Hindawi
Complexity
Volume 2019, Article ID 4096981, 12 pages
https://doi.org/10.1155/2019/4096981

Research Article

Optimizing Pinned Nodes to Maximize the Convergence Rate of Multiagent Systems with Digraph Topologies

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Received 3 August 2018; Accepted 5 December 2018; Published 10 January 2019

Academic Editor: Danilo Comminiello

This paper investigates how to choose pinned node set to maximize the convergence rate of multiagent systems under digraph topologies in cases of sufficiently small and large pinning strength. In the case of sufficiently small pinning strength, perturbation methods are employed to derive formulas in terms of asymptotics that indicate that the left eigenvector corresponding to eigenvalue zero of the Laplacian measures the importance of node in pinning control multiagent systems if the underlying network has a spanning tree, whereas for the network with no spanning trees, the left eigenvectors of the Laplacian matrix corresponding to eigenvalue zero can be used to select the optimal pinned node set. In the case of sufficiently large pinning strength, by the similar method, a metric based on the smallest real part of eigenvalues of the Laplacian submatrix corresponding to the unpinned nodes is used to measure the stabilizability of the pinned node set. Different algorithms that are applicable for different scenarios are develped. Several numerical simulations are given to verify theoretical results.

1. Introduction

Control problems in multiagent systems and complex networks are widely studied in recent years. In multiagent systems, consensus means that all agents will converge to some common state. Many algorithms are proposed to assure consensus [1–9]. Among them, the following linear interaction rule is commonly used:

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{m} a_{ij}\left[x_j(t) - x_i(t)\right], \quad i = 1, \ldots, m \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i and $a_{ij} \geq 0$ is the coupling strength from agent j to i. By graph theory, $A = [a_{ij}]$ can be seen as the weight matrix of a weighted directed graph. In most of the existing literatures, the concept of spanning tree is widely used to describe the communicability among agents in networks that can guarantee consensus [5–7]. It was proven that when the underlying graph has a spanning tree, the agents will agree on a common value, which is a linear combination of the initial states of all agents [8, 9]. However, in some cases, it is desirable to steer the state to a prescribed value s. For this purpose, the pinning control strategy could be applied:

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{m} a_{ij}\left[x_j(t) - x_i(t)\right] - \kappa d_i\left[x_i(t) - s\right], \qquad i = 1, \dots, m \tag{2}$$

where κ is the pinning strength; d_i takes value 1 if agent i is pinned and 0 otherwise.

In [10], it was proven that a pinned node set can stabilize a directed network to some unstable trajectories if and only if the pinned node set can access all the other vertices in the digraph. In [11], it was proven that a single controller can pin a coupled complex network to a homogenous solution.

These studies mainly concern how to choose pinned node(s) to stabilize the network. However, the pinned node set that can stabilize the network is not unique, and the stability performance under various pinned node sets can be different. Therefore, the following question is raised: given the number of feedback controllers, which nodes should be pinned to stabilize the network best?

Up unitl now, many centrality measures (see [12]), such as degree, closeness, betweenness, and eigenvector centrality, have been proposed to assess the influence of nodes in the network. In [13–15], it was concluded that, for heterogeneous networks, pinning nodes with high degree or betweenness centrality perform better than the randomly pinning scheme, whereas, for homogeneous networks, there is no significant difference between random pinning and selective pinning schemes.

However, when the number of pinned nodes is sufficiently large, it was shown in [16] that pinning nodes that are adjacent to those with highest degree have good performance, whereas, for scale-free networks, it was observed in [15, 17] that pinning nodes with small degree centrality or randomly pinning perform better than pinning nodes with high degree centrality. In [18, 19], a metric based on the Laplacian eigenratio was proposed to quatify local controllability of the network. A similar metric was also used for pinning controllability of undirected and unweighted networks in [20], where, by sensitivity analysis of the eigenratio, it was shown that the magnitude of elements in the eigenvector corresponds to the largest eigenvalue of the Laplacian that can be used to assess the importance of nodes. In [21], a new centrality named ControlRank (CR) was proposed for strongly connected networks. The CR is a dual form of PageRank, which can be seen as a variation of eigenvector centrality. But, as pointed out in [21], such a centrality cannot be used for disconnected networks.

In this paper, we search for the pinned nodes that maximize the convergence rate of multiagent systems in cases of sufficiently small and large pinning strength. In the case of sufficently small pinning strength, perturbation methods are employed for analysis; we show that the left eigenvector(s) corresponding to eigenvalue 0 of the Laplacian matrix can be used to select the optimal pinned node set. In the case of sufficiently large pinning strength, a metric based on the smallest real part of eigenvalues of the Laplacian submatrix corresponding to the unpinned nodes is used to measure the controllability of pinned node set, which leads to a strategy to obtain the optimal pinned node set.

2. Preliminary

Given a matrix A, denote a_{ij} as the element of A on the i-th row and j-th column. Let $\mathbf{1}$ and $\mathbf{0}$ denote the column vectors with each element being 1 and 0, respectively. For any vector x, $\mathrm{diag}(x)$ denotes the diagonal matrix with its i-th diagonal element being the i-th element of x; $[x]_i$ denotes the i-the element of x; $x^\top$ denotes the transpose of x. For a matrix $A \in \mathbb{R}^{m\times m}$, denote its i-th smallest eigenvalue (in real part) by $\lambda_i(A)$, $i = 1, \dots, m$. I_n denotes the $n \times n$ identity matrix.

A weighted directed graph $\mathcal{G}$ consists of a node set $\mathcal{V}(\mathcal{G})$, numbered by $\{1, \dots, m\}$, a directed edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$, where $e(i, j) \in \mathcal{E}(\mathcal{G})$ if and only if there exists an edge from node j to node i, and a weighted matrix $A = [a_{ij}]$, where $a_{ij} > 0$ denotes the weight of edge $e(i, j)$. The graph is said to have a spanning tree if there is a node called *root* such that, for any other node j, there is a path from the root to j. Define $L = [l_{ij}]$ by $l_{ij} = -a_{ij}$ if $i \neq j$ and $l_{ii} = -\sum_{j=1, j\neq i}^{m} l_{ij}$. We call L the Laplacian matrix of the graph $\mathcal{G}$.

Denote $y_i = x_i(t) - s$, $y = [y_1(t), \dots, y_m(t)]^\top$, and $D = \mathrm{diag}([d_1, \dots, d_m])$. System (2) can be rewritten as

$$\dot{y}(t) = -[L + \kappa D]\, y(t). \tag{3}$$

Denote by $\mathcal{D}$ the pinned node set in the following. Hence, $i \in \mathcal{D}$ is equivalent to $d_i = 1$.

Noting that pinning consensus is a special case of pinning synchronization, we can apply the result given in [10, 11] to system (3). Then we have the following Lemma.

Lemma 1 (see [10, 11]). *All the eigenvalues of matrix $L + \kappa D$ have positive real parts if and only if the pinned node set $\mathcal{D}$ can access all the other vertices.*

In this paper, we aim to find a set of p nodes whose pinning increases the convergence rate of the system maximally. It is known that the eigenvalue of $L + \kappa D$ with the smallest real part, denoted by $\lambda_1(L + \kappa D)$, gives the lower bound of the convergence rate of system (3). Therefore, the optimization problem in this paper is formalized as follows: *given the Laplacian matrix L, the pinning strength κ, and the number of pinned nodes p, find a pinned node set $\mathcal{D}$ that reaches the following maximum:*

$$\max_{\mathcal{D}:|\mathcal{D}|=p} \mathrm{Re}\left(\lambda_1(L + \kappa D)\right). \tag{4}$$

Our investigation considers two extreme situations when the pinning strength κ is sufficiently small or sufficiently large.

