Hindawi
Complexity
Volume 2019, Article ID 2709820, 10 pages
https://doi.org/10.1155/2019/2709820

Research Article

The Synchronization of *N* Cascade-Coupled Chaotic Systems

Pengyu Li, Juan Du, Shouliang Li, Yazhao Zheng, and Bowen Jia

School of Information Science and Engineering, Lanzhou University, Lanzhou 730000, China

Correspondence should be addressed to Juan Du; duj@lzu.edu.cn

Received 12 September 2019; Accepted 27 November 2019; Published 12 December 2019

Guest Editor: Mahendra K. Gupta

In this paper, we investigate a novel synchronization method, which consists of n $(n \geq 2)$ cascade-coupled chaotic systems. Furthermore, as the number of chaotic systems decreases from n to 2, the proposed synchronization will transform into bidirectional coupling synchronization. Based on Lyapunov stability theory, a general criterion is proposed for choosing the appropriate coupling parameters to ensure cascading synchronization. Moreover, 4 Lü systems are taken as an example and the corresponding numerical simulations demonstrate the effectiveness of our idea.

1. Introduction

Since Pecora and Carrol reported the discovery of synchronization for two chaotic systems by circuits implementation [1], the exciting phenomenon has gained much attention. In various fields, such as secure communication, signal processing, and life sciences, many types of chaotic synchronization are proving to be increasingly useful [2–5]. The design of novel synchronization models is necessary in such applications and thus motivating researchers to develop it, such as phase synchronization [6], projective synchronization [7], generalized synchronization [8], complex synchronization [9], modulus synchronization [10], and hybrid dislocated synchronization [11].

It is known that chaotic systems are extremely sensitive to initial values [12]. For two chaotic systems with the same structure, minor deviations in their initial values will lead to significantly different chaotic state [13]. After the transmission through the cascade-coupled systems, the difference becomes more difficult to be analyzed. In these cases, when a signal to be transmitted is loaded into a cascade of chaotic systems, the security of the transmitted information and the difficulty of being deciphered will be greatly enhanced. However, the existing synchronization has been acquired for two chaotic systems, and the synchronization problems of n chaotic systems are yet to be investigated.

In fact, the synchronization for two chaotic systems can be defined as the following two types: unidirectional coupling synchronization and bidirectional coupling synchronization [14]. Unidirectional coupling synchronization includes a drive system and a response system. The two systems are synchronized by introducing the state of the drive system on the response system [15–17]. However, bidirectional coupling synchronization, also known as mutual coupling synchronization, can be obtained by introducing the state of a system to that of another system, in which each system can be considered as drive system or response system [18–20]. When compared with unidirectional coupling synchronization, bidirectional (or mutual) coupling synchronization is more attractive due to the fact that it is ready to implement in practice. Recently, Sivaganesh et al. proposed the synchronization of a network of mutually coupled chaotic systems, in which the synchronization behaviour changes as the coupling parameters change [21].

Interestingly, Carroll and Pecora not only discovered chaos synchronization phenomenon but also studied cascading synchronized chaotic systems [22]. Their model consists of a 3D drive system and two 2-D subresponse systems, where the subresponse systems can be constructed by copying two of the expressions of the drive systems. By combining the cascading synchronized chaotic system with projective synchronization schemes, An and Chen reported

the function cascade synchronization method [23, 24], which presented 3 chaotic systems. Indeed, the two 2-D response systems are not chaotic systems. And, the status signal of the response system is not fed back to the drive system.

By the motivation of the above discussion, a novel synchronization method is investigated in this paper, which consists of n $(n \geq 2)$ cascade-coupled chaotic systems. This implies that the chaotic systems are in linear topology and cascade-coupled with each other. For example, the i-th system will be synchronized with the $(i-1)$-th and $(i+1)$-th systems $(2 \leq i \leq (n-1))$. Further, as the number of chaotic systems decreases from n to 2, the proposed synchronization will transform into bidirectional coupling synchronization. Based on global control strategy, the synchronization scheme can be realized by choosing the appropriate coupling parameters, and the corresponding numerical simulations are presented to verify the effectiveness of our idea. In short, the proposed synchronization of n cascade-coupled chaotic systems can provide a novel choice for secure communication and signal processing.

The reminder of this paper is organized as follows. In Section 2, the principle of the synchronization for n cascade-coupled chaotic systems is introduced. And 4 Lü systems are chosen as an example to illustrate the effectiveness of our idea in Section 3. Conclusions are drawn in Section 4.

2. Model Description of the Synchronization of N Cascade-Coupled Chaotic Systems

This section mainly introduces the principle of the synchronization of N cascade-coupled chaotic systems.

We consider the following n chaotic systems.

The first system can be depicted as

$$\dot{X}_1 = AX_1 + f(X_1) - D_1(X_1 - X_2). \tag{1}$$

The second system can be depicted as

$$\dot{X}_2 = AX_2 + f(X_2) - D_1'(X_1 - X_2) - D_2(X_2 - X_3). \tag{2}$$

The third system can be depicted as

$$\begin{aligned} \dot{X}_3 &= AX_3 + f(X_3) - D_2'(X_2 - X_3) - D_3(X_3 - X_4), \\ &\vdots \end{aligned} \tag{3}$$

The $(n-1)$-th system can be depicted as

$$\begin{aligned} \dot{X}_{n-1} = AX_{n-1} &+ f(X_{n-1}) - D_{n-2}'(X_{n-2} - X_{n-1}) \\ &- D_{n-1}(X_{n-1} - X_n). \end{aligned} \tag{4}$$

The n-th system can be depicted as

$$\dot{X}_n = AX_n + f(X_n) - D_{n-1}'(X_{n-1} - X_n), \tag{5}$$

where X_1, X_2, $\ldots$, and X_n denote m-dimensional state vectors of the chaotic systems. A denotes an $m \times m$ parameters matrix. $f(X_1)$, $f(X_2)$, $\ldots$, $f(X_{n-1})$, and $f(X_n)$ denote $m \times 1$ continuous vector functions. D_1, D_1', D_2, D_2', D_3, D_3', $\ldots$, D_{n-1} and D_{n-1}' are diagonal matrices which rule the feedback gain.

Definition 1. For the n chaotic systems, our goal is that the trajectory of X_1 synchronizes with that of X_2, and the trajectory of X_2 synchronizes with that of X_3, $\ldots$, finally the trajectory of X_{n-1} synchronizes with that of X_n. Then

$$\lim_{t \longrightarrow \infty} \|e(t)\| = 0, \tag{6}$$

where $e(t)$ is a $(m \times (n-1))$-dimensional column vector and $e(t) = (e_1(t), e_2(t), \ldots, e_{n-1}(t))^{\mathrm{T}}$, with $e_1(t) = X_1 - X_2, e_2(t) = X_2 - X_3, \ldots, e_{n-1}(t) = X_{n-1} - X_n$, and $\|\cdot\|$ represents the Euclidean norm.

