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Research Article

Master-Slave Synchronization of Chaotic Φ^6 Duffing Oscillators by Linear State Error Feedback Control

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This paper is concerned with master-slave synchronization of chaotic Φ^6 Duffing oscillators by using linear state error feedback control. Compared with some existing methods and results, this paper estimates the bound of the first trajectory (variable) of the controlled slave system and uses this bound to derive synchronization criteria for two chaotic Φ^6 Duffing oscillators. The effectiveness of synchronization criteria is illustrated by three simulation examples.

1. Introduction

Synchronization of chaotic systems has received considerable attention due to its theoretical importance and practical applications in secure communication and signal processing (see for example, [1–31] and references therein).

As is well known, some models for damped and driven oscillators, such as stiffening springs, beam bulking, and superconducting Josephson parametric amplifiers, can be described as Φ^6 Duffing oscillators which have been widely used in mechanical and electrical systems [1, 9–11, 32–36]. With proper parameters, Duffing oscillators have exhibited chaotic behaviors. For chaotic Φ^6 Duffing oscillators, Njah [10, 11] used the active control to achieve master-slave synchronization, in which the active control removed all nonlinear terms of the error system. For chaotic Φ^4 Duffing oscillators which is the special case of Φ^6 Duffing oscillators, synchronization criteria were derived by the active control in [32–34, 37] and [35] in which the linear error system and synchronization criteria were derived. It should be pointed out that chaotic Φ^6 Duffing oscillators are nonlinear systems in which the nonlinear terms play a key role in the generation of chaotic attractors. Thus, how to use the nonlinear properties of the error system and how to use linear state error feedback control to derive synchronization criteria for chaotic Φ^6 Duffing oscillators is one motivation of this paper.

The bounds of trajectories of the master system and slave system have been widely used to derive the synchronization criteria for chaotic systems (see for example, [36, 38–40]). But it was difficult to estimate the bounds of slave systems. Therefore, how to derive the bound of some (not all) trajectories of the controlled slave system before the master system and the slave system achieve synchronization and how to use the derived bound to achieve synchronization criteria for the chaotic Φ^6 Duffing oscillators is another motivation of this paper.

In this paper, we will construct a master-slave synchronization scheme for chaotic Φ^6 Duffing oscillators by using linear state error feedback control. We will use the linear state error feedback control to derive the bound of the first trajectory of the slave system before the master system and the slave system achieve synchronization and use this bound to obtain synchronization criteria. Moreover, we will use three examples to illustrate the effectiveness of our synchronization criteria.

The rest of this paper is as follows. In Section 2, the related problems and concepts will be introduced. In Section 3, the synchronization results for chaotic Φ^6 Duffing oscillators will be given. As applications, the synchronization results for classic Φ^4 Duffing oscillators and parametrically excited Φ^4 Duffing oscillators will be provided in Section 4. In Section 5, three simulation results will be given. Conclusions and future works will be presented in Section 6.

2. Problem Statement

The mathematical model of Φ^6 Duffing oscillator is

$$\ddot{x}(t) = -c\dot{x}(t) - dx(t) - lx^3(t) - ax^5(t) + q\cos\omega t, \quad (1)$$

where $x(t)$ is the displacement of rotation angle; $\dot{x}(t) = dx(t)/dt$; $\ddot{x}(t) = d^2(x(t))/dt^2$; a, c, d, l, q, and ω are constants; $q\cos\omega t$ is the excitation; $c\dot{x}(t)$ is the linear damping term; $dx(t) + lx^3(t)$ is a nonlinear force; and the initial condition is $x(0) = x_0$ and $\dot{x}(0) = x_0'$. The potential of (1) is $W_6(x) = (1/2)dx^2(t) + (1/4)lx^4(t) + (1/6)ax^6(t)$, which is the reason why system (1) is called Φ^6 Duffing oscillator.

Remark 1. If $a = 0$, system (1) reduces to the following Φ^4 Duffing oscillator:

$$\ddot{x}(t) = -c\dot{x}(t) - dx(t) - lx^3(t) + q\cos\omega t, \quad (2)$$

with the potential $W_4(x) = (1/2)dx^2(t) + (1/4)lx^4(t)$.

Let $y_1(t) = x(t)$ and $y_2(t) = \dot{y}_1(t)$. The nonautonomous system (1) can be written as the following dimensionless system:

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = -dy_1(t) - cy_2(t) + g(y_1(t)) + p(t), \end{cases} \quad (3)$$

where

$$p(t) = q\cos\omega t,$$
$$g(y_1(t)) = -ly_1^3(t) - ay_1^5(t). \quad (4)$$

The initial condition of system (3) is given by $y_1(0) = y_{1_0}$, $y_2(0) = y_{2_0}$.

Let $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathbb{R}^2$. Write the system described by (3) as

$$\dot{y}(t) = Ay(t) + \varphi(y(t)) + r(t), \quad (5)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -d & -c \end{pmatrix},$$
$$r(t) = \begin{pmatrix} 0 \\ p(t) \end{pmatrix}, \quad (6)$$
$$\varphi(y(t)) = \begin{pmatrix} 0 \\ g(y_1(t)) \end{pmatrix}.$$

Let $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \in \mathbb{R}^2$. We can construct the following synchronization scheme for the system described by (5):

$$\mathscr{M} : \dot{y}(t) = Ay(t) + \varphi(y(t)) + r(t), \quad (7)$$

$$\mathscr{S} : \dot{z}(t) = Az(t) + \varphi(z(t)) + r(t) + u(t), \quad (8)$$

$$\mathscr{C} : u(t) = K(y(t) - z(t)), \quad (9)$$

with the master system described by $\mathscr{M}$ and the slave system described by $\mathscr{S}$, where $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathbb{R}^2$ is the controller and $K = \begin{pmatrix} k_1 & 1 \\ k_2 & k_3 \end{pmatrix}$ in which $k_1 > 0$, k_2, and k_3 are gains which can be determined later. The initial condition of system (8) is given by $z_1(0) = z_{1_0}$ and $z_2(0) = z_{2_0}$.