3. Small Pinning Strength

3.1. Network Has a Spanning Tree. Suppose that $\mathcal{G}$ has a spanning tree and L is the Laplacian matrix. By the Gerschgorin theorem [22] and the Perron-Frobenius theorem [23], we have that 0 is the smallest eigenvalue of L and its associated left eigenvector, denoted by ξ, is nonnegative or nonpositive. With loss of generality, we assume that $\sum_{i=1}^{m}[\xi]_i = 1$ and $[\xi]_i \geq 0$ holds for $\forall i$. Moreover, it can be seen that $\mathbf{1}$ is the right eigenvector of L corresponding to eigenvalue 0. By the perturbation theory [24, 25], the smallest eigenvalue of $L+\kappa D$ has the form

$$\lambda_1(L+\kappa D) = 0 + \kappa \cdot \lambda^{(1)}(D) + \kappa^2 \cdot \lambda^{(2)}(D) + o\left(\kappa^2\right) \quad (5)$$

and its associated right eigenvector, denoted by $\eta_1(L+\kappa D)$, satisfies

$$\eta_1(L+\kappa D) = \mathbf{1} + \kappa \cdot \eta^{(1)}(D) + \kappa^2 \cdot \eta^{(2)}(D) + o\left(\kappa^2\right) \quad (6)$$

as $\kappa \longrightarrow 0$, where $o(s)$ denotes terms that satisfy $\lim_{s\longrightarrow 0}|o(s)|/s = 0$. By $(L+\kappa D)\cdot\eta_1(L+\kappa D) = \lambda_1(L+\kappa D)\cdot\eta_1(L+\kappa D)$, we have

$$\begin{aligned}
&L\mathbf{1} + \kappa\left[D\mathbf{1} + L\eta^{(1)}(D)\right] + \kappa^2\left[D\eta^{(1)}(D) + L\eta^{(2)}(D)\right] \\
&\quad + o\left(\kappa^2\right) \\
&= \kappa\lambda^{(1)}(D)\,\mathbf{1} + \kappa^2\left[\lambda^{(1)}(D)\,\eta^{(1)}(D) + \lambda^{(2)}(D)\,\mathbf{1}\right] \\
&\quad + o\left(\kappa^2\right)
\end{aligned} \quad (7)$$

which leads to

$$D\mathbf{1} + L\eta^{(1)}(D) = \lambda^{(1)}(D)\,\mathbf{1}, \quad (8)$$

$$D\eta^{(1)}(D) + L\eta^{(2)}(D) = \lambda^{(1)}(D)\,\eta^{(1)}(D) + \lambda^{(2)}(D)\,\mathbf{1}. \quad (9)$$

Multiplying the vector $\xi^\top$ from left to (8) on both sides leads to

$$\xi^\top D\mathbf{1} = \lambda^{(1)}(D)\,\xi^\top\mathbf{1} \quad (10)$$

since $\xi^\top L = \mathbf{0}^\top$. Then we have that

$$\lambda^{(1)}(D) = \frac{\xi^\top D\mathbf{1}}{\xi^\top\mathbf{1}} = \xi^\top D\mathbf{1} = \sum_{i=1}^{m}[\xi]_i\, d_i. \quad (11)$$

Suppose that p is the number of feedback controllers and $j_1,\ldots,j_m$ is a permutation of $1,\ldots,m$ such that $[\xi]_{j_1} \geq [\xi]_{j_2} \geq \cdots \geq [\xi]_{j_m}$. Then we get

$$\lambda^{(1)}(D) = \sum_{i=1}^{m}[\xi]_i\, d_i \leq \sum_{k=1}^{p}[\xi]_{j_k} \quad (12)$$

and the equality sign holds if $\mathscr{D} = \{j_1,\ldots,j_p\}$. Let $\mathscr{D}_0 = \{j_1,\ldots,j_p\}$. Apparently, if $[\xi]_{j_p} > [\xi]_{j_{p+1}}$, $\mathscr{D}_0$ is the unique set that maximizes $\lambda^{(1)}(D)$. However, if $[\xi]_{j_p} = [\xi]_{j_{p+1}}$, the set $\mathscr{D}_1 = \{j_1,\ldots,j_{p-1},j_{p+1}\}$ also maximizes $\lambda^{(1)}(D)$. Now we need to compare $\mathscr{D}_0$ with other sets that also maximize $\lambda^{(1)}(D)$ and find the unqiue one that maximizes $\mathrm{Re}(\lambda^{(2)}(D))$.

Analogously, multiplying the vector $\xi^\top$ from left to (9) on both sides and employing the equalities $\xi^\top L = \mathbf{0}^\top$, $\xi^\top\mathbf{1} = 1$, we get

$$\lambda^{(2)}(D) = \left[\xi^\top D - \lambda^{(1)}(D)\,\xi^\top\right]\eta^{(1)}(D). \quad (13)$$

Next, we need to compute $\eta^{(1)}(D)$. Suppose that $J = Q^{-1}LQ$ is a Jordan form matrix of L with $Q = [\mathbf{1},\widehat{Q}]$ and $J = \mathrm{diag}\{J_1, J_2,\ldots,J_K\}$, where J_k is a Jordan block, $k = 1,\ldots,K$. Notice that 0 is a simple eigenvalue of L; we get that $J_1 = 0$ and $J_2,\ldots,J_K$ are nonsingular. Let $Q^{-1} = [\xi,\widehat{P}^\top]^\top$ and $J = \mathrm{diag}\{0,\widehat{J}\}$. It can be derived from (8) that

$$\eta^{(1)}(D) = a\mathbf{1} - \widehat{Q}\widehat{J}^{-1}\widehat{P}D\mathbf{1} \quad (14)$$

where $a \in \mathbb{C}$ is some constant number. Substituting (14) into (13), we get

$$\lambda^{(2)}(D) = -\xi^\top D\widehat{Q}\widehat{J}^{-1}\widehat{P}D\mathbf{1}. \quad (15)$$

Now we have the following result.

Theorem 2. *Suppose that L has a spanning tree, $\xi \in \mathbb{R}^m$ with $\sum_{i=1}^{m}[\xi]_i = 1$ is the left eigenvector of L associated with eigenvalue 0, and p is the number of pinned nodes. Let $j_1,\ldots,j_m$ be a permutation of $1,\ldots,m$ such that $[\xi]_{j_1} \geq \cdots \geq [\xi]_{j_m}$.*

(1) *If $[\xi]_{j_p} > [\xi]_{j_{p+1}}$,*

$$\begin{aligned}
&\max_{\mathscr{D}:|\mathscr{D}|=p}\mathrm{Re}\left(\lambda_1(L+\kappa D)\right) = \kappa\cdot\max_{\mathscr{D}:|\mathscr{D}|=p}\lambda^{(1)}(D) + o(\kappa) \\
&\quad = \kappa\cdot\sum_{k=1}^{p}[\xi]_{j_k} + o(\kappa)
\end{aligned} \quad (16)$$

as $\kappa \longrightarrow 0$, and the maximum is reached when $\mathscr{D} = \{j_1,\ldots,j_p\}$ and it is the unique set that maximizes $\lambda^{(1)}(D)$.