Remark 1. The error dynamical system can be acquired from equations (1)–(6):

$$\begin{aligned} \dot{e}_1(t) &= \dot{X}_1 - \dot{X}_2 = \left[A + H_{X_1,X_2} - (D_1 - D_1')\right]e_1(t) + D_2 e_2(t), \\ \dot{e}_2(t) &= \dot{X}_2 - \dot{X}_3 = \left[A + H_{X_2,X_3} - (D_2 - D_2')\right]e_2(t) - D_1' e_1(t) + D_3 e_3(t), \\ &\vdots \\ \dot{e}_{n-2}(t) &= \dot{X}_{n-2} - \dot{X}_{n-1} = \left[A + H_{X_{n-2},X_{n-1}} - (D_{n-2} - D_{n-2}')\right] e_{n-2}(t) - D_{n-3}' e_{n-3}(t) + D_{n-1} e_{n-1}(t), \\ \dot{e}_{n-1}(t) &= \dot{X}_{n-1} - \dot{X}_n = \left[A + H_{X_{n-1},X_n} - (D_{n-1} - D_{n-1}')\right]e_{n-1}(t) - D_{n-2}' e_{n-2}(t), \end{aligned} \tag{7}$$

where $f(X_1) - f(X_2) = H_{X_1,X_2} \cdot e_1(t)$, $f(X_2) - f(X_3) = H_{X_2,X_3} \cdot e_2(t), \cdots$, and $f(X_{n-1}) - f(X_n) = H_{X_{n-1},X_n} \cdot e_{n-1}(t)$. It is clear that $f(X_1)$, $f(X_2)$, $f(X_3)$, $\ldots$, $f(X_{n-1})$, $f(X_n)$ are bounded matrices, and thus H_{X_1,X_2}, H_{X_2,X_3}, $\ldots$, H_{X_{n-1},X_n} are also bounded matrices. A general condition for achieving the synchronization of n cascade-coupled chaotic systems is given in the following theorem.

Theorem 1. *If there exists a positive definite symmetric matrix P, and $n-1$ constants $\lambda_1 > 0$, $\lambda_2 > 0$, $\ldots$, $\lambda_{n-1} > 0$, we have that*

$$\begin{cases} \left(A + H_{X_1,X_2} - (D_1 - D_1')\right)^{\mathrm{T}} P + P\left(A + H_{X_1,X_2} - (D_1 - D_1')\right) \le -\lambda_1 I, \\ \left(A + H_{X_2,X_3} - (D_2 - D_2')\right)^{\mathrm{T}} P + P\left(A + H_{X_2,X_3} - (D_2 - D_2')\right) \le -\lambda_2 I, \\ \vdots \\ \left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_{n-1}')\right)^{\mathrm{T}} P + P\left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_{n-1}')\right) \le -\lambda_{n-1} I, \end{cases} \tag{8}$$

where I is the identity matrix. Then the synchronization for the cascade-coupled systems (1)–(5) is realized.

Proof. Construct a Lyapunov function as

$$V(t) = e^{\mathrm{T}}(t) P_{n-1} e(t), \tag{9}$$

where $P_{n-1} = \mathrm{diag}(P, P, \ldots, P)$, with the total is $n-1$. Then it is easily known that $V(t) \ge 0$. And the time derivative of $V(t)$ is given as

$$\begin{aligned} \dot{V}(t) &= \dot{e}_1^T(t) P e_1(t) + e_1^T(t) P \dot{e}(t)_1 + \dot{e}(t)_2^T P e_2(t) + e_2^T(t) P \dot{e}_2(t) + \cdots + \dot{e}_{n-1}^T(t) P e_{n-1}(t) + e_{n-1}^T P \dot{e}_{n-1}(t) \\ &= e_1^T(t)\left(A + H_{X_1,X_2} - (D_1 - D_1')\right)^T P e_1(t) + e_1^T(t) P\left(A + H_{X_1,X_2} - (D_1 - D_1')\right) e_1(t) + e_2^T(t) D_2^T P e_1(t) + e_1^T(t) P D_2 e_2(t) \\ &\quad + e_2^T(t)\left(A + H_{X_2,X_3} - (D_2 - D_2')\right)^T P e_2(t) + e_2^T(t) P\left(A + H_{X_2,X_3} - (D_2 - D_2')\right) e_2(t) - e_1^T(t)(D_1')^T P e_2(t) \\ &\quad + e_3^T(t) D_3^T P e_2(t) - e_2^T(t) P D_1' e_1(t) + e_2^T(t) P D_3 e_3(t) + \cdots + e_{n-1}^T(t)\left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_{n-1}')\right)^T P e_{n-1}(t) \\ &\quad + e_{n-1}^T(t) P\left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_{n-1}')\right) e_{n-1}(t) - e_{n-2}^T(t)(D_{n-2}')^T P e_{n-1}(t) - e_{n-1}^T(t) P D_{n-2}' e_{n-2}(t) \\ &= e_1^T(t)\left[\left(A + H_{X_1,X_2} - (D_1 - D_2)\right)^T P + P\left(A + H_{X_1,X_2} - (D_1 - D_2)\right)\right] e_1(t) + e_2^T(t)\left[\left(A + H_{X_2,X_3} - (D_2 - D_3)\right)^T P \right. \\ &\quad \left. + P\left(A + H_{X_2,X_3} - (D_2 - D_3)\right)\right] e_2(t) + \cdots + e_{n-1}^T(t)\left[\left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_n)\right)^T P + P\left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_n)\right)\right] \\ &\quad \cdot e_{n-1}(t) + e_1^T(t)\left[P(D_2 - D_1') + (D_2 - D_1')^T P\right] e_2 + e_2^T(t)\left[P(D_3 - D_2') + (D_3 - D_2')^T P\right] e_3 + \cdots \\ &\quad + e_{n-2}^T(t)\left[P(D_{n-1} - D_{n-2}') + (D_{n-1} - D_{n-2}')^T P\right] e_{n-1}. \end{aligned} \tag{10}$$

As long as

$$\begin{cases} D_1' = D_2, \\ D_2' = D_3, \\ \vdots \\ D_{n-2}' = D_{n-1}. \end{cases} \tag{11}$$