Defining a signal $e(t) = y(t) - z(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \in \mathbb{R}^2$, one can obtain the error system

$$\begin{cases} \dot{e}_1(t) = -k_1e_1(t), \\ \dot{e}_2(t) = -(k_2 + d)e_1(t) - (k_3 + c)e_2(t) + g(y_1(t)) \\ \quad -g(z_1(t)). \end{cases} \quad (10)$$

In view of differential mean theorem, one can have

$$g(y_1(t)) - g(z_1(t)) = g'(\xi(t))(y_1(t) - z_1(t)), \quad (11)$$

where

$$g'(\xi(t)) = \left.\frac{dg(\rho)}{d\rho}\right|_{\rho=\xi(t)} = -\left(3l\xi^2(t) + 5a\xi^4(t)\right), \quad (12)$$

for $\xi(t) \in (\min\{y_1(t), z_1(t)\}, \max\{y_1(t), z_1(t)\})$, which results in

$$\dot{e}(t) = \widehat{K}(t)e(t), \quad (13)$$

where

$$\widehat{K}(t) = \begin{pmatrix} -k_1 & 0 \\ -(k_2 + d) + g'(\xi(t)) & -(k_3 + c) \end{pmatrix}. \quad (14)$$

The initial condition of system (10) is $e_1(0) = y_{1_0} - z_{1_0}$, $e_2(0) = y_{2_0} - z_{2_0}$.

Notice that the master system described by (7) is chaotic. Thus, there exist two scales $m_1 > 0$ and $m_2 > 0$ for any y_{1_0} and y_{2_0} in the attracting area such that

$$|y_i(t)| \le m_i, \quad i = 1, 2, \ \forall t > 0. \quad (15)$$

From the first equation of system described by (10), we have

$$e_1(t) = \left(y_{1_0} - z_{1_0}\right)\exp(-k_1t), \quad (16)$$

which indicates that

$$z_1(t) = y_1(t) - \left(y_{1_0} - z_{1_0}\right)\exp(-k_1t). \quad (17)$$

From the equation described by (17), we have

$$\begin{aligned} |z_1(t)| &\le |y_1(t)| + |y_{1_0} - z_{1_0}| \\ &\le m_1 + |y_{1_0} - z_{1_0}|, \quad \forall t > 0, k_1 > 0. \end{aligned} \tag{18}$$

From inequalities (15) and (18) and equation (11), there exists a scale $\lambda > 0$ such that

$$|g'(\xi(t))| \le \lambda, \tag{19}$$

where

$$\begin{aligned} &\lambda = (3|l| + 5|a|\sigma)\sigma, \\ &\sigma = \max\{\zeta^2(t)\}, \\ &\forall \zeta(t) \in (\min\{y_1(t), z_1(t)\}, \max\{y_1(t), z_1(t)\}). \end{aligned} \tag{20}$$

Remark 2. Since the bounds of $y_1(t)$ and $z_1(t)$ for any $t > 0$ are given by (15) and (18), respectively, and $g(\cdot)$ and $g'(\cdot)$ are defined and differentiable, the bound of $|g'(\xi(t))|$ for $\xi(t) \in (\min\{y_1(t), z_1(t)\}, \max\{y_1(t), z_1(t)\})$ can be estimated by (19).

Remark 3. For the cascaded system described by

$$\dot{\zeta}_1(t) = \phi_1(t, \zeta_1(t)) + \chi(t, \zeta_1(t), \zeta_2(t)), \tag{21}$$

$$\dot{\zeta}_2(t) = \phi_2(t, \zeta_2(t)), \tag{22}$$

where ϕ_1, ϕ_2, and χ are smooth, the origin $(0, 0)$ is uniformly globally asymptotically stable if $\dot{\zeta}_1(t) = \phi_1(t, \zeta_1(t))$ and $\dot{\zeta}_2(t) = \phi_2(t, \zeta_2(t))$ are uniformly globally asymptotically stable and the solutions of (21) and (22) are uniformly globally bounded (Lemma 2, [41]). The system described by (10) can be regarded as a cascaded system. Although, it is easy to obtain the conditions to ensure that $\dot{\zeta}_1(t) = -k_1 e_1(t)$ and $\dot{\zeta}_2(t) = -(k_3 + c)e_2(t)$ are uniformly globally asymptotically stable, we cannot directly claim that the solutions of (10) are uniformly globally bounded. Thus, we cannot directly use Lemma 2 of [41] to study the stability of the error system (10).

The purpose of this paper is to investigate the master-slave synchronization for the system described by (1) and to find the controller gain K, such that the system described by (10) is globally asymptotically stable, which indicates that the system described by (7)–(9) synchronizes.

3. Main Results: Master-Slave Synchronization Criteria

In this section, we give some stability criteria for the error system described by (10), which ensures that the system described by (7)–(9) synchronizes.

Choosing the following Lyapunov function:

$$V(t) = e^T(t) P e(t), \tag{23}$$

where $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is a real positive matrix, we state and establish the following result.

Proposition 1. *The error system described by (10), (15), and (19) is globally asymptotically stable, i.e., the master system described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} \Theta_1 = -(k_2 + d)p_{12} - p_{11}k_1 + \lambda |p_{12}| < 0, \\ \Theta_2 = -p_{22}(k_3 + c) < 0, \\ \left(|-(k_1 + k_3 + c)p_{12} - (k_2 + d)p_{22}| + \lambda p_{22}\right)^2 \le 4\Theta_1\Theta_2. \end{cases} \tag{24}$$

Proof. Taking the derivative of $V(t)$ with respect to t along the trajectory of (10) yields

$$\dot{V}(t) = e^T(t) L(t) e(t), \tag{25}$$

where

$$L(t) = \widehat{K}^T(t) P + P\widehat{K}(t) = \begin{pmatrix} l_{11}(t) & l_{12}(t) \\ l_{12}(t) & l_{22}(t) \end{pmatrix}, \tag{26}$$

with

$$\begin{aligned} l_{11}(t) &= 2\left((g'(\xi(t)) - k_2 - d)p_{12} - p_{11}k_1\right), \\ l_{12}(t) &= -(k_1 + k_3 + c)p_{12} + (g'(\xi(t)) - k_2 - d)p_{22}, \\ l_{22}(t) &= -2p_{22}(k_3 + c). \end{aligned} \tag{27}$$