(2) *If $[\xi]_{j_p} = [\xi]_{j_{p+1}}$, let*

$$\mathscr{P} = \left\{\mathscr{D} : |\mathscr{D}| = p,\ \sum_{i\in\mathscr{D}}[\xi]_i = \sum_{k=1}^{p}[\xi]_{j_k}\right\}. \quad (17)$$

Then

$$\begin{aligned}
&\max_{\mathscr{D}:|\mathscr{D}|=p}\mathrm{Re}\left(\lambda_1(L+\kappa D)\right) \\
&\quad = \kappa\cdot\max_{\mathscr{D}:|\mathscr{D}|=p}\left[\lambda^{(1)}(D) + \kappa\cdot\mathrm{Re}\left(\lambda^{(2)}(D)\right)\right] + o\left(\kappa^2\right) \\
&\quad = \kappa\cdot\sum_{k=1}^{p}[\xi]_{j_k} + \kappa^2 \\
&\quad\quad \cdot\max_{\mathscr{D}:\mathscr{D}\in\mathscr{P}}\mathrm{Re}\left(-\xi^\top D\widehat{Q}\widehat{J}^{-1}\widehat{P}D\mathbf{1}\right) + o\left(\kappa^2\right)
\end{aligned} \quad (18)$$

as $\kappa \longrightarrow 0$, where $J = Q^{-1}LQ$ is a Jordan form matrix of L with $Q = [\mathbf{1},\widehat{Q}]$, $\widehat{P} = [\mathbf{0}, I_{m-1}]Q^{-1}$, and $\widehat{J} = [\mathbf{0}, I_{m-1}]J[\mathbf{0}, I_{m-1}]^\top$.

Remark 3. Theorem 2 indicates that the magnitude of elements in the left eigenvector corresponding to eigenvalue 0 of the Laplacian matrix can be used to measure pinned nodes' effect on the convergence rate of the system as $\kappa \longrightarrow 0$.

3.2. Network without Spanning Trees. Without loss of generality, we suppose that L has the following form [26]:

$$L = \begin{bmatrix} L_{11} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & L_{K_1K_1} & 0 & \cdots & 0 \\ L_{K_1+1,1} & \cdots & L_{K_1+1,K_1} & L_{K_1+1,K_1+1} & \cdots & L_{K_1+1,K_1+K_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L_{K_1+K_2,1} & \cdots & L_{K_1+K_2,K_1} & L_{K_1+K_2,K_1+1} & \cdots & L_{K_1+K_2,K_1+K_2} \end{bmatrix} \tag{19}$$

Here, each $L_{ii} \in \mathbb{R}^{l_i,l_i}$, $i = 1,\ldots,K_1$ is a Laplacian matrix and its corresponding subgraph has a spanning tree (isolated node can be seen as a subgraph that has a spanning tree with itself being the root). In [26], the subgraphs corresponding to L_{ii}, $i = 1,\ldots,K_1$, were called primary layer subgraphs and an algorithm was given to extract them. Denote by $\xi_{(i)}$ the left eigenvector of L_{ii}, $i = 1,\ldots,K_1$, corresponding to 0 and $\sum_j[\xi_{(i)}]_j = 1$. Without loss of generality, we assume that $[\xi_{(i)}]_1 \geq \cdots \geq [\xi_{(i)}]_{l_i}$; otherwise, we can permutate the indices of nodes of the graph topology to get the above ordering.

Meanwhile, $\forall i \in \{K_1+1,\ldots,K_1+K_2\}$, the subgraph corresponding to L_{ii} has a spanning tree and there exists $j \neq i$ such that $L_{ij} \neq 0$. Denote

$$\widehat{L} = \begin{bmatrix} L_{K_1+1,K_1+1} & \cdots & L_{K_1+1,K_1+K_2} \\ \vdots & \ddots & \vdots \\ L_{K_1+K_2,K_1+1} & \cdots & L_{K_1+K_2,K_1+K_2} \end{bmatrix}. \tag{20}$$

By Gershgorin circle theorem [22], we get that all eigenvalues of $\widehat{L}$ have positive real parts. Therefore, 0 is the smallest eigenvalue of L and has K_1 different associated eigenvectors. Denote by $\{\eta_i, i = 1,\ldots,K_1\}$ and $\{\xi_i, i = 1,\ldots,K_1\}$ the right and left eigenspaces of L corresponding to eigenvalue 0. Suppose that

$$\eta_i = \left[\left\{\mathbf{0}^{l_1+\cdots+l_{i-1}}\right\}^\top, \left\{\mathbf{1}^{l_i}\right\}^\top, \left\{\mathbf{0}^{l_{i+1}+\cdots,l_{K_1}}\right\}^\top, \#\right]^\top, \tag{21}$$

$$\xi_i = \left[\left\{\mathbf{0}^{l_1+\cdots+l_{i-1}}\right\}^\top, \xi_{(i)}^\top, \left\{\mathbf{0}^{m-l_1-\cdots-l_{K_1}}\right\}^\top\right]^\top, \quad i = 1,\ldots,K_1. \tag{22}$$

Denote by $\widetilde{\lambda}_i(\kappa,D)$, $i = 1,\ldots,K_1$, the K_1 smallest eigenvalues of $L+\kappa D$ and by $\widetilde{\eta}_i(\kappa,D)$, $i = 1,\ldots,K_1$, the associated right eigenvectors. For any fixed D, we regard $\widetilde{\lambda}_i(\kappa,D)$ and $\widetilde{\eta}_i(\kappa,D)$ as functions of κ with $\widetilde{\lambda}_i(0,D) = 0$, $\widetilde{\eta}_i(0,D) = \eta_i$, $i = 1,\ldots,K_1$. By a perturbation expansion [24, 25], we have

$$\widetilde{\lambda}_i(\kappa,D) = 0 + \kappa\lambda_i^{(1)} + o(\kappa), \quad i = 1,\ldots,K_1 \tag{23}$$

$$\widetilde{\eta}_i(\kappa,D) = \eta_i + \kappa\eta_i^{(1)} + o(\kappa), \quad i = 1,\ldots,K_1 \tag{24}$$

as $\kappa \longrightarrow 0$. Then by

$$\begin{aligned}(L+\kappa D)\cdot\left[\eta_i + \kappa\eta_i^{(1)} + o(\kappa)\right] \\ = \left[\kappa\lambda_i^{(1)} + o(\kappa)\right]\cdot\left[\eta_i + \kappa\eta_i^{(1)} + o(\kappa)\right],\end{aligned} \tag{25}$$

we have

$$L\eta_i^{(1)} + D\eta_i = \lambda_i^{(1)}\eta_i. \tag{26}$$

Multiplyinig the vector $\xi_i^\top$ from left to the above equation, we have

$$\xi_i^\top D\eta_i = \lambda_i^{(1)}\xi_i^\top\eta_i \tag{27}$$

since $\xi_i^\top L = \mathbf{0}$. According to the partition of L, we divide D as follows:

$$D = \operatorname{diag}\left\{D_{11},\ldots,D_{K_1+K_2,K_1+K_2}\right\}, \quad \text{where } D_{ii} \in \mathbb{R}^{l_i,l_i}, \ \forall i. \tag{28}$$

Then by (21), (22), and $\xi_{(i)}^\top\cdot\mathbf{1} = 1$, we have

$$\lambda_i^{(1)} = \frac{\xi_i^\top D\eta_i}{\xi_i^\top\eta_i} = \xi_{(i)}^\top D_{ii}\mathbf{1}, \quad i = 1,\ldots,K_1. \tag{29}$$

Substituting $\lambda_i^{(1)}$ into (23) gives

$$\widetilde{\lambda}_i(\kappa,D) = \kappa\cdot\xi_{(i)}^\top D_{ii}\mathbf{1} + o(\kappa), \quad i = 1,\ldots,K_1. \tag{30}$$

Apparently, the smallest eigenvalue $\lambda_1(L+\kappa D)$ lies in $\widetilde{\lambda}_i(\kappa,D)$, $i = 1,\ldots,K_1$. Hence, the optimal pinned node set can maximize

$$\begin{aligned}\operatorname{Re}\left(\lambda_1(L+\kappa D)\right) &= \min_i \operatorname{Re}\left(\widetilde{\lambda}_i(\kappa,D)\right) \\ &= \kappa\cdot\min_i \xi_{(i)}^\top D_{ii}\mathbf{1} + o(\kappa).\end{aligned} \tag{31}$$

Then we have the following result.

