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Research Article

PD^{ϑ} Control Strategy for a Fractional-Order Chaotic Financial Model

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In this article, based on the previous works, a new fractional-order financial model is put up. The chaotic behavior of the fractional-order financial model is suppressed by designing an appropriate PD^{ϑ} controller. By choosing the delay as the bifurcation parameter, we establish the sufficient condition to guarantee the stability and the existence of Hopf bifurcation of fractional-order financial model. Also, the influence of the delay and the fractional order on the stability and the existence of Hopf bifurcation of fractional-order financial model is revealed. An example is given to confirm the effectiveness of the analysis results. The main findings of this article play an important role in maintaining economic stability.

1. Introduction

Establishing financial models to investigate the complex dynamical behavior of economic society has attracted more attention of scholars in numerous areas. To grasp the law of operation accurately, various financial models have been established to reveal the inherent characteristics of economic development and numerous fruitful results are achieved. For instance, Zhang et al. [1] discussed the stability of a financial hyperchaotic model, Serletic [2] investigated the chaos in economic system, Lin et al. [3] made an detailed analysis on chaotic behavior of a financial complex model, and Gao and Ma [4] studied the chaotic phenomenon and bifurcation of a finance model. For more information on financial models, one can see [5–9].

In many cases, chaotic behavior often happens in financial models. Chaotic phenomenon will have a serious effect on man's everyday life. Thus the research on chaos control of financial models becomes a hot issue in financial community. The appearance of chaotic phenomenon in economic system implies that the macroeconomic operation has its inherent indefiniteness and complexity. Of course, government departments can take measures for control or interference, but the effect is very limited [10]. Thus, it is worthwhile to deal with the control of chaos in financial systems by theoretical analysis.

In 2001, Ma and Chen [11, 12] studied the following financial model:

$$\begin{aligned} \frac{du_1}{dt} &= u_3 + (u_2 - p_1) u_1, \\ \frac{du_2}{dt} &= 1 - p_2 u_2 - u_1^2, \\ \frac{du_3}{dt} &= -u_1 - p_3 u_3, \end{aligned} \tag{1}$$

where $p_1 \geq 0$ denotes the saving amount, $p_2 \geq 0$ denotes the cost per investment, $p_3 \geq 0$ denotes the elasticity of demand of commercial markets, u_1 represents interest rate, u_2 represents investment demand, and u_3 represents price index. In 2008, Chen [13] designed a suitable time delayed feedback controller to control the chaotic phenomena of model (1); computer simulations are presented to illustrate the effectiveness of designed controller. In 2011, Son and Park [14] further dealt with the chaos control issue of model (1) by applying delayed feedback method. Detailed theoretical analysis and numerical simulations are carried out to check the correctness of the controller. In 2009, Gao and Ma [4] discussed the chaos control of model (1) by adding a time delay feedback term to the second equation of system (1). Also the sufficient condition to guarantee the stability and the existence of Hopf bifurcation of involved controlled financial model is established. Considering the effect of time delay on the financial phenomena, Zhang and Zhu [15], Chen et al. [16], Mircea et al. [17], and Zhang [18] established different delayed financial models and analyzed their Hopf bifurcation or chaos control issue. In 2014, Zhao et al. [19] investigated the the anticontrol of Hopf bifurcation and chaos control of model (1) by applying delayed washout filters. For details, one can see [5–7, 20–30].

Here we would like to point out that all the above works are only concerned with the integer-order differential systems. Nowadays, numerous scholars have found that fractional calculus, which is a generalization of ordinary differentiation and integration, has potential applications in numerous fields such as economics, physics, heat transfer, and chemical engineer [31–38]. Many researchers argued that it is more reasonable to describe the object natural phenomena by fractional-order differential equations than integer-order differential equations, since fractional-order differential equations can better describe the memory characteristics and historical dependence. Noticing that financial coefficients possess very long memory and the variation of financial coefficients has close connection with previous and current time, we think that it is important for us to establish fractional-order financial systems. In recent years, there are numerous articles that investigate the fractional-order financial systems. One can see [11, 12, 39–47].

In view of the above analysis and based on system (1), we modify system (1) as the following fractional-order financial model:

$$\begin{aligned}\frac{d^{\vartheta}u_1}{dt^{\vartheta}} &= u_3 + (u_2 - p_1)\,u_1,\\ \frac{d^{\vartheta}u_2}{dt^{\vartheta}} &= 1 - p_2u_2 - u_1^2,\\ \frac{d^{\vartheta}u_3}{dt^{\vartheta}} &= -u_1 - p_3u_3,\end{aligned} \tag{2}$$

where $0 < \vartheta < 1$ stands for the fractional order. The study reveals that chaotic phenomenon will appear if $\vartheta = 0.73$ and $p_1 = 3$, $p_2 = 0.1$, $p_3 = 1$. The results can be shown in Figures 1–10.

Our key task is concerned with two topics: (i) designing a suitable PD^{ϑ} controller to suppress the chaotic behavior of system (2) and (2) seeking the influence of delay and fractional order on the stability and bifurcation phenomenon of controlled system.

The superiority of this article is stated as follows:

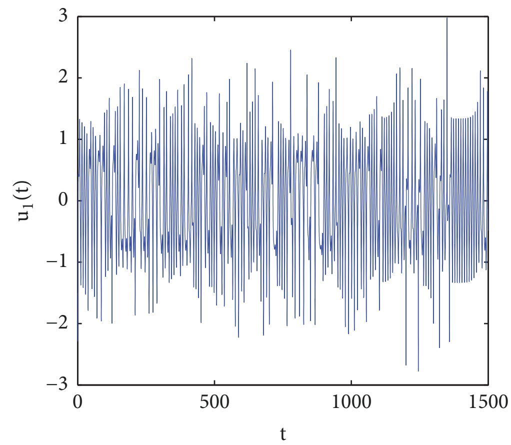

Figure 1: The relation of t and u_1 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

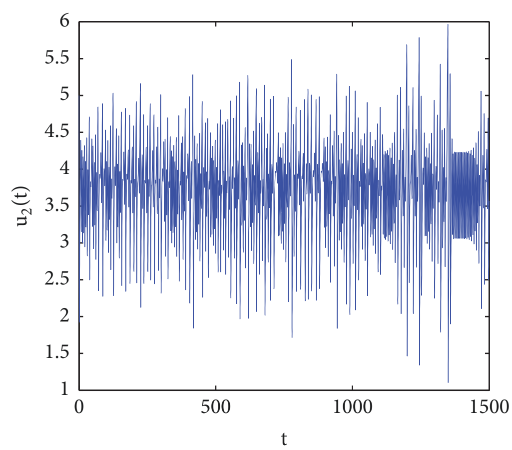

Figure 2: The relation of t and u_2 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

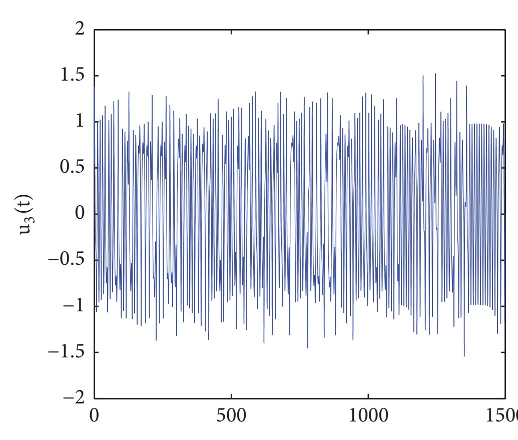

Figure 3: The relation of t and u_3 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

(I) A new fractional-order financial model is established.

