Research Article
Solutions to All-Colors Problem on Graph Cellular Automata

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The All-Ones Problem comes from the theory of $\sigma^+$-automata, which is related to graph dynamical systems as well as the Odd Set Problem in linear decoding. In this paper, we further study and compute the solutions to the “All-Colors Problem,” a generalization of “All-Ones Problem,” on some interesting classes of graphs which can be divided into two subproblems: Strong-All-Colors Problem and Weak-All-Colors Problem, respectively. We also introduce a new kind of All-Colors Problem, $k$-Random Weak-All-Colors Problem, which is relevant to both combinatorial number theory and cellular automata theory.

1. Introduction

A graph dynamical system (GDS) is a dynamical system constructed over a graph whose vertices can have different states, such that all these states together at a given time constitute a state of the system which can evolve according to an updating scheme \cite{1, 2}. The states of the vertices are commonly modeled by the Boolean values 0 and 1, while the updating scheme consists of as many local functions as vertices and a series of rules that indicate the order in which the local functions act. GDS can be divided into two categories: parallel (PDS) \cite{3, 4} and sequential (SDS) \cite{5, 6} when all the local functions act synchronously or follow an order to act, respectively. In the specific literature, other related topics appeared previously, such as Boolean networks (BN) \cite{7} and cellular automata (CA) \cite{8}, which are, in fact, particular cases of GDS. In this paper we will consider “All-Colors Problem,” which is concerned with graph cellular automata as one kind of dynamical systems on networks.

A graph is a pair $G = (V, E)$ where $E \subseteq V \times V$. These structures allow for self-loops. In the theory of automata, we will insist that $G$ is local finite; i.e., every vertex in $G$ is adjacent to only finite many vertices. In this paper, we only consider $G$ as a finite simple graph. It is convenient to identify $E$ with the adjacency matrix of $G$ constructed as a matrix in $\prod_{V \times V} F_2$. Here $F_2 = \{0, 1\}$ is the two-element field and there is a 1 in the $u^{th}$ row and $v^{th}$ column of $E$ if and only if there is a directed edge from vertex $u$ to vertex $v$ in $G$.

A vertex $u$ is a predecessor of $v$ if there exists an edge $(u, v)$ in $G$. The collection of all predecessors of $v$ will be denoted by

$$\Gamma_G(v) = \{ u \in V \mid (u, v) \in E \}. \tag{1}$$

Note that $\Gamma_G(v)$ may or may not include $v$ depending on whether $v$ has a self-loop in $G$ or not. We will refer to $\Gamma_G(v)$ as the neighborhood of $v$ in $G$. A configuration of $G$ is a function

$$X : V \rightarrow F_2. \tag{2}$$

The collection of all configurations of $G$ will be denoted by $C_G$. Define the transition rule $\sigma_G : C_G \rightarrow C_G$ by

$$\sigma(X)(v) = \sum_{u \in \Gamma(v)} X(u) \mod 2. \tag{3}$$

$A = (G, \sigma_G)$ is called the $\sigma$-automaton on $G$.

Configurations are conveniently identified with subsets of $V$; i.e., $X$ is identified with $\{ v \in V \mid X(v) = 1 \}$. Observe that algebraically $C_G$ is a vector space over $F_2$, $C_G = \prod_v F_2$; addition here amounts to take symmetric differences. We will call this space the configuration space. Furthermore, $\sigma$ is a linear map from the configuration space to itself (such rules are called additive in \cite{9}). If one thinks of configuration $X$ as
a column vector over $F_2$, it is obvious from the definition that $\sigma(X) = E \cdot X$ where $E$ is the adjacency matrix of $G$.

A $\sigma$-automaton $A = (G, \sigma_G)$ is symmetric if and only if the adjacency matrix $E$ of $G$ is symmetric; thus symmetric $\sigma$-automata arise from undirected graphs. In the following sections, we will consider only symmetric $\sigma$-automata on undirected graphs.

Let $G$ be an undirected graph without self-loops and $D$ a subset of $V$. Define $G(D)$ to be the graph obtained from $G$ by adding self-loops at all vertices in $D$. $\sigma$-automata of the form $(G, \sigma_{G(V)})$ or $(G, \sigma_{G(0)})$ are called Lindenmayer automata on $G$.

To lighten notation we will usually omit the subscript $G$ and write $\Gamma(v)$, $\sigma$ and $(G, \sigma)$, and so forth. Also we will write $\sigma^+$ for $\sigma_{G(V)}$ and $\sigma^-$ for $\sigma_{G(0)}$. We will write $0$ for the empty set and $I$ for $V$ as members of $C_G$, so $0(v) = 0$ and $I(v) = 1$ for all $v \in V$.

The term All-Ones Problem was introduced by Sutner; see [10]. It has applications in linear cellular automata; see [11] and the references therein. The problem is cited as follows: suppose each of the vertices of an undirected graph with $n$ vertices is equipped with an indicator light and a button. If a vertex is pressed one time, then the color values of the vertex and its neighbors are added by 1 under the meaning of modular $m$. If the initial status is that the color value of every vertex is 0, then how to press some vertices (maybe many times) to make the color value of every vertex equal to 1 under the meaning of modular $m$?