Theorem 4. *Suppose that L has the form in (19), $\xi_{(i)}$ is the left eigenvector of L_{ii} corresponding to 0 with $\sum_j[\xi_{(i)}]_j = 1$, $i = 1,\ldots,K_1$, and p is the number of pinned nodes. Then*

$$\begin{aligned}\max_{\mathscr{D}:|\mathscr{D}|=p} \operatorname{Re}\left(\lambda_1(L+\kappa D)\right) \\ = \kappa\cdot\max_{\mathscr{D}:|\mathscr{D}|=p}\min_{i=1,\ldots,K_1}\xi_{(i)}^\top D_{ii}\mathbf{1} + o(\kappa)\end{aligned} \tag{32}$$

as $\kappa \longrightarrow 0$, where D has the form in (28).

Lemma 1 implies that, to stabilize system (3), the pinned node set $\mathscr{D}$ should contain at least one root of every subgraph $\mathscr{G}(L_{ii})$, $i = 1,\ldots,K_1$. Then we have the following lemma.

Lemma 5. *Suppose that L has the form in (19). Then $\max_{\mathscr{D}:|\mathscr{D}|=p}\min_{i=1,\ldots,K_1}\xi_{(i)}^\top D_{ii}\mathbf{1}$ is positive if and only if the number of pinned nodes satisfies $p \geq K_1$.*

Input:
the Laplacian matrix L, the pinning strength κ, the number of pinned nodes p
Output: the pinned node set $\mathscr{D}_O$; $\min_{i=1,\dots,K_1} \xi_{(i)}^{\top} D_{ii} \mathbf{1}$
1: permute the indices of the nodes of graph $\mathscr{G}$ to transform L into the form (19)
2: compute the left eigenvectors $\xi_{(i)}$ of L_{ii}, $i = 1, \dots, K_1$ corresponding to 0 and normalize them with 1-norm
3: permute the indices of the nodes in the subgraphs $\mathscr{G}(L_{ii})$ to get $[\xi_{(i)}]_1 \geq \cdots \geq [\xi_{(i)}]_{l_i}$, $i = 1, \dots, K_1$
4: initialize $\mathscr{D}_i = \{l_{i-1} + 1\}$ and $S_i = [\xi_{(i)}]_{l_{i-1}+1}$, $i = 1, \dots, K_1$ with $l_0 = 0$
5: **while** $\sum_{i=1}^{K_1} |\mathscr{D}_i| < p$ **do**
6: find the smallest $S_{i_0} = \min_i \{S_i\}$ and add the node $l_{i_0-1} + |\mathscr{D}_{i_0}| + 1$ to the node set $\mathscr{D}_{i_0}$: $\mathscr{D}_{i_0} \longleftarrow \mathscr{D}_{i_0} \cup \{l_{i_0-1} + |\mathscr{D}_{i_0}| + 1\}$
7: update $S_{i_0} = S_{i_0} + [\xi_{(i_0)}]_{l_{i_0-1}+|\mathscr{D}_{i_0}|+1}$
8: **end while**
9: **return** $\mathscr{D}_O = \bigcup_{i=1}^{K_1} \mathscr{D}_i$, $\min_{i=1,\dots,K_1} \xi_{(i)}^{\top} D_{ii} \mathbf{1}$

Algorithm 1: Searching for the pinned node set $\mathscr{D}_O$ that maximizes $\min_{i=1,\dots,K_1} \xi_{(i)}^{\top} D_{ii} \mathbf{1}$.

Remark 6. Assume that $[\xi_{(i)}]_1 \geq \cdots \geq [\xi_{(i)}]_{l_i}$, $i = 1, \dots, K_1$. If $[\xi_{(i)}]_1 > [\xi_{(i)}]_2$ holds $\forall i$, then the optimal pinned node set should contain nodes $1, l_1 + 1, l_1 + l_2 + 1, \dots, \sum_{k=1}^{K_1-1} l_k + 1$.

In this section, we focus on the first-order approximation of $\lambda_1(L + \kappa D)$ and search for the pinned node set that maximizes the first-order term of the approximation.

Remark 7. Notice that the pinned node set that maximizes $\min_{i=1,\dots,K_1} \xi_{(i)}^{\top} D_{ii} \mathbf{1}$ may not be unique; one may need to do the second-order approximation of $\lambda_1(L+\kappa D)$ to find the optimal pinned node set. Specifically, among the sets that maximize $\min_{i=1,\dots,K_1} \xi_{(i)}^{\top} D_{ii} \mathbf{1}$, find the one that maximizes $\lambda^{(2)}(D)$, which is the second-order term of the approximation.

Note that when $p > \sum_{i=1}^{K} l_i$, all nodes in the primary layer subgraphs should be pinned to maximize $\min_{i=1,\dots,K} \xi_{(i)}^{\top} D_{ii} \mathbf{1}$. But the selection of the rest of pinned nodes leads to a different optimization problem. Therefore, in this section, we focus on the case $p \leq \sum_{i=1}^{K} l_i$. The investigation of the case $p > \sum_{i=1}^{K} l_i$ is left as an open problem for future research.

In this section, we need to solve the optimization problem:

$$\max_{\mathscr{D}:|\mathscr{D}|=p} \min_{i=1,\dots,K_1} \xi_{(i)}^{\top} D_{ii} \mathbf{1}. \tag{33}$$

Based on the above theoretical analysis, Algorithm 1 is presented to search for the pinned node set $\mathscr{D}_O$ of cardinality p ($p \geq K_1$) that maximizes $\min_{i=1,\dots,K_1} \xi_{(i)}^{\top} D_{ii} \mathbf{1}$ when $\kappa \ll 1$.