(II) A PD^{ϑ} controller is designed to suppress the chaos of the fractional-order financial model. Also, a set of

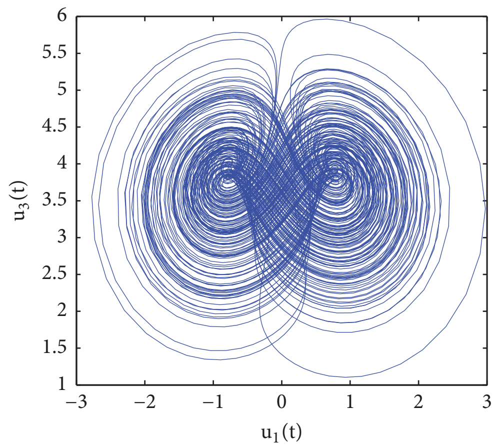

FIGURE 4: The relation of u_1 and u_2 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

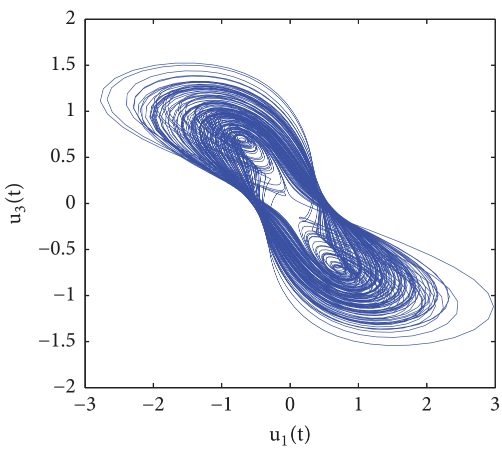

FIGURE 5: The relation of u_1 and u_3 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

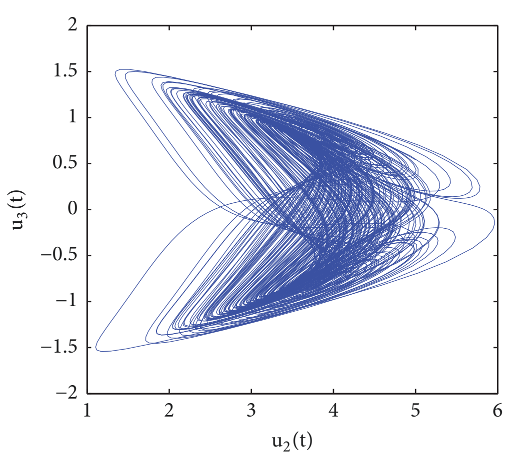

FIGURE 6: The relation of u_2 and u_3 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

new sufficient conditions to guarantee the stability and the existence of Hopf bifurcation of fractional-order financial model are obtained. In addition, the

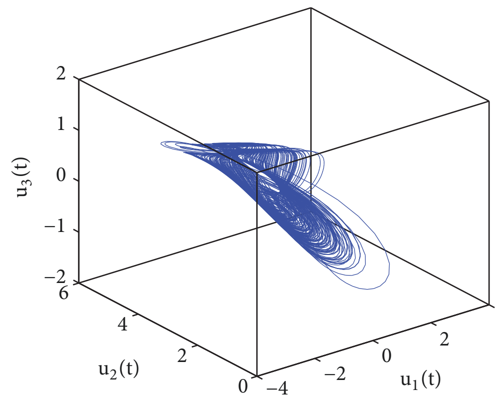

FIGURE 7: The relation of u_1, u_2, and u_3 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

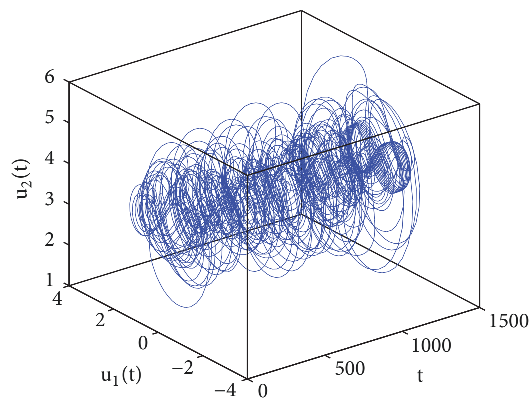

FIGURE 8: The relation of t, u_1, and u_2 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

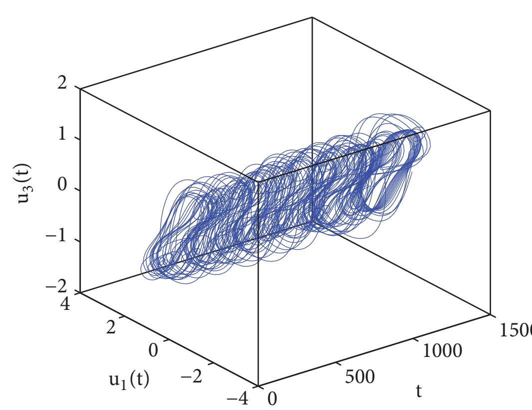

FIGURE 9: The relation of t, u_2, and u_3 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

effect of the delay and fractional order on the dynamics of fractional-order financial model is displayed.

(III) To the best of our knowledge, it is the first time to control chaos of fractional-order financial model by applying PD^{ϑ} controller.

We organize this article as follows. In Section 2, some basic knowledge on fractional calculus is presented. In Section 3,

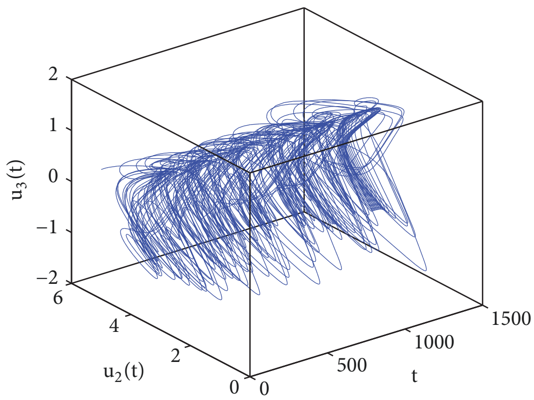

FIGURE 10: The relation of t, u_2, and u_3 in model (2) when $\vartheta = 0.73$, $p_1 = 3$, $p_2 = 0.1$, and $p_3 = 1$.

PD^ϑ controller is designed to control chaos of fractional-order financial model. In Section 4, an example is given to show the effectiveness of the main findings. A conclusion is presented in Section 5.

2. Basic Results

In this section, we give some basic results on fractional calculus.

Definition 1 (see [48]). Define Caputo fractional-order derivative as follows:

$$\mathbf{D}^\vartheta p(\iota) = \frac{1}{\Gamma(n-\vartheta)} \int_{\iota_0}^{\iota} \frac{p^{(m)}(s)}{(\iota - s)^{\vartheta - m + 1}} ds, \quad (3)$$

where $p(\iota) \in ([\iota_0, \infty), R)$, $\Gamma(s) = \int_0^\infty \iota^{s-1} e^{-\iota} d\iota$. $\iota \geq \iota_0$ and m is a positive integer such that $m - 1 \leq \vartheta < m$. If $0 < \vartheta < 1$, then

$$\mathbf{D}^\vartheta p(\iota) = \frac{1}{\Gamma(1-\vartheta)} \int_{\iota_0}^{\iota} \frac{p'(s)}{(\iota - s)^{\vartheta}} ds. \quad (4)$$

Definition 2 (see [49]). The point (u_1^*, u_2^*, u_3^*) is called an equilibrium point if the equations

$$\begin{aligned} u_3^* + (u_2^* - p_1) u_1^* &= 0, \\ 1 - p_2 u_2^* - (u_1^*)^2 &= 0, \\ -u_1^* - p_3 u_3^* &= 0 \end{aligned} \quad (5)$$

hold.

Lemma 3 (see [50]). *Let λ_i $(i = 1, 2, \ldots, n)$ be the root of the characteristic equation of the autonomous system $\mathbf{D}^\vartheta u = \mathbf{B}u$, $u(0) = u_0$, where $0 < \vartheta < 1$, $u \in R^n$, $\mathbf{B} \in R^{n\times n}$. Then system $\mathbf{D}^\vartheta u = \mathbf{B}u$ is asymptotically stable if and only if $|\arg(\lambda_i)| > \vartheta\pi/2$ $(i = 1, 2, \ldots, n)$. The system $\mathbf{D}^\vartheta u = \mathbf{B}u$ is stable if and only if $|\arg(\lambda_i)| > \vartheta\pi/2$ $(i = 1, 2, \ldots, n)$ and those critical eigenvalues that satisfy $|\arg(\lambda_i)| = \vartheta\pi/2$ $(i = 1, 2, \ldots, n)$ have geometric multiplicity one.*

Lemma 4 (see [51]). *The system $\mathbf{D}^\vartheta v(t) = \mathbf{A}_1 v(t) + \mathbf{A}_2 v(t-\rho)$, where $v(t) = \phi(t)$, $t \in [-\rho, 0]$, $\vartheta \in (0, 1]$, $v \in R^n$, $\mathbf{A}_1, \mathbf{A}_2 \in R^{n\times n}$, $\vartheta \in R^{+(n\times n)}$. Then the characteristic equation of the system is $\det|s^\vartheta I - \mathbf{A}_1 - \mathbf{A}_2 e^{-s\rho}| = 0$. If all the roots of the characteristic equation of the system have negative real roots, then the zero solution of the system is asymptotically stable.*