Conditions

$$\begin{aligned} &l_{11}(t) < 0, \\ &l_{22}(t) < 0, \\ &l_{12}^2(t) < l_{11}(t) l_{22}(t), \end{aligned} \tag{28}$$

can ensure

$$\dot{V}(t) < 0, \quad \forall e_1(t), e_2(t) \ne 0. \tag{29}$$

It follows from (19) and (27) that

$$\begin{aligned} \frac{l_{11}(t)}{2} &= g'(\xi(t))p_{12} - (k_2 + d)p_{12} - p_{11}k_1 \\ &\le -(k_2 + d)p_{12} - p_{11}k_1 + |g'(\xi(t))||p_{12}| \\ &\le -(k_2 + d)p_{12} - p_{11}k_1 + \lambda|p_{12}| \\ &= \Theta_1, \\ \frac{l_{22}(t)}{2} &= -p_{22}(k_3 + c) = \Theta_2, \\ l_{12}^2(t) &= \left(-(k_1 + k_3 + c)p_{12} + (g'(\xi(t)) - k_2 - d)p_{22}\right)^2 \\ &\le \left(|-(k_1 + k_3 + c)p_{12} - (k_2 + d)p_{22}| + |g'(\xi(t))|p_{22}\right)^2 \\ &\le \left(|-(k_1 + k_3 + c)p_{12} - (k_2 + d)p_{22}| + \lambda p_{22}\right)^2. \end{aligned} \tag{30}$$

From (30), one can see that inequalities (24) can guarantee inequalities (28). Thus, it follows from inequalities (19), (24), and (29) that the error system described by (10) is globally asymptotically stable. This completes the proof. Q.E.D.

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. One can derive the following synchronization result by Proposition 1.

Proposition 2. *The error system described by (10), (15), and (19) is globally asymptotically stable, i.e., the master system described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} k_1 > 0, c + k_3 > 0, \\ 2\sqrt{k_1(c + k_3)} - |d + k_2| > \lambda. \end{cases} \tag{31}$$

If $k_1 = k_3 = k$, we have the following corollary.

Corollary 1. *The error system described by (10), (15), and (19) is globally asymptotically stable, i.e., the master system described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} k > 0, c + k > 0, \\ 2\sqrt{k(c + k)} - |d + k_2| > \lambda. \end{cases} \tag{32}$$

If $k_1 = k_3 = k$ and $k_2 = 0$, the following result is obtained.

Corollary 2. *The error system described by (10), (15), and (19) is globally asymptotically stable, i.e., the master system described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} k > 0, c + k > 0, \\ 2\sqrt{k(c + k)} - |d| > \lambda. \end{cases} \tag{33}$$

In some applications, one can only measure the position variables for a chaotic system, which means that we only use $y_1(t) - z_1(t)$ in the feedback control. In this situation, one can obtain $k_3 = 0$. The corresponding result is given as follows.

Corollary 3. *The error system described by (10), (15), and (19) is globally asymptotically stable, i.e., the master system described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} k_1 > 0, c > 0, \\ 2\sqrt{k_1 c} - |d + k_2| > \lambda. \end{cases} \tag{34}$$

Furthermore, in the case of $k_2 = k_3 = 0$, one can have the following result.

Corollary 4. *The error system described by (10), (15), and (19) is globally asymptotically stable, i.e., the master system described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} k_1 > 0, c > 0, \\ 2\sqrt{k_1 c} - |d| > \lambda. \end{cases} \tag{35}$$

Remark 4. Njah [10, 11] constructed the master-slave synchronization scheme for the Φ^6 Duffing equation and studied master-slave synchronization by the active control, in which the active controller $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathbb{R}^2$ was

$$\begin{cases} u_1(t) = -\hat{a}e_1(t) - \hat{b}e_2(t), \\ u_2(t) = -a\left(y_1^5(t) - z_1^5(t)\right) - l\left(y_1^3(t) - z_1^3(t)\right) \\ \quad -\eta_1 e_1(t) - \eta_2 e_2(t), \end{cases} \tag{36}$$

where $\hat{a}, \hat{b}, \eta_1$, and η_2 are parameters for control inputs. Then, one can have the error system

$$\begin{cases} \dot{e}_1(t) = \hat{a}e_1(t) + (1 + \hat{b})e_2(t), \\ \dot{e}_2(t) = -(d - \eta_1)e_1(t) - (c - \eta_2)e_2(t). \end{cases} \tag{37}$$

Obviously, a linear error system described by (37) can be perfectly derived by the control (36) in [10, 11] which removed all nonlinear terms of the error system, and the stability criterion for the linear error system described by (37) can be easily obtained. However, the original Duffing oscillator (1) was completely canceled. Compared with control (36) in [10, 11], control (9) in this paper has two advantages. The first advantage is that the nonlinear term $y_1^3(t) - z_1^3(t)$ of the error system described by (10) is kept which means that the error system described by (10) is a nonlinear system, rather than a linear error system (37) in [10, 11]. The second advantage is that it is easy to estimate the bounds for $z_1(t)$ and $g'(\xi(t))$ by using (18) and (19), respectively, which are necessary for deriving the stability criterion for the error system described by (10).

Remark 5. In this paper, we only use the bound of $z_1(t)$ because it can be estimated by (18).