Then, equation (12) can be represented as

$$\begin{aligned} \dot{V}(t) &= \dot{e}_1^{\mathrm{T}}(t) P e_1(t) + e_1^{\mathrm{T}}(t) P \dot{e}(t)_1 + \dot{e}(t)_2^{\mathrm{T}} P e_2(t) + e_2^{\mathrm{T}}(t) P \dot{e}_2(t) \\ &\quad + \cdots + \dot{e}_{n-1}^{\mathrm{T}}(t) P e_{n-1}(t) + e_{n-1}^{\mathrm{T}} P \dot{e}_{n-1}(t) \\ &= e_1^{\mathrm{T}}(t)\left[\left(A + H_{X_1,X_2} - (D_1 - D_2)\right)^{\mathrm{T}} P \right. \\ &\quad \left. + P\left(A + H_{X_1,X_2} - (D_1 - D_2)\right)\right] e_1(t) \\ &\quad + e_2^{\mathrm{T}}(t)\left[\left(A + H_{X_2,X_3} - (D_2 - D_3)\right)^{\mathrm{T}} P \right. \\ &\quad \left. + P\left(A + H_{X_2,X_3} - (D_2 - D_3)\right)\right] e_2(t) + \cdots \\ &\quad + e_{n-1}^{\mathrm{T}}(t)\left[\left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_n)\right)^{\mathrm{T}} P \right. \\ &\quad \left. + P\left(A + H_{X_{n-1},X_n} - (D_{n-1} - D_n)\right)\right] e_{n-1}(t) \\ &\le \left(-\lambda_1 e_1^{\mathrm{T}}(t) e_1(t)\right) + \left(-\lambda_2 e_2^{\mathrm{T}}(t) e_2(t)\right) + \cdots \\ &\quad + \left(-\lambda_{n-1} e_{n-1}^{\mathrm{T}}(t) e_{n-1}(t)\right) < 0. \end{aligned} \tag{12}$$

Equation (12) holds if and only if $e(t) \ne 0$. Based on Lyapunov stability theory, system (7) is globally asymptotically stable. This implies that the synchronization of n cascade-coupled chaotic systems is achieved. The proof is completed. □

Remark 2. From Theorem 1, we obtain that $e_1(t) = X_1 - X_2 = 0$, $e_2(t) = X_2 - X_3 = 0, \cdots$, and $e_{n-1}(t) = X_{n-1} - X_n = 0$ as $t \longrightarrow \infty$, i.e., $X_1 = X_2 = \cdots = X_n$. This implies that any two systems in the n chaotic systems can be synchronized with each other. The interesting phenomenon is demonstrated in Figure 1. Take the first and second system in Figure 1 as an example. Firstly, we get the error signal $e_1(t) = X_1 - X_2$ through an adder. Then, $e_1(t)$ is injected into the first system via multiplier $-D_1$, while being injected into the second system via multiplier $-D_1'$. The coupling cases between the subsequent systems are similar.

Remark 3. If the n chaotic systems are degraded to 2 chaotic systems, then synchronization problem in this paper will change into bidirectional coupling synchronization problem, which has been intensively revealed in previous researches.

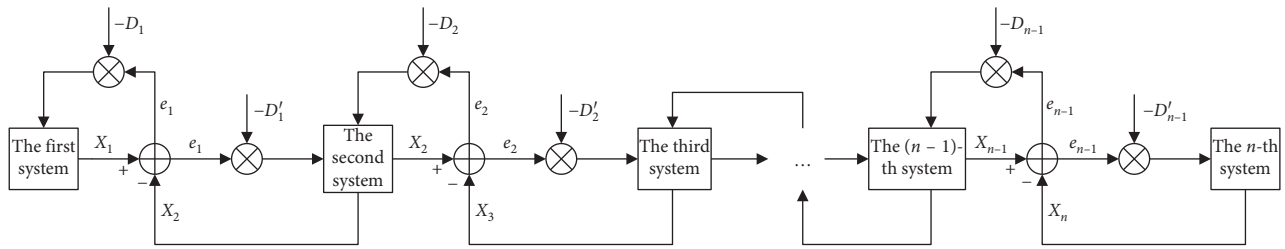

Figure 1: The schematic representation of (n) cascade-coupled chaotic systems.

3. Synchronization of 4 Cascade-Coupled Lü Systems

This section demonstrates the effectiveness and flexibility of the model by adopting 4 Lü systems as an example.

The first system can be presented as

$$\begin{cases} \dot{x}_{11} = a(x_{12} - x_{11}) - d_{11}(x_{11} - x_{21}), \\ \dot{x}_{12} = cx_{12} - x_{11}x_{13} - d_{12}(x_{12} - x_{22}), \\ \dot{x}_{13} = x_{11}x_{12} - bx_{13} - d_{13}(x_{13} - x_{23}). \end{cases} \tag{13}$$

The second system can be presented as

$$\begin{cases} \dot{x}_{21} = a(x_{22} - x_{21}) - d_{11}'(x_{11} - x_{21}) - d_{21}(x_{21} - x_{31}), \\ \dot{x}_{22} = cx_{22} - x_{21}x_{23} - d_{12}'(x_{12} - x_{22}) - d_{22}(x_{22} - x_{32}), \\ \dot{x}_{23} = x_{21}x_{22} - bx_{23} - d_{13}'(x_{13} - x_{23}) - d_{23}(x_{23} - x_{33}). \end{cases} \tag{14}$$

The third system can be presented as

$$\begin{cases} \dot{x}_{31} = a(x_{32} - x_{31}) - d_{21}'(x_{21} - x_{31}) - d_{31}(x_{31} - x_{41}), \\ \dot{x}_{32} = cx_{32} - x_{31}x_{33} - d_{22}'(x_{22} - x_{32}) - d_{32}(x_{32} - x_{42}), \\ \dot{x}_{33} = x_{31}x_{32} - bx_{33} - d_{23}'(x_{23} - x_{33}) - d_{33}(x_{33} - x_{43}). \end{cases} \tag{15}$$

The fourth system can be presented as

$$\begin{cases} \dot{x}_{41} = a(x_{42} - x_{41}) - d_{31}'(x_{31} - x_{41}), \\ \dot{x}_{42} = cx_{42} - x_{41}x_{43} - d_{32}'(x_{32} - x_{42}), \\ \dot{x}_{43} = x_{41}x_{42} - bx_{43} - d_{33}'(x_{33} - x_{43}), \end{cases} \tag{16}$$

where $X_1 = (x_{11}, x_{12}, x_{13})^{\mathrm{T}}$, $X_2 = (x_{21}, x_{22}, x_{23})^{\mathrm{T}}$, $X_3 = (x_{31}, x_{32}, x_{33})^{\mathrm{T}}$, and $X_4 = (x_{41}, x_{42}, x_{43})^{\mathrm{T}}$ are the state vectors of 4 Lü systems; d_{ij} and d_{ij}' $(i = 1, 2, 3;\ j = 1, 2, 3)$ are coupling parameters.