3. Chaos Control by PD^ϑ Controller

If $p_2 + p_1 p_2 p_3 - p_3 > 0$, then system (2) has a unique equilibrium point:

$$(u_{10}, u_{20}, u_{30}) = \left(0, \frac{1}{p_2}, 0\right). \quad (6)$$

If $p_2 + p_1 p_2 p_3 - p_3 < 0$, then model (2) has three equilibrium points:

$$\begin{aligned} &(u_{10}, u_{20}, u_{30}) = \left(0, \frac{1}{p_2}, 0\right), \\ &(u_1^*, u_2^*, u_3^*) = \left(\frac{\sqrt{p_3 - p_2 - p_1 p_2 p_3}}{\sqrt{p_3}}, \frac{1 + p_1 p_3}{p_3}, \right. \\ &\qquad \left. -\frac{\sqrt{p_3 - p_2 - p_1 p_2 p_3}}{\sqrt[3]{p_3^2}}\right), \\ &(u_1^{**}, u_2^{**}, u_3^{**}) = \left(-\frac{\sqrt{p_3 - p_2 - p_1 p_2 p_3}}{\sqrt{p_3}}, \frac{1 + p_1 p_3}{p_3}, \right. \\ &\qquad \left. \frac{\sqrt{p_3 - p_2 - p_1 p_2 p_3}}{\sqrt[3]{p_3^2}}\right). \end{aligned} \quad (7)$$

During the past several decades, many different control strategies have been applied to control the chaos and bifurcation behavior. But they only involve the integer-order differential systems. Applying PD^ϑ control strategy to control the chaotic behavior of fractional-order chaotic system is rare. To make up for the deficiency, we try to design a PD^ϑ feedback controller to suppress the chaotic phenomenon of model (2). In this paper, we only consider the equilibrium point (u_1^*, u_2^*, u_3^*). The other equilibrium points can be discussed in the same way. Here we omit it. Throughout this paper, we always assume that $p_2 + p_1 p_2 p_3 - p_3 < 0$.

Add the following PD^ϑ feedback controller to the first equation of system (2):

$$\varrho(t) = \kappa_p (u_1(t-\rho) - u_1^*) + \kappa_d \frac{d^\vartheta}{dt^\vartheta} (u_1(t) - u_1^*), \quad (8)$$

where κ_p and κ_d are the proportional control parameter and the derivative control parameter, respectively, and ρ is time delay; then system (2) becomes

$$\begin{aligned}\frac{d^{\vartheta}u_1}{dt^{\vartheta}} &= u_3 + (u_2 - p_1) u_1 + \varrho(t), \\ \frac{d^{\vartheta}u_2}{dt^{\vartheta}} &= 1 - p_2 u_2 - u_1^2, \\ \frac{d^{\vartheta}u_3}{dt^{\vartheta}} &= -u_1 - p_3 u_3.\end{aligned} \tag{9}$$

That is,

$$\begin{aligned}\frac{d^{\vartheta}u_1}{dt^{\vartheta}} &= u_3 + (u_2 - p_1) u_1 + \kappa_p (u_1(t-\rho) - u_1^*) \\ &\quad + \kappa_d \frac{d^{\vartheta}}{dt^{\vartheta}} (u_1(t) - u_1^*), \\ \frac{d^{\vartheta}u_2}{dt^{\vartheta}} &= 1 - p_2 u_2 - u_1^2, \\ \frac{d^{\vartheta}u_3}{dt^{\vartheta}} &= -u_1 - p_3 u_3.\end{aligned} \tag{10}$$

Let $v_i(t) = u_i(t) - u_i(t)$ $(i = 1, 2, 3)$; then linear system of system (10) takes the form

$$\begin{aligned}&\frac{d^{\vartheta}v_1}{dt^{\vartheta}} \\ &\quad = \frac{1}{1-\kappa_d}\left[(u_2^* - p_1) v_1 + v_2 + u_1^* v_3 + \kappa_p v_1(t-\rho)\right], \\ &\frac{d^{\vartheta}v_2}{dt^{\vartheta}} = 2u_1^* v_1 - p_2 v_2, \\ &\frac{d^{\vartheta}v_3}{dt^{\vartheta}} = -v_1 - p_3 v_3.\end{aligned} \tag{11}$$

The corresponding characteristic equation of (11) is given by

$$\det \begin{bmatrix} s^{\vartheta} - \dfrac{u_2^* - p_1}{1-\kappa_d} - \dfrac{\kappa_p}{1-\kappa_d} e^{-s\rho} & -\dfrac{1}{1-\kappa_d} & -\dfrac{u_1^*}{1-\kappa_d} \\ 2u_1^* & s^{\vartheta} + p_2 & 0 \\ 1 & 0 & s^{\vartheta} + p_3 \end{bmatrix}. \tag{12}$$

Then

$$\mathcal{Q}_1(s) + \mathcal{Q}_2(s) e^{-s\rho} = 0, \tag{13}$$

where

$$\begin{aligned}\mathcal{Q}_1(s) &= s^{3\vartheta} + \left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right) s^{2\vartheta} \\ &\quad + \left[p_2 p_3 - \frac{(u_2^* - p_1)(p_2 + p_3)}{1-\kappa_d}\right] s^{\vartheta} \\ &\quad - \frac{(u_2^* - p_1)(p_2 + p_3)}{1-\kappa_d}, \\ \mathcal{Q}_2(s) &= -\frac{\kappa_p}{1-\kappa_d}\left(s^{2\vartheta} + (p_2 + p_3) s^{\vartheta} + p_2 p_3\right).\end{aligned} \tag{14}$$

Assume that $s = i\zeta = \zeta(\cos(\pi/2) + i\sin(\pi/2))$ is a root of (13); then one has

$$\begin{aligned}\mathcal{Q}_{2R}(\zeta)\cos\zeta\rho + \mathcal{Q}_{2I}(\phi)\sin\zeta\rho &= -\mathcal{Q}_{1R}(\zeta), \\ \mathcal{Q}_{2I}(\zeta)\cos\zeta\rho - \mathcal{Q}_{2R}(\phi)\sin\zeta\rho &= -\mathcal{Q}_{1I}(\zeta),\end{aligned} \tag{15}$$

where

$$\begin{aligned}\mathcal{Q}_{1R}(\zeta) &= \zeta^{3\vartheta}\cos\frac{3\vartheta\pi}{2} + \left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right)\zeta^{2\vartheta} \\ &\quad \cdot \cos\vartheta\pi + \left[p_2 p_3 - \frac{(u_2^* - p_1)(p_2 + p_3)}{1-\kappa_d}\right]\zeta^{\vartheta}\cos\frac{\vartheta\pi}{2} \\ &\quad - \frac{(u_2^* - p_1)(p_2 + p_3)}{1-\kappa_d}, \\ \mathcal{Q}_{1I}(\zeta) &= \zeta^{3\vartheta}\sin\frac{3\vartheta\pi}{2} + \left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right)\zeta^{2\vartheta} \\ &\quad \cdot \sin\vartheta\pi + \left[p_2 p_3 - \frac{(u_2^* - p_1)(p_2 + p_3)}{1-\kappa_d}\right]\zeta^{\vartheta}\sin\frac{\vartheta\pi}{2}, \\ \mathcal{Q}_{2R}(\zeta) &= -\frac{\kappa_p}{1-\kappa_d}\left(\vartheta^{2\vartheta}\cos\vartheta\pi + (p_2 + p_3)\zeta^{\vartheta}\cos\frac{\vartheta\pi}{2}\right. \\ &\quad \left. + p_2 p_3\right), \\ \mathcal{Q}_{2I}(\zeta) &= -\frac{\kappa_p}{1-\kappa_d}\left(\vartheta^{2\vartheta}\sin\vartheta\pi + (p_2 + p_3)\zeta^{\vartheta}\sin\frac{\vartheta\pi}{2}\right).\end{aligned} \tag{16}$$

By (15), one has

$$\begin{aligned}\cos\zeta\rho &= -\frac{\mathcal{Q}_{1R}(\zeta)\mathcal{Q}_{2R}(\zeta) + \mathcal{Q}_{1I}(\zeta)\mathcal{Q}_{2I}(\zeta)}{\mathcal{Q}_{2R}^2(\zeta) + \mathcal{Q}_{2I}^2(\zeta)}, \\ \sin\zeta\rho &= -\frac{\mathcal{Q}_{1R}(\zeta)\mathcal{Q}_{2I}(\zeta) - \mathcal{Q}_{1I}(\zeta)\mathcal{Q}_{2R}(\zeta)}{\mathcal{Q}_{2R}^2(\zeta) + \mathcal{Q}_{2I}^2(\zeta)}.\end{aligned} \tag{17}$$