4. Large Pinning Strength

Now, we consider the case of very large pinning strength κ. Let $\epsilon = 1/\kappa$; $L + \kappa D = \kappa(\epsilon L + D)$. Notice that D is a diagonal matrix with diagonal elements being 0 or 1. Hence, D has eigenvalues 0 and 1, and the smallest eigenvalue (in real part) of $\epsilon L + D$ is a perturbation near zero eigenvalues of D in terms of ϵ. Let u be a right eigenvector of D corresponding to 0 and let $\lambda(\epsilon)$ and $\eta(\epsilon)$ be the perturbed eigenvalue and the associated right eigenvector of $\epsilon L + D$ with $\lambda(0) = 0, \eta(0) = u$. By a perturbation expansion, we have

$$\begin{aligned} \lambda(\epsilon) &= \epsilon \lambda^{(1)} + o(\epsilon) \\ \eta(\epsilon) &= u + \epsilon \eta^{(1)} + o(\epsilon) \end{aligned} \tag{34}$$

and

$$\begin{aligned} &(\epsilon L + D) \cdot \left[u + \epsilon \eta^{(1)} + o(\epsilon)\right] \\ &\quad = \left(\epsilon \lambda^{(1)} + o(\epsilon)\right) \cdot \left[u + \epsilon \eta^{(1)} + o(\epsilon)\right] \end{aligned} \tag{35}$$

as $\epsilon \longrightarrow 0$. By comparing the coefficients of ϵ in (35), we have

$$Lu + D\eta^{(1)} = \lambda^{(1)} u \tag{36}$$

as $\epsilon \longrightarrow 0$. Suppose that P is the permutation matrix such that

$$P^{\top} D P = \begin{bmatrix} I_{|\mathscr{D}|} & \\ & 0 \end{bmatrix}. \tag{37}$$

By $Du = \mathbf{0}$, we have $\mathbf{0} = P^{\top} D u = P^{\top} D P \cdot P^{\top} u$, which implies that $P^{\top} u = [\mathbf{0}_{|\mathscr{D}|}^{\top}, u_1^{\top}]^{\top}$ for some $u_1 \in \mathbb{R}^{m-|\mathscr{D}|}$. Denote

$$P^{\top} L P = \begin{bmatrix} L_{11}^{\mathscr{D}} & L_{12}^{\mathscr{D}} \\ L_{21}^{\mathscr{D}} & L_{22}^{\mathscr{D}} \end{bmatrix}. \tag{38}$$

Multiplying the permutation matrix $P^{\top}$ from left to (36) on both sides, we have

$$P^{\top} L P \cdot P^{\top} u + P^{\top} D P \cdot P^{\top} \eta^{(1)} = \lambda^{(1)} P^{\top} u \tag{39}$$

as $\epsilon \longrightarrow 0$. Substituting (37) and (38) into the above equation, we get

$$L_{22}^{\mathscr{D}} \cdot u_1 = \lambda^{(1)} \cdot u_1 \tag{40}$$

Input:
the Laplacian matrix L, the pinning strength κ, the cardinality of pinned node set p,
the generation number G_0, the population size NP
Output: the pinned node set $\mathscr{D}_O$, $\mathrm{Re}(\lambda_1(L_{22}^{\mathscr{D}_O}))$
1: initialize $G = 0$ and individuals $Q_{i,G} = [Q_{i,G}^1, \dots, Q_{i,G}^p]$, $i = 1, \dots, NP$ as $p * 1$ vectors by randomly choosing p integers from $\{1 \dots, m\}$, where m is the network size
2: **for** $G = 1 : G_0$ **do**
3: mutation step: generate a mutated vector $V_{i,G}$ for each individual i, $i = 1, \dots, NP$
4: crossover step: generate a trial vector $U_{i,G}$ for each individual i, $i = 1, \dots, NP$
5: selection step: Let $Q_{i,G+1} = U_{i,G}$ if $\mathrm{Re}(\lambda_1(L_{22}^{U_{i,G}})) > \mathrm{Re}(\lambda_1(L_{22}^{Q_{i,G}}))$, otherwise $Q_{i,G+1} = Q_{i,G}$, $i = 1, \dots, NP$.
6: **end for**
7: set $i_0 = \arg\max_{i=1,\dots,NP} \mathrm{Re}(\lambda_1(L_{22}^{Q_{i,G_0}}))$
8: **return** $\mathscr{D}_O = \{Q_{i_0 G_0}^j, j = 1 \dots, p\}$, $\mathrm{Re}(\lambda_1(L_{22}^{\mathscr{D}_O}))$

ALGORITHM 2: Searching for the pinned node set $\mathscr{D}_O$ that maximizes $\mathrm{Re}(\lambda_1(L_{22}^{\mathscr{D}}))$.

as $\epsilon \longrightarrow 0$, which implies that $\lambda^{(1)}$ is an eigenvalue of $L_{22}^{\mathscr{D}}$ as $\epsilon \longrightarrow 0$. Then we get that the smallest eigenvalue of $\epsilon L + D$ satisfies

$$\lambda_1 (\epsilon L + D) = \epsilon \cdot \lambda_1 \left(L_{22}^{\mathscr{D}}\right) + o(\epsilon) \tag{41}$$

as $\epsilon \longrightarrow 0$. Multiplying $\kappa = 1/\epsilon$ on both sides of the above inequality yields

$$\lambda_1 (L + \kappa D) = \lambda_1 \left(L_{22}^{\mathscr{D}}\right) + o(1) \tag{42}$$

as $\kappa \longrightarrow +\infty$, where $o(1)$ denotes terms that satisfy $\lim_{\kappa \longrightarrow \infty} o(1) = 0$. In other words, in the limit of large pinning strength, the real part of $\lambda_1(L_{22}^{\mathscr{D}})$, which is used to measure the stabilizability of the pinned network in [10], gives the lower bounds of the convergence rate of the pinned network.

Hence, we have the following result.

Theorem 8. *Suppose that p is the number of pinned nodes. Then*

$$\begin{aligned}&\max_{\mathscr{D}:|\mathscr{D}|=p} \mathrm{Re}\left(\lambda_1 (L + \kappa D)\right) \\ &\quad = \max_{\mathscr{D}:|\mathscr{D}|=p} \mathrm{Re}\left(\lambda_1 \left(L_{22}^{\mathscr{D}}\right)\right) + o(1)\end{aligned} \tag{43}$$

holds as $\kappa \longrightarrow +\infty$, where $L_{22}^{\mathscr{D}}$ satisfies (38) and $\lambda_1(L_{22}^{\mathscr{D}})$ denotes the eigenvalue of $L_{22}^{\mathscr{D}}$ with the smallest real part.

Remark 9. For pinning synchronization problem of linearly coupled complex networks, the smallest eigenvalue of $L + \kappa D$ also plays a dominate role in the stability analysis [10, 11]. Therefore, our theoretical results can be generalized to pinning synchronization problem.

Remark 10. In [25], the stability of multiagent systems with transimission and pinning delays was studied. The dominant eigenvalue was estimated in the limit of small and large pinning strengths, when the underlying graph was strongly connected. From our analysis and [25], one can reveal the dependence of the dominant eigenvalue on the pinned node set and generalize Theorems 2, 4, and 8 to time-delayed systems in [25].

Inspired by [27, 28], we employ generic algorithm to solve the optimization problem (43) and search for the optimal node set. The algorithm contains mutation, crossover, and selection operations. Firstly, after initializing the population size and individual vectors, mutation operation is employed to generate a mutation vector associated with each individual. Mutation is applied to the current best individual vector and realized by randomly picking elements with probability 0.5 and replacing them by other randomly selected elements. Secondly, crossover is realized by randomly picking elements from each individual and its mutation vector to generate a new trial vector. At last, fitness of the generated trial vector is calculated and compared to that of its associated individual vector. Choose the vector with better fitness to be the individual vector of the next generation. At each generation, its best individual vector is the one with best fitness value. Here the fitness is defined by $\mathrm{Re}(\lambda_1(L_{22}^{\mathscr{D}}))$.

