetilde{\delta}_{\varphi}(p_1, x, p_2), \widetilde{\delta}_{\psi}(q_1, y, q_2)\right\} \\ &\left(\omega_{\varphi} \times \omega_{\psi}\right)((p_1, q_1), (x, y), (p_2, q_2)) \\ &\quad = \max\left\{\omega_{\varphi}(p_1, x, p_2), \omega_{\psi}(q_1, y, q_2)\right\}\end{aligned} \tag{70}$$

for all (p_1, q_1) and $(p_2, q_2) \in \mathscr{P} \times \mathscr{Q}$ and $(x, y) \in X \times Y$.

Proposition 36. *The "direct product" of two cubic fssms is a cubic fssm.*

Proof. It is straightforward. □

Definition 37. Let $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ be the cubic subsystems of cubic fssms $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, respectively, and let $X \cap Y = \emptyset$. The "direct product" of $\widehat{\varphi}$ and $\widehat{\psi}$ is denoted by

$$\widehat{\varphi} \times \widehat{\psi} = \langle \mathscr{P} \times \mathscr{Q}, \widehat{\varphi} \times \widehat{\psi}, X \times Y, \varphi \times \psi \rangle \tag{71}$$

and is defined as follows:

(i)

$$\begin{aligned}\left(\widetilde{\delta}_{\widehat{\varphi}} \times \widetilde{\delta}_{\widehat{\psi}}\right)(p_1, q_1) &= r \min\left\{\widetilde{\delta}_{\widehat{\varphi}}(p_1), \widetilde{\delta}_{\widehat{\psi}}(q_1)\right\} \\ \left(\omega_{\widehat{\varphi}} \times \omega_{\widehat{\psi}}\right)(p_1, q_1) &= \max\left\{\omega_{\widehat{\varphi}}(p_1), \omega_{\widehat{\psi}}(q_1)\right\}\end{aligned} \tag{72}$$

for all $(p_1, q_1) \in \mathscr{P} \times \mathscr{Q}$.

(ii)

$$\begin{aligned}\left(\widetilde{\delta}_{\varphi} \times \widetilde{\delta}_{\psi}\right)((p_1, q), x, (p_2, q)) &= \widetilde{\delta}_{\varphi}(p_1, x, p_2) \\ \left(\omega_{\varphi} \times \omega_{\psi}\right)((p_1, q), x, (p_2, q)) &= \omega_{\varphi}(p_1, x, p_2)\end{aligned} \tag{73}$$

for all $p_1, p_2 \in \mathscr{P}$, $q \in \mathscr{Q}$ and $x \in X$.

(iii)

$$\begin{aligned}\left(\widetilde{\delta}_{\varphi} \times \widetilde{\delta}_{\psi}\right)((p, q_1), y, (p, q_2)) &= \widetilde{\delta}_{\psi}(q_1, y, q_2) \\ \left(\omega_{\varphi} \times \omega_{\psi}\right)((p, q_1), y, (p, q_2)) &= \omega_{\psi}(q_1, y, q_2)\end{aligned} \tag{74}$$

for all $p \in \mathscr{P}$, $q_1, q_2 \in \mathscr{Q}$ and $y \in Y$.

(iv)

$$\begin{aligned}&\left(\widetilde{\delta}_{\varphi} \times \widetilde{\delta}_{\psi}\right)((p_1, q_1), (x, y), (p_2, q_2)) \\ &\quad = r \min\left\{\widetilde{\delta}_{\varphi}(p_1, x, p_2), \widetilde{\delta}_{\psi}(q_1, y, q_2)\right\} \\ &\left(\omega_{\varphi} \times \omega_{\psi}\right)((p_1, q_1), (x, y), (p_2, q_2)) \\ &\quad = \max\left\{\omega_{\varphi}(p_1, x, p_2), \omega_{\psi}(q_1, y, q_2)\right\}\end{aligned} \tag{75}$$

for all (p_1, q_1) and $(p_2, q_2) \in \mathscr{P} \times \mathscr{Q}$ and $(x, y) \in X \times Y$.