The All-Ones Problem is $\sigma^+$-rule on graphs, which means that a button lights not only its neighbors but also its own light $L_i$. If a button lights only its neighbors but not its own light, this rule on graphs is called $\sigma^-$-rule. There were many publications on the All-Ones Problem; see Sutner [14, 15], Barua et al. [16], and the references therein. Using linear algebra, Sutner [17] proved that it is always possible to light every lamp in any graphs by $\sigma^+$-rule. Losseres [18] gave another beautiful proof also by using linear algebra. A graph-theoretic proof was given by Eriksson et al. [19]. More results and related references can be referred to [20–25].

In graph-theoretic terminology, a solution to the All-Ones Problem with $\sigma^+$-rule can be stated as follows: given a graph $G = (V, E)$, where $V$ and $E$ denote the vertex-set and the edge-set of $G$, respectively, a subset $X$ of $V$ is a solution if and only if for every vertex $v$ of $G$ the number of vertices in $X$ adjacent to or equal to $v$ is odd. Such a subset $X$ is called an odd parity cover in [17].

The All-Ones Problem can be formulated as follows: given a graph $G = (V, E)$, does a subset $X$ of $V$ exist such that, for any vertex $v \in V - X$, the number of vertices in $X$ adjacent to $v$ is odd, while for any vertex $v \in X$, the number of vertices in $X$ adjacent to $v$ is even?

Sutner [10] proposed the question whether there is a graph-theoretic method to find a solution for the All-Ones Problem for trees. Galvin [26] solved this question.

The “All-Colors Problem” can be considered to find the preimages of configuration 1 under parallel map defined by local rule $\sigma$. In this paper, we further study the “All-Colors Problem,” a natural generalization of “All-Ones Problem,” which can be divided into two subproblems: Strong-All-Colors Problem and Weak-All-Colors Problem, respectively.

This paper is organized as follows. In Section 2, we introduce the preliminary and definitions of the “All-Colors Problem” under $\sigma^+$-rule. The “All-Colors Problem” can be divided into two subproblems: Strong-All-Colors Problem and Weak-All-Colors Problem which are studied in Sections 3 and 4, respectively. In the end of Section 4, we also introduce a new kind of All-Color Problem, $k$-Random Weak-All-Colors Problem, which is relevant to both combinatorial number theory and cellular automata theory.

2. Preliminary of All-Colors Problem

Here we introduce a natural generalization for the All-Ones Problem—All-Colors Problem. When we discuss the All-Colors Problem, we need a positive integer $m > 2$ and a graph $G$ each of whose vertices has a color value between 0 and $m - 1$. There are two kinds of All-Colors Problem: Strong-All-Colors Problem and Weak-All-Colors Problem. We give their accurate definitions as follows.

Definition 1 (strong-all-colors problem under $\sigma^+$-rule on $Z_m$).

If a vertex is pressed one time, then the color values of the vertex and its neighbors are added by 1 under the meaning of modular $m$. If the initial status is that the color value of every vertex is 0, then how to press some vertices (maybe many times) to make the color value of every vertex equal to 1 under the meaning of modular $m$?

Definition 2 (weak-all-colors problem under $\sigma^+$-rule on $Z_m$).

If a vertex is pressed one time, then the color values of the vertex and its neighbors are added by 1 under the meaning of modular $m$. If the initial status is that the color value of every vertex is 0, then how to press some vertices (maybe many times) to make the color value of every vertex not equal to 0 under the meaning of modular $m$?

A solution to the Strong-All-Colors Problem on a graph $G$ under $\sigma^+$-rule is a configuration $f : V(G) \rightarrow Z_m$ such that, for any $v \in V(G)$,

$$\sum_{w \in N(v) \cup \{v\}} f(w) = 1 \mod m, \quad (4)$$
where \( N(v) \) is denoted as the set of vertices of \( G \) which are adjacent to \( v \). Correspondingly, a solution to the Weak-All-Colors Problem on a graph \( G \) under \( \sigma^+\)-rule is a configuration \( f : V(G) \rightarrow Z_m \) such that, for any \( v \in V(G) \),

\[
\sum_{w \in N(v) \cup \{v\}} f(w) \not\equiv 0 \mod m. \tag{5}
\]

It is worth noting that we only need to study the Strong-All-Colors Problem and the Weak-All-Colors Problem in the case that \( m \) is a power of a prime. Indeed, let \( m = \prod_{i=1}^{r} p_i^{e_i} \) be the prime factorization of \( m \). Suppose that there exists a solution \( c_i : V(G) \rightarrow Z_{p_i^{e_i}} \) to the Strong-All-Colors Problem under \( \sigma^+ \) on \( Z_{p_i^{e_i}} \) for any \( i \). That is, for any \( v \in V(G) \),

\[
\sum_{w \in N(v) \cup \{v\}} c_i(w) = 1 \mod p_i^{e_i}. \tag{6}
\]

By the Chinese Remainder Theorem, there exists \( c : V(G) \rightarrow Z_m \) such that

\[
c(v) = c_i(v) = 1 \mod p_i^{e_i} \tag{7}
\]

for any \( i \in \{1, \ldots, r\} \) and \( v \in V(G) \). It follows that

\[
\sum_{w \in N(v) \cup \{v\}} c(w) = 1 \mod p_i^{e_i}, \tag{8}
\]

for any \( v \). By the Chinese Remainder Theorem again, one has

\[
\sum_{w \in N(v) \cup \{v\}} c(w) = 1 \mod m. \tag{9}
\]

That is, \( c \) is a solution to the Strong-All-Colors Problem on \( Z_m \). The discussion of the Weak-All-Colors Problem is similar.