4. Applications to Master-Slave Synchronization of Chaotic Φ^4 Duffing Oscillators

4.1. Master-Slave Synchronization of Classic Φ^4 Duffing Oscillators. Now, we can study the synchronization of classic Φ^4 Duffing oscillator (2). Let $y_1(t) = x(t)$ and $y_2(t) = \dot{y}_1(t)$. One can derive the dimensionless system (3), in which $g(y_1(t))$ is replaced by $-ly_1^3(t)$. From (19), the bound of $g'(\cdot)$ can be estimated as

$$\left|g'(\xi(t))\right| \le \tilde{\lambda}, \tag{38}$$

where $\tilde{\lambda} = 3|l|\sigma$ in which $\sigma = \max\{\zeta^2(t)\}$, $\forall \zeta(t) \in (\min\{y_1(t), z_1(t)\}, \max\{y_1(t), z_1(t)\})$. One can construct the synchronization scheme described by (7)–(9) for the system described by (2) and derive the error system described by (13), where $g(y_1(t))$ is replaced by $-ly_1^3(t)$. Employing the Lyapunov function described by (23), one can have the following synchronization criterion for the Φ^4 Duffing oscillator described by (2).

Proposition 3. *The error system described by (13), (15), and (38) is globally asymptotically stable, i.e., the master system* *described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} \Theta_1 = -(k_2 + d)p_{12} - p_{11}k_1 + \tilde{\lambda}|p_{12}| < 0, \\ \Theta_2 = -p_{22}(k_3 + c) < 0, \\ \left(|-(k_1 + k_3 + c)p_{12} - (k_2 + d)p_{22}| + \tilde{\lambda}p_{22}\right)^2 \le 4\Theta_1\Theta_2. \end{cases} \tag{39}$$

If $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *in (23), one can obtain the following synchronization criterion for* Φ^4 *Duffing oscillator (2).*

Proposition 4. *The error system described by (13), (15), and (38) is globally asymptotically stable, i.e., the master system described by (7) and the slave system described by (8) achieve synchronization if*

$$\begin{cases} k_1 > 0, c + k_3 > 0, \\ 2\sqrt{k_1(c + k_3)} - |d + k_2| > \tilde{\lambda}. \end{cases} \tag{40}$$

Remark 6. Jiang [3] and Nijmeijer and Berghuis [9] studied the tracking control for Duffing oscillators which can be equivalent to master-slave synchronization for Duffing oscillators. The stability criteria for the error system were derived by using the control $u(t) = \tilde{K}e(t) - \varsigma(t)$ [3], where $\tilde{K} = \begin{pmatrix} 0 & 0 \\ k_4 & k_5 \end{pmatrix}$ and $\varsigma(t) = \begin{pmatrix} 0 \\ 3k_6 z_1^2(t)e_1(t) \end{pmatrix}$ in which k_4, k_5, and k_6 are gains, and the control $u(t) = \tilde{K}e(t) - v(t)$ [9], where $\tilde{K} = \begin{pmatrix} 0 & 0 \\ k_7 & k_8 \end{pmatrix}$ and $v(t) = \begin{pmatrix} 0 \\ 3y_1(t)z_1(t)e_1(t) \end{pmatrix}$ in which k_7 and k_8 are gains. It should be pointed out that those controls in [3, 9] were nonlinear feedback controls. Our control (9) $u(t) = K(y(t) - z(t))$ is a linear feedback control.

Remark 7. Han et al. [32] and Njah and Vincent [37] used the active control to derive synchronization criteria for chaotic Φ^4 Duffing oscillators, in which the active controller $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathbb{R}^2$ was

$$\begin{cases} u_1(t) = -z_1(t) + y_1(t) + \breve{k}_1 e_1(t), \\ u_2(t) = dz_1(t) + cz_2(t) + lz_1^3(t) - dy_1(t) - cy_2(t) \\ \quad -ly_1^3(t) + \breve{k}_2 e_2(t), \end{cases} \tag{41}$$

where $\breve{k}_1$ and $\breve{k}_2$ are feedback gains. Thus, the error system was

$$\begin{cases} \dot{e}_1(t) = -\breve{k}_1 e_1(t), \\ \dot{e}_2(t) = -\breve{k}_2 e_2(t). \end{cases} \tag{42}$$

As discussed in Remark 4, the error system described by (42) was a linear system. Compared with the control method in [32, 37], we keep the linear and nonlinear terms and fully use the term $-dy_1(t) - cy_2(t) - ly_1^3(t)$ to derive the synchronization criterion Proposition 3.

4.2. Master-Slave Synchronization of Parametrically Excited Φ^4 Duffing Oscillators.

Wu et al. [36] studied the following parametrically excited Φ^4 Duffing oscillator:

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_1(t) = (1 + \mu \sin \omega t)y_1(t) - y_1^3(t) - cy_2(t), \end{cases} \tag{43}$$

where μ and ω are constants, which can be rewritten as

$$\dot{y}(t) = \breve{A}(t)y(t) + \breve{\varphi}(y(t)), \tag{44}$$

where

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathbb{R}^2, \quad \breve{A}(t) = \begin{pmatrix} 0 & 1 \\ 1 + \mu \sin \omega t & -c \end{pmatrix}, \quad \breve{\varphi}(y(t)) = \begin{pmatrix} 0 \\ -y_1^3(t) \end{pmatrix}. \tag{45}$$

In [36], the master-slave scheme was constructed as follows:

$$\mathcal{M} : \dot{y}(t) = \breve{A}(t)y(t) + \breve{\varphi}(y(t)), \tag{46}$$

$$\mathcal{S} : \dot{z}(t) = \breve{A}(t)z(t) + \breve{\varphi}(z(t)) + u(t), \tag{47}$$

$$\mathcal{C} : u(t) = \breve{K}(y(t) - z(t)), \tag{48}$$

where $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \in \mathbb{R}^2$ and $\breve{K} = \begin{pmatrix} \breve{k}_{11} & \breve{k}_{12} \\ \breve{k}_{21} & \breve{k}_{22} \end{pmatrix}$ is a gain matrix which can be determined. The initial conditions of the master and slave system were $y_1(0) = y_{1_0}, y_2(0) = y_{2_0}$, and $z_1(0) = z_{1_0}, z_2(0) = z_{2_0}$, respectively. The error system was

$$\dot{e}(t) = (\breve{A}(t) + M(t) - \breve{K})e(t), \tag{49}$$

where

$$e(t) = y(t) - z(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = \begin{pmatrix} y_1(t) - y_2(t) \\ z_1(t) - z_2(t) \end{pmatrix},$$
$$M(t)e(t) = \breve{\varphi}(y(t)) - \breve{\varphi}(z(t)) \text{ with } M(t) = \begin{pmatrix} 0 & 0 \\ -F(t) & 0 \end{pmatrix},$$
$$F(t) = \left(y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)\right). \tag{50}$$

The initial conditions of the error system were $e_1(0) = e_{1_0} = y_1(0) - z_1(0), e_2(0) = e_{2_0} = y_2(0) - z_2(0)$.