Let $e_{1i} = x_{1i} - x_{2i}$, $e_{2i} = x_{2i} - x_{3i}$, and $e_{3i} = x_{3i} - x_{4i}$, with $i = 1, 2, 3$. From Remark 1, the error dynamical system can be described as

$$\begin{aligned} \dot{e}_1(t) &= \dot{X}_1 - \dot{X}_2 = \left[A + H_{X_1,X_2} - (D_1 - D_1')\right] e_1 + D_2 e_2, \\ \dot{e}_2(t) &= \dot{X}_2 - \dot{X}_3 = \left[A + H_{X_2,X_3} - (D_2 - D_2')\right] e_2 - D_1' e_1 + D_3 e_3, \\ \dot{e}_3(t) &= \dot{X}_3 - \dot{X}_4 = \left[A + H_{X_3,X_4} - (D_3 - D_3')\right] e_3 - D_2' e_2, \end{aligned} \tag{17}$$

where $e_1 = (e_{11}, e_{12}, e_{13})^{\mathrm{T}}$, $e_2 = (e_{21}, e_{22}, e_{23})^{\mathrm{T}}$, and $e_3 = (e_{31}, e_{32}, e_{33})^{\mathrm{T}}$.

The n systems model, which was presented in Section 2, illustrates that A, $f(X_1)$, $f(X_2)$, $f(X_3)$, $f(X_4)$, H_{X_1,X_2}, H_{X_2,X_3}, H_{X_3,X_4}, D_1, D_2, D_3, and, D_4 can be described as

$$\begin{aligned} A &= \begin{pmatrix} -a & a & 0 \\ 0 & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \\ f(X_1) &= \begin{pmatrix} 0 \\ -x_{11}x_{13} \\ x_{11}x_{12} \end{pmatrix}, \\ f(X_2) &= \begin{pmatrix} 0 \\ -x_{21}x_{23} \\ x_{21}x_{22} \end{pmatrix}, \\ f(X_3) &= \begin{pmatrix} 0 \\ -x_{31}x_{33} \\ x_{31}x_{32} \end{pmatrix}, \\ f(X_4) &= \begin{pmatrix} 0 \\ -x_{41}x_{43} \\ x_{41}x_{42} \end{pmatrix}, \\ H_{X_1,X_2} &= \begin{pmatrix} 0 & 0 & 0 \\ -x_{13} & 0 & -x_{21} \\ x_{12} & x_{21} & 0 \end{pmatrix}, \\ H_{X_2,X_3} &= \begin{pmatrix} 0 & 0 & 0 \\ -x_{23} & 0 & -x_{31} \\ x_{22} & x_{31} & 0 \end{pmatrix}, \\ H_{X_3,X_4} &= \begin{pmatrix} 0 & 0 & 0 \\ -x_{33} & 0 & -x_{41} \\ x_{32} & x_{41} & 0 \end{pmatrix}, \\ D_1 &= \mathrm{diag}\{d_{11}, d_{12}, d_{13}\}, \\ D_1' &= \mathrm{diag}\{d_{11}', d_{12}', d_{13}'\}, \\ D_2 &= \mathrm{diag}\{d_{21}, d_{22}, d_{23}\}, \\ D_2' &= \mathrm{diag}\{d_{21}', d_{22}', d_{23}'\}, \\ D_3 &= \mathrm{diag}\{d_{31}, d_{32}, d_{33}\}, \\ D_3' &= \mathrm{diag}\{d_{31}', d_{32}', d_{33}'\}. \end{aligned} \tag{18}$$

When the coupling parameter matrices are set as $D_1 = D_1' = D_2 = D_2' = D_3 = D_3' = 0$ and the initial condition of systems (13)–(16) are chosen as $(x_{11}, x_{12}, x_{13}) = (1, 1, 1)$, $(x_{21}, x_{22}, x_{23}) = (5, 5, 5)$, $(x_{31}, x_{32}, x_{33}) = (10, 10, 10)$, $(x_{41}, x_{42}, x_{43}) = (20, 20, 20)$. Figure 2 indicates that systems (13)–(16) do not synchronize with each other.

Based on global control strategy, we choose the positive definite symmetric constant matrix $P = \mathrm{diag}(p_1, p_2, p_3)$, with $p_i > 0$ $(i = 1, 2, 3)$, and positive constants $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, then we have