Note that we can also define the corresponding Strong-All-Colors Problem under \( \sigma^- \)-rule on \( Z_m \) and Weak-All-Colors Problem under \( \sigma^- \)-rule on \( Z_m \), however, they are more difficult to study.

In the following sections, we will discuss the All-Colors Problem under \( \sigma^- \)-rule in detail. We need to use two definitions equivalent to Definitions 1 and 2 for convenience. Suppose \( G \) is a simple undirected graph; each vertex of \( G \) has a color value on \( Z_m \) which can change with time. These changes abide by the following rules (\( \sigma^- \)-rule): If, for \( v \in V(G), c_0(v) \) is the initial color value of \( v \) at time \( t = 0 \), then, at time \( t = 1 \), the color value of \( v \) has changed to \( c_1(v) = \sum_{u \in N(v) \cup \{v\}} c_0(u) \), where \( N(v) \) is the set of vertices which are adjacent to \( v \).

The Strong-All-Colors Problem under \( \sigma^- \)-rule can be redefined as follows: to find the initial color values of all the vertices of \( G \) at time \( t = 0 \) such that any vertex of \( G \) has a color value equal to \( 1(\mod m) \) at time \( t = 1 \) under \( \sigma^- \)-rule.

The Weak-All-Colors Problem can be redefined as follows: to find the initial color values of all the vertices of \( G \) at time \( t = 0 \) such that no vertex of \( G \) has a color value equal to \( 0(\mod m) \) at time \( t = 1 \) under \( \sigma^- \)-rule.

For the sake of simplicity, the Strong-All-Colors Problem is denoted as SACP from now on. Correspondingly, the Weak-All-Colors Problem is denoted as WACP.

### 3. Strong-All-Colors Problem under \( \sigma^+ \)-Rule on \( Z_m \)

First of all, for a general graph, the SACP with \( \sigma^+ \)-rule on \( Z_m \) may have no solution. For example, there is no solution to the SACP under \( \sigma^+ \)-rule on \( Z_m \) for a circle \( C_n \) when \( 3 \nmid n \) and \( 3 \mid m \). Particularly, by observing the following tree \( T_0 \) with 6 vertices, in Figure 1, we will discover surprisingly that there is no solution to the SACP under \( \sigma^+ \)-rule on \( Z_m \) for any \( m \geq 3 \).

Then a question arises naturally: How to determine whether a graph has a solution to the SACP with \( \sigma^+ \)-rule on \( Z_m \)?

It is easy to find an algebraic method to solve this problem. Suppose \( G \) is a graph with \( n \) vertices. The following theorem can be obtained easily and we omit the proof.

**Theorem 3.** There exists a solution to the SACP with \( \sigma^+ \)-rule on \( Z_m(m \geq 2) \) for a graph \( G \) if and only if the following system of linear equations has a solution on \( Z_m \):

\[
(A + I) X = (1)(\mod m), \quad (10)
\]

where \( A \) is the adjacent matrix of \( G \), \( I \) is the identity matrix, \( X \) is an \( n \times 1 \) vector of variables, and \((1)(\mod m)\) is an \( n \times 1 \) vector each element of which is \( 1(\mod m) \).

**Example 4.** Consider the tree \( T_0 \) at the beginning of this section. Its adjacent matrix is

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}. \quad (11)
\]

It is easy to verify that

\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= \begin{pmatrix}
1 \mod m \\
1 \mod m \\
\vdots \\
1 \mod m
\end{pmatrix}, \quad (12)
\]
is equivalent to
\[
\begin{align*}
x_1 + x_3 &= 1 \mod m \quad (1) \\
x_2 + x_3 &= 1 \mod m \quad (2) \\
x_1 + x_2 + x_3 + x_4 &= 1 \mod m \quad (3) \\
x_3 + x_4 + x_5 + x_6 &= 1 \mod m \quad (4) \\
x_4 + x_5 &= 1 \mod m \quad (5) \\
x_4 + x_6 &= 1 \mod m \quad (6)
\end{align*}
\]

Let us operate the 6 equations as follows.
\[
(1) + (2) + (5) + (6) - (3) - (4),
\]
so we get
\[
0 = 2 \mod m \quad (15)
\]
It is obvious that equation (15) holds if and only if \(m = 2\). So \(T_0\) has no solution to the SACP under \(\sigma^+\)-rule on \(Z_m\) for \(m \geq 3\), which we have just declared.

The system of linear equations over \(Z_m\) could be solved. Let \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times 1}\), and \(X = (x_1, \ldots, x_n)^T\). We need to solve the equation \(AX = B\). If \(R\), as \(Z_m\) above, is a principle idea ring, there exist invertible matrices \(U, V \in M_n(R)\) such that \(A = UCV\), where
\[
C = \\
\begin{pmatrix}
    d_1 & 0 & \cdots & 0 \\
    0 & d_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & d_n
\end{pmatrix},
\]
where \(d_i \mid d_{i+1}\) and \(C\) is the Smith normal form of \(A\). We can compute \(U\) and \(V\) if \(R = \mathbb{Z}_m\). Then \(AX = B\) is equivalent to \(CVX = U^{-1}B\). We set \(Y = VX\) and we can solve \(CY = U^{-1}B\) since \(C\) is a diagonal matrix. Hence, we obtain \(X = V^{-1}Y\).