Proposition 38. *Let $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ be two subsystems (resp., good subsystems, strong subsystems) of cubic fssms $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, respectively. Then $\widehat{\varphi} \times \widehat{\psi}$ is a subsystem (resp., good subsystem, strong subsystem) of cubic fssm $\mathscr{T} \times \mathscr{V}$.*

Proof. It is straightforward. □

Theorem 39. *If $\widehat{\varphi} \times \widehat{\psi}$ is a good subsystem of cubic fssm $\mathscr{T} \times \mathscr{V}$, then at least $\widehat{\varphi}$ or $\widehat{\psi}$ must be good subsystem.*

Proof. Similar to Theorem 30. □

Theorem 40. *If $\widehat{\varphi} \times \widehat{\psi}$ is a strong subsystem of cubic fssm $\mathscr{T} \times \mathscr{V}$, then at least $\widehat{\varphi}$ or $\widehat{\psi}$ must be strong subsystem.*

Proof. It is similar to Theorem 34. □

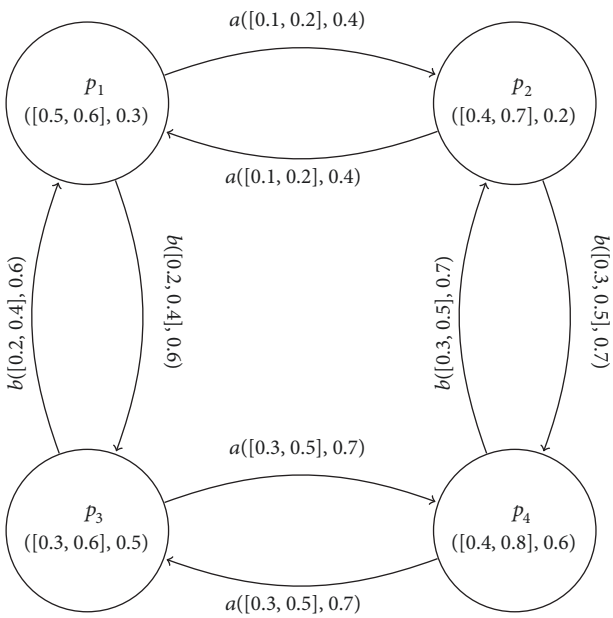

Figure 12: A subsystem $\widehat{\varphi}$ of cubic fssm $\mathscr{T}$.

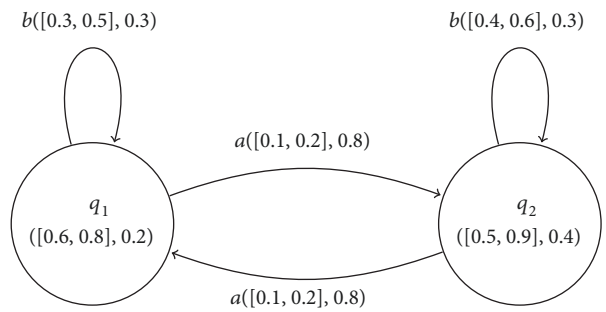

Figure 13: A subsystem $\widehat{\psi}$ of cubic fssm $\mathscr{V}$.

5. Homomorphism Problems in Cubic fssm

Definition 41. Let $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$ be two cubic fssms. A pair (Φ, Ψ) of mappings $\Phi : \mathscr{P} \longrightarrow \mathscr{Q}$ and $\Psi : X \longrightarrow Y$ is called homomorphism, written as $(\Phi, \Psi) : \mathscr{T} \longrightarrow \mathscr{V}$, if it satisfies

$$\begin{aligned} \widetilde{\delta}_{\varphi}(p_1, x, p_2) &\preceq \widetilde{\delta}_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)), \\ \omega_{\varphi}(p_1, x, p_2) &\geq \omega_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)), \end{aligned} \tag{76}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