Let

$$\begin{aligned}a_1 &= \cos\frac{3\vartheta\pi}{2}, \\ a_2 &= \left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right)\cos\vartheta\pi,\end{aligned}$$

$$
\begin{aligned}
a_3 &= \left[p_2 p_3 - \frac{(u_2^* - p_1)(p_2 + p_3)}{1 - \kappa_d}\right] \cos\frac{\vartheta\pi}{2}, \\
a_4 &= -\frac{(u_2^* - p_1)(p_2 + p_3)}{1 - \kappa_d}, \\
a_5 &= \sin\frac{3\vartheta\pi}{2}, \\
a_6 &= \left(p_2 + p_3 - \frac{u_2^* - p_1}{1 - \kappa_d}\right) \sin\vartheta\pi, \\
a_7 &= \left[p_2 p_3 - \frac{(u_2^* - p_1)(p_2 + p_3)}{1 - \kappa_d}\right] \sin\frac{\vartheta\pi}{2}, \\
a_8 &= -\frac{\kappa_p \cos\vartheta\pi}{1 - \kappa_d}, \\
a_9 &= -\frac{\kappa_p (p_2 + p_3)}{1 - \kappa_d}, \\
a_{10} &= -\frac{\kappa_p p_2 p_3}{1 - \kappa_d}, \\
a_{11} &= -\frac{\kappa_p \sin\vartheta\pi}{1 - \kappa_d}, \\
a_{12} &= -\frac{\kappa_p (p_2 + p_3) \sin(\vartheta\pi/2)}{1 - \kappa_d}.
\end{aligned} \tag{18}
$$

Then

$$
\begin{aligned}
\mathcal{Q}_{1R}(\zeta) &= a_1 \zeta^{3\vartheta} + a_2 \zeta^{2\vartheta} + a_3 \zeta^{\vartheta} + a_4, \\
\mathcal{Q}_{1I}(\zeta) &= a_5 \zeta^{3\vartheta} + a_6 \zeta^{2\vartheta} + a_7 \zeta^{\vartheta}, \\
\mathcal{Q}_{2R}(\zeta) &= a_8 \zeta^{2\vartheta} + a_9 \zeta^{\vartheta} + a_{10}, \\
\mathcal{Q}_{2I}(\zeta) &= a_{11} \zeta^{2\vartheta} + a_{12} \zeta^{\vartheta}.
\end{aligned} \tag{19}
$$

It follows from (17) that

$$
\begin{aligned}
&\left[\mathcal{Q}_{1R}\mathcal{Q}_{2R} + \mathcal{Q}_{1I}\mathcal{Q}_{2I}\right]^2 + \left[\mathcal{Q}_{1R}\mathcal{Q}_{2I} - \mathcal{Q}_{1I}\mathcal{Q}_{2R}\right]^2 \\
&\quad = \left[\mathcal{Q}_{2R}^2(\phi) + \mathcal{Q}_{2I}^2(\phi)\right]^2.
\end{aligned} \tag{20}
$$

Notice that

$$
\begin{aligned}
&\left[\mathcal{Q}_{1R}(\zeta)\mathcal{Q}_{2R}(\zeta) + \mathcal{Q}_{1I}(\zeta)\mathcal{Q}_{2I}(\zeta)\right]^2 \\
&\quad = \left(b_1 \phi^{5\vartheta} + b_2 \phi^{4\vartheta} + b_3 \zeta^{3\vartheta} + b_4 \zeta^{2\vartheta} + b_5 \zeta^{\vartheta} + b_6\right)^2, \\
&\left[\mathcal{Q}_{1R}(\zeta)\mathcal{Q}_{2I}(\zeta) - \mathcal{Q}_{1I}(\zeta)\mathcal{Q}_{2R}(\zeta)\right]^2 \\
&\quad = \left(b_7 \phi^{5\vartheta} + b_8 \phi^{4\vartheta} + b_9 \phi^{3\vartheta} + b_{10} \zeta^{2\vartheta} + b_{11} \zeta^{\vartheta}\right)^2, \\
&\left[\mathcal{Q}_{2R}^2(\zeta) + \mathcal{Q}_{2I}^2(\zeta)\right]^2 \\
&\quad = \left(b_{12} \zeta^{4q} + b_{13} \zeta^{3\vartheta} + b_{14} \zeta^{2\vartheta} + b_{15} \zeta^{\vartheta}\right)^2,
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
b_1 &= a_1 a_8 + a_5 a_{11}, \\
b_2 &= a_2 a_8 + a_1 a_9 + a_6 a_{11} + a_5 a_{12}, \\
b_3 &= a_3 a_8 + a_2 a_9 + a_1 a_{10} + a_7 a_{11} + a_6 a_{12}, \\
b_4 &= a_4 a_8 + a_3 a_9 + a_2 a_{10} + a_7 a_{12}, \\
b_5 &= a_4 a_9 + a_3 a_{10}, \\
b_6 &= a_4 a_{10}, \\
b_7 &= a_1 a_{11} - a_5 a_8, \\
b_8 &= a_2 a_{11} + a_1 a_{12} - a_5 a_9 - a_6 a_8, \\
b_9 &= l a_3 a_{11} + a_2 a_{12} - a_5 a_{10} - a_6 a_9 - a_7 a_8, \\
b_{10} &= a_4 a_{11} + a_3 a_{12} - a_6 a_{10} - a_7 a_9, \\
b_{11} &= a_4 a_{12} - a_7 a_{10}, \\
b_{12} &= a_8^2 + a_{11}^2, \\
b_{13} &= 2\left(a_8 a_9 + a_{11} a_{12}\right), \\
b_{14} &= 2 a_8 a_{10} + a_{12}^2, \\
b_{15} &= 2 a_9 a_{10}.
\end{aligned} \tag{22}
$$

By (20), one has

$$
\begin{aligned}
&\varrho_1 \zeta^{10\vartheta} + \varrho_2 \zeta^{9\vartheta} + \varrho_3 \zeta^{8\vartheta} + \varrho_4 \zeta^{7\vartheta} + \varrho_5 \zeta^{6\vartheta} + \varrho_6 \zeta^{5\vartheta} + \varrho_7 \zeta^{4\vartheta} \\
&\quad + \varrho_8 \zeta^{3\vartheta} + \varrho_9 \zeta^{2\vartheta} + \varrho_{10} \zeta^{\vartheta} + \varrho_{11} = 0,
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
\varrho_1 &= b_1^2 + b_7^2, \\
\varrho_2 &= 2 b_1 b_2 + 2 b_7 b_8, \\
\varrho_3 &= b_2^2 + b_8^2 - b_{12}^2 + 2 b_1 b_3 + 2 b_7 b_9, \\
\varrho_4 &= 2\left(b_2 b_3 + b_1 b_2 + b_7 b_{10} + b_8 b_9 - b_{12} b_{13}\right), \\
\varrho_5 &= b_3^2 + 2 b_1 b_5 + 2 b_2 b_4 + b_9^2 + 2 b_7 b_{11} + 2 b_8 b_{10} - b_{13}^2 \\
&\quad - 2 b_{12} b_{14}, \\
\varrho_6 &= 2 b_1 b_6 + 2 b_2 b_5 + 2 b_3 b_4 + 2 b_9 b_{10} - 2 b_{12} b_{15} \\
&\quad - 2 b_{13} b_{14}, \\
\varrho_7 &= b_4^2 + 2 b_3 b_5 + b_{10}^2 + 2 b_8 b_{11} + 2 b_9 b_{11} - b_{14}^2 \\
&\quad - 2 b_{13} b_{15}, \\
\varrho_8 &= 2 b_3 b_6 + 2 b_4 b_5 + 2 b_{10} b_{11} - 2 b_{14} b_{15}, \\
\varrho_9 &= b_5^2 + 2 b_4 b_6 + b_{11}^2 - b_{15}^2, \\
\varrho_{10} &= 2 b_5 b_6, \\
\varrho_{11} &= b_6^2.
\end{aligned} \tag{24}
$$