Although this algebraic method is quite succinct, it is interesting to study the SACP under \(\sigma^+\) rule on \(Z_m\) over some classes of graphs.

In the following, we consider the SACP under \(\sigma^+\)-rule on \(Z_m\) for trees. Note that we do all the operations “+” and “−” on the ring \(Z_m\). Consider the special case when the tree is a “caterpillar.” A tree \(T\) is called a caterpillar if and only if the remainder of \(T\) is a path after removing all the leaves of \(T\). Figure 2 shows such a caterpillar.

We can find the solution to the SACP under \(\sigma^+\)-rule on \(Z_m\) for a caterpillar \(T\) according to the following process.

Suppose \(T\) is a caterpillar with the following shape of Figure 2. First we assume that the initial color value of vertex \(v_1\) is \(x\), which will be determined in the end. It is easy to see that the initial color values of the leaves of vertex \(v_1\) must be \(1-x\) in order to make the color values of vertices \(u_{11}, u_{12}, \ldots, u_{1r_1}\) changing to 1 by the meaning of modular \(m\) under \(\sigma^+\)-rule. Then the initial color value of \(v_2\) would be \(-(r_1 - 1)(1-x)\) to make the color value of \(v_1\) equal to 1 under \(\sigma^+\)-rule. Repeat this step until we have determined the initial color value of \(v_n\), denoted by \(f_n(x)\), which is a linear function of \(x\). From this procedure we can see that there is only one equation to be satisfied; i.e.,
\[
f_{n-1}(x) + f_n(x) + r_n \cdot (1 - f_n(x)) = 1 \mod m, \quad (17)
\]
where \(f_{n-1}(x)\) is the initial color value of vertex \(v_{n-1}\).

So we have the following.

**Theorem 5.** A caterpillar \(T\) has a solution to the SACP under \(\sigma^+\)-rule on \(Z_m\) if and only if the equation (17) has a solution on \(Z_m\).

We give two examples with solutions to the SACP under \(\sigma^+\)-rule on \(Z_m\). Figure 3 shows such a caterpillar. All the negative integers that appeared in the following figures are under the meaning of modular \(m\).

As an application of Theorem 5, we have the following.

**Corollary 6.** If \(T\) is a path with \(n\) vertices, then \(T\) has a solution to the SACP under \(\sigma^+\)-rule on \(Z_m\) for any \(m \geq 2\).

**Proof.** Let \(x_1, x_2, \ldots, x_n\) a sequence of integers on \(Z_m\), represent the initial color values of vertices of \(T\). It is easy to check that
\[
x_1, x_2, \ldots, x_n = \begin{cases} 
100100 \cdots 10010, & \text{if } n \equiv 2 \pmod{3} \\
100100 \cdots 1001, & \text{if } n \equiv 1 \pmod{3} \\
010010 \cdots 010, & \text{if } n \equiv 0 \pmod{3}
\end{cases}
\]

is a solution to the SACP under \(\sigma^+\)-rule on \(Z_m\) for any \(m \geq 2\). \(\square\)
Corollary 7. If $T$ is a tree with the shape as shown in Figure 4, then

(1) when $k \equiv 2 \pmod{3}$, $T$ has a solution as in Figure 5
(2) when $k \equiv 1 \pmod{3}$, $T$ has a solution if and only if $(pq - 1)x \equiv q - 1 \pmod{m}$ has a solution on $Z_m$, which is equivalent to gcd$(pq - 1, m) \mid (q - 1)$. Here $x$ is the initial color value of vertices $u_1, \ldots, u_p$
(3) when $k \equiv 0 \pmod{3}$, $T$ has a solution if and only if $(pq - p - q)x \equiv -q \pmod{m}$ has a solution on $Z_m$, which is equivalent to gcd$(pq - p - q, m) \mid q$. The meaning of $x$ is the same to above

Now we show a reduction technique. First we introduce a new concept.

Definition 8. Suppose $G$ is a graph. If there are 3 distinct vertices $v_1, v_2, v_3$ of $G$ satisfying

(1) $v_3$ is a leaf
(2) $v_2$ is exactly adjacent to $v_1$ and $v_3$

then $\{(v_1, v_2, v_3), (v_1, v_2, v_3), (v_1, v_2, v_3)\}$ is called a 3-connected path and denoted by $(v_1, v_2, v_3)$ simply.

Example 9. For example, Figure 6 shows a tree which has a 3-connected path marked by an ellipse.

Theorem 10 (deleting a 3-connected path). Suppose that $G$ is a graph with $n$ vertices $v_1, v_2, \ldots, v_n$. If $G$ contains a 3-connected path $(v_{i_1}, v_{i_2}, v_{i_3})$, then $G$ has a solution to the SACP under $\sigma^+$-rule on $Z_m$ if and only if $G \setminus \{v_{i_1}, v_{i_2}, v_{i_3}\}$ has a solution to the SACP under $\sigma^+$-rule on $Z_m$, where $G \setminus \{v_{i_1}, v_{i_2}, v_{i_3}\}$ is a tree or a forest.

Proof. Suppose $T$ has a solution to the SACP under $\sigma^+$-rule on $Z_m$. Assume the initial color value of vertex $v_j$ is $x_j (1 \leq i \leq n)$. We have $x_{i_1} = 0 \pmod{m}$ because it requires $x_2 + x_3 = 1 \pmod{m}$ and $x_1 + x_2 + x_3 = 1 \pmod{m}$ to ensure the color value of $v_2$ and $v_3$ to be changed to 1 under $\sigma^+$-rule. Then we can say that $(x_4, x_5, \ldots, x_n)$ is a solution to the SACP under $\sigma^+$-rule on $Z_m$ for $G \setminus \{v_1, v_2, v_3\}$.

