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Research Article

Solutions to All-Colors Problem on Graph Cellular Automata

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The All-Ones Problem comes from the theory of σ^+-automata, which is related to graph dynamical systems as well as the Odd Set Problem in linear decoding. In this paper, we further study and compute the solutions to the "All-Colors Problem," a generalization of "All-Ones Problem," on some interesting classes of graphs which can be divided into two subproblems: Strong-All-Colors Problem and Weak-All-Colors Problem, respectively. We also introduce a new kind of All-Colors Problem, k-Random Weak-All-Colors Problem, which is relevant to both combinatorial number theory and cellular automata theory.

1. Introduction

A graph dynamical system (GDS) is a dynamical system constructed over a graph whose vertices can have different states, such that all these states together at a given time constitute a state of the system which can evolve according to an updating scheme [1, 2]. The states of the vertices are commonly modeled by the Boolean values 0 and 1, while the updating scheme consists of as many local functions as vertices and a series of rules that indicate the order in which the local functions act. GDS can be divided into two categories: parallel (PDS) [3, 4] and sequential (SDS) [5, 6] when all the local functions act synchronously or follow an order to act, respectively. In the specific literature, other related topics appeared previously, such as Boolean networks (BN) [7] and cellular automata (CA) [8], which are, in fact, particular cases of GDS. In this paper we will consider "All-Colors Problem," which is concerned with graph cellular automata as one kind of dynamical systems on networks.

A graph is a pair $G = (V, E)$ where $E \subseteq V \times V$. These structures allow for self loops. In the theory of automata, we will insist that G is local finite; i.e., every vertex in G is adjacent to only finite many vertices. In this paper, we only consider G as a finite simple graph. It is convenient to identify E with the adjacency matrix of G constructed as a matrix in $\prod_{V\times V} F_2$. Here $F_2 = \{0, 1\}$ is the two-element field and there is a 1 in the uth row and v^{th} column of E if and only if there is a directed edge from vertex u to vertex v in G.

A vertex u is a predecessor of v if there exists an edge (u, v) in G. The collection of all predecessors of v will be denoted by

$$\Gamma_G(v) = \{u \in V \mid (u, v) \in E\}. \tag{1}$$

Note that $\Gamma_G(v)$ may or may not include v depending on whether v has a self-loop in G or not. We will refer to $\Gamma_G(v)$ as the *neighborhood* of v in G. A *configuration* of G is a function

$$X : V \longrightarrow F_2. \tag{2}$$

The collection of all configurations of G will be denoted by C_G. Define the *transition rule* $\sigma_G : C_G \longrightarrow C_G$ by

$$\sigma(X)(v) := \sum_{u\in\Gamma(v)} X(u) \mod 2. \tag{3}$$

$A = (G, \sigma_G)$ is called the *σ-automaton* on G.

Configurations are conveniently identified with subsets of V; i.e., X is identified with $\{v \in V \mid X(v) = 1\}$. Observe that algebraically C_G is a vector space over F_2, $C_G = \prod_V F_2$; addition here amounts to take symmetric differences. We will call this space the *configuration space*. Furthermore, σ is a linear map from the configuration space to itself (such rules are called additive in [9]). If one thinks of configuration X as

a column vector over F_2, it is obvious from the definition that $\sigma(X) = E \cdot X$ where E is the adjacency matrix of G.

A σ-automaton $A = (G, \sigma_G)$ is *symmetric* if and only if the adjacency matrix E of G is symmetric; thus symmetric σ-automata arise from undirected graphs. In the following sections, we will consider only symmetric σ-automata on undirected graphs.

Let G be an undirected graph without self-loops and D a subset of V. Define $G(D)$ to be the graph obtained from G by adding self-loops at all vertices in D. σ-automata of the form $(G, \sigma_{G(V)})$ or $(G, \sigma_{G(\emptyset)})$ are called *Lindenmayer automata* on G.

To lighten notation we will usually omit the subscript G and write $\Gamma(v)$, σ and (G, σ), and so forth. Also we will write σ^+ for $\sigma_{G(V)}$ and σ^- for $\sigma_{G(\emptyset)}$. We will write $\mathbf{0}$ for the empty set and $\mathbf{1}$ for V as members of C_G, so $\mathbf{0}(v) = 0$ and $\mathbf{1}(v) = 1$ for all v in V.

The term *All-Ones Problem* was introduced by Sutner; see [10]. It has applications in linear cellular automata; see [11] and the references therein. The problem is cited as follows: suppose each of the vertices of an undirect graph with n vertices is equipped with an indicator light and a button. If the button of a vertex is pressed, the light of that vertex will change from off to on and vice versa; the same happens to the lights of all the edge-adjacent vertices. Initially all lights are off. Now, consider the following questions: is it possible to press a sequence of buttons in such a way that in the end all lights are on? This is referred as the *All-Ones Problem*. If there is such a way, how to find a such way? Here and in what follows, we consider connected simple undirected graphs only. When disconnected graphs must be concerned, we can deal with them component by component. For all terminology and notations on graphs, we refer to [12]. An equivalent version of the All-Ones Problem was proposed by Peled in [13], where it was called the *Lamp Lighting Problem*. We cite it as follows.

Light bulbs $L_1, L_2, \ldots, L_n$ are controlled by switches $S_1, S_2, \ldots, S_n$. Switch S_i changes the on/off status of light L_i and possibly the status of some other lights. Assume that if S_i changes the status of light L_j, then S_j changes the status of light L_i. Initially all the lights are off. Prove that it is possible to operate the switches in such a way that all the lights are on.

The rule of the All-Ones Problem is σ^+-rule on graphs, which means that a button lights not only its neighbors but also its own light L_i. If a button lights only its neighbors but not its own light, this rule on graphs is called σ^--rule. There were many publications on the All-Ones Problem; see Sutner [14, 15], Barua et al. [16], and the references therein. Using linear algebra, Sutner [17] proved that it is always possible to light every lamp in any graphs by σ^+ rule. Lossers [18] gave another beautiful proof also by using linear algebra. A graph-theoretic proof was given by Erikisson et al. [19]. More results and related references can be referred to [20–25].

In graph-theoretic terminology, a solution to the All-Ones Problem with σ^+-rule can be stated as follows: given a graph $G = (V, E)$, where V and E denote the vertex-set and the edge-set of G, respectively, a subset X of V is a solution if and only if for every vertex v of G the number of vertices in X adjacent to or equal to v is odd. Such a subset X is called an *odd parity cover* in [17]. So, the All-Ones Problem can be formulated as follows: given a graph $G = (V, E)$, does a subset X of V exist such that, for any vertex $v \in V - X$, the number of vertices in X adjacent to v is odd, while for any vertex $v \in X$, the number of vertices in X adjacent to v is even?

Sutner [10] proposed the question whether there is a graph-theoretic method to find a solution for the All-Ones Problem for trees. Galvin [26] solved this question.