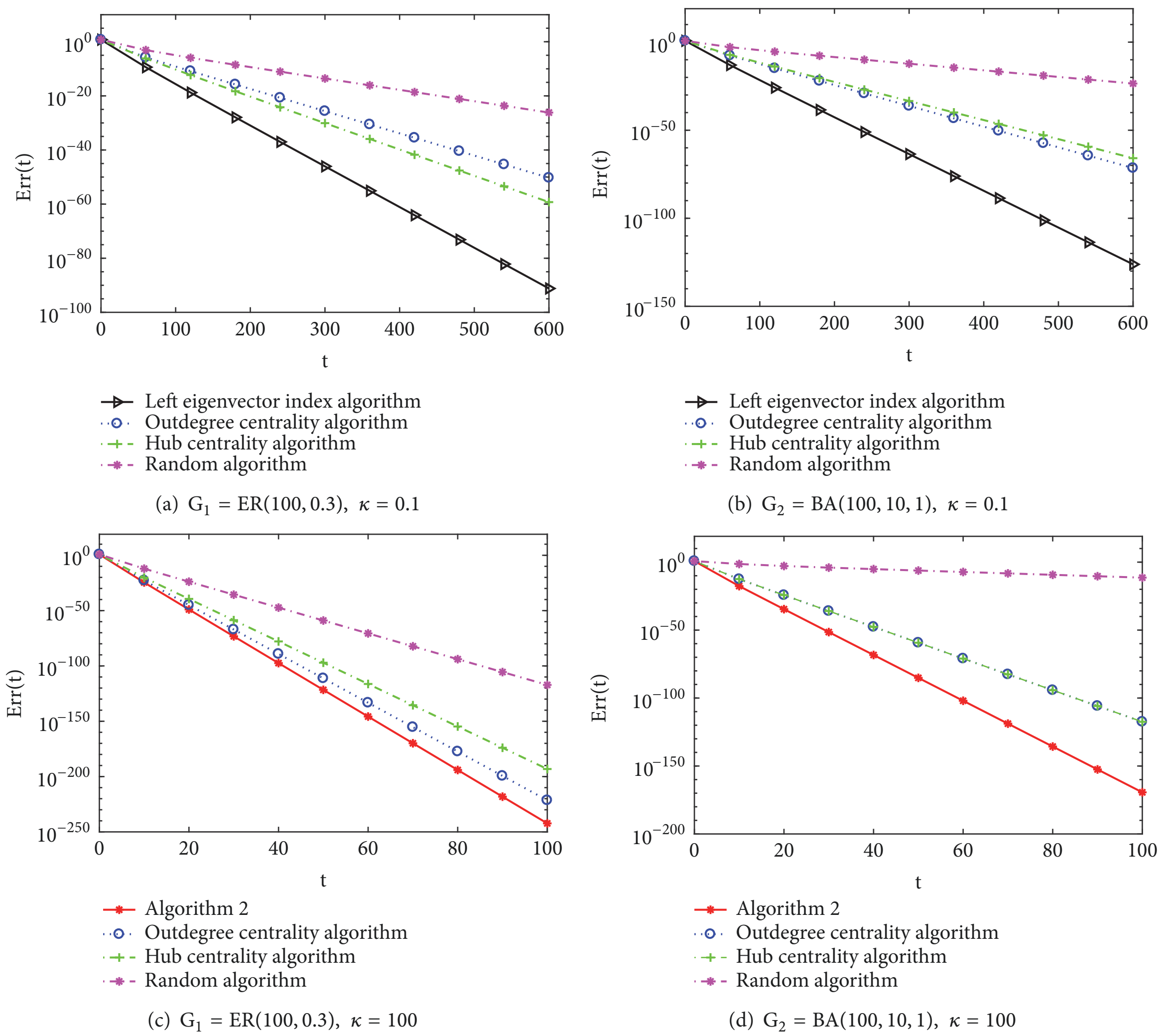
The specific algorithm is presented in Algorithm 2.

5. Numerical examples

In this section, we compare the proposed pinning strategies with the following ones: (i) pinning nodes with highest outdegrees, (ii) pinning nodes with highest hub centraility, and (iii) randomly selecting nodes to be pinned.

In the following, the left eigenvector index algorithm applies the strategy of pinning nodes with the largest magnitude of elements in the left eigenvector corresponding to 0 of the Laplacian matrix.

Denote by $\mathrm{ER}(n, p_1)$ an Erdös-Rényi random network with n nodes and linking probability p_1. For each added linking, its direction is randomly chosen. Denote by $\mathrm{BA}(n, m, n_0)$ a weighted Barabási-Albert scale-free network of n nodes. The network begins with a complete connected network of n_0 nodes and m new edges are added at each time step. Inspired



(a) $G_1 = \mathrm{ER}(100, 0.3)$, $\kappa = 0.1$

(b) $G_2 = \mathrm{BA}(100, 10, 1)$, $\kappa = 0.1$

(c) $G_1 = \mathrm{ER}(100, 0.3)$, $\kappa = 100$

(d) $G_2 = \mathrm{BA}(100, 10, 1)$, $\kappa = 100$

Figure 1: Dynamics of $\mathrm{Err}(t)$ with different pinning algorithms. The left eigenvector index algorithm applies the strategy of pinning nodes associated with the largest elements in the left eigenvector corresponding to 0 of the Laplacian matrix. The pinned node number is fixed to be 3. Averaged over 50 times.

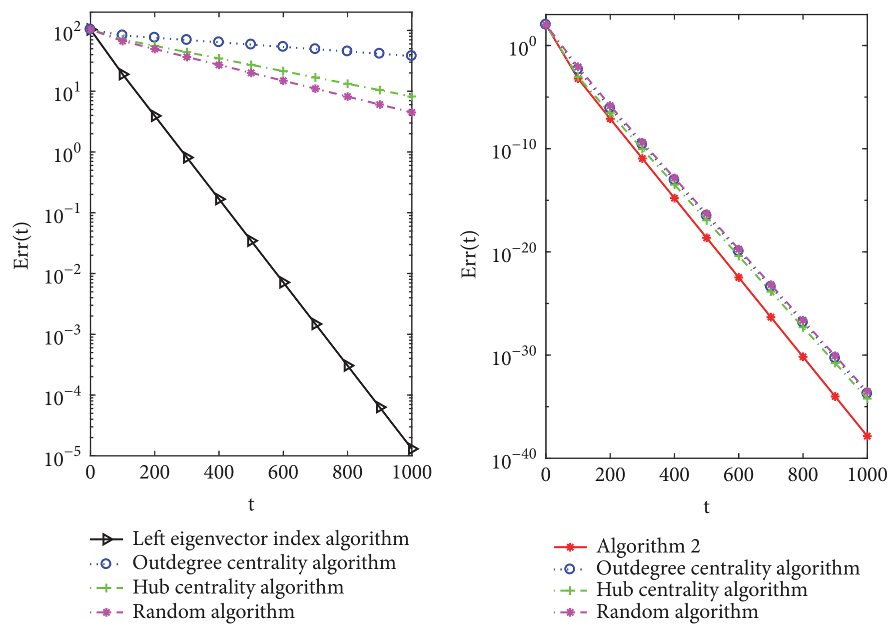
by the method to generate the asymmetry of the coupling matrix in [29], here we let $l_{ij} = 1$ for $i > j$ and $l_{ji} = 2$ for $i < j$ in BA networks if there is a linking between i and j. In this section, pinning consensus is measured by the variance

$$\mathrm{Err}(t) = \sqrt{\sum_{i=1}^{m} \left| x_i(t) - s \right|}. \tag{44}$$

5.1. Strongly Connected Networks. Figures 1(a) and 1(b) plot the dynamics of $\mathrm{Err}(t)$ under different pinning algorithms. The numerical result shows a good agreement with our theoretical result that, in the case of sufficiently small pinning strength, the left eigenvector corresponding to eigenvalue 0 of the Laplacian matrix measures the importance of nodes best in comparison to other centrality algorithms.