Let

$$\Phi(\zeta) = \varrho_1\zeta^{10\vartheta} + \varrho_2\zeta^{9\vartheta} + \varrho_3\zeta^{8\vartheta} + \varrho_4\zeta^{7\vartheta} + \varrho_5\zeta^{6\vartheta} + \varrho_6\zeta^{5\vartheta} + \varrho_7\zeta^{4\vartheta} + \varrho_8\zeta^{3\vartheta} + \varrho_9\zeta^{2\vartheta} + \varrho_{10}\zeta^{\vartheta} + \varrho_{11} \tag{25}$$

and

$$\gamma(\nu) = \varrho_1\nu^{10} + \varrho_2\nu^9 + \varrho_3\nu^8 + \varrho_4\nu^7 + \varrho_5\nu^6 + \varrho_6\nu^5 + \varrho_7\nu^4 + \varrho_8\nu^3 + \varrho_9\nu^2 + \varrho_{10}\nu + \varrho_{11}. \tag{26}$$

Lemma 5. *(a) If $\varrho_i > 0$ $(i = 1,2,3,\ldots,11)$, then (13) has no root with zero real parts; (b) if $\varrho_{11} > 0$ and there exists a constant $\nu_0 > 0$ such that $\gamma(\nu_0) < 0$, then (13) has at least two pairs of purely imaginary roots.*

Proof. (a) It follows from (25) that

$$\begin{aligned}\frac{d\Phi(\zeta)}{d\zeta} &= 10\vartheta\varrho_1\zeta^{10\vartheta-1} + 9\vartheta\varrho_2\zeta^{9\vartheta-1} + 8\vartheta\varrho_3\zeta^{8\vartheta-1} \\ &\quad + 7\vartheta\varrho_4\zeta^{7\vartheta-1} + 6\vartheta\varrho_5\zeta^{6\vartheta-1} + 5\vartheta\varrho_6\zeta^{5\vartheta-1} \\ &\quad + 4\vartheta\varrho_7\zeta^{4\vartheta-1} + 3\vartheta\varrho_8\zeta^{3\vartheta-1} + 2\vartheta\varrho_9\zeta^{2\vartheta-1} \\ &\quad + \vartheta\varrho_{10}\zeta^{\vartheta-1}.\end{aligned} \tag{27}$$

Notice that $\varrho_i > 0$ $(i = 1,2,3,\ldots,10)$; then $d\Phi(\zeta)/d\zeta > 0$, $\forall\zeta > 0$. By $\Phi(0) = \varrho_{11} > 0$, one knows that (26) has no positive real root. In addition, $s = 0$ is not the root of (13). We complete the proof of (a).

(b) Notice that $\gamma(0) = \varrho_{11} > 0$, $\gamma(\eta_0) < 0$ $(\eta_0 > 0)$ and $\lim_{\mu\longrightarrow+\infty}(\gamma(\mu)/d\mu) = +\infty$; then $\exists\eta_{01} \in (0,\eta_0)$ and $\varepsilon_{02} \in (\eta_0,+\infty)$ such that $\gamma(\eta_{01}) = \gamma(\eta_{02}) = 0$, and then (25) has at least two positive real roots. Thus (13) has at least two pairs of purely imaginary roots. We complete the proof of (b). □

Without loss of generality, one can assume that (23) has ten positive real roots ζ_j $(j = 1,2,\ldots,10)$. By (13), we obtain

$$\rho_j^l = \frac{1}{\zeta_j}\left[\arccos\left(-\frac{\mathcal{Q}_{1R}(\zeta_j)\mathcal{Q}_{2R}(\zeta_j) + \mathcal{Q}_{1I}(\zeta_j)\mathcal{Q}_{2I}(\zeta_j)}{\mathcal{Q}_{2R}^2(\zeta_j) + \mathcal{Q}_{2I}^2(\zeta_j)}\right) + 2l\pi\right], \tag{28}$$

where $l = 0,1,2,\ldots,$ $j = 1,2,\ldots,10$. Then $\pm i\zeta_j$ is a pair of purely imaginary roots of (10) when $\rho = \rho_j^l$. Let

$$\begin{aligned}\rho_0 &= \min_{j=1,2,\ldots,11}\left\{\rho_j^0\right\}, \\ \zeta_0 &= \zeta\big|_{\rho=\rho_0}.\end{aligned} \tag{29}$$

Now we give the following assumption:

$(\mathscr{A}1)$ $\mathscr{U}_1\mathscr{V}_1 + \mathscr{U}_2\mathscr{V}_2 > 0$, where

$$\begin{aligned}
\mathscr{U}_1 &= 3\vartheta\zeta_0^{3\vartheta-1}\cos\frac{(3\vartheta-1)\pi}{2} + 2\vartheta\left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right)\zeta_0^{2\vartheta-1}\cos\frac{(2\vartheta-1)\pi}{2} + \vartheta\left[p_2p_3 - \frac{(u_2^*-p_1)(p_2+p_3)}{1-\kappa_d}\right]\zeta_0^{\vartheta-1}\cos\frac{(\vartheta-1)\pi}{2} \\
&\quad - \frac{\kappa_p}{1-\kappa_d}\left[2\vartheta\zeta_0^{2\vartheta-1}\cos\frac{(2\vartheta-1)\pi}{2} + \vartheta(p_2+p_3)\zeta_0^{\vartheta-1}\cos\frac{(\vartheta-1)\pi}{2}\right]\cos\zeta_0\rho_0 \\
&\quad - \frac{\kappa_p}{1-\kappa_d}\left[2\vartheta\zeta_0^{2\vartheta-1}\sin\frac{(2\vartheta-1)\pi}{2} + \vartheta(p_2+p_3)\zeta_0^{\vartheta-1}\sin\frac{(\vartheta-1)\pi}{2}\right]\sin\zeta_0\rho_0, \\
\mathscr{U}_2 &= 3\vartheta\zeta_0^{3\vartheta-1}\sin\frac{(3\vartheta-1)\pi}{2} + 2\vartheta\left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right)\zeta_0^{2\vartheta-1}\sin\frac{(2\vartheta-1)\pi}{2} + \vartheta\left[p_2p_3 - \frac{(u_2^*-p_1)(p_2+p_3)}{1-\kappa_d}\right]\zeta_0^{\vartheta-1}\sin\frac{(\vartheta-1)\pi}{2} \\
&\quad + \frac{\kappa_p}{1-\kappa_d}\left[2\vartheta\zeta_0^{2\vartheta-1}\cos\frac{(2\vartheta-1)\pi}{2} + \vartheta(p_2+p_3)\zeta_0^{\vartheta-1}\cos\frac{(\vartheta-1)\pi}{2}\right]\sin\zeta_0\rho_0 \\
&\quad - \frac{\kappa_p}{1-\kappa_d}\left[2\vartheta\zeta_0^{2\vartheta-1}\sin\frac{(2\vartheta-1)\pi}{2} + \vartheta(p_2+p_3)\zeta_0^{\vartheta-1}\sin\frac{(\vartheta-1)\pi}{2}\right]\cos\zeta_0\rho_0, \\
\mathscr{V}_1 &= -\frac{\kappa_p}{1-\kappa_d}\left[\zeta_0^{2\vartheta}\cos\vartheta\pi + (p_2+p_3)\zeta_0^{\vartheta}\cos\frac{\vartheta\pi}{2} + p_2p_3\right]\zeta_0\sin\zeta_0\rho_0 + \frac{\kappa_p}{1-\kappa_d}\left[\zeta_0^{2\vartheta}\sin\vartheta\pi + (p_2+p_3)\zeta_0^{\vartheta}\sin\frac{\vartheta\pi}{2} + p_2p_3\right]\zeta_0\cos\zeta_0\rho_0, \\
\mathscr{V}_2 &= -\frac{\kappa_p}{1-\kappa_d}\left[\zeta_0^{2\vartheta}\cos\vartheta\pi + (p_2+p_3)\zeta_0^{\vartheta}\cos\frac{\vartheta\pi}{2} + p_2p_3\right]\zeta_0\cos\zeta_0\rho_0 - \frac{\kappa_p}{1-\kappa_d}\left[\zeta_0^{2\vartheta}\sin\vartheta\pi + (p_2+p_3)\zeta_0^{\vartheta}\sin\frac{\vartheta\pi}{2} + p_2p_3\right]\zeta_0\sin\zeta_0\rho_0.
\end{aligned} \tag{30}$$

Lemma 6. *If $s(\rho) = \xi(\rho) + i\psi(\rho)$ is the root of (13) near $\rho = \rho_0$ which satisfies $\xi(\rho_0) = 0$, $\psi(\rho_0) = \psi_0$, then* $\operatorname{Re}[ds/d\rho]|_{\rho=\rho_0,\psi=\psi_0} > 0$.