The "All-Ones Problem" can be considered to find the preimages of configuration $\mathbf{1}$ under parallel map defined by local rule σ. In this paper, we further study the "All-Colors Problem," a natural generalization of "All-Ones Problem," which can be divided into two subproblems: Strong-All-Colors Problem and Weak-All-Colors Problem, respectively.

This paper is organized as follows. In Section 2, we introduce the preliminary and definitions of the "All-Colors Problem" under σ^+-rule. The "All-Colors Problem" can be divided into two subproblems: Strong-All-Colors Problem and Weak-All-Colors Problem which are studied in Sections 3 and 4, respectively. In the end of Section 4, we also introduce a new kind of All-Color Problem, k-Random Weak-All-Colors Problem, which is relevant to both combinatorial number theory and cellular automata theory.

2. Preliminary of All-Colors Problem

Here we introduce a natural generalization for the All-Ones Problem—*All-Colors Problem*. When we discuss the All-Colors Problem, we need a positive integer $m > 2$ and a graph G each of whose vertices has a color value between 0 and $m - 1$. There are two kinds of All-Colors Problem: Strong-All-Colors Problem and Weak-All-Colors Problem. We give their accurate definitions as follows.

Definition 1 (strong-all-colors problem under σ^+-rule on Z_m). If a vertex is pressed one time, then the color values of the vertex and its neighbors are added by 1 under the meaning of modular m. If the initial status is that the color value of every vertex is 0, then how to press some vertices (maybe many times) to make the color value of every vertex equal to 1 under the meaning of modular m.

Definition 2 (weak-all-colors problem under σ^+-rule on Z_m). If a vertex is pressed one time, then the color values of the vertex and its neighbors are added by 1 under the meaning of modular m. If the initial status is that the color value of every vertex is 0, then how to press some vertices (maybe many times) to make the color value of every vertex not equal to 0 under the meaning of modular m.

A solution to the Strong-All-Colors Problem on a graph G under σ^+-rule is a configuration $f : V(G) \longrightarrow Z_m$ such that, for any $v \in V(G)$,

$$\sum_{w \in N(v) \cup \{v\}} f(w) = 1 \mod m, \tag{4}$$

where $N(v)$ is denoted as the set of vertices of G which are adjacent to w. Correspondingly, a solution to the Weak-All-Colors Problem on a graph G under σ^+-rule is a configuration $f : V(G) \longrightarrow Z_m$ such that, for any $v \in V(G)$,

$$\sum_{w \in N(v) \cup \{v\}} f(w) \neq 0 \mod m. \tag{5}$$

It is worth noting that we only need to study the Strong-All-Colors Problem and the Weak-All-Colors Problem in the case that m is a power of a prime. Indeed, let $m = \prod_{i=1}^{r} p_i^{e_i}$ be the prime factorization of m. Suppose that there exists a solution $c_i : V(G) \longrightarrow Z_{p_i^{e_i}}$ to the Strong-All-Colors Problem under σ^+ on $Z_{p_i^{e_i}}$ for any i. That is, for any $v \in V(G)$,

$$\sum_{w \in N_G(v) \cup \{v\}} c_i(w) = 1 \mod p_i^{e_i}. \tag{6}$$

By the Chinese Remainder Theorem, there exists $c : V(G) \longrightarrow Z_m$ such that

$$c(v) = c_i(v) = 1 \mod p_i^{e_i} \tag{7}$$

for any $i \in \{1, \ldots, r\}$ and $v \in V(G)$. It follows that

$$\sum_{w \in N_G(v) \cup \{v\}} c(w) = 1 \mod p_i^{e_i}, \tag{8}$$

for any v. By the Chinese Remainder Theorem again, one has

$$\sum_{w \in N_G(v) \cup \{v\}} c(w) = 1 \mod m. \tag{9}$$

That is, c is a solution to the Strong-All-Colors Problem on Z_m. The discussion of the Weak-All-Colors Problem is similar.

Note that we can also define the corresponding *Strong-All-Colors Problem* under σ^--rule on Z_m and *Weak-All-Colors Problem* under σ^--rule on Z_m; however, they are more difficult to study.

In the following sections, we will discuss the All-Colors Problem under σ^+-rule in detail. We need to use two definitions equivalent to Definitions 1 and 2 for convenience. Suppose G is a simple undirected graph; each vertex of G has a color value on Z_m which can change with time. These changes abide by the following rules (σ^+-rule): If, for $\forall v \in G$, $c_0(v)$ is the initial color value of v at time $t = 0$, then, at time $t = 1$, the color value of v has changed to $c_1(v) = \sum_{u \in N(v) \cup \{v\}} c_0(u)$, where $N(v)$ is the set of vertices which are adjacent to v.

The Strong-All-Colors Problem under σ^+-rule can be redefined as follows: *to find the initial color values of all the vertices of G at time $t = 0$ such that any vertex of G has a color value equal to $1 (\mathrm{mod}\, m)$ at time $t = 1$ under σ^+-rule.*

The Weak-All-Colors Problem can be redefined as follows: *to find the initial color values of all the vertices of G at time $t = 0$ such that no vertex of G has a color value equal to $0 (\mathrm{mod}\, m)$ at time $t = 1$ under σ^+-rule.*

For the sake of simplicity, the Strong-All-Colors Problem is denoted as SACP from now on. Correspondingly, the Weak-All-Colors Problem is denoted as WACP.

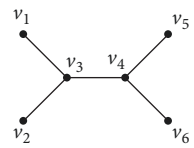

Figure 1: A tree without solution to the SACP for $m \geq 3$.

3. Strong-All-Colors Problem under σ^+-Rule on Z_m

First of all, for a general graph, the SACP with σ^+-rule on Z_m may have no solution. For example, there is no solution to the SACP under σ^+-rule on Z_m for a circle C_n when $3 \nmid n$ and $3 \mid m$. Particularly, by observing the following tree T_0 with 6 vertices, in Figure 1, we will discover surprisingly that there is no solution to the SACP under σ^+-rule on Z_m for any $m \geq 3$.

Then a question arises naturally: How to determine whether a graph has a solution to the SACP with σ^+-rule on Z_m?

It is easy to find an algebraic method to solve this problem. Suppose G is a graph with n vertices. The following theorem can be obtained easily and we omit the proof.

Theorem 3. *There exists a solution to the SACP with σ^+-rule on $Z_m (m \geq 2)$ for a graph G if and only if the following system of linear equations has a solution on Z_m.*

$$(A + I) X = (\mathbf{1}) (\mathrm{mod}\, m), \tag{10}$$

where A is the adjacent matrix of G, I is the identity matrix, X is an $n \times 1$ vector of variables, and $(\mathbf{1})(\mathrm{mod}\, m)$ is an $n \times 1$ vector each element of which is $1 (\mathrm{mod}\, m)$.