Definition 42. Let $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ be two subsystems of cubic fssms $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, respectively. A pair (Φ, Ψ) of mappings $\Phi : \mathscr{P} \longrightarrow \mathscr{Q}$ and $\Psi : X \longrightarrow Y$ is called homomorphism, written as $(\Phi, \Psi) : \widehat{\varphi} \longrightarrow \widehat{\psi}$, if it satisfies

$$\begin{aligned} \widetilde{\delta}_{\widehat{\varphi}}(p_1) &\preceq \widetilde{\delta}_{\widehat{\psi}}(\Phi(p_1)), \\ \omega_{\widehat{\varphi}}(p_1) &\geq \omega_{\widehat{\psi}}(\Phi(p_1)), \end{aligned} \tag{77}$$

also

$$\begin{aligned} \widetilde{\delta}_{\varphi}(p_1, x, p_2) &\preceq \widetilde{\delta}_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)), \\ \omega_{\varphi}(p_1, x, p_2) &\geq \omega_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)), \end{aligned} \tag{78}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

Example 43. Let $\widehat{\varphi} = \langle \widetilde{\delta}_{\widehat{\varphi}}, \omega_{\widehat{\varphi}} \rangle$ and $\widehat{\psi} = \langle \widetilde{\delta}_{\widehat{\psi}}, \omega_{\widehat{\psi}} \rangle$ be two subsystems of cubic fssms $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$, respectively, as shown in Figures 12 and 13.

We define $(\Phi, \Psi) : \widehat{\varphi} \longrightarrow \widehat{\psi}$ by

$$\begin{aligned} \Phi : \mathscr{P} \longrightarrow \mathscr{Q} \mid p \longrightarrow \Phi(p) &= \begin{cases} q_1 & \text{if } p \in \{p_1, p_2\} \\ q_2 & \text{if } p \in \{p_3, p_4\}, \end{cases} \\ \Psi : X \longrightarrow Y \mid x \longrightarrow \Psi(x) &= \begin{cases} a & \text{if } x = b \\ b & \text{if } x = a \end{cases} \end{aligned} \tag{79}$$

Then by the routine calculation, we can see that (Φ, Ψ) is a homomorphism from $\widehat{\varphi}$ to $\widehat{\psi}$.

Definition 44. Let $\mathscr{T} = \langle \mathscr{P}, X, \varphi \rangle$ and $\mathscr{V} = \langle \mathscr{Q}, Y, \psi \rangle$ be two cubic fssms. A pair (Φ, Ψ) of mappings $\Phi : \mathscr{P} \longrightarrow \mathscr{Q}$ and $\Psi : X \longrightarrow Y$ is called strong homomorphism, written as $(\Phi, \Psi) : \mathscr{T} \longrightarrow \mathscr{V}$, if it satisfies

$$\begin{aligned} &\widetilde{\delta}_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)) \\ &\quad = \bigvee \left\{ \widetilde{\delta}_{\varphi}(p_1, x, p) \mid p \in \mathscr{P},\ \Phi(p) = \Phi(p_1) \right\}, \\ &\omega_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)) \\ &\quad = \bigwedge \left\{ \omega_{\varphi}(p_1, x, p) \mid p \in \mathscr{P},\ \Phi(p) = \Phi(p_1) \right\}, \end{aligned} \tag{80}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

If $X = Y$ and Ψ is the identity map, then we simply write $\Phi : \mathscr{T} \longrightarrow \mathscr{Q}$ and say that Φ is a homomorphism or strong homomorphism accordingly. If (Φ, Ψ) is a strong homorphism with Φ is one-one, then

$$\begin{aligned} \widetilde{\delta}_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)) &= \widetilde{\delta}_{\varphi}(p_1, x, p_2), \\ \omega_{\psi}(\Phi(p_1), \Psi(x), \Phi(p_2)) &= \omega_{\varphi}(p_1, x, p_2) \end{aligned} \tag{81}$$

for all $p_1, p_2 \in \mathscr{P}$ and $x \in X$.