For the converse, suppose $(x_4, x_5, \ldots, x_n)$ is a solution to the SACP under $\sigma^+$-rule on $Z_m$ for $G \setminus \{v_1, v_2, v_3\}$. Let $N(v_j)$ denote the set of vertices $G \setminus \{v_1, v_2, v_3\}$ that are adjacent to $v_j$ in $T$. It is easy to check $(x_4', x_5', \ldots, x_n')$ is a solution to the SACP under $\sigma^+$-rule on $Z_m$ for $T$, where

$$x_j' = \begin{cases} x_j & 1 \leq j \leq n \text{ and } j \neq 1, 2, 3 \\ 1 - \sum_{u \in N(v_j)} x_u & j = 2 \\ \sum_{u \in N(v_j)} x_u & j = 3 \end{cases}$$

This completes the Proof.

It is worth noting that we can use this reduction technique repeatedly. For example, the tree in Example 9 can be reduced continuously. The reduction process and its inverse, which can help us to get the solution to the SACP, are shown in Figure 7.

In fact, there are some other complex structures which can be reduced. But they are commonly relevant to $Z_m$. For example, when $m = 3$, the structures shown in Figure 8 can be removed from a tree, just like the 3-connected path.

Next, we study another special kind of trees—radioactive trees. A tree $T$ is called a radioactive tree if and only if there is only at most one vertex of $T$ which has a degree bigger than 2. In fact, a radioactive tree can be viewed as several paths adhering to a common vertex.

Theorem 11. If $T$ is a radioactive tree, then $T$ has a solution to the SACP under $\sigma^+$-rule on $Z_m$ for any $m \geq 2$.

Proof. Assume that $T$ has the shape shown in Figure 9.
By Theorem 10 we can delete 3-connected path continuously. Without loss of generality we may assume that at last $T$ has changed to $T'$ that has $k'$ paths from the root. It is enough to prove that $T'$ has a solution to the SACP. Suppose $r_i = 1$ for $1 \leq i \leq s$ and $r_i = 2$ for $s + 1 \leq i \leq k'$ in $T'$. If $s = k'$, then $x_v = 1$, $x_{u_{s,1}} = 0$ for $1 \leq i \leq s = k'$ is a solution. If $s < k'$, then $x_v = 0$, $x_{u_{s,1}} = 1$ for $1 \leq i \leq s$, $x_{u_{s,2}} = 0$ and $x_{u_{s,2}} = 1$ for $s + 1 \leq i < k'$, $x_{u_{k',1}} = 1 - s (mod m)$ and $x_{u_{k',2}} = s (mod m)$ is a solution to the SACP. The schematic diagram for the latter case is shown in Figure 10.

Actually, we can also prove Theorem 11 with the help of our reduction technique introduced before. The detail of the proof is omitted here.

4. Weak-All-Colors Problem

In this section, we will study the WACP under $\sigma^+$-rule on $Z_m$, $m \geq 2$. It is worth noting when $m = 2$, the WACP becomes the All-Ones Problem which we have discussed before. So the WACP is another generalized form of All-Ones Problem. The WACP has corresponding linear algebraic representation. If $G$ is a graph with $n$ vertices and $A$ is the adjacent matrix of $G$, then the WACP is equivalent to the next algebraic question:

Find the values of $n$ variables $x_1, x_2, \ldots, x_n$ on $Z_m$ such that no component of the $n \times 1$ result vector of

$$ (A + I) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} $$

is equal to 0.

It is known that any graph $G$ has a solution to the All-Ones Problem under $\sigma^+$-rule. We conjecture that it also holds for the WACP.

**Conjecture 12.** For any graph $G$, there is a solution to the WACP under $\sigma^+$-rule on $Z_m (m \geq 2)$.

We should point out that the original proof to the existence of a solution to the All-Ones Problem for any graph $G$ cannot do any effects on the WACP on $Z_m$ when $m \geq 3$. So it seems that we need to find another approach to solve this conjecture.

In this section, we emphasize the trees by using the technique of increasing a leaf. By Theorem 14, we show that Conjecture 12 is valid for trees. By Theorem 15, it also holds for cycles. Using the increasing technique and Theorem 15, we prove that Conjecture 12 is valid for unicyclic graphs, which is Corollary 16. Furthermore, by Theorem 17, it is interesting that we can find a solution, which contains only 0 and 1, to the WACP under $\sigma^+$-rule on $Z_3$ for trees. Correspondingly another Conjecture 18 is proposed.

**Lemma 13** (increasing a leaf). If $G$ is a graph and $v$ is a vertex of $G$ whose degree is 1, i.e., $v$ is a leaf, then $G$ has a solution to the WACP under $\sigma^+$-rule on $Z_m (m \geq 3)$ if $G \setminus \{v\}$ does too.
Proof. Assume $G \setminus \{v\}$ has a solution $c_0$ to the WACP under $\sigma^1$-rule on $Z_m$. Suppose $v$ is only adjacent to $w$ in $G$. Then let
\[
\begin{align*}
c'_0(v) & \in Z_m \setminus \left\{ \sum_{u \in N(v) \cup \{w\}} c_0(u) - c_0(w) \right\}, \\
c'_0(u) & = c_0(u), \quad \text{if } u \neq v.
\end{align*}
\] (21)

Because $m \geq 3$, so we could find a $c'_0(v)$ in $Z_m \setminus \left\{ \sum_{u \in N(v) \cup \{w\}} c_0(u) - c_0(w) \right\}$. It is easy to check that $c'_0$ is a solution to the WACP under $\sigma^1$-rule on $Z_m$.