Proof. By (13), one has

$$\frac{\mathcal{Q}_1(s)}{d\rho} + \frac{\mathcal{Q}_2(s)}{d\rho}e^{-s\rho} - e^{-s\rho}\left(\frac{ds}{d\rho}\rho + s\right)\mathcal{Q}_2(s) = 0. \tag{31}$$

Notice that

$$\begin{aligned}\frac{\mathcal{Q}_1(s)}{d\rho} &= 3\vartheta s^{3\vartheta-1}\frac{ds}{d\rho} + 2\vartheta\left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right) \\ &\cdot s^{2\vartheta-1}\frac{ds}{d\rho} + \vartheta\left[p_2p_3 - \frac{(u_2^* - p_1)(p_2+p_3)}{1-\kappa_d}\right] \\ &\cdot s^{\vartheta-1}\frac{ds}{d\rho} = \left\{3\vartheta s^{3\vartheta-1}\right. \\ &+ 2\vartheta\left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right)s^{2\vartheta-1} \\ &\left.+ \vartheta\left[p_2p_3 - \frac{(u_2^* - p_1)(p_2+p_3)}{1-\kappa_d}\right]s^{\vartheta-1}\right\}\frac{ds}{d\rho}, \\ \frac{\mathcal{Q}_2(s)}{d\rho} &= -\frac{\kappa_p}{1-\kappa_d}\left[2\vartheta s^{2\vartheta-1} + \vartheta(p_2+p_3)s^{\vartheta-1}\right]\frac{ds}{d\rho},\end{aligned} \tag{32}$$

and then one has

$$\left[\frac{ds}{d\rho}\right]^{-1} = \frac{\mathcal{F}_1(\rho)}{\mathcal{F}_2(\rho)} - \frac{\rho}{s}, \tag{33}$$

where

$$\begin{aligned}\mathcal{F}_1(s) &= 3\vartheta s^{3\vartheta-1} + 2\vartheta\left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right)s^{2\vartheta-1} \\ &+ \vartheta\left[p_2p_3 - \frac{(u_2^* - p_1)(p_2+p_3)}{1-\kappa_d}\right]s^{\vartheta-1} \\ &- \frac{\kappa_p}{1-\kappa_d}\left[2\vartheta s^{2\vartheta-1} + \vartheta(p_2+p_3)s^{\vartheta-1}\right]e^{-s\rho}, \\ \mathcal{F}_2(s) &= \mathcal{Q}_2(s)se^{-s\rho}.\end{aligned} \tag{34}$$

Then

$$\operatorname{Re}\left\{\left[\frac{ds}{d\rho}\right]^{-1}\right\} = \operatorname{Re}\left\{\frac{\mathcal{F}_1(\rho)}{\mathcal{F}_2(\rho)}\right\}. \tag{35}$$

Thus

$$\begin{aligned}\operatorname{Re}\left\{\frac{ds}{d\rho}\right\}\Bigg|_{\rho=\rho_0,\psi=\psi_0} &= \operatorname{Re}\left\{\frac{F_1(\rho)}{F_2(\rho)}\right\}\Bigg|_{\rho=\rho_0,\psi=\psi_0} \\ &= \frac{\mathcal{U}_1\mathcal{V}_1 + \mathcal{U}_2\mathcal{V}_2}{\mathcal{V}_1^2 + \mathcal{V}_2^2}.\end{aligned} \tag{36}$$

It follows from ($\mathcal{A}1$) that

$$\operatorname{Re}\left\{\left[\frac{ds}{d\rho}\right]^{-1}\right\}\Bigg|_{\rho=\rho_0,\psi=\psi_0} > 0. \tag{37}$$

Th proof of Lemma 6 is completed. ☐

Let

$$\begin{aligned}\epsilon_1 &= \left(p_2 + p_3 - \frac{u_2^* - p_1}{1-\kappa_d}\right) - \frac{\kappa_p}{1-\kappa_d}, \\ \epsilon_2 &= p_2p_3 - \frac{(u_2^* - p_1 - \kappa_p)(p_2+p_3)}{1-\kappa_d}, \\ \epsilon_3 &= -\frac{\kappa_p p_2 p_3 + (u_2^* - p_1)(p_2+p_3)}{1-\kappa_d}.\end{aligned} \tag{38}$$

Next we give an assumption as follows:

($\mathcal{A}2$) $\epsilon_1 > 0$, $\epsilon_1\epsilon_2 > 2\epsilon_3$, $\epsilon_3 > 0$.

Lemma 7. *If $\rho = 0$ and ($\mathcal{A}2$) holds true, then system (9) is locally asymptotically stable.*

Proof. If $\rho = 0$, then (13) takes the form

$$\lambda^3 + \epsilon_1\lambda^2 + \epsilon_2\lambda + \epsilon_3 = 0. \tag{39}$$

By ($\mathcal{A}2$), all roots λ_i of (39) satisfy $|\arg(\lambda_i)| > \vartheta\pi/2$ ($i = 1, 2, 3$). By Lemma 3, we know that the conclusion of Lemma 4.3 holds. The proof of Lemma 3.4 is completed. ☐

According to the analysis above, we have the following conclusion.

Theorem 8. *In addition to condition (b) of Lemma 6. If ($\mathcal{A}1$) and ($\mathcal{A}2$) are fulfilled, then the equilibrium point (u_1^*, u_2^*, u_3^*) of system (9) is locally asymptotically stable when $\rho \in [0, \rho_0)$ and a Hopf bifurcation will appear around the equilibrium point E_1 when $\rho = \rho_0$.*

Remark 9. In [4–7, 13–20], the authors studied the various dynamics of integer-order financial models. They did not involve the fractional-order forms. In [39], Bhalekar and Gejji considered the chaos of fractional-order financial model by predictor-corrector method. In this article, we control the chaos of fractional-order financial model by applying PD^ϑ control strategy. All the derived results and analysis ways of [4–7, 13–20, 39] can not be transferred to (2) to control the chaotic behavior. Based on these viewpoints, the fruits of this paper are entirely innovative and supplement the previous publications.

4. An Example

We give the following controlled financial system:

$$\begin{aligned}\frac{d^\vartheta u_1}{dt^\vartheta} &= u_3 + (u_2 - 3)u_1 + \kappa_p(u_1(t-\rho) - u_1^*) \\ &+ \kappa_d\frac{d^\vartheta}{dt^\vartheta}(u_1(t) - u_1^*), \\ \frac{d^\vartheta u_2}{dt^\vartheta} &= 1 - 0.1u_2 - u_1^2, \\ \frac{d^\vartheta u_3}{dt^\vartheta} &= -u_1 - u_3.\end{aligned} \tag{40}$$

Table 1: The relation of parameters ϑ and ρ_0 of model (40).

ϑ	ρ_0
0.195	0.1968
0.267	0.2569
0.384	0.3450
0.558	0.4593
0.672	0.5262
0.736	0.5615
0.829	0.6104

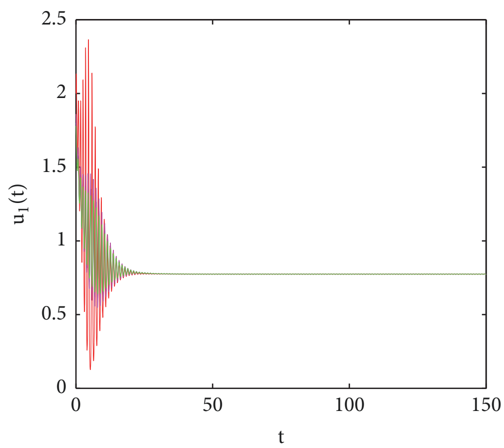

Figure 11: The relation of t and u_1 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

Clearly, system (40) has an equilibrium point (0.6831, 2.6667, −0.4554). Let $\kappa_p = 0.07$, $\kappa_p = 0.32$, $\vartheta = 0.672$. Then the critical frequency $\zeta_0 = 0.8653$ and the bifurcation point $\rho_0 = 0.5262$. We can check that the assumptions in Theorem 8 hold true. Figures 11–20 indicate that when $\rho \in [0, 0.5262)$, the equilibrium point (0.6831, 2.6667, −0.4554) of model (40) is locally asymptotically stable. From the financial point of view, it means that as time goes on, the interest rate will tend to the constant 0.6831, investment demand will tend to the constant 2.6667, and price index will tend to the constant -0.4554. Figures 21–30 indicate that $\rho \in [0.5262, +\infty)$, system (40) becomes unstable, and a Hopf bifurcation emerges. From the financial point of view, it means that as time goes on, the interest rate, investment demand, and price index will keep a periodic cycle. In addition, we show the relation of parameters ϑ and ρ_0 of (40) with Table 1.