Example 4. Consider the tree T_0 at the beginning of this section. Its adjacent matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{11}$$

It is easy to verify that

$$(A + I) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \mod m \\ 1 \mod m \\ \vdots \\ 1 \mod m \end{pmatrix} \tag{12}$$

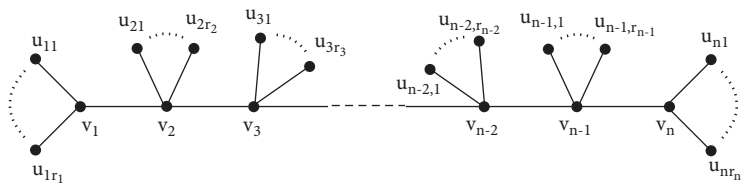

Figure 2: A caterpillar T.

is equivalent to

$$\begin{array}{lllllll} x_1 & & +x_3 & & & & = 1 \mod m & (1) \\ & x_2 & +x_3 & & & & = 1 \mod m & (2) \\ x_1 & +x_2 & +x_3 & +x_4 & & & = 1 \mod m & (3) \\ & & x_3 & +x_4 & +x_5 & +x_6 & = 1 \mod m & (4) \\ & & & x_4 & +x_5 & & = 1 \mod m & (5) \\ & & & x_4 & & +x_6 & = 1 \mod m & (6) \end{array} \tag{13}$$

Let us operate the 6 equations as follows.

$$(1) + (2) + (5) + (6) - (3) - (4), \tag{14}$$

so we get

$$0 = 2 \mod m \tag{15}$$

It is obvious that equation (15) holds if and only if $m = 2$. So T_0 has no solution to the SACP under σ^+-rule on Z_m for $m \geq 3$, which we have just declared.

The system of linear equations over Z_m could be solved. Let $A \in R^{n\times n}$, $B \in R^{n\times 1}$, and $X = (x_1, \ldots, x_n)^T$. We need to solve the equation $AX = B$. If R, as Z_m above, is a principle idea ring, there exist invertible matrices $U, V \in M_n(R)$ such that $A = UCV$, where

$$C = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}, \tag{16}$$

where $d_i \mid d_{i+1}$ and C is the Smith normal form of A. We can compute U and V if $R = Z_m$. Then $AX = B$ is equivalent to $CVX = U^{-1}B$. We set $Y = VX$ and we can solve $CY = U^{-1}B$ since C is a diagonal matrix. Hence, we obtain $X = V^{-1}Y$.

Although this algebraic method is quite succinct, it is interesting to study the SACP under σ^+ rule on Z_m over some classes of graphs.

In the following, we consider the SACP under σ^+-rule on Z_m for trees. Note that we do all the operations "+" and "−" on the ring Z_m. Consider the special case when the tree is a "caterpillar." A tree T is called a *caterpillar* if and only if the remainder of T is a path after removing all the leaves of T. Figure 2 shows such a caterpillar.

We can find the solution to the SACP under σ^+-rule on Z_m for a caterpillar T according to the following process.

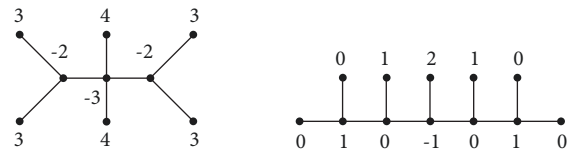

Figure 3: Two examples with solutions to the SACP on Z_m.

Suppose T is a caterpillar with the following shape of Figure 2. First we assume that the initial color value of vertex v_1 is x, which will be determined in the end. It is easy to see that the initial color values of the leaves of vertex v_1 must be $1 - x$ in order to make the color values of vertices $u_{11}, u_{12}, \ldots, u_{1r_1}$ changing to 1 by the meaning of modular m under σ^+-rule. Then the initial color value of v_2 would be $-(r_1 - 1)(1 - x)$ to make the color value of v_1 equal to 1 under σ^+-rule. Repeat this step until we have determined the initial color value of v_n, denoted by $f_n(x)$, which is a linear function of x. From this procedure we can see that there is only one equation to be satisfied; i.e.,

$$f_{n-1}(x) + f_n(x) + r_n \cdot (1 - f_n(x)) = 1 \mod m, \tag{17}$$

where $f_{n-1}(x)$ is the initial color value of vertex v_{n-1}.

So we have the following.

Theorem 5. *A caterpillar T has a solution to the SACP under σ^+-rule on Z_m if and only if the equation (17) has a solution on Z_m.*

We give two examples with solutions to the SACP under σ^+-rule on Z_m in Figure 3. All the negative integers that appeared in the following figures are under the meaning of modular m.

As an application of Theorem 5, we have the following.

Corollary 6. *If T is a path with n vertices, then T has a solution to the SACP under σ^+-rule on Z_m for any $m \geq 2$.*

Proof. Let $x_1 x_2 \cdots x_n$, a sequence of integers on Z_m, represent the initial color values of vertices of T. It is easy to check that

$$x_1 x_2 \cdots x_n = \begin{cases} 100100\cdots10010, & \text{if } n \equiv 2 \pmod 3 \\ 100100\cdots1001, & \text{if } n \equiv 1 \pmod 3 \\ 010010\cdots010, & \text{if } n \equiv 0 \pmod 3 \end{cases} \tag{18}$$

is a solution to the SACP under σ^+-rule on Z_m for any $m \geq 2$. □

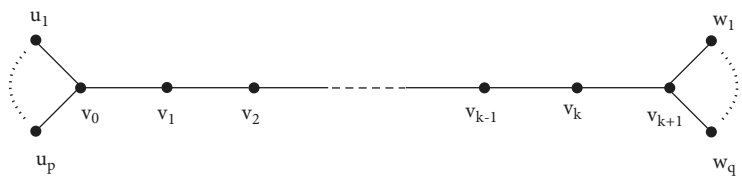

Figure 4: The shape of a special kind of caterpillars.

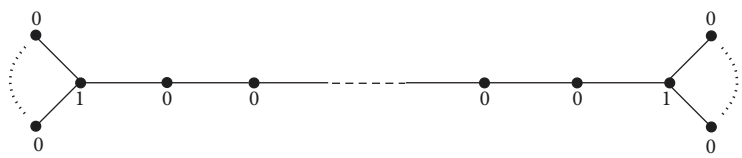

Figure 5: A solution for T when $k \equiv 2 (\bmod 3)$.

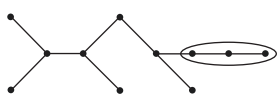

Figure 6: An example of a 3-connected path.