Choosing our method, one can use the control as follows:

$$\mathcal{C} : u(t) = \overline{K}(y(t) - z(t)), \tag{51}$$

where $\overline{K} = \begin{pmatrix} \overline{k}_1 & 1 \\ \overline{k}_2 & \overline{k}_3 \end{pmatrix}$ in which $\overline{k}_1 > 0, \overline{k}_2$, and $\overline{k}_3$ can be determined later. From master-slave schemes (46), (47), and (51), one can have the following error system:

$$\begin{cases} \dot{e}_1(t) = -\overline{k}_1 e_1(t), \\ \dot{e}_2(t) = \left(1 + \mu \sin \omega t - \overline{k}_2\right) e_1(t) + \left(\overline{g}(y_1(t)) - \overline{g}(z_1(t))\right) \\ \quad -\left(c + \overline{k}_3\right) e_2(t), \end{cases} \tag{52}$$

where

$$\overline{g}(y_1(t)) = -y_1^3(t), \tag{53}$$

and the initial condition is $e_1(0) = e_{1_0}$ and $e_2(0) = e_{2_0}$. From (19), one can obtain

$$\begin{aligned} \overline{g}(y_1(t)) - \overline{g}(z_1(t)) &= -y_1^3(t) + z_1^3(t) = \overline{g}'(\xi(t)) \\ &\quad \cdot (y_1(t) - z_1(t)), \\ \overline{g}'(\xi(t)) &= -3\xi^2(t), \xi(t) \in (\min\{y_1(t), z_1(t)\}, \\ &\quad \max\{y_1(t), z_1(t)\}). \end{aligned} \tag{54}$$

Then, the error system described by (52) can be rewritten as

$$\dot{e}(t) = \acute{K}(t) e(t), \tag{55}$$

where

$$\acute{K}(t) = \begin{pmatrix} -\overline{k}_1 & 0 \\ 1 + \mu \sin \omega t - \overline{k}_2 + \overline{g}'(\xi(t)) - c - \overline{k}_3 \end{pmatrix}. \tag{56}$$

It follows from (19) that there exists a scale $\overline{\lambda} > 0$ such that

$$\left|\overline{g}'(\xi(t))\right| \le \overline{\lambda}, \tag{57}$$

where

$$\begin{aligned} \overline{\lambda} &= 3\sigma, \\ \sigma &= \max\left\{\zeta^2(t)\right\}, \end{aligned} \tag{58}$$

$\forall \zeta(t) \in (\min\{y_1(t), z_1(t)\}, \max\{y_1(t), z_1(t)\})$. By using the similar proof of Proposition 1, one can have the synchronization result for (43).

Proposition 5. *The error system described by (52) is globally asymptotically stable, i.e., the master system described by (46) and the slave system described by (47) achieve synchronization if*

$$\begin{cases} \overline{k}_1 > 0, c + \overline{k}_3 > 0, \\ 2\sqrt{\overline{k}_1\left(c + \overline{k}_3\right)} - \left|1 + |\mu| + \overline{k}_2\right| > \widetilde{\lambda}. \end{cases} \tag{59}$$

5. Simulation Study

5.1. Simulation for Chaotic Φ^6 Duffing Oscillators. Consider the chaotic Φ^6 Duffing oscillator (1) with $a = 0.1$, $c = 0.4$, $d = 1.1$, $l = 0.4$, $q = 1.8$, and $\omega = 2.1$. If we choose the initial condition of (1) as $y_{1_0} = 0$ and $z_{1_0} = -1$, there is an attractor which is demonstrated in Figure 1. From Figure 1, one can obtain that the bound of $|y_1(t)|$ is 1.4, i.e., $m_1 = 1.4$.

For the master-slave scheme (7)–(9) with (1), one can choose the initial condition of the slave system as $z_{1_0} = 0.1$ and $z_{2_0} = -1.5$. By virtue of (18), one can obtain $|z_1(t)| \le 1.4 + 0.1 = 1.5$. It follows from (19) that $|g'(\xi(t))| \le 5.2313 = \lambda$. Let $k_2 = 0$ and $k_1 = k_3$. From Corollary 2, we have $\sqrt{k_1(c + k_1)} > ((\lambda + d)/2)$, which implies that $k_1 \ge 2.97$. We choose $k_1 = 2.98$.

Figures 2–4 give the simulation results for the master system, the slave system, and the error system with $k_1 = 2.98$, $k_2 = 0$, and $k_3 = 2.98$, respectively, from which one can see that the error system (10) is globally asymptotically stable; i.e., the master-slave synchronization scheme described by (7)–(9) indeed achieves synchronization.

5.2. Simulation for Classic Φ^4 Duffing Oscillators. For the classic Φ^4 Duffing oscillator (2), parameters c, d, l, q, and ω are the same as those defined in the abovementioned Φ^6 Duffing oscillators. If we choose the initial condition of (1) as $y_{1_0} = 0$ and $z_{1_0} = -1$, there is an attractor which is demonstrated in Figure 5. From Figure 5, we have that the bound of $|y_1(t)|$ is 1.4, i.e., $m_1 = 1.4$.

Consider the master-slave scheme (7)–(9) with (2), where the initial condition of the slave system is $z_{1_0} = 0.1$ and $z_{2_0} = 1.5$. From (18), we have $|z_1(t)| \le 1.4 + 0.1 = 1.5$. It follows from (38) that $|g'(\xi(t))| \le 2.7 = \widetilde{\lambda}$. Let $k_2 = 0$ and $k_1 = k_3$. From Proposition 4, we have $\sqrt{k_1(c + k_1)} > ((\lambda + d)/2)$, which implies that $k_1 \ge 1.71$. We choose $k_1 = 1.72$.