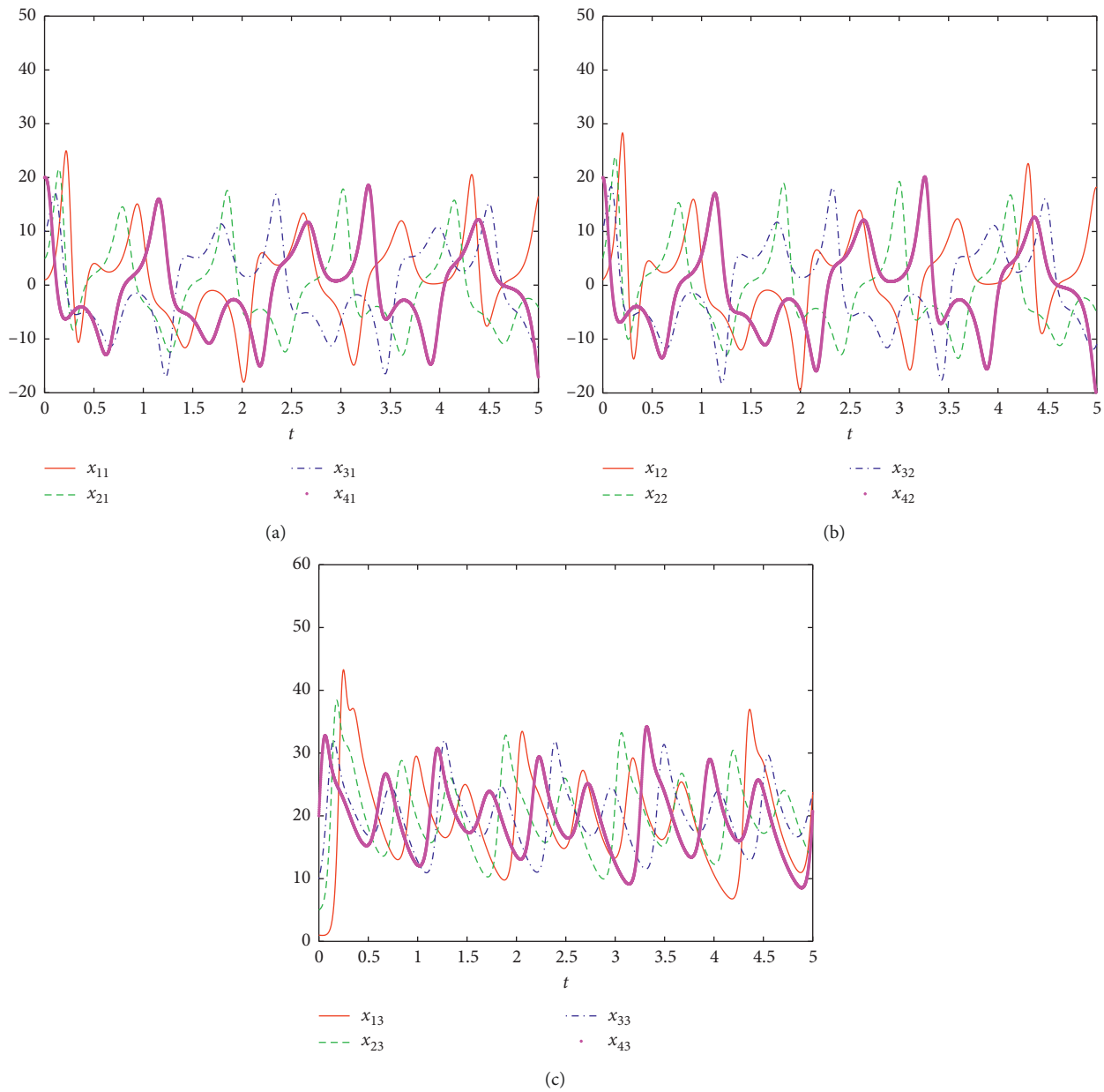

FIGURE 2: The graph of synchronization when the coupling parameters are all zero. (a) x_{11}, x_{21}, x_{31} and x_{41} vs. t. (b) x_{12}, x_{22}, x_{32} and x_{42} vs. t. (c) x_{13}, x_{23}, x_{33} and x_{43} vs. t.

$$\begin{aligned}
&\left(A + H_{x_1,x_2} - (D_1 - D_1')\right)^{\mathrm{T}} P + P\left(A + H_{x_1,x_2} - (D_1 - D_1')\right) + \lambda_1 I \\
&= \begin{pmatrix} -2p_1\left(a + d_{11} - d_{11}' - \dfrac{\lambda_1}{2p_1}\right) & p_1 a - p_2 x_{13} & p_3 x_{12} \\ p_1 a - p_2 x_{13} & 2p_2\left(c - d_{12} + d_{12}' + \dfrac{\lambda_1}{2p_2}\right) & (p_3 - p_2)x_{21} \\ p_3 x_{12} & (p_3 - p_2)x_{21} & -2p_3\left(b + d_{13} - d_{13}' - \dfrac{\lambda_1}{2p_3}\right) \end{pmatrix}.
\end{aligned} \tag{19}$$

According to the basic algebraic theory, the matrix (19) is negative definite as long as

$$\Delta_1 = -2p_1\left(a + d_{11} - d'_{11} - \frac{\lambda_1}{2p_1}\right) < 0, \tag{20}$$

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} -2p_1\left(a + d_{11} - d'_{11} - \frac{\lambda_1}{2p_1}\right) & p_1 a - p_2 x_{13} \\ p_1 a - p_2 x_{13} & 2p_2\left(c - d_{12} + d'_{12} + \frac{\lambda_1}{2p_2}\right) \end{vmatrix} \\ &= -4p_1 p_2 \cdot \left(a + d_{11} - d'_{11} - \frac{\lambda_1}{2p_1}\right)\left(c - d_{12} + d'_{12} + \frac{\lambda_1}{2p_2}\right) - (p_1 a - p_2 x_{13})^2 > 0,\end{aligned} \tag{21}$$

$$\begin{aligned}\Delta_3 &= \begin{vmatrix} -2p_1\left(a + d_{11} - d'_{11} - \frac{\lambda_1}{2p_1}\right) & p_1 a - p_2 x_{13} & p_3 x_{12} \\ p_1 a - p_2 x_{13} & 2p_2\left(c - d_{12} + d'_{12} + \frac{\lambda_1}{2p_2}\right) & (p_3 - p_2)x_{21} \\ p_3 x_{12} & (p_3 - p_2)x_{21} & -2p_3\left(b + d_{13} - d'_{13} - \frac{\lambda_1}{2p_3}\right) \end{vmatrix} \\ &= -2p_3 \cdot \left(b + d_{13} - d'_{13} - \frac{\lambda_1}{2p_3}\right)\Delta_2 - (p_3 x_{12})^2 \cdot [2p_2(c - d_{12} + d'_{12} + \lambda)_1] + (p_3 - p_2)x_{21} \cdot [(2p_1(a + d_{11} - d'_{11}) - \lambda_1) \\ &\quad \cdot (p_3 - p_2)x_{21} + 2p_3 x_{12}(p_1 a - p_2 x_{13})] < 0.\end{aligned} \tag{22}$$

From equations (20)–(22), we can get

$$\begin{cases} d_{11} - d'_{11} > -a + \dfrac{\lambda_1}{2p_1}, \\ d_{12} - d'_{12} > \dfrac{-(p_1 a - p_2 x_{13})^2}{4p_1 p_2 \cdot (a + d_{11} - d'_{11} - (\lambda_1/2p_1))} + c + \dfrac{\lambda_1}{2p_2}, \\ d_{13} - d'_{13} > -b + \dfrac{\lambda_1}{2p_3} - \dfrac{p_3 x_{12}^2 \cdot [2p_2(c - d_{12} + d'_{12}) + \lambda_1]}{2\Delta_2} \\ \quad + \dfrac{(p_3 - p_2)x_{21} \cdot [(2p_1(a + d_{11} - d'_{11}) - \lambda_1)(p_3 - p_2)x_{21} + 2p_3 x_{12}(p_1 a - p_2 x_{13})]}{2p_3 \Delta_2}. \end{cases} \tag{23}$$

The synchronization can be achieved easily as long as the inequalities (23) holds, ie., we need to choose the appropriate coupling parameters d_{ij} and d'_{ij}, with $i = 1, 2, 3$ and $j = 1, 2, 3$.