Corollary 7. *If T is a tree with the shape as shown in Figure 4, then*

(1) *when $k \equiv 2 (\bmod 3)$, T has a solution as in Figure 5*

(2) *when $k \equiv 1 (\bmod 3)$, T has a solution if and only if $(pq-1)x \equiv q-1 (\bmod m)$ has a solution on Z_m, which is equivalent to $\gcd(pq-1, m) \mid (q-1)$. Here x is the initial color value of vertices $u_1, \ldots, u_p$*

(3) *when $k \equiv 0 (\bmod 3)$, T has a solution if and only if $(pq-p-q)x \equiv -q (\bmod m)$ has a solution on Z_m, which is equivalent to $\gcd(pq-p-q, m) \mid q$. The meaning of x is the same to above*

Now we show a reduction technique. First we introduce a new concept.

Definition 8. Suppose G is a graph. If there are 3 distinct vertices v_1, v_2, v_3 of G satisfying

(1) v_3 is a leaf

(2) v_2 is exactly adjacent to v_1 and v_3

then $(\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_3)\})$ is called a 3-*connected path* and denoted by (v_1, v_2, v_3) simply.

Example 9. For example, Figure 6 shows a tree which has a 3-connected path marked by an ellipse.

Theorem 10 (deleting a 3-connected path). *Suppose that G is a graph with n vertices $v_1, v_2, \ldots, v_n$. If G contains a 3-connected path $(v_{i_1}, v_{i_2}, v_{i_3})$, then G has a solution to the SACP under σ^+-rule on Z_m if and only if $G \setminus \{v_{i_1}, v_{i_2}, v_{i_3}\}$ has a solution to the SACP under σ^+-rule on Z_m, where $G \setminus \{v_{i_1}, v_{i_2}, v_{i_3}\}$ is a tree or a forest.*

Proof. Suppose T has a solution to the SACP under σ^+-rule on Z_m. Assume the initial color value of vertex v_i is $x_i (1 \le i \le n)$. We have $x_{i_1} = 0 (\bmod m)$ because it requires $x_2 + x_3 = 1 (\bmod m)$ and $x_1 + x_2 + x_3 = 1 (\bmod m)$ to ensure the color value of v_2 and v_3 to be changed to 1 under σ^+-rule. Then we can say that $(x_4, x_5, \ldots, x_n)$ is a solution to the SACP under σ^+-rule on Z_m for $G \setminus \{v_1, v_2, v_3\}$.

For the converse, suppose $(x_4, x_5, \ldots, x_n)$ is a solution to the SACP under σ^+-rule on Z_m for $G \setminus \{v_1, v_2, v_3\}$. Let $\widetilde{N}(v_{i_1})$ denote the set of vertices $G \setminus \{v_1, v_2, v_3\}$ that are adjacent to v_{i_1} in T. It is easy to check $(x_1', x_2', \ldots, x_n')$ is a solution to the SACP under σ^+-rule on Z_m for T, where

$$x_j' = \begin{cases} x_j & 1 \le j \le n \text{ and } j \ne 1, 2, 3 \\ 0 & j = 1 \\ 1 - \sum\limits_{u \in \widetilde{N}(v_{i_1})} x_u & j = 2 \\ \sum\limits_{u \in \widetilde{N}(v_{i_1})} x_u & j = 3 \end{cases} \tag{19}$$

This completes the Proof. □

It is worth noting that we can use this reduction technique repeatedly. For example, the tree in Example 9 can be reduced continuously. The reduction process and its inverse, which can help us to get the solution to the SACP, are shown in Figure 7.

In fact, there are some other complex structures which can be reduced. But they are commonly relevant to Z_m. For example, when $m = 3$, the structures shown in Figure 8 can be removed from a tree, just like the 3-connected path.

Next, we study another special kind of trees—radioactive trees. A tree T is called a *radioactive tree* if and only if there is only at most one vertex of T which has a degree bigger than 2. In fact, a radioactive tree can be viewed as several paths adhering to a common vertex.

Theorem 11. *If T is a radioactive tree, then T has a solution to the SACP under σ^+-rule on Z_m for any $m \ge 2$.*

Proof. Assume that T has the shape shown in Figure 9.

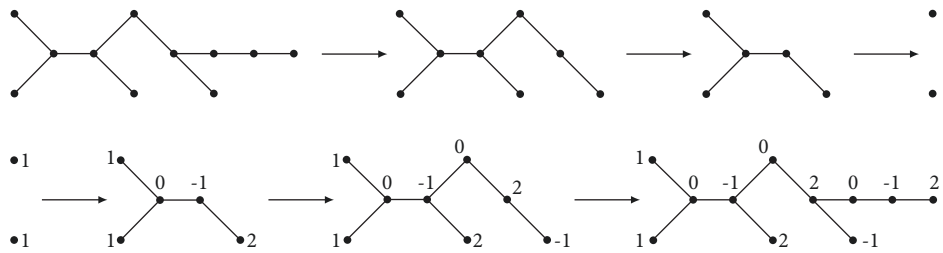

Figure 7: Reduction to get the solution.

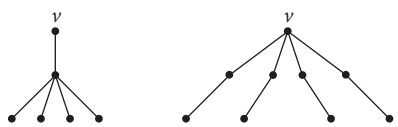

Figure 8: Complex structures that can be reduced when $m = 3$.

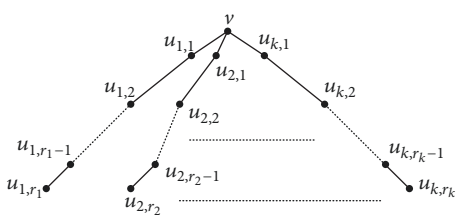

Figure 9: A radioactive tree.

By Theorem 10 we can delete 3-connected path continuously. Without loss of generality we may assume that at last T has changed to T' that has $k' \le k$ paths from the root. It is enough to prove that T' has a solution to the SACP. Suppose $r_i = 1$ for $1 \le i \le s$ and $r_i = 2$ for $s+1 \le i \le k'$ in T'. If $s = k'$, then $x_v = 1$, $x_{u_{i,1}} = 0$ for $1 \le i \le s = k'$ is a solution. If $s < k'$, then $x_v = 0$, $x_{u_{i,1}} = 1$ for $1 \le i \le s$, $x_{u_{i,1}} = 0$ and $x_{u_{i,2}} = 1$ for $s+1 \le i < k'$, $x_{u_{k',1}} = 1 - s(\mathrm{mod}\, m)$ and $x_{u_{k',2}} = s(\mathrm{mod}\, m)$ is a solution to the SACP. The schematic diagram for the latter case is shown in Figure 10. □

Actually, we can also prove Theorem 11 with the help of our reduction technique introduced before. The detail of the proof is omitted here.