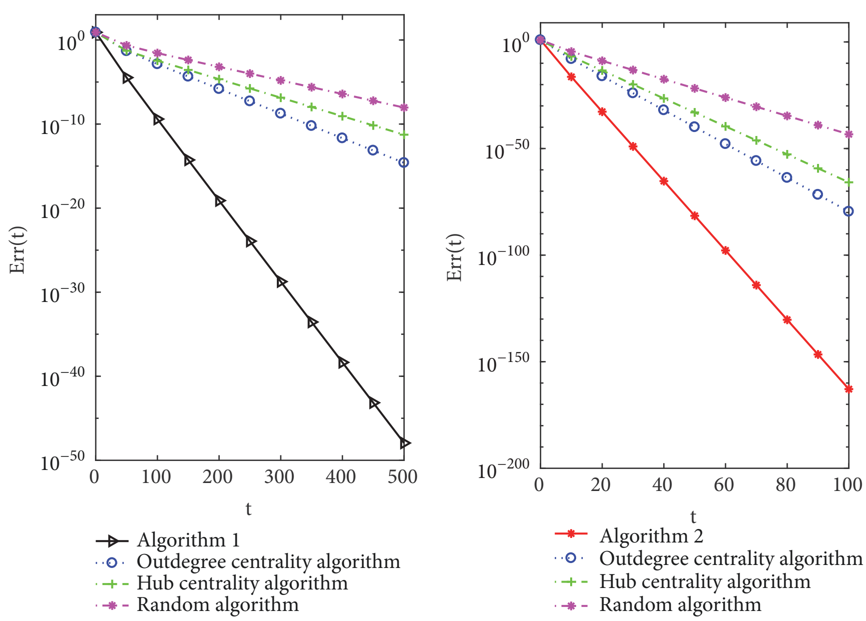
Next, in order to verify the effectiveness of Algorithm 2, Figures 1(c) and 1(d) plot the dynamics of $\mathrm{Err}(t)$ under different pinning algorithms in the case $\kappa = 100$. The simulation result shows that Algorithm 2 performs best as compared to random pinning algorithm and other pinning algorithms based on centrality measures.

To illustrate the effectiveness of our algorithm in real-word networks, we use Epinions dataset gathered from Stanford Large Network Dataset Collection (http://snap.stanford.edu/data/soc-Epinions1.html). Epinions.com is a general consumer review site and its members can decide whether to trust each other based upon their reviews. The members and their trust relationship specify a real-world trust network with 75879 nodes and 508837 edges. Since Epinions social network is not strongly connected, here we consider its subnetwork corresponding to the largest strongly connected component, which contains 32223 nodes and 443506 edges. In this example, $p = 1000$ nodes are selected to be pinned. Figures 2(a) and 2(b) plot the dynamics of $\mathrm{Err}(t)$ under different pinning algorithms with pinning strength $\kappa = 0.1$



(a) Epinions social network, $\kappa = 0.1$ (b) Epinions social network, $\kappa = 100$

FIGURE 2: Dynamics of Err(t) with different pinning algorithms. The pinned node number is 1000.

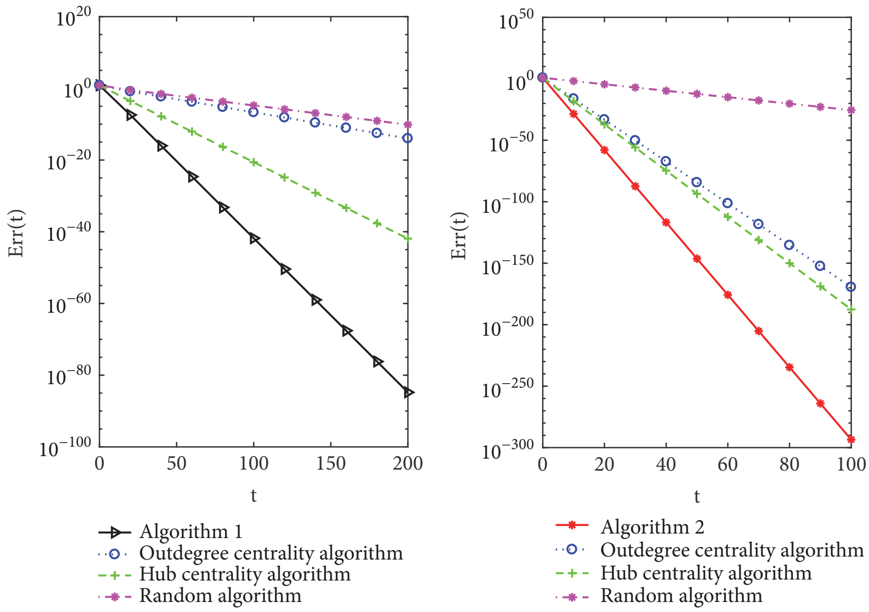


(a) $G_3 = \mathrm{ER}(100, 100, 0.3, 0.3)$, $\kappa = 0.1$ (b) $G_3 = \mathrm{ER}(100, 100, 0.3, 0.3)$, $\kappa = 100$

FIGURE 3: Dynamics of Err(t) with different pinning algorithms. The pinned node number is fixed to be 5. Averaged over 20 times.

and $\kappa = 100$, respectively, which shows a good agreement with our theoretical result.