Figures 6–8 give the simulation results for the master system, the slave system, and the error system with $k_1 = 1.72$, $k_2 = 0$, and $k_3 = 1.72$, respectively. It follows from Figures 6–8 that the error system (10) is globally asymptotically stable; i.e., the master-slave synchronization scheme described by (7)–(9) indeed achieves synchronization.

5.3. Simulation for Parametrically Excited Φ^4 Duffing Oscillators. Now, we study the synchronization of parametrically excited Φ^4 Duffing oscillators (43) where $c = 0.2$, $\mu = 0.5$, and $\omega = 1$. The initial conditions of master system (46) and slave system (47) are $y_1(0) = 1.6$, $y_2(0) = 0.2$ and $z_1(0) = 2$, $z_2(0) = 1.4$, respectively. It follows from Figure 9 that the up bound of $|y_1(t)|$ is 1.66. However, we can see that the up bound of $|z_1(t)|$ is larger than 2 because $z_1(0) = 2$.

It follows from (57) that $|z_1(t)| \le 1.66 + 0.4 = 2.06$ and $|\overline{g}'(\xi(t))| \le 3 \times 2.06^2 = 12.7308 = \overline{\lambda}$. Let $\overline{k}_1 = \overline{k}_3$ and $\overline{k}_2 = 0$. Using Proposition 5, one can have $\overline{k}_1 = \overline{k}_3 > 7.01$. Let $\overline{k}_1 = \overline{k}_3 = 7.1$. Figure 10 illustrates that the error system is globally asymptotically stable; i.e., the master-slave

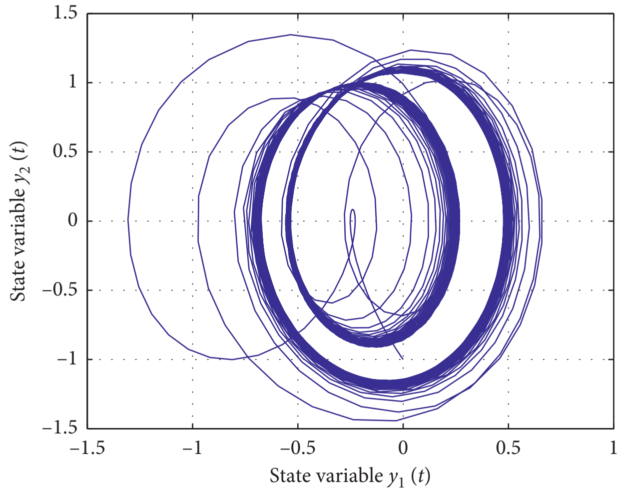

Figure 1: The phase diagram of the chaotic Φ^6 Duffing oscillator.

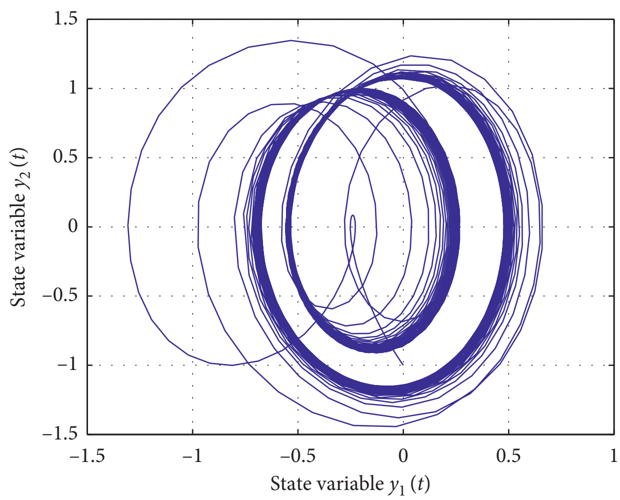

Figure 2: The simulation result for the master system.

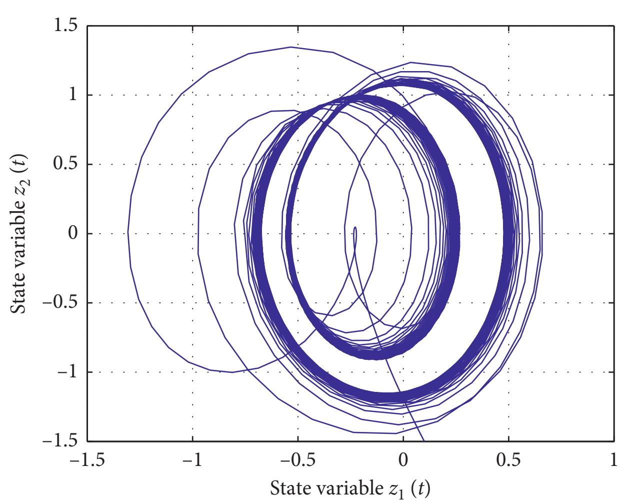

Figure 3: The simulation result for the slave system with $k_1 = 2.98, k_2 = 0$, and $k_3 = 2.98$.

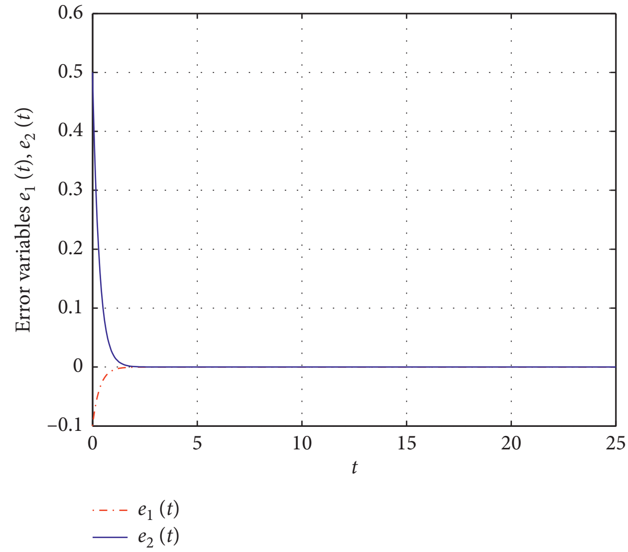

Figure 4: The simulation result for the error system with $k_1 = 2.98, k_2 = 0$, and $k_3 = 2.98$.