The proof is completed. \(\square\)

From Lemma 13, we can obtain Theorem 14 easily.

**Theorem 14.** Let $T$ be a tree and $v \in V(T)$. Then $T$ has a solution $c_0$ to the WACP under $\sigma^1$-rule on $Z_m (m \geq 3)$ satisfying $c_0(v) = 1$.

Proof. We use the technique of increasing a leaf. Firstly let $T_1 = \{v\}$ be a one-vertex tree, and let $T'_1(v) = 1$, which is a solution to the WACP under $\sigma^1$-rule on $Z_m$ satisfying $T'_1(v) = 1$. Secondly we can add another vertex $u$, which is adjacent to $v$, and the edge $(u, v)$ into $T_1$. Let $T_2$ denote the new tree. It is easy to see that there is a solution $\hat{T}_2$ to the WACP under $\sigma^1$-rule on $Z_m$ satisfying $\hat{T}_2(v) = 1$. Repeat these steps by adding an appropriate vertex each time, until at last we get a solution $\bar{T}_n$ to the WACP under $\sigma^1$-rule on $Z_m$ for $T$ satisfying $\bar{T}_n(v) = 1$, which is just $c_0$ that we want. We conclude the proof. \(\square\)

Since it is not hard to construct a solution, we can obtain the following theorem straightforwardly.

**Theorem 15.** There is a solution to the WACP under $\sigma^1$-rule on $Z_m (m \geq 3)$ for any cycle.

In fact we can extend the result to “unicyclic graphs.” Recall that a graph $G$ is called unicyclic if it contains a unique cycle. In other words, we can regard a unicyclic graph as a cycle attached with each vertex a rooted tree.

**Corollary 16.** If $G$ is a unicyclic graph, then there is a solution to the WACP under $\sigma^1$-rule on $Z_m (m \geq 3)$ for any cycle.

Proof. From Theorem 15, a cycle has a solution to the WACP under $\sigma^1$-rule on $Z_m$. Using the increasing technique repeatedly, we can prove that this statement holds. \(\square\)

Next, we prove that there is a special solution to the WACP on $Z_2$ for any tree such that all the color values of vertices are 0 and 1.

**Theorem 17.** If $T$ is a tree, then there is a solution $c_0$ to the WACP under $\sigma^1$-rule on $Z_2$ such that $c_0(v) = 0$ or 1 for any $v \in V(T)$.

Proof. Suppose $T$ is a rooted tree with root $t$. We need to introduce three new concepts.

The small WACP is to find an initial color value array $c$ on $T$ such that
\begin{align*}
(1) & \quad c(v) = 0 \text{ or } 1, \forall v \in T \\
(2) & \quad \sum_{u \in N(v) \cup \{t\}} c(v) \neq 0 (\mod 3), \forall u \neq t \\
(3) & \quad \sum_{u \in N(t) \cup \{t\}} c(v) = 0 \text{ or } 1 (\mod 3)
\end{align*}

Then the initial color value array $c$ will be called a solution to the small WACP.

Correspondingly, the positive WACP is to find an initial color value array $c$ on $T$ such that
\begin{align*}
(1) & \quad c(v) = 0 \text{ or } 1, \forall v \in T \\
(2) & \quad \sum_{u \in N(v) \cup \{t\}} c(v) \neq 0 (\mod 3), \forall u \neq t \\
(3) & \quad \sum_{u \in N(t) \cup \{t\}} c(v) = 0 \text{ or } 2 (\mod 3)
\end{align*}

Then the initial color value array $c$ will be called a solution to the positive WACP.

In fact a solution to the positive WACP is also a solution we required in this proposition. So it is sufficient for us to prove that there is a solution to the positive WACP for any rooted tree.

Suppose $T$ is a rooted tree with root $t$. If there is a solution $c$ to the small WACP such that $c(t) = 0$, we say that 0 is a permissible color value of root $t$ to the small WACP. Accordingly, if there is a solution $c$ to the small WACP such that $c(t) = 1$, then 1 is a permissible color value of root $t$ to the small WACP. We name the set of all permissible color values of root $t$ as the permission set of root $t$ to the small WACP and denote it by $S$.

Similarly, we can define the permission set of root $t$ to the positive WACP, denoted by $P$, and the permission set of root $t$ to the even WACP, denoted by $E$. Then for the rooted tree $T$ with root $t$, $(S, P, E)$ will be called the type of root $t$ and corresponding rooted tree $T$. For example, the type of a rooted tree shown in Figure 11 is $\{(1), (0, 1), (0, 1)\}$, or denoted by $(1, 01, 01)$ for simplicity.