4. Weak-All-Colors Problem

In this section, we will study the WACP under σ^+-rule on Z_m, $m \ge 2$. It is worth noting when $m = 2$, the WACP becomes the All-Ones Problem which we have discussed before. So the WACP is another generalized form of All-Ones Problem. The WACP has corresponding linear algebraic representation. If G is a graph with n vertices and A is the adjacent matrix of G, then the WACP is equivalent to the next algebraic question:

Find the values of n variables $x_1, x_2, \ldots, x_n$ on Z_m such that no component of the $n \times 1$ result vector of

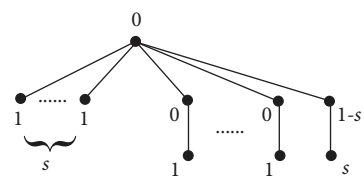

Figure 10: The answer to the SACP for simplified tree T' when $s < k'$.

$$(A + I)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{20}$$

is equal to 0.

It is known that any graph G has a solution to the All-Ones Problem under σ^+-rule. We conjecture that it also holds for the WACP.

Conjecture 12. *For any graph G, there is a solution to the WACP under σ^+-rule on $Z_m (m \ge 2)$.*

We should point out that the original proof to the existence of a solution to the All-Ones Problem for any graph G cannot do any effects on the WACP on Z_m when $m \ge 3$. So it seems that we need to find another approach to solve this conjecture.

In this section, we emphasize the trees by using the technique of increasing a leaf. By Theorem 14, we show that Conjecture 12 is valid for trees. By Theorem 15, it also holds for cycles. Using the increasing technique and Theorem 15, we prove that Conjecture 12 is valid for unicyclic graphs, which is Corollary 16. Furthermore, by Theorem 17, it is interesting that we can find a solution, which contains only 0 and 1, to the WACP under σ^+-rule on Z_3 for trees. Correspondingly another Conjecture 18 is proposed.

Lemma 13 (increasing a leaf). *If G is a graph and v is a vertex of G whose degree is 1, i.e., v is a leaf, then G has a solution to the WACP under σ^+-rule on $Z_m (m \ge 3)$ if $G \setminus \{v\}$ does too.*

Proof. Assume $G \setminus \{v\}$ has a solution c_0 to the WACP under σ^+-rule on Z_m. Suppose v is only adjacent to w in G. Then let

$$c_0'(v) \in Z_m \setminus \left\{ -\sum_{u \in N(w) \cup \{w\}} c_0(u), -c_0(w) \right\}, \qquad c_0'(u) = c_0(u), \text{ if } u \neq v. \tag{21}$$

Because $m \geq 3$, so we could find a $c_0'(v)$ in $Z_m \setminus \{-\sum_{u \in N(w) \cup \{w\}} c_0(u), -c_0(w)\}$. It is easy to check that c_0' is a solution to the WACP under σ^+-rule on Z_m.

The proof is completed. □

From Lemma 13, we can obtain Theorem 14 easily.

Theorem 14. *Let T be a tree and $v \in V(T)$. Then T has a solution c_0 to the WACP under σ^+-rule on $Z_m (m \geq 3)$ satisfying $c_0(v) = 1$.*

Proof. We use the technique of increasing a leaf. Firstly let $T_1 = \{v\}$ be a one-vertex tree, and let $\widehat{c}_1(v) = 1$, which is a solution to the WACP under σ^+-rule on Z_m satisfying $\widehat{c}_1(v) = 1$. Secondly we can add another vertex u, which is adjacent to v, and the edge (u, v) into T_1. Let T_2 denote the new tree. It is easy to see that there is a solution $\widehat{c}_2$ to the WACP under σ^+-rule on Z_m satisfying $\widehat{c}_2(v) = 1$. Repeat these steps by adding an appropriate vertex each time, until at last we get a solution $\widehat{c}_n$ to the WACP under σ^+-rule on Z_m for T satisfying $\widehat{c}_n(v) = 1$, which is just c_0 that we want. We conclude the proof. □

Since it is not hard to construct a solution, we can obtain the following theorem straightforwardly.

Theorem 15. *There is a solution to the WACP under σ^+-rule on $Z_m (m \geq 3)$ for any cycle.*

In fact we can extend the result to "unicyclic graphs." Recall that a graph G is called unicyclic if it contains a unique cycle. In other words, we can regard a unicyclic graph as a cycle attached with each vertex a rooted tree.

Corollary 16. *If G is a unicyclic graph, then there is a solution to the WACP under σ^+-rule on $Z_m (m \geq 3)$ for G.*

Proof. From Theorem 15, a cycle has a solution to the WACP under σ^+-rule on Z_m. Using the increasing technique repeatedly, we can prove that this statement holds. □

Next, we prove that there is a special solution to the WACP on Z_3 for any tree such that all the color values of vertices are 0 and 1.

Theorem 17. *If T is a tree, then there is a solution c_0 to the WACP under σ^+-rule on Z_3 such that $c_0(v) = 0$ or 1 for any $v \in V(T)$.*

Proof. Suppose T is a rooted tree with root t. We need to introduce three new concepts.

The *small WACP* is to find an initial color value array c on T such that

(1) $c(v) = 0$ or $1, \forall v \in T$

(2) $\sum_{v \in N(u) \cup \{u\}} c(v) \neq 0 (\text{mod } 3), \forall u \neq t$

(3) $\sum_{v \in N(t) \cup \{t\}} c(v) = 0$ or $1 (\text{mod } 3)$

Then the initial color value array c will be called a solution to the small WACP.

Correspondingly, the *positive WACP* is to find an initial color value array c on T such that

(1) $c(v) = 0$ or $1, \forall v \in T$

(2) $\sum_{v \in N(u) \cup \{u\}} c(v) \neq 0 (\text{mod } 3), \forall u \neq t$

(3) $\sum_{v \in N(t) \cup \{t\}} c(v) = 1$ or $2 (\text{mod } 3)$

Then the initial color value array c will be called a solution to the positive WACP.

The *even WACP* is to find an initial color value array c on T such that

(1) $c(v) = 0$ or $1, \forall v \in T$

(2) $\sum_{v \in N(u) \cup \{u\}} c(v) \neq 0 (\text{mod } 3), \forall u \neq t$

(3) $\sum_{v \in N(t) \cup \{t\}} c(v) = 0$ or $2 (\text{mod } 3)$

Then the initial color value array c will be called a solution to the even WACP.

In fact a solution to the positive WACP is also a solution we required in this proposition. So it is sufficient for us to prove that there is a solution to the positive WACP for any rooted tree.

Suppose T is a rooted tree with root t. If there is a solution c to the small WACP such that $c(t) = 0$, we say that 0 is a permissible color value of root t to the small WACP. Accordingly, if there is a solution c to the small WACP such that $c(t) = 1$, then 1 is a permissible color value of root t to the small WACP. We name the set of all permissible color values of root t as the *permission set of root t to the small WACP* and denote it by S.