Theorem 45. *Let $\mathscr{T} = (\mathscr{P}, X, \varphi)$ and $\mathscr{Q} = (\mathscr{Q}, Y, \psi)$ be two cubic fsms. Let $(\Phi, \Psi) : \mathscr{T} \longrightarrow \mathscr{Q}$ be an onto strong homomorphism. If $\mathscr{T}$ is a cubic fssm, then so is $\mathscr{Q}$.*

Proof. Let $\mathscr{T} = (\mathscr{P}, X, \varphi)$ be a cubic fssm. Let $q_1, q_2 \in \mathscr{Q}$. Then there are $p_1, p_2 \in \mathscr{P}$ such that $\Phi(p_1) = q_1$ and $\Phi(p_2) = q_2$. Let

$y_1, y_2 \in Y$. Then there exists $x_1, x_2 \in X$ such that $\Psi(x_1) = y_1$ and $\Psi(x_2) = y_2$. Since $\mathcal{T}$ is commutative, we have

$$\begin{aligned}
&\widetilde{\delta}^*_\psi(q_1, y_1y_2, q_2)\\
&\quad= \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi(x_1)\Psi(x_2), \Phi(p_2))\\
&\quad= \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi(x_1x_2), \Phi(p_2))\\
&\quad= \bigvee\{\widetilde{\delta}^*_\varphi(p_1, x_1x_2, p) \mid p \in \mathcal{P},\ \Phi(p) = \Phi(p_2)\}\\
&\quad= \bigvee\{\widetilde{\delta}^*_\varphi(p_1, x_2x_1, p) \mid p \in \mathcal{P},\ \Phi(p) = \Phi(p_2)\}\\
&\quad= \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi(x_2x_1), \Phi(p_2))\\
&\quad= \widetilde{\delta}^*_\psi(q_1, y_2y_1, q_2),
\end{aligned} \tag{82}$$

and

$$\begin{aligned}
&\omega^*_\psi(q_1, y_1y_2, q_2)\\
&\quad= \omega^*_\psi(\Phi(p_1), \Psi(x_1)\Psi(x_2), \Phi(p_2))\\
&\quad= \omega^*_\psi(\Phi(p_1), \Psi(x_1x_2), \Phi(p_2))\\
&\quad= \bigwedge\{\omega^*_\varphi(p_1, x_1x_2, p) \mid p \in \mathcal{P},\ \Phi(p) = \Phi(p_2)\}\\
&\quad= \bigwedge\{\omega^*_\varphi(p_1, x_2x_1, p) \mid p \in \mathcal{P},\ \Phi(p) = \Phi(p_2)\}\\
&\quad= \omega^*_\psi(\Phi(p_1), \Psi(x_2x_1), \Phi(p_2))\\
&\quad= \omega^*_\psi(\Phi(p_1), \Psi(x_2)\Psi(x_1), \Phi(p_2))\\
&\quad= \omega^*_\psi(q_1, y_2y_1, q_2).
\end{aligned} \tag{83}$$

Hence $\mathcal{Q}$ is a commutative cubic fsm. Similarly we can show that $\mathcal{Q}$ is switching. Hence $\mathcal{Q} = (\mathcal{Q}, Y, \psi)$ is also a cubic fssm. □

Theorem 46. *Let $\mathcal{T} = (\mathcal{P}, X, \varphi)$ and $\mathcal{Q} = (\mathcal{Q}, Y, \psi)$ be two cubic fssms. Let $(\Phi, \Psi) : \mathcal{T} \longrightarrow \mathcal{Q}$ be a homomorphism. Then*

$$\begin{aligned}
\widetilde{\delta}^*_\varphi(p_1, x, p_2) &\preceq \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)),\\
\omega^*_\varphi(p_1, x, p_2) &\geq \omega^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)),
\end{aligned} \tag{84}$$

for all $p_1, p_2 \in \mathcal{P}$ and $x \in X^*$.

Proof. Let $p_1, p_2 \in \mathcal{P}$ and $x \in X^*$. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \xi$ and so $\Psi^*(x) = \Psi^*(\xi) = \xi$. If $p_1 = p_2$, then

$$\begin{aligned}
\widetilde{\delta}^*_\varphi(p_1, x, p_2) = \widetilde{\delta}^*_\varphi(p_1, \xi, p_2) &= [1, 1]\\
&= \widetilde{\delta}^*_\psi(\Phi(p_1), \xi, \Phi(p_2))\\
&= \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)),
\end{aligned} \tag{85}$$

and

$$\begin{aligned}
\omega^*_\varphi(p_1, x, p_2) = \omega^*_\varphi(p_1, \xi, p_2) &= [0, 0]\\
&= \omega^*_\psi(\Phi(p_1), \xi, \Phi(p_2))\\
&= \omega^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)).
\end{aligned} \tag{86}$$