For convenience, let the matrix $p_2 = p_3$, then we have

$$\begin{cases} d_{11} - d'_{11} > -a + \dfrac{\lambda_1}{2p_1}, \\ d_{12} - d'_{12} > \dfrac{-(p_1 a - p_2 x_{13})^2}{4p_1 p_2 \cdot (a + d_{11} - d'_{11} - (\lambda_1/2p_1))} + c + \dfrac{\lambda_1}{2p_2}, \\ d_{13} - d'_{13} > -b + \dfrac{\lambda_1}{2p_3} - \dfrac{p_3 x_{12}^2 \cdot [2p_2(c - d_{12} + d'_{12}) + \lambda_1]}{2\Delta_2}. \end{cases} \tag{24}$$

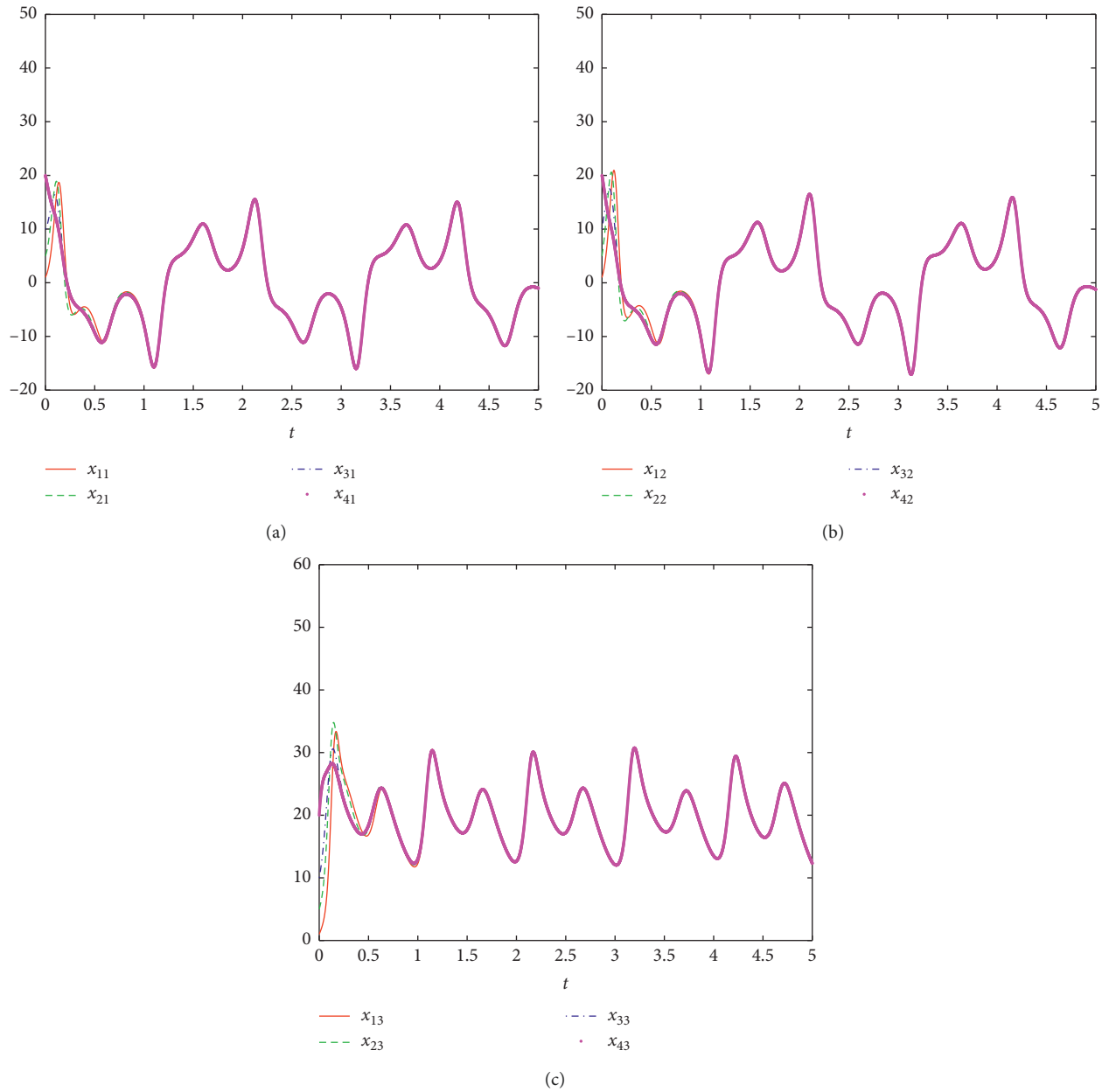

FIGURE 3: The graph of synchronization. (a) x_{11}, x_{21}, x_{31} and x_{41} vs. t. (b) x_{12}, x_{22}, x_{32} and x_{42} vs. t. (c) x_{13}, x_{23}, x_{33} and x_{43} vs. t.

In a similar way, we can get the following two inequalities:

$$\begin{cases} d_{21} - d'_{21} > -a + \dfrac{\lambda_2}{2p_1}, \\ d_{22} - d'_{22} > \dfrac{-(p_1 a - p_2 x_{23})^2}{4p_1 p_2 \cdot (a + d_{21} - d'_{21} - (\lambda_2/2p_1))} + c + \dfrac{\lambda_2}{2p_2}, \\ d_{23} - d'_{23} > -b + \dfrac{\lambda_2}{2p_3} - \dfrac{p_3 x_{22}^2 \cdot [2p_2(c - d_{22} + d'_{22}) + \lambda_2]}{2\Delta'_2}. \end{cases} \tag{25}$$

$$\begin{cases} d_{31} - d'_{31} > -a + \dfrac{\lambda_3}{2p_1}, \\ d_{32} - d'_{32} > \dfrac{-(p_1 a - p_2 x_{33})^2}{4p_1 p_2 \cdot (a + d_{31} - d'_{31} - (\lambda_3/2p_1))} + c + \dfrac{\lambda_3}{2p_2}, \\ d_{33} - d'_{33} > -b + \dfrac{\lambda_3}{2p_3} - \dfrac{p_3 x_{32}^2 \cdot [2p_2(c - d_{32} + d'_{32}) + \lambda_3]}{2\Delta''_2}, \end{cases} \tag{26}$$

where $\Delta'_2 = -4p_1 p_2 \cdot (a + d_{21} - d'_{21} - (\lambda_2/2p_1))(c - d_{22} + d'_{22} + (\lambda_2/2p_2)) - (p_1 a - p_2 x_{23})^2$, and $\Delta''_2 = -4p_1 p_2 \cdot (a + d_{31} - d'_{31} - (\lambda_3/2p_1))(c - d_{32} + d'_{32} + (\lambda_3/2p_2)) - (p_1 a - p_2 x_{33})^2$.