Similarly, we can define *the permission set of root t to the positive WACP*, denoted by P, and *the permission set of root t to the even WACP*, denoted by E. Then for the rooted tree T with root t, (S, P, E) will be called the *type* of root t and corresponding rooted tree T. For example, the type of a rooted tree shown in Figure 11 is $(\{1\}, \{0, 1\}, \{0, 1\})$, or denoted by $(1, 01, 01)$ for simplicity.

Next we need to define "*Add Into Operation*." Suppose T_1 and T_2 are two rooted trees with roots t_1 and t_2, respectively. We will get a new rooted tree with root t_2 by adding T_1 into T_2 with t_1 as a child of t_2. This operation is called an *Add Into Operation on T_1 to T_2*. If the result tree is T_3, then we denote the operation as $T_1 > T_2 \longrightarrow T_3$. It is not difficult to show that each rooted tree can be derived by doing an Add Into Operation on one smaller rooted tree to another smaller rooted tree.

Furthermore, if the color value array $c^1_{(0,S)}$ on T_1 is a solution to the small WACP for T_1 and $c^2_{(1,S)}$ on T_2 is a solution to the small WACP for T_2, then the color value array $c^3 = c^1_{(0,S)} \cup c^2_{(1,S)}$ will be a solution to the small WACP for $T_1 > T_2$,

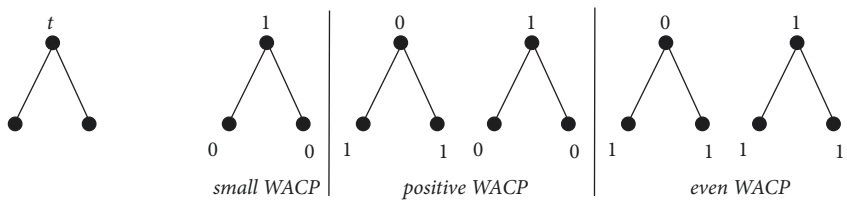

Figure 11: A little tree with root t and its solutions to the three kinds of WACP, which show that the type of it is $(1, 01, 01)$.

i.e., a new tree with root t_2, so we write c^3 as $c^3_{(1,S)}$. Then we denote it as the equation

$$c^1_{(0,S)} > c^2_{(1,S)} \longrightarrow c^3_{(1,S)}. \tag{22}$$

Similarly, we can get other equations like this. Altogether there are 15 equations as follows.

$$\begin{aligned}
&c^1_{(0,S)} > c^2_{(1,S)} \longrightarrow c^3_{(1,S)},\\
&c^1_{(0,S)} > c^2_{(1,P)} \longrightarrow c^3_{(1,P)},\\
&c^1_{(0,S)} > c^2_{(1,E)} \longrightarrow c^3_{(1,E)},\\
&c^1_{(1,S)} > c^2_{(1,S)} \longrightarrow c^3_{(1,P)},\\
&c^1_{(1,S)} > c^2_{(1,P)} \longrightarrow c^3_{(1,E)},\\
&c^1_{(1,S)} > c^2_{(1,E)} \longrightarrow c^3_{(1,S)},\\
&c^1_{(0,P)} > c^2_{(0,S)} \longrightarrow c^3_{(0,S)},\\
&c^1_{(0,P)} > c^2_{(0,P)} \longrightarrow c^3_{(0,P)},\\
&c^1_{(0,P)} > c^2_{(0,E)} \longrightarrow c^3_{(0,E)},\\
&c^1_{(1,P)} > c^2_{(0,S)} \longrightarrow c^3_{(0,P)},\\
&c^1_{(1,P)} > c^2_{(0,P)} \longrightarrow c^3_{(0,E)},\\
&c^1_{(1,P)} > c^2_{(0,E)} \longrightarrow c^3_{(0,S)},\\
&c^1_{(0,E)} > c^2_{(1,S)} \longrightarrow c^3_{(1,S)},\\
&c^1_{(0,E)} > c^2_{(1,P)} \longrightarrow c^3_{(1,P)},\\
&c^1_{(0,E)} > c^2_{(1,E)} \longrightarrow c^3_{(1,E)}.
\end{aligned} \tag{23}$$

From the above equations, we can calculate the type of the rooted tree T_3 obtained from the Add Into Operation on a rooted tree T_1 to another rooted tree T_2, as long as we have known the types of T_1 and T_2. For example, if the root of T_1 has the type $(01, 1, 0)$ and the root of T_2 has the type $(01, 01, 1)$, then the type of the new result rooted tree derived by doing an Add Into Operation on T_1 to T_2 will be $(1, 01, 01)$. The validity lies in the following 5 equations.

$$\begin{aligned}
&c^1_{(0,S)} > c^2_{(1,S)} \longrightarrow c^3_{(1,S)},\\
&c^1_{(1,P)} > c^2_{(0,S)} \longrightarrow c^3_{(0,P)},\\
&c^1_{(0,E)} > c^2_{(1,P)} \longrightarrow c^3_{(1,P)},\\
&c^1_{(1,P)} > c^2_{(0,P)} \longrightarrow c^3_{(0,E)},\\
&c^1_{(0,E)} > c^2_{(1,E)} \longrightarrow c^3_{(1,E)}.
\end{aligned} \tag{24}$$

Then we want to list all possible types of rooted trees. In fact, there are only 8 distinct types of rooted trees. Each type and one of its representative rooted trees are shown in Figure 12.

In order to show that there are only 8 distinct types, we need to show that the following assertion is correct.

Assertion. If each of the two rooted trees T_1, T_2 with roots t_1 and t_2, respectively, belongs to one of the 8 types, then the rooted tree T derived by doing an Add Into Operation on T_1 to T_2 belongs to one of the 8 types.

We have mentioned before how to calculate the type of the result rooted tree obtained by doing the Add Into Operation on a rooted tree to another if their types are known. Since all rooted trees can be derived by doing an Add Into Operation on a smaller rooted tree to another smaller one, we can observe that all the types of rooted trees are contained in the set of 8 distinct types above. The proof of the Assertion exists in Table 1. Since the original table is too wide, we divide it into two tables as in Tables 1(a) and 1(b).

We have showed that any rooted tree belongs to one of the 8 types and, in each of the 8 types, the permission set to the positive WACP is not empty. So any rooted tree would have a solution c to the positive WACP, which finishes the proof. □

From Theorem 17, we will think whether there is a solution to the WACP when $m \geq 4$, which is the following conjecture.

Conjecture 18. *Let T be a tree. Then T has a solution to the WACP under σ^+-rule on $Z_m (m \geq 4)$ such that $c_0(v) = 0$ or 1, $\forall v \in T$.*

In the following, we will put forward a special WACP under σ^+-rule on complete graphs.