If $p_1 \neq p_2$, then

$$\begin{aligned}
\widetilde{\delta}^*_\varphi(p_1, x, p_2) = \widetilde{\delta}^*_\varphi(p_1, \xi, p_2) &= [0, 0]\\
&\preceq \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)),
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
\omega^*_\varphi(p_1, x, p_2) = \omega^*_\varphi(p_1, \xi, p_2) &= [1, 1]\\
&\geq \omega^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)).
\end{aligned} \tag{88}$$

Therefore the result is true for $n = 0$. Let us assume that the result is true for all $b \in X^*$ such that $|b| = n - 1, n > 0$. Let $x = bc$, where $b \in X^*, c \in X$ and $|b| = n - 1$. Then

$$\begin{aligned}
&\widetilde{\delta}^*_\varphi(p_1, x, p_2) = \widetilde{\delta}^*_\varphi(p_1, bc, p_2) = \bigvee_{p \in \mathcal{P}} [\widetilde{\delta}^*_\varphi(p_1, b, p)\\
&\quad\wedge \widetilde{\delta}^*_\varphi(p, c, p_2)] \preceq \bigvee_{p \in \mathcal{P}} [\widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(b), \Phi(p))\\
&\quad\wedge \widetilde{\delta}^*_\psi(\Phi(p), \Psi(c), \Phi(p_2))]\\
&\quad\preceq \bigvee_{p^\circ \in \mathcal{P}} [\widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(b), p^\circ)\\
&\quad\wedge \widetilde{\delta}^*_\psi(p^\circ, \Psi(c), \Phi(p_2))] = \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(b)\\
&\quad\cdot \Psi(c), \Phi(p_2)) = \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(bc), \Phi(p_2))\\
&\quad= \widetilde{\delta}^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)),
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
&\omega^*_\varphi(p_1, x, p_2) = \omega^*_\varphi(p_1, bc, p_2) = \bigwedge_{p \in \mathcal{P}} [\omega^*_\varphi(p_1, b, p)\\
&\quad\vee \omega^*_\varphi(p, c, p_2)] \geq \bigwedge_{p \in \mathcal{P}} [\omega^*_\psi(\Phi(p_1), \Psi^*(b), \Phi(p))\\
&\quad\vee \omega^*_\psi(\Phi(p), \Psi(c), \Phi(p_2))]\\
&\quad\geq \bigwedge_{p^\circ \in \mathcal{P}} [\omega^*_\psi(\Phi(p_1), \Psi^*(b), p^\circ)\\
&\quad\vee \omega^*_\psi(p^\circ, \Psi(c), \Phi(p_2))] = \omega^*_\psi(\Phi(p_1), \Psi^*(b)\\
&\quad\cdot \Psi(c), \Phi(p_2)) = \omega^*_\psi(\Phi(p_1), \Psi^*(bc), \Phi(p_2))\\
&\quad= \omega^*_\psi(\Phi(p_1), \Psi^*(x), \Phi(p_2)).
\end{aligned} \tag{90}$$

This is the required proof. □

6. Conclusion

A finite switchboard state machine is a specialized finite state machines. We applied the notion of cubic sets to switchboard state machines. We introduced the concepts of (resp., good, strong) subsystem of cubic fssms. We proved that the "Cartesian composition" of two (resp., good, strong) subsystems of cubic fssm is (resp., good, strong) subsystem of cubic fssms. Similarly, the product of two (resp., good, strong) subsystems of cubic fssm is (resp. good, strong) subsystem of cubic fssms. We defined the homomorphism between two of cubic fssms and proved some related results. We provided many examples on each case.

The construction of "P-union, P-intersection, R-union, and R-intersection" of subsystems of cubic fssms is still open.

Abbreviations

Cubic fsm:	Cubic finite state machine
Cubic fsms:	Cubic finite state machines
IVF-set:	Interval valued fuzzy set
Cubic fssm:	Cubic finite switchboard state machine
Cubic fssms:	Cubic finite switchboard state machines.