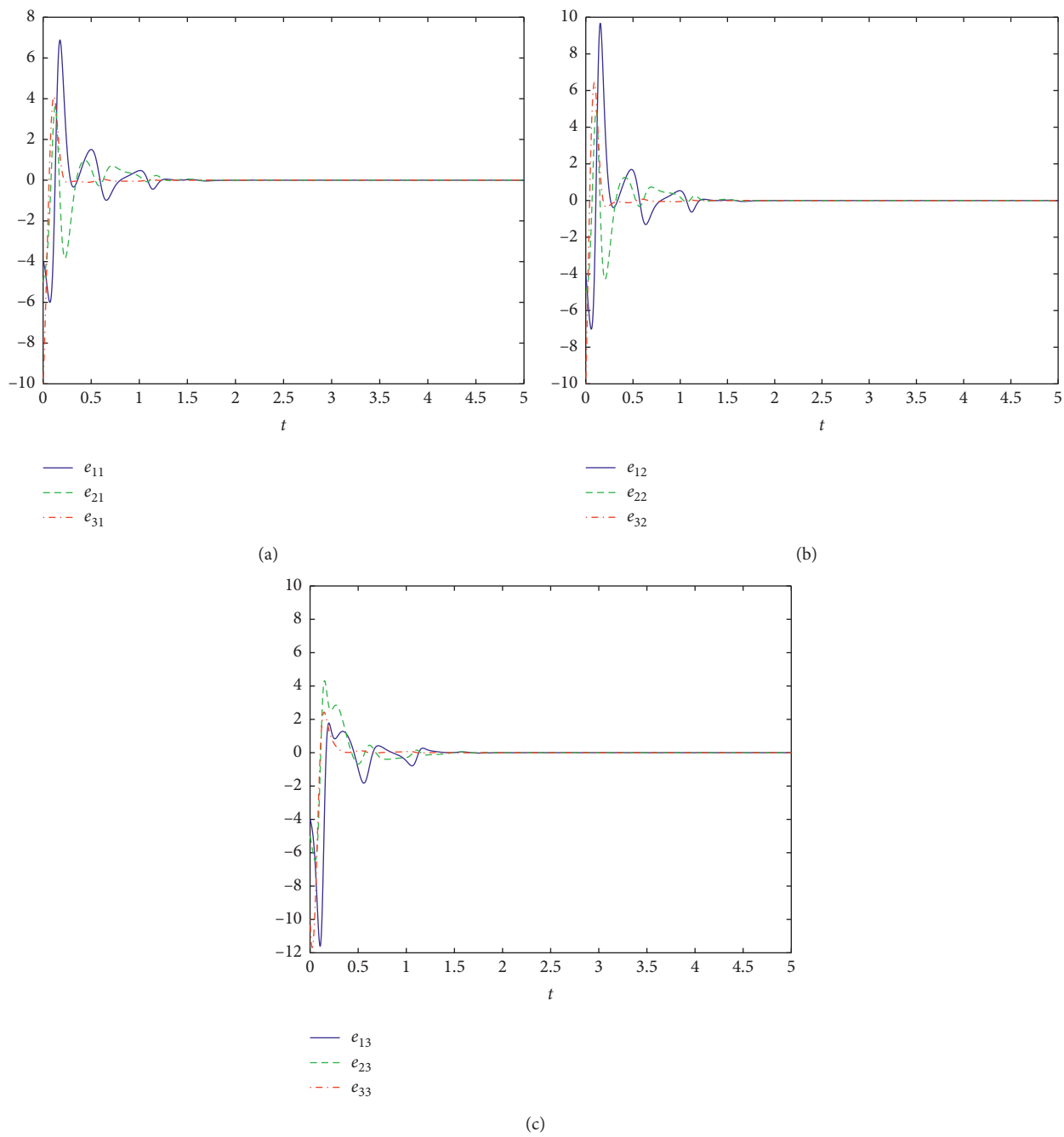

FIGURE 4: The errors of synchronization. (a) e_{11}, e_{21} and e_{31} vs. t. (b) e_{12}, e_{22} and e_{32} vs. t. (c) e_{13}, e_{23} and e_{33} vs. t.

The following numerical simulations are presented to demonstrate the effectiveness of our idea. Let $a = 36, b = 3$, $c = 20$, $P = \mathrm{diag}\{1, 1, 1\}$. $\lambda_j = 60$, with $j = 1, 2, 3$. For the purpose of satisfying above inequalities (24)–(26), the coupling parameters are set as $d_{11} = d_{12} = d_{13} = 10$, $d'_{11} = d'_{12} = d'_{13} = 5$, $d_{21} = d_{22} = d_{23} = 5$, $d'_{21} = d'_{22} = d'_{23} = 5$, $d_{31} = d_{32} = d_{33} = 1$, $d'_{31} = d'_{32} = d'_{33} = -10$. The corresponding numerical results are illustrated in Figures 3 and 4. Figure 3 indicates that the trajectory of 4 Lü systems can be synchronized with each other. Figure 4 illustrates that the errors $e_1 = (e_{11}, e_{12}, e_{13})^{\mathrm{T}}$, $e_2 = (e_{21}, e_{22}, e_{23})^{\mathrm{T}}$, $e_3 = (e_{31}, e_{32}, e_{33})^{\mathrm{T}}$ are finally stabilized to 0 as $t \longrightarrow \infty$, i.e., the synchronization for 4 cascade-coupled Lü systems is realized.

Remark 4. When fixing the above coupling parameters except d_{11}, then we obtain $d_{11} > -1$. Further, we select $d_{11} \in (-10, 40)$ to observe the nature of the dynamical state of system (13) before and after synchronization. Here, we note that the largest Lyapunov exponent of system (13) is always greater than zero, which demonstrates that system (13) operates in chaotic state within a large range of coupling parameters (Figure 5).

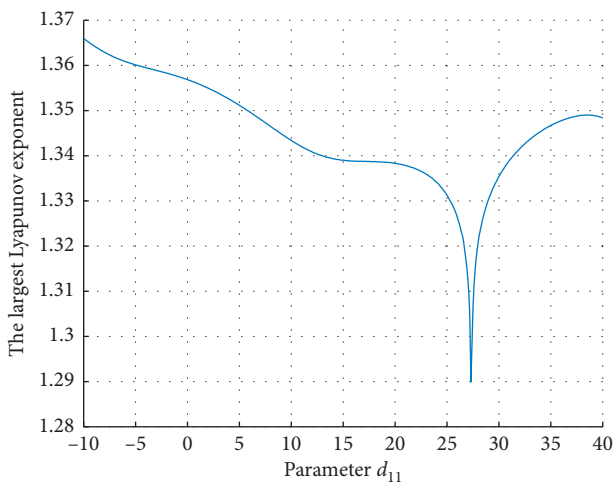

Figure 5: The largest Lyapunov exponent of system (13).