Let K_n be a complete graph with n vertices $v_1, v_2, \ldots, v_n$. Each vertex v_i has an initial color value $c_0(v_i)$ on Z_m at time $t = 0$. Suppose, at time $t = 1$, the color value of each vertex v_i, denoted by $c_1(v_i)$, is equal to the sum of the color values of random m vertices under the meaning of modular m, i.e.,

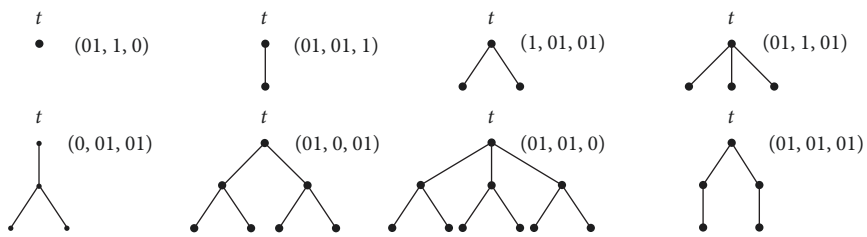

Figure 12: All possible types of rooted trees.

Table 1: The result table of types of a new rooted tree derived by doing an Add Into Operation on a rooted tree T_1 which has the type in the first left column to a rooted tree T_2 which has the type in the first top row.

(a)

	(01, 1, 0)	**(01, 01, 1)**	**(1, 01, 01)**	**(01, 1, 01)**
(01, 1, 0)	(01, 01, 1)	(1, 01, 01)	(01, 1, 01)	(01, 01, 1)
(01, 01, 1)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(1, 01, 01)	(0, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(01, 1, 01)	(01, 01, 1)	(1, 01, 01)	(01, 1, 01)	(01, 01, 1)
(0, 01, 01)	(01, 01, 0)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(01, 0, 01)	(01, 1, 01)	(01, 01, 1)	(1, 01, 01)	(01, 1, 01)
(01, 01, 0)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)

(b)

	(0, 01, 01)	**(01, 0, 01)**	**(01, 01, 0)**	**(01, 01, 01)**
(01, 1, 0)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(01, 01, 1)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(1, 01, 01)	(01, 0, 01)	(01, 01, 0)	(0, 01, 01)	(01, 01, 01)
(01, 1, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(0, 01, 01)	(0, 01, 01)	(01, 0, 01)	(01, 01, 0)	(01, 01, 01)
(01, 0, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(01, 01, 0)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)
(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)	(01, 01, 01)

$$c_1(v_i) \in \{c_0(v_{j_1}) + c_0(v_{j_2}) + \cdots + c_0(v_{j_m}) \bmod m \mid j_1, j_2, \ldots, j_m \text{ are distinct}\}. \tag{25}$$

If there is no possibility for v_i to have the color value $c_1(v_i)$ equal to 0, i.e.,

$$0 \notin \{c_0(v_{j_1}) + c_0(v_{j_2}) + \cdots + c_0(v_{j_m}) \bmod m \mid j_1, j_2, \ldots, j_m \text{ are distinct}\}, \tag{26}$$

then we call v_i 0-*avoidable*. If all vertices are 0-avoidable, then we say that the initial color value array $c_0 = (c_0(v_1), c_0(v_2), \ldots, c_0(v_n))$ is *good*. If c_0 has k distinct elements under the meaning of modular m, we call c_0 *k-good*. The problem to find a k-good initial color value array c_0 is called the *k-Random WACP under σ^+-rule on Z_m*. Note that we always assume that $n \geq m$ and $k \leq m$ when we discuss on the k-Random WACP on Z_m.

We need to point out that if n is big enough, the k-Random WACP may have no solution for any k. For example, it is easy to verify that when $m = 2$, $n = 4$, K_4 has no k-good initial color value array for any $1 \leq k \leq 4$; i.e., the k-Random WACP has no solution. In fact, for any 4 integers x_1, x_2, x_3, x_4 such that $x_i = 0$ or 1, $1 \leq i \leq 4$, the set

$$\{x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4\} \tag{27}$$

must have a 2-element subset such that the sum of its elements is equal to 0 under the meaning of modular 2.

In the following we will try to find when the k-Random WACP has a solution and how to construct a solution by the help of combinatorial number theory. We need to represent the k-Random WACP by using the terminology in combinatorial number theory as follows:

Whether there exists $A = (a_1, a_2, \ldots, a_n)$, a sequence of elements of Z_m of length n such that the number of distinct a_i's is equal to k, and the sum of any m elements of A is not equal to 0 under the meaning of modular m.

It is necessary to introduce Bialostocki Number $f(m, k)$ from combinatorial number theory. Suppose m, k are positive integers. Denote by $f(m, k)$ the least integer g for which

the following holds: If $A = (a_1, a_2, \ldots, a_g)$ is a sequence of elements of Z_m of length g such that the number of distinct a_i's is equal to k, then there are m indices $i_1, \ldots, i_m$ belonging to $\{1, \ldots, g\}$ such that $a_{i_1} + \ldots + a_{i_m} = 0 (\mathrm{mod}\, m)$. It is easy to see that there is a solution to the k-Random WACP for K_n if and only if $n < f(m, k)$.

How to calculate $f(m, k)$ is still an open problem in combinatorial number theory. Until now only part of Bialostocki Numbers $f(n, k)$ have been derived. In [27], Gallardo, Grekos, and Pihko proved the following.

(1) If m is odd, then $f(m, m) = m$; if m is even, then $f(m, m) = m + 1$.

(2) If $m \geq 5$ and $1 + m/2 < k \leq m - 1$, then $f(m, k) = m + 2$.

In [28], Wang proved the following.

(1) If $k \geq 3$ is odd and $m \geq \max\{(k-1)^2 - 4, (k-1)(k+5)/8 + 2\}$, then

$$f(m, k) = 2n - \frac{(k-1)^2}{4} - 1. \tag{28}$$

(2) If $k \geq 2$ is even and $m \geq \max\{k(k-2) - 4, k(k+2)/8 + 1\}$, then

$$f(m, k) = 2m - \frac{k(k-2)}{4} - 1. \tag{29}$$

Now return to the k-Random WACP. We have the following results.

Theorem 19. *Suppose m and k are two given positive integers and $k \leq m$.*

(1) *When $k = m$ and m is odd, there is no solution to the k-Random WACP on Z_m for K_n.*

(2) *When $k = m$ and m is even, there is a solution to the k-Random WACP on Z_m for K_n if and only if $n = m$, and the initial color value array $c_0 = (0, 1, \ldots, m-1)$ is one solution.*

(3) *When $1 + m/2 < k < m$, there is a solution to the k-Random WACP on Z_m for K_n if and only if $m \leq n \leq m + 1$.*

(4) *When $k \geq 3$ is odd and $m \geq \max\{(k-1)^2 - 4, (k-1)(k+5)/8 + 2\}$, there is a solution to the k-Random WACP on Z_m for K_n if and only if $m \leq n \leq 2m - (k-1)^2/4 - 2$, and the initial color value array*

$$c_0 = \left(\underbrace{0, \ldots, 0}_{x}, \underbrace{1, \ldots, 1}_{y}, 2, 3, \ldots, \frac{k-1}{2}, m - \frac{k-1}{2}, m - \frac{k-1}{2} + 1, \ldots, m-1\right) \tag{30}$$

satisfying $1 \leq x \leq m - (k-1)(k+3)/8$, $1 \leq y \leq m - (k-1)(k+1)/8$, $x + y = n - k + 2$, is one solution.