5. Conclusions

In this paper, we propose a new fractional-order financial system. To control the chaotic behavior of the fractional-order financial system, we successfully design a PD^{ϑ} controller to achieve our goal. By adjusting the proportional and derivative parameters, we can change the stability and Hopf bifurcation character of the considered fractional-order financial system. By regarding the time delay as bifurcation parameter, we have established a new sufficient condition to ensure the stability

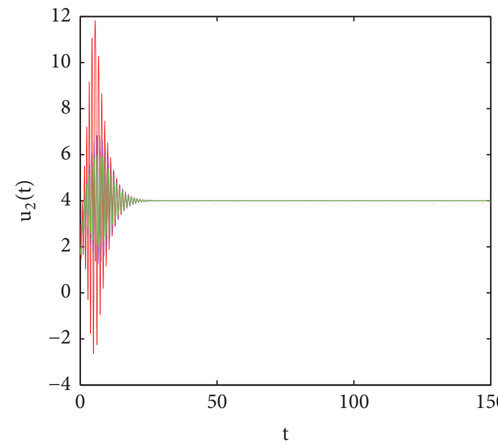

Figure 12: The relation of t and u_2 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

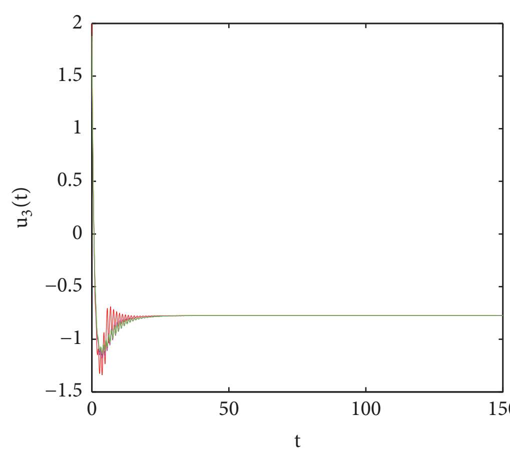

Figure 13: The relation of t and u_3 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

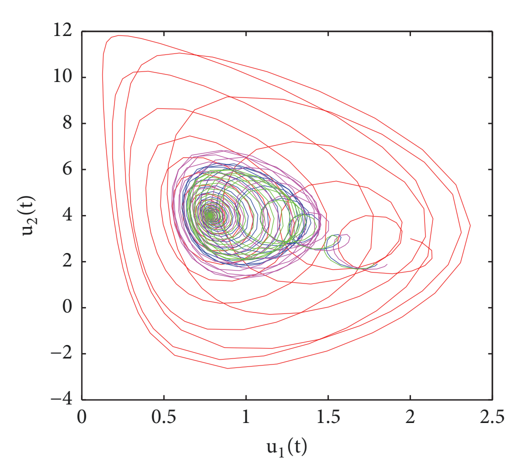

Figure 14: The relation of u_1 and u_2 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

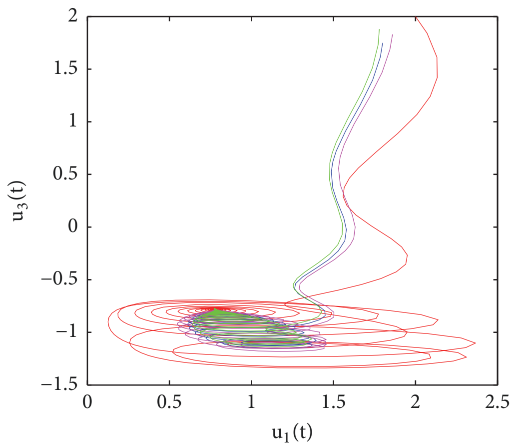

Figure 15: The relation of u_1 and u_3 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

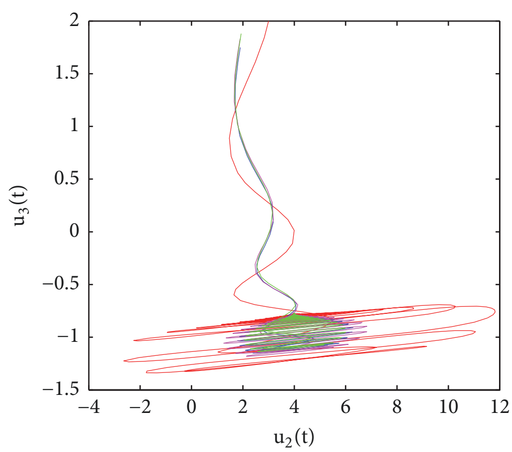

Figure 16: The relation of u_2 and u_3 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

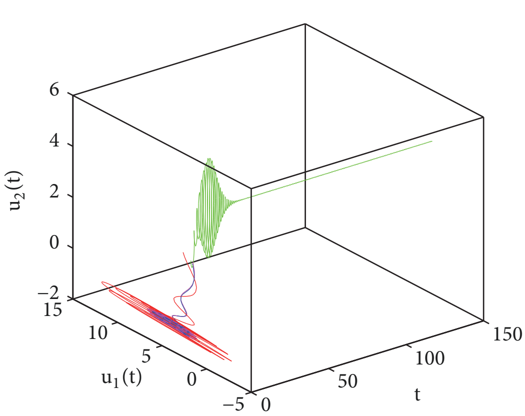

Figure 17: The relation of u_1, u_2, and u_3 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

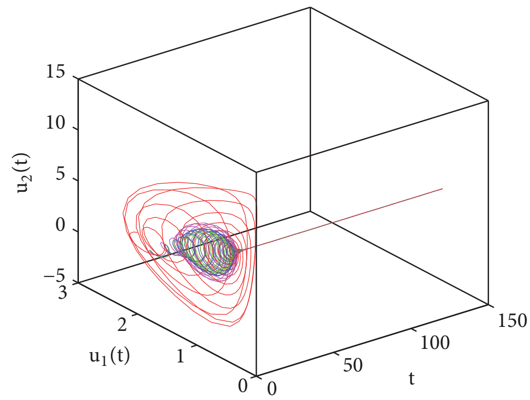

Figure 18: The relation of t, u_1, and $u2$ in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

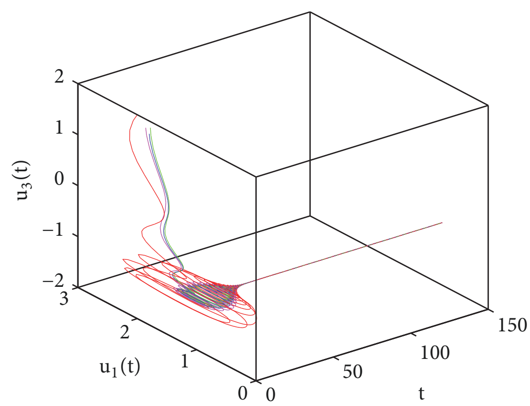

Figure 19: The relation of t, u_2, and u_3 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

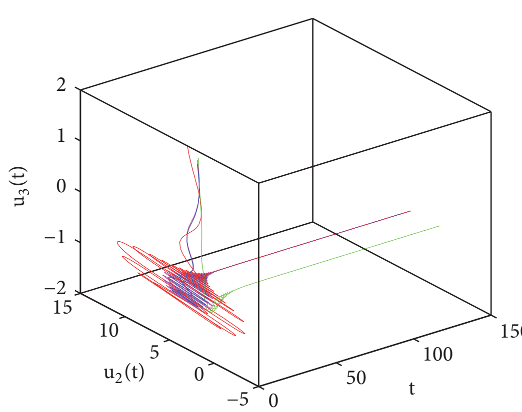

Figure 20: The relation of t, u_2, and u_3 in model (40) when $\rho = 0.4237 < \rho_0 = 0.5262$.

and the existence of Hopf bifurcation of the fractional-order financial model. Also, the effect of the fractional order and delay on the stability and Hopf bifurcation is revealed. The research idea and the obtained theoretical