4. Conclusions

Recently, the chaos synchronization has been a subject of increasing research interest due to its potential application in fields of secure communication, signal processing, and life sciences. Numerous researchers have extensively explored the synchronization of two chaotic systems. This paper firstly proposes cascade-coupled synchronization of n $(n \geq 2)$ chaotic systems, which is proved by rigorous mathematical analysis. Based on Lyapunov stability theory, a general condition is presented and applied to 4 Lü systems. This validates the effectiveness of our idea and then motivates us to develop cascade synchronization in different cases, such as complex chaotic systems, fractional-order chaotic systems, and time-delayed chaotic systems. Moreover, the other coupling parameter types are also worthy of further investigation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities (lzujbky-2019-91).

References

[1] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Physical Review Letters*, vol. 64, no. 8, pp. 821–824, 1990.

[2] J. Liu, Z. Wang, M. Shu, F. Zhang, S. Leng, and X. Sun, "Secure communication of fractional complex chaotic systems based on fractional difference function synchronization," *Complexity*, vol. 2019, Article ID 7242791, 10 pages, 2019.

[3] S. A. Rahman, M. Zribi, and N. Smaoui, "Secure communications based on the projective synchronization of four-dimensional hyperchaotic systems," *Mathematical Problems in Engineering*, vol. 2019, Article ID 2491850, 16 pages, 2019.

[4] W. Tan, F. Jiang, C. Huang, and L. Zhou, "Synchronization for a class of fractional-order hyperchaotic system and its application," *Complexity*, vol. 2012, Article ID 974639, 11 pages, 2012.

[5] H. Yau, Y. Pu, and S. Li, "An FPGA-based PID controller design for chaos synchronization by evolutionary programming," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 516031, 11 pages, 2011.

[6] A. S. Pikovsky, M. G. Rosenblum, G. V. Osipov, and J. Kurths, "Phase synchronization of chaotic oscillators by external driving," *Physica D: Nonlinear Phenomena*, vol. 104, no. 3-4, pp. 219–238, 1997.

[7] D. Xu and Z. Li, "Controlled projective synchronization in nonpartially-linear chaotic systems," *International Journal of Bifurcation and Chaos*, vol. 12, no. 6, pp. 1395–1402, 2002.

[8] X.-S. Yang, "On the existence of generalized synchronizor in unidirectionally coupled systems," *Applied Mathematics and Computation*, vol. 122, no. 1, pp. 71–79, 2001.

[9] A. Ouannas, X. Wang, V. T. Wang, and T. Ziar, "Dynamic analysis of complex synchronization schemes between integer order and fractional order chaotic systems with different dimensions," *Complexity*, vol. 2017, Article ID 4948392, 12 pages, 2017.

[10] P. Li, J. Du, S. Li, and Y. Zheng, "Modulus synchronization of a novel hyperchaotic real system and its corresponding complex system," *IEEE Access*, vol. 7, pp. 109577–109584, 2019.

[11] J. Chen, J. Sun, M. Chi, and X.-M. Cheng, "A novel scheme adaptive hybrid dislocated synchronization for two identical and different memristor chaotic oscillator systems with uncertain parameters," *Abstract and Applied Analysis*, vol. 2014, Article ID 675840, 10 pages, 2014.

[12] J. Sun, Y. Shen, and X. Zhang, "Modified projective and modified function projective synchronization of a class of real nonlinear systems and a class of complex nonlinear systems," *Nonlinear Dynamics*, vol. 78, no. 3, pp. 1755–1764, 2014.

[13] W. Kinzel, A. Englert, and I. Kanter, "On chaos synchronization and secure communication," *Philosophical Transactions of the Royal Society A–Mathematical Physical and Engineerig Sciences*, vol. 368, no. 1911, pp. 379–389, 2010.

[14] F. Liu, X. Zang, and J. Song, "Anti-synchronism of the bidirectional coupled chaotic system," *Acta Physica Sinica*, vol. 58, no. 6, pp. 3765–3771, 2009.

[15] S. Kuntanapreeda, "Synchronization of uncertain fractional-order hyperchaotic systems via unidirectional linear error feedback coupling scheme," *Journal of Chaos*, vol. 2013, Article ID 512403, 6 pages, 2013.

[16] Z.-M. Ge and Y.-S. Chen, "Adaptive synchronization of unidirectional and mutual coupled chaotic systems," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 881–888, 2005.

[17] J. M. Cruz, M. Rivera, and P. Parmananda, "Chaotic synchronization under unidirectional coupling: numerics and experiments," *The Journal of Physical Chemistry A*, vol. 113, no. 32, pp. 9051–9056, 2009.

[18] Y. Yu and S. Zhang, "The synchronization of linearly bidirectional coupled chaotic systems," *Chaos, Solitons & Fractals*, vol. 22, no. 1, pp. 189–197, 2004.

[19] M. A. Khan and S. Poria, "Projective synchronization of chaotic systems with bidirectional nonlinear coupling," *Pramana*, vol. 81, no. 3, pp. 395–406, 2013.

[20] Y. Yu, "The synchronization for time-delay of linearly bi-directional coupled chaotic system," *Chaos, Solitons & Fractals*, vol. 33, no. 4, pp. 1197–1203, 2007.

[21] G. Sivaganesh, A. Arulgnanam, and A. N. Seethalakshmi, "Numerical studies on the synchronization of a network of mutually coupled simple chaotic systems," in *Proceedings of the Conference on Nonlinear Systems and Dynamics, (CNSD)*, pp. 95–99, Kanpur, India, Septemper 2019.

[22] T. L. Carroll and L. M. Pecora, "Cascading synchronized chaotic systems," *Physica D: Nonlinear Phenomena*, vol. 67, no. 1–3, pp. 126–140, 1993.

[23] H.-L. An and Y. Chen, "A function cascade synchronization method with unknown parameters and applications," *Chinese Physics B*, vol. 17, no. 1, pp. 98–104, 2008.

[24] H.-L. An and Y. Chen, "The function cascade synchronization method and applications," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 10, pp. 2246–2255, 2008.