Data Availability

Not applicable because no data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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References

[1] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," *Information Sciences*, vol. 8, pp. 199–249, 1975.

[2] N. Abughazalah and N. Yaqoob, "Applications of cubic structures to subsystems of finite state machines," *Symmetry*, vol. 10, no. 11, article 598, 2018.

[3] S. Abdullah, R. Naz, and W. Pedrycz, "Cubic finite state machine and cubic transformation semigroups," *New Trends in Mathematical Science*, vol. 5, no. 4, pp. 24–39, 2017.

[4] W. Dorfler, "The Cartesian composition of automata," *Mathematical Systems Theory*, vol. 11, no. 3, pp. 239–257, 1978.

[5] D. S. Malik, J. N. Mordeson, and M. K. Sen, "On subsystems of a fuzzy finite state machine," *Fuzzy Sets and Systems*, vol. 68, no. 1, pp. 83–92, 1994.

[6] D. S. Malik, J. N. Mordeson, and M. K. Sen, "The Cartesian composition of fuzzy finite state machines," *Kybernetes*, vol. 24, no. 4, pp. 3, 98–110, 1995.

[7] D. S. Malik, J. N. Mordeson, and M. K. Sen, "Products of fuzzy finite state machines," *Fuzzy Sets and Systems*, vol. 92, no. 1, pp. 95–102, 1997.

[8] N. A. Ebas, N. S. Hamzah, K. Jacob et al., "Algebraic properties of finite switchboard state machine," *AIP Conference Proceedings*, vol. 1974, no. 1, article 020058, 2018.

[9] H. V. Kumbhojkar and S. R. Chaudhari, "On covering of products of fuzzy finite state machines," *Fuzzy Sets and Systems*, vol. 125, no. 2, pp. 215–222, 2002.

[10] T. Mahmood and Q. Khan, "Interval neutrosophic finite switchboard state machine," *Afrika Matematika*, vol. 27, no. 7-8, pp. 1361–1376, 2016.

[11] A. Hussain and M. Shabbir, "Soft finite state machine," *Journal of Intelligent and Fuzzy Systems*, vol. 29, no. 4, pp. 1635–1641, 2015.

[12] Y. B. Jun, "Intuitionistic fuzzy finite state machines," *Journal of Applied Mathematics and Computing*, vol. 17, no. 1-2, pp. 109–120, 2005.

[13] Y. B. Jun, "Intuitionistic fuzzy finite switchboard state machines," *Journal of Applied Mathematics & Computing*, vol. 20, no. 1-2, pp. 315–325, 2006.

[14] Y. B. Jun, "Quotient structures of intuitionistic fuzzy finite state machines," *Information Sciences*, vol. 177, no. 22, pp. 4977–4986, 2007.

[15] Y. B. Jun and J. Kavikumar, "Bipolar fuzzy finite state machines," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 34, no. 1, pp. 181–188, 2011.

[16] S. Subramaniyan and M. Rajasekar, "Subsystem in bipolar fuzzy finite state machines," *International Journal of Computer Applications*, vol. 90, no. 14, pp. 1–6, 2014.

[17] J. Kavikumar, A. B. Khamis, and R. B. Roslan, "Bipolar-valued fuzzy finite switchboard state machines," in *Proceedings of the World Congress on Engineering and Computer Science*, p. 6, October 2012.

[18] Y. B. Jun, K. J. Lee, and M. S. Kang, "Cubic structures applied to ideals of BCI-algebras," *Computers & Mathematics with Applications*, vol. 62, no. 9, pp. 3334–3342, 2011.

[19] Y. B. Jun, G. Muhiuddin, M. A. Oztürk, and E. H. Roh, "Cubic soft ideals in BCK/BCI-algebras," *Journal of Computational Analysis and Applications*, vol. 22, no. 5, pp. 929–940, 2017.

[20] J. Liu, Z.-w. Mo, D. Qiu, and Y. Wang, "Products of Mealy-type fuzzy finite state machines," *Fuzzy Sets and Systems*, vol. 160, no. 16, pp. 2401–2415, 2009.