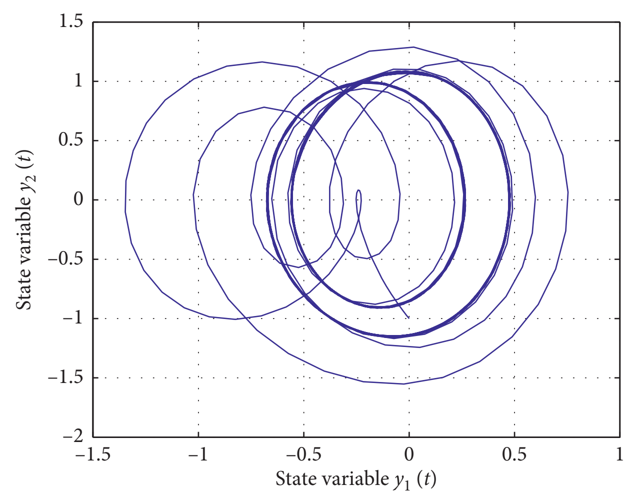

Figure 5: The phase diagram of the chaotic Φ^4 Duffing oscillator.

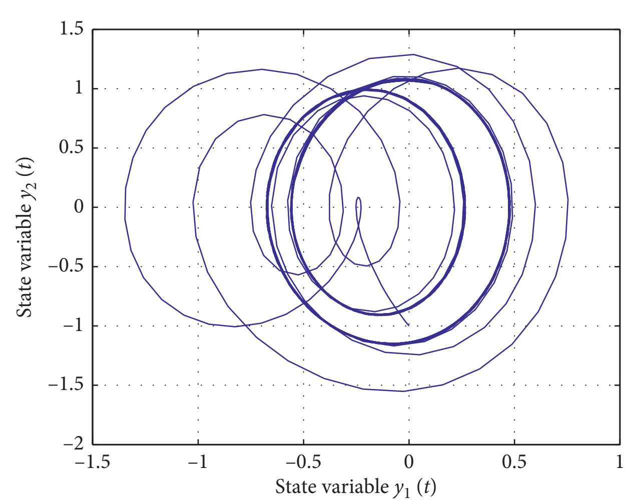

Figure 6: The simulation result for the master system.

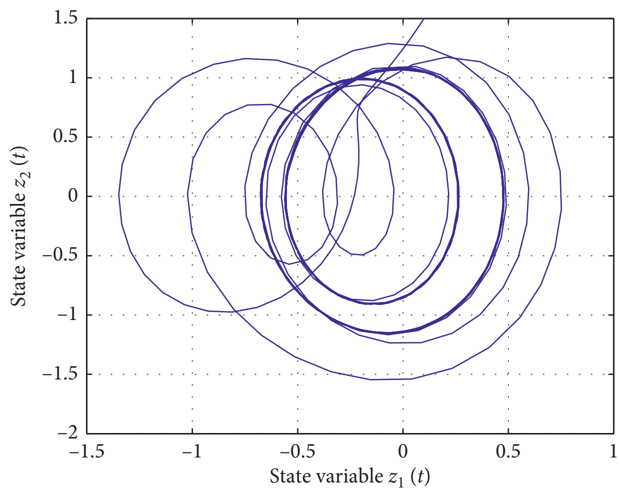

Figure 7: The simulation result for the slave system with $k_1 = 1.72, k_2 = 0$, and $k_3 = 1.72$.

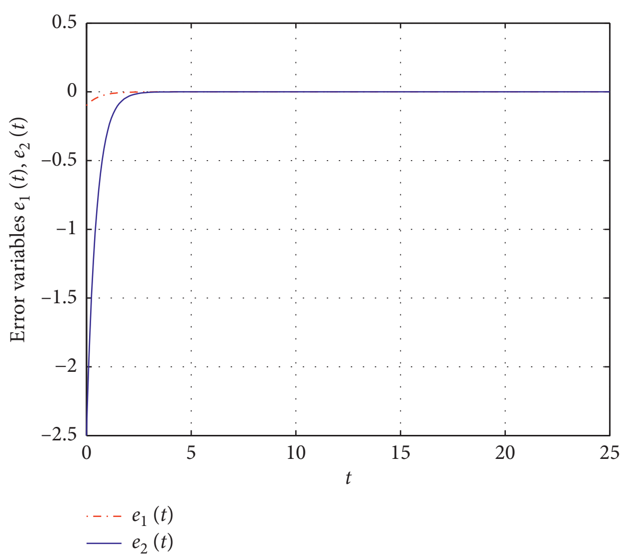

Figure 8: The simulation result for error system with $k_1 = 1.72, k_2 = 0$, and $k_3 = 1.72$.

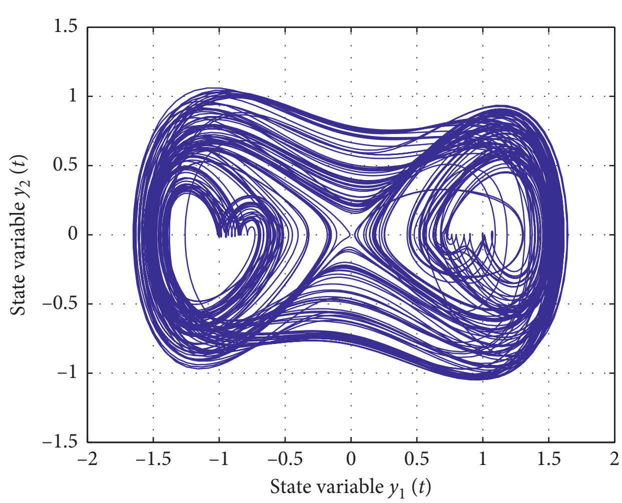

Figure 9: The simulation result for the master system.