Next we need to define “Add Into Operation.” Suppose $T_1$ and $T_2$ are two rooted trees with roots $t_1$ and $t_2$, respectively. We will get a new rooted tree with root $t_2$ by adding $T_1$ into $T_2$ with $t_1$ as a child of $t_2$. This operation is called an Add Into Operation on $T_1$ to $T_2$. If the result tree is $T_3$, then we denote the operation as $T_1 \nrightarrow T_2 \rightarrow T_3$. It is not difficult to show that each rooted tree can be derived by doing an Add Into Operation on one smaller rooted tree to another smaller rooted tree.

Furthermore, if the color value array $c_1^{(0, 5)}$ on $T_1$ is a solution to the small WACP for $T_1$ and $c_1^{(1, 5)}$ on $T_2$ is a solution to the small WACP for $T_2$, then the color value array $c^3 = c_1^{(0, 5)} \cup c_1^{(1, 5)}$ will be a solution to the small WACP for $T_1 \nrightarrow T_2', T_1 \nrightarrow T_2$, and $T_2 \rightarrow T_3$. We can use the following procedure to check this statement:
then we know the types of equations.
Similarly, we can get other equations like this. Altogether there are 15 equations as follows.

\[
\begin{align*}
\textbf{Equation 1:} & \quad c_1^{1,0,5} > c_2^{1,0,5} \rightarrow c_3^{3,0,5}, \\
\textbf{Equation 2:} & \quad c_1^{1,1,0} > c_2^{1,1,0} \rightarrow c_3^{3,1,0}, \\
\textbf{Equation 3:} & \quad c_1^{1,0,2} > c_2^{1,0,2} \rightarrow c_3^{3,0,2}, \\
\textbf{Equation 4:} & \quad c_1^{1,1,2} > c_2^{1,1,2} \rightarrow c_3^{3,1,2}. 
\end{align*}
\]

Then we want to list all possible types of rooted trees. In fact, there are only 8 distinct types of rooted trees. Each type and one of its representative rooted trees are shown in Figure 12.

In order to show that there are only 8 distinct types, we need to show that the following assertion is correct.

**Assertion.** If each of the two rooted trees \(T_1, T_2\) with roots \(t_1\) and \(t_2\), respectively, belongs to one of the 8 types, then the rooted tree \(T\) derived by doing an Add Into Operation on \(T_1\) to \(T_2\) belongs to one of the 8 types.

We have mentioned before how to calculate the type of the result rooted tree obtained by doing the Add Into Operation on a rooted tree to another if their types are known. Since all rooted trees can be derived by doing an Add Into Operation on a smaller rooted tree to another smaller one, we can observe that all the types of rooted trees are contained in the set of 8 distinct types above. The proof of the Assertion exists in Table 1. Since the original table is too wide, we divide it into two tables as in Tables 1(a) and 1(b).

We have showed that any rooted tree belongs to one of the 8 types and, in each of the 8 types, the permission set to the positive WACP is not empty. So any rooted tree would have a solution \(\varepsilon\) to the positive WACP, which finishes the proof.

From Theorem 17, we will think whether there is a solution to the WACP when \(m \geq 4\), which is the following conjecture.

**Conjecture 18.** Let \(T\) be a tree. Then \(T\) has a solution to the WACP under \(\sigma^+\)-rule on \(Z_m(m \geq 4)\) such that \(c_0(v) = 0\) or 1, \(\forall v \in T\).

In the following, we will put forward a special WACP under \(\sigma^+\)-rule on complete graphs.

Let \(K_m\) be a complete graph with \(n\) vertices \(v_1, v_2, \ldots, v_n\). Each vertex \(v_i\) has an initial color value \(c_i(v)\) on \(Z_m\) at time \(t = 0\). Suppose, at time \(t = 1\), the color value of each vertex \(v_i\), denoted by \(c_i(v_i)\), is equal to the sum of the color values of random \(m\) vertices under the meaning of modular \(m\), i.e.,
Random WACP may have no solution for any $\sigma^t$-rule on the rooted tree $T_1$, which has the type in the first left column to a rooted tree $T_2$ which has the type in the first top row.

Table 1: The result table of types of a new rooted tree derived by doing an AddIntoOperation on a rooted tree $T_1$ which has the type in the first left column to a rooted tree $T_2$ which has the type in the first top row.

(a)

<table>
<thead>
<tr>
<th>$(01, 1, 0)$</th>
<th>$(01, 01, 1)$</th>
<th>$(1, 01, 01)$</th>
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If there is no possibility for $v_i$ to have the color value $c_i(v_i)$ equal to 0, i.e.,

$$0 \notin \{c_0(v_{j_1}) + c_0(v_{j_2}) + \cdots + c_0(v_{j_m}) \mod m : j_1, j_2, \ldots, j_m \text{ are distinct}\},$$

then we call $v_i$ 0-avoidable. If all vertices are 0-avoidable, then we say that the initial color value array $c_0 = (c_0(v_1), c_0(v_2), \ldots, c_0(v_n))$ is good. If $c_0$ has $k$ distinct elements under the meaning of modular $m$, we call $c_0$ $k$-good. The problem to find a $k$-good initial color value array $c_0$ is called the $k$-Random WACP under $\sigma^t$-rule on $Z_m$. Note that we always assume that $n \geq m$ and $k \leq m$ when we discuss on the $k$-Random WACP on $Z_m$.