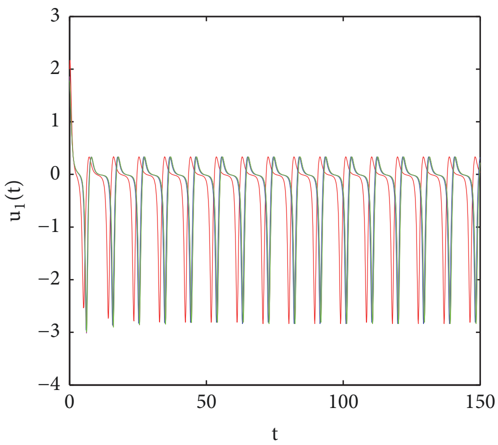

FIGURE 21: The relation of t and u_1 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

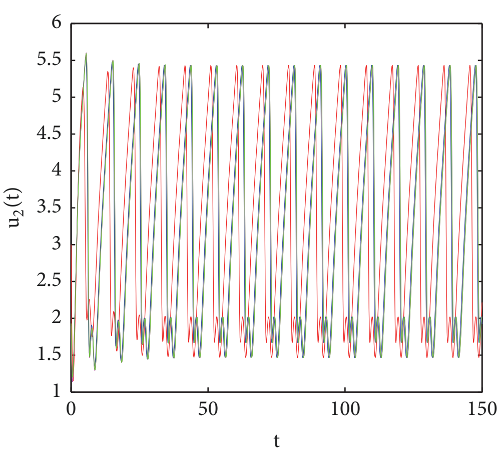

FIGURE 22: The relation of t and u_2 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

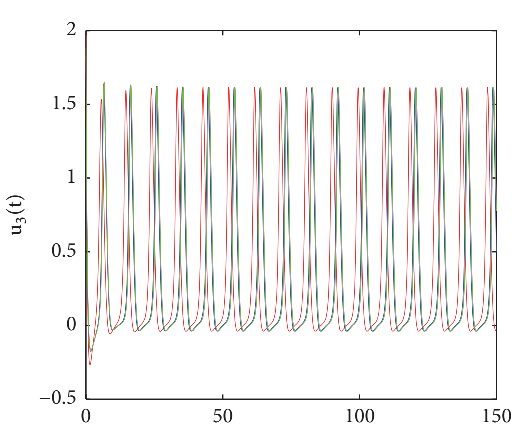

FIGURE 23: The relation of t and u_3 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

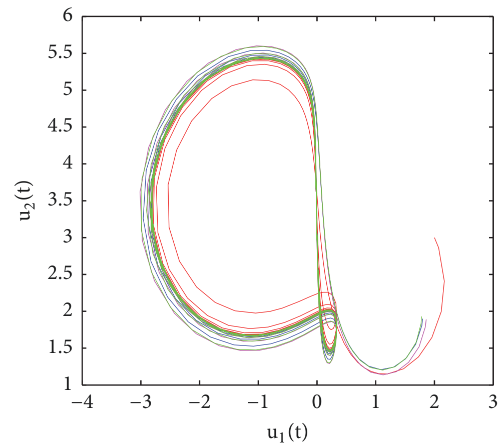

FIGURE 24: The relation of u_1 and u_2 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

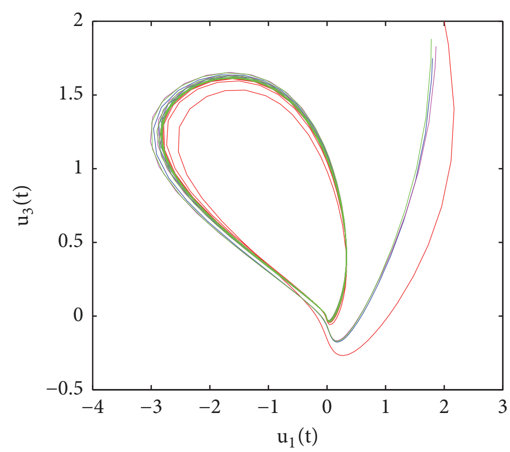

FIGURE 25: The relation of u_1 and u_3 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

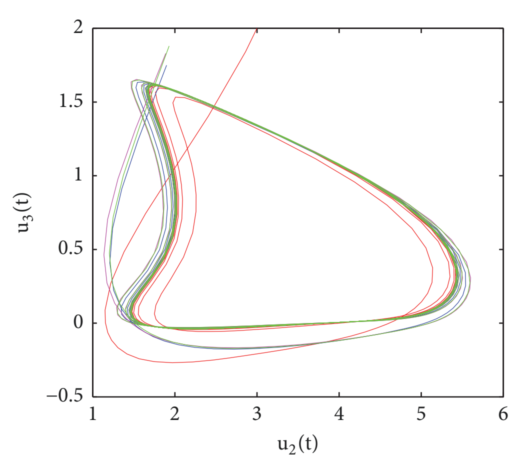

FIGURE 26: The relation of u_2 and u_3 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

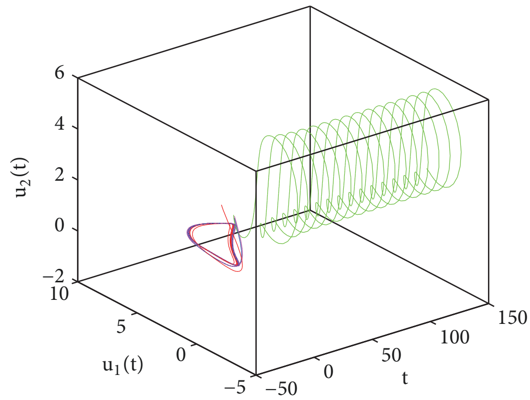

FIGURE 27: The relation of u_1, u_2, and u_3 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

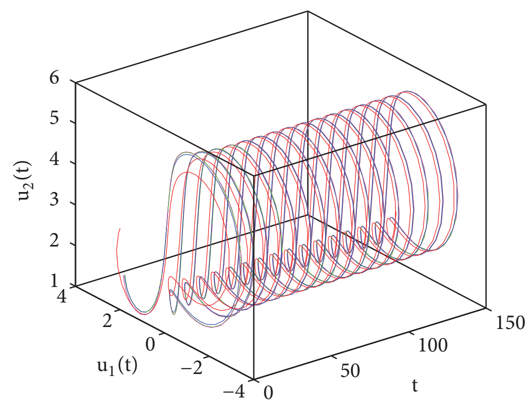

FIGURE 28: The relation of t, u_1, and u_2 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

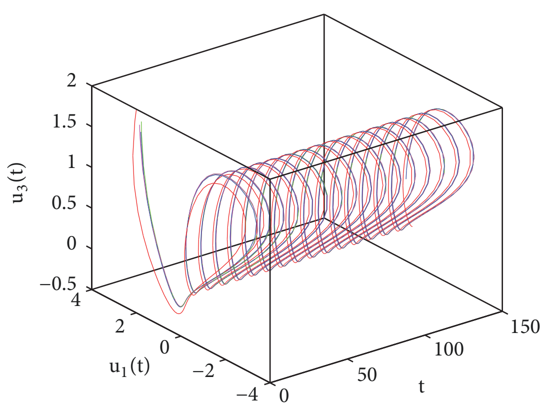

FIGURE 29: The relation of t, u_2, and u_3 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

results of this article enrich and develop the bifurcation and control theory of fractional-order differential equations. The obtained results can provide useful guidance to people in financial community. We can properly adjust the parameter of the PD^{ϑ} controller to apply the suggested fractional-order financial models to deal with financially chaotic problems. In

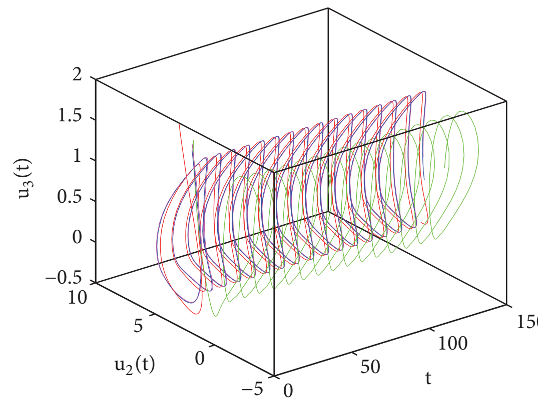

FIGURE 30: The relation of t, u_2, and u_3 in model (40) when $\rho = 0.8345 > \rho_0 = 0.5262$.

addition, we point out that although the PD^{ϑ} controller can effectively control the chaos of the fractional-order financial model, it involves multiple parameters: proportional control parameter κ_p, the derivative control parameter κ_d, time delay ρ, and fractional order ϑ. In the future, we will seek some more simple controllers with less parameters to suppress the chaotic behavior.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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