(5) *When $k \geq 2$ is even and $m \geq \max\{k(k-2) - 4, k(k+2)/8 + 1\}$, there is a solution to the k-Random WACP on Z_m for K_n if and only if $m \leq n \leq 2m - k(k-2)/4 - 2$, and the initial color value array*

$$c_0 = \left(\underbrace{0, \ldots, 0}_{x}, \underbrace{1, \ldots, 1}_{y}, 2, 3, \ldots, \frac{k-1}{2}, m - \frac{k-1}{2}, m - \frac{k-1}{2} + 1, \ldots, m-1\right) \tag{31}$$

satisfying $1 \leq x \leq m - k(k+2)/8$, $1 \leq y \leq m - k(k+2)/8$, $x + y = n - k + 2$, is one solution.

The proof and corresponding solutions constructed can be referred to [27, 28].

5. Conclusion

The All-Ones Problem comes from the theory of σ^+-automata, which is related to the graph dynamical system. In this paper, we introduce and study the generalization form of the All-Ones Problem, named "All-Colors Problem." The "All-Colors Problem" can be divided into Strong-All-Colors Problem and Weak-All-Colors Problem, respectively. We analyze and compute the solutions to these two problems on some interesting classes of graphs. At last we also introduce a new kind of Weak-All-Color Problem, k-Random Weak-All-Colors Problem, which is relevant to both combinatorial number theory and σ^+-automata.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References

[1] J. A. Aledo, S. Martinez, and J. C. Valverde, "Graph dynamical systems with general boolean states," *Applied Mathematics & Information Sciences*, vol. 9, no. 4, pp. 1803–1808, 2015.

[2] M. Macauley and H. S. Mortveit, "Cycle equivalence of graph dynamical systems," *Nonlinearity*, vol. 22, no. 2, pp. 421–436, 2009.

[3] J. A. Aledo, S. Martínez, F. L. Pelayo, and J. C. Valverde, "Parallel discrete dynamical systems on maxterm and minterm Boolean functions," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 666–671, 2012.

[4] J. A. Aledo, S. Martínez, and J. C. Valverde, "Parallel discrete dynamical systems on independent local functions," *Journal of Computational and Applied Mathematics*, vol. 237, no. 1, pp. 335–339, 2013.

[5] J. A. Aledo, S. Martínez, and J. C. Valverde, "Updating method for the computation of orbits in parallel and sequential dynamical systems," *International Journal of Computer Mathematics*, vol. 90, no. 9, pp. 1796–1808, 2013.

[6] C. L. Barrett, W. Y. Chen, and M. J. Zheng, "Discrete dynamical systems on graphs and Boolean functions," *Mathematics and Computers in Simulation*, vol. 66, no. 6, pp. 487–497, 2004.

[7] T. Mihaela, T. Matache, and J. Heidel, "Asynchronous random Boolean network model based on elementary cellular automata rule," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 71, no. 2, pp. 1–13, 2005.

[8] J. Kari, "Theory of cellular automata: a survey," *Theoretical Computer Science*, vol. 334, no. 1–3, pp. 3–33, 2005.

[9] K. Sutner, "Additive automata on graphsa," *Complex Systems*, vol. 2, no. 1, pp. 1–28, 1988.

[10] K. Sutner, "Problem 88-8," *Mathematical Intelligencer*, vol. 10, 1988.

[11] K. Sutner, "On σ-automata," *Complex Systems*, vol. 2, no. 1, pp. 1–28, 1988.

[12] J. A. Bondy and U. S. R. Murty, *Graph Theory with Application*, Macmillan, London, UK, 1976.

[13] U. Peled, "Problem 10197," *American Mathematical Monthly*, vol. 99, article 162, 1992.

[14] K. Sutner, "The σ-game and cellular automata," *The American Mathematical Monthly*, vol. 97, no. 1, pp. 24–34, 1990.

[15] K. Sutner, "σ-automata and Chebyshev-polynomials," *Theoretical Computer Science*, vol. 230, no. 1-2, pp. 49–73, 2000.

[16] R. Barua and S. Ramakrishnan, "σ-game, $\sigma+$-game and two-dimensional additive cellular automata," *Theoretical Computer Science*, vol. 154, no. 2, pp. 349–366, 1996.

[17] K. Sutner, "Linear cellular automata and the Garden-of-Eden," *The Mathematical Intelligencer*, vol. 11, no. 2, pp. 49–53, 1989.

[18] O. P. Lossers, "Solution to problem 10197," *American Mathematical Monthly*, vol. 100, pp. 806-807, 1993.

[19] H. Eriksson, K. Eriksson, and J. Sjöstrand, "Note on the lamp lighting problem," *Advances in Applied Mathematics*, vol. 27, no. 2-3, pp. 357–366, 2001.

[20] B. Curtis, J. Earl, D. Livingston, and B. Shader, "Resolution of conjectures related to Lights out! and cartesian products," *Electronic Journal of Linear Algebra*, vol. 34, pp. 718–724, 2018.

[21] R. Fleischer and J. Yu, "A survey of the game Lights Out!, space-efficient data structures, streams, and algorithms," vol. 8066 of *Part of the Lecture Notes in Computer Science book series*, pp. 176–198, 2013.

[22] H.-w. Huang, "Two-lit trees for lit-only σ-game," *Linear Algebra and its Applications*, vol. 438, no. 3, pp. 1057–1066, 2013.

[23] J. Hughes, "Random lights out processes on graphs," *Advances in Applied Mathematics*, vol. 51, no. 2, pp. 254–265, 2013.

[24] X. Wang and Y. Wu, "Lit-only σ-game on pseudo-trees," *Discrete Applied Mathematics*, vol. 158, no. 17, pp. 1945–1952, 2010.

[25] Y. Wu, "Lit-only sigma game on a line graph," *European Journal of Combinatorics*, vol. 30, no. 1, pp. 84–95, 2009.

[26] F. Galvin, "Solution to problem 88-8," *Mathematical Intelligencer*, vol. 11, pp. 31-32, 1989.

[27] L. Gallardo, G. Grekos, and J. Pihko, "On a variant of the Erdös-Ginzburg-Ziv problem," *Acta Arithmetica*, vol. 89, no. 4, pp. 331–336, 1999.

[28] C. Wang, "Note on a variant of the Erdös-Ginzburg-Ziv problem," *Acta Arithmetica*, vol. 108, no. 1, pp. 53–59, 2003.