We need to point out that if $n$ is big enough, the $k$-Random WACP may have no solution for any $k$. For example, it is easy to verify that when $m = 2$, $n = 4$, $K_4$ has no $k$-good initial color value array for any $1 \leq k \leq 4$; i.e., the $k$-Random WACP has no solution. In fact, for any 4 integers $x_1, x_2, x_3, x_4$ such that $x_i = 0$ or 1, $1 \leq i \leq 4$, the set

$$\{x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4\}$$

must have a 2-element subset such that the sum of its elements is equal to 0 under the meaning of modular 2.

In the following we will try to find when the $k$-Random WACP has a solution and how to construct a solution by the help of combinatorial number theory. We need to represent the $k$-Random WACP by using the terminology in combinatorial number theory as follows:

Whether there exists $A = (a_1, a_2, \ldots, a_n)$, a sequence of elements of $Z_m$ of length $n$ such that the number of distinct $a_i$'s is equal to $k$, and the sum of any $m$ elements of $A$ is not equal to 0 under the meaning of modular $m$.

It is necessary to introduce Bialostocki Number $f(m, k)$ from combinatorial number theory. Suppose $m, k$ are positive integers. Denote by $f(m, k)$ the least integer $g$ for which
the following holds: If $A = (a_1, a_2, \ldots, a_g)$ is a sequence of elements of $Z_m$ of length $g$ such that the number of distinct $a_i$’s is equal to $k$, then there are $m$ indices $i_1, \ldots, i_m$ belonging to $\{1, \ldots, g\}$ such that $a_{i_1} + \ldots + a_{i_m} = 0 \pmod{m}$. It is easy to see that there is a solution to the $k$-Random WACP on $K_n$ if and only if $n < f(m, k)$.

How to calculate $f(m, k)$ is still an open problem in combinatorial number theory. Until now only part of Bialostocki Numbers $f(n, k)$ have been derived. In [27], Gallardo, Grekos, and Pihko proved the following.

(1) If $m$ is odd, then $f(m, m) = m$; if $m$ is even, then $f(m, m) = m + 1$.

(2) If $m \geq 5$ and $1 + m/2 < k \leq m - 1$, then $f(m, k) = m + 2$.

In [28], Wang proved the following.

(1) If $k \geq 3$ is odd and $m \geq \max[(k - 1)^2 - 4, (k - 1)(k + 5)/8 + 2]$, then

$$f(m, k) = 2n - \frac{(k - 1)^2}{4} - 1. \quad (28)$$

(2) If $k \geq 2$ is even and $m \geq \max[k(k - 2) - 4, k(k + 2)/8 + 1]$, then

$$f(m, k) = 2n - \frac{k(k - 2)}{4} - 1. \quad (29)$$

Now return to the $k$-Random WACP. We have the following results.

**Theorem 19.** Suppose $m$ and $k$ are two given positive integers and $k \leq m$.

(1) When $k = m$ and $m$ is odd, there is no solution to the $k$-Random WACP on $Z_m$ for $K_n$.

(2) When $k = m$ and $m$ is even, there is a solution to the $k$-Random WACP on $Z_m$ for $K_n$ if and only if $n = m$, and the initial color value array $c_0 = (0, 1, \ldots, m - 1)$ is one solution.

(3) When $1 + m/2 < k < m$, there is a solution to the $k$-Random WACP on $Z_m$ for $K_n$ if and only if $m \leq n \leq m + 1$.

(4) When $k \geq 3$ is odd and $m \geq \max[(k - 1)^2 - 4, (k - 1)(k + 5)/8 + 2]$, there is a solution to the $k$-Random WACP on $Z_m$ for $K_n$ if and only if $m \leq n \leq 2m - (k - 1)^2/4 - 2$, and the initial color value array

$$c_0 = \left( \begin{array}{c} 0, \ldots, 0, 1, \ldots, 1, 2, 3, \ldots, \frac{k - 1}{2}, m - \frac{k - 1}{2}, m \\ x, y \end{array} \right) \quad (30)$$

satisfying $1 \leq x \leq m - (k - 1)(k + 3)/8$, $1 \leq y \leq m - (k - 1)(k + 1)/8$, $x + y = n - k + 2$, is one solution.

(5) When $k \geq 2$ is even and $m \geq \max[k(k - 2) - 4, k(k + 2)/8 + 1]$, there is a solution to the $k$-Random WACP on $Z_m$ for $K_n$ if and only if $m \leq n \leq 2m - (k - 2)/4 - 2$, and the initial color value array

$$c_0 = \left( \begin{array}{c} 0, \ldots, 0, 1, \ldots, 1, 2, 3, \ldots, \frac{k - 1}{2}, m - \frac{k - 1}{2}, m \\ x, y \end{array} \right) - \frac{k - 1}{2} + 1, \ldots, m - 1 \right) \quad (31)$$

satisfying $1 \leq x \leq m - (k + 2)/8$, $1 \leq y \leq m - (k + 2)/8$, $x + y = n - k + 2$, is one solution.

The proof and corresponding solutions constructed can be referred to [27, 28].

### 5. Conclusion

The All-Ones Problem comes from the theory of $\sigma^+$-automata, which is related to the graph dynamical system. In this paper, we introduce and study the generalization form of the All-Ones Problem, named “All-Colors Problem.” The “All-Colors Problem” can be divided into Strong-All-Colors Problem and Weak-All-Colors Problem, respectively. We analyze and compute the solutions to these two problems on some interesting classes of graphs. At last we also introduce a new kind of Weak-All-Color Problem, $k$-Random Weak-All-Colors Problem, which is relevant to both combinatorial number theory and $\sigma^+$-automata.

### Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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