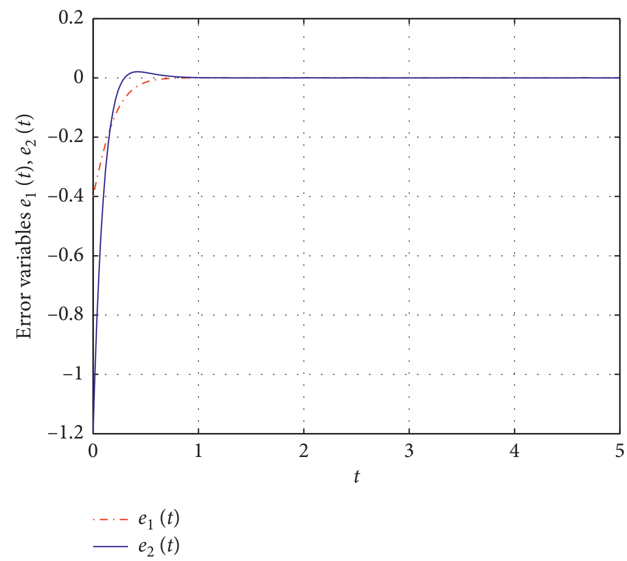

Figure 10: The simulation result for the error system with $k_1 = 7.1, k_2 = 0$, and $k_3 = 7.1$.

synchronization scheme described by (46), (47), and (51) indeed achieves synchronization.

6. Conclusion

We have constructed a master-slave synchronization scheme for chaotic Φ^6 Duffing oscillators by using linear feedback control. By estimating the first trajectory of the controlled slave system and keeping the nonlinear property of the error system, we have derived some synchronization criteria. Then, we have used three examples to illustrate the effectiveness of synchronization criteria for Duffing oscillators. In this paper, master and slave systems are all Φ^6 Duffing oscillators. The synchronization between Φ^6 and Φ^4 Duffing oscillators and the synchronization between Φ^6 Duffing oscillators with different parameters are our future research interests. Moreover, how to design the time delayed feedback control to achieve synchronization between Φ^6 Duffing oscillators with different parameters can be our future research interest as well.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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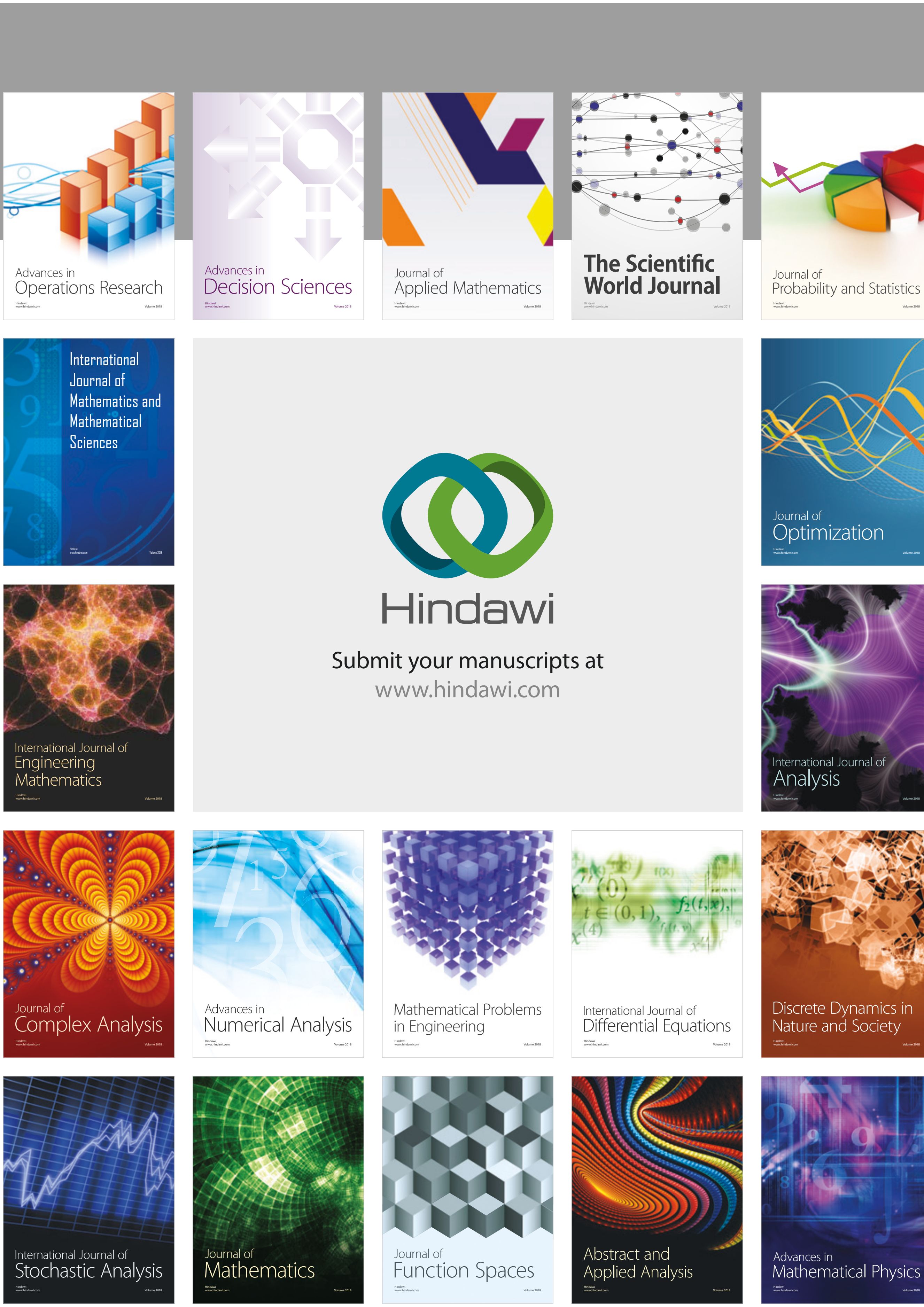
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