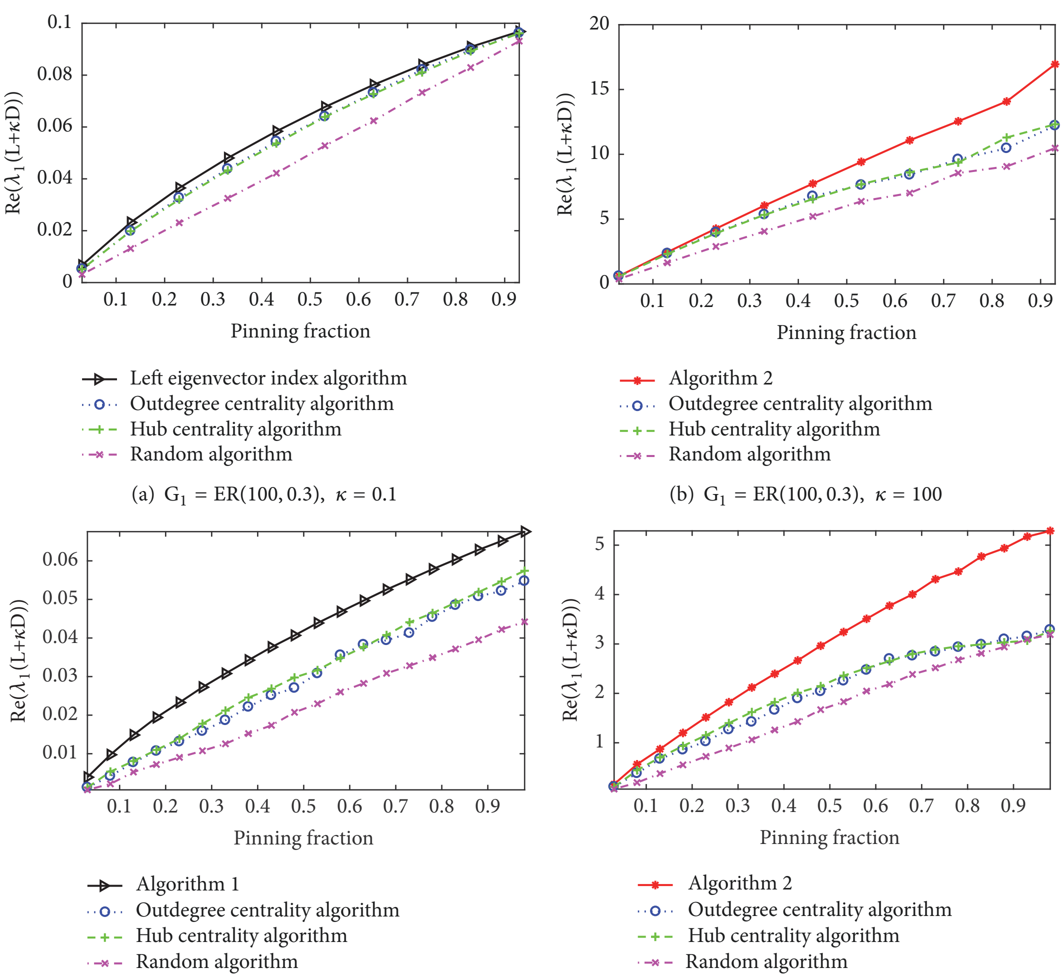
5.2. Disconnected Networks. We firstly construct a disconnected network $\mathrm{ER}(n_1, n_2, p_1, p_2)$ by integrating two mutually independent ER random networks $\mathrm{ER}(n_1, p_1)$ and $\mathrm{ER}(n_2, p_2)$ into a larger one. Each ER random network is a strongly connected component of $\mathrm{ER}(n_1, n_2, p_1, p_2)$, and there is no linking between these two components. Similary, $\mathrm{BA}(n_1, n_2, m_1, m_2, n_{01}, n_{02})$ represents a disconnected network that contains two mutually independent BA networks $\mathrm{BA}(n_1, m_1, n_{01})$ and $\mathrm{BA}(n_2, m_2, n_{02})$. We verify the effectiveness of Algorithms 1 and 2 by comparing them with other pinning algorithms. Figure 3 plots the dynamics of Err(t), which shows that the pinned node set obtained



(a) Enron email network, $\kappa = 0.1$

(b) Enron email network, $\kappa = 100$

FIGURE 4: Dynamics of Err(t) with different pinning algorithms. The pinned node number is fixed to be 6.



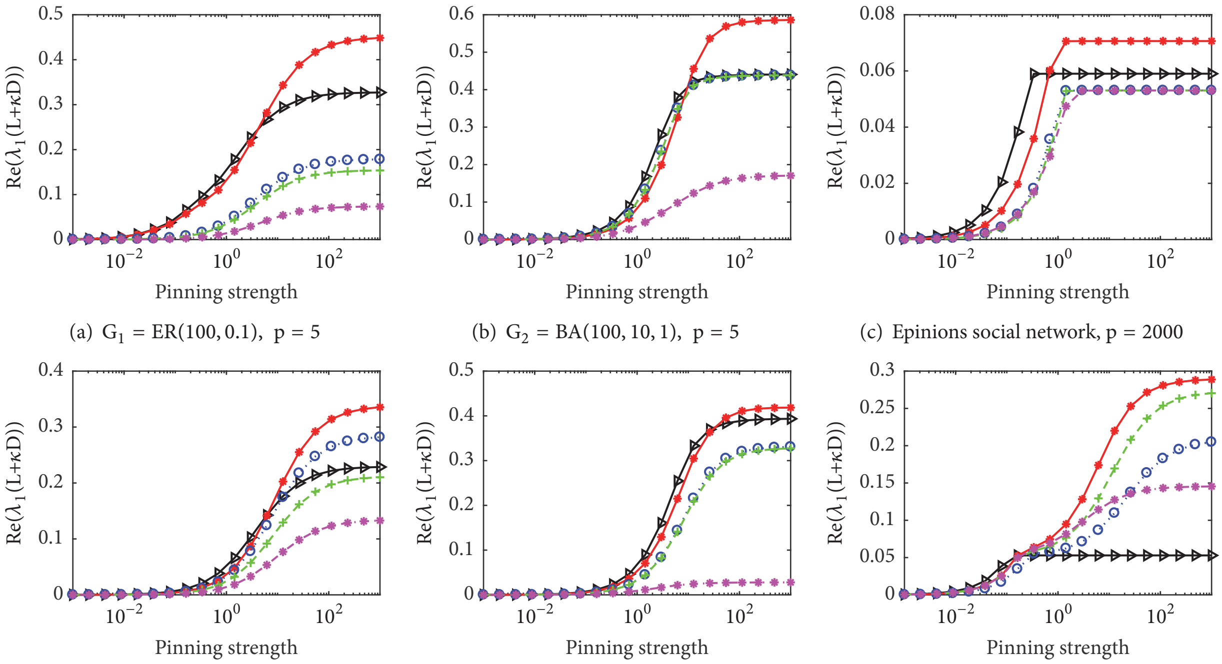
(a) $G_1 = ER(100, 0.3)$, $\kappa = 0.1$

(b) $G_1 = ER(100, 0.3)$, $\kappa = 100$

(c) $G_3 = ER(100, 100, 0.2, 0.2)$, $\kappa = 0.1$

(d) $G_3 = ER(100, 100, 0.2, 0.2)$, $\kappa = 100$

FIGURE 5: Variation of Re($\lambda_1(L + \kappa D)$) with respect to pinning fraction.



(a) $G_1 = \mathrm{ER}(100, 0.1)$, $p = 5$ (b) $G_2 = \mathrm{BA}(100, 10, 1)$, $p = 5$ (c) Epinions social network, $p = 2000$

(d) $G_3 = \mathrm{ER}(100, 100, 0.2, 0.2)$, $p = 5$ (e) $G_4 = \mathrm{BA}(100, 100, 10, 10, 1, 1)$, $p = 5$ (f) Enron email network, $p = 5$

Figure 6: Variation of $\mathrm{Re}(\lambda_1(L + \kappa D))$ with respect to pinning strength. The solid red curve represents Algorithm 2, the dotted blue curve represents the outdegree centrality algorithm, the dashed green curve represents the hub centrality algorithm, and the dashed purple curve represents random algorithm. In (a), (b), and (c), the solid black curve represents the left eigenvector index pinning algorithm, whereas in (d), (e), and (f), it represents Algorithm 1.

from Algorithms 1 or 2 performs best, as compared to other pinning algorithms.

Next, we consider the Enron email dataset collected from 1998 to 2002 and adopt the dataset downloaded from http://www.cis.jhu.edu/~parky/Enron/, which considers 184 users by counting the number of unique addresses. Let E_{ij} be the number of mails sent from user i to user j and let C_{ij} be the number of mails cc-ed or bcc-ed to j from i. Define the weight of the edge from i to j as

$$a_{ij} = \left\lfloor \frac{E_{ij} + 0.5 * C_{ij}}{25} \right\rfloor . \tag{45}$$

And remove the users who have no linkings with other users. Then we get a network with 136 nodes and 472 edges, denoted by G_5. The pinned node number is 6 in this simulation.

By the primary layer detection algorithm in [26], we find that there are 4 primary layer subgraphs in G_5. Denote the root set of these 4 primary layer subgraphs by $\mathcal{S}_i$, $i = 1, 2, 3, 4$. As pointed out in Lemma 1, at least one root in every primary layer subgraph should be pinned to reach consensus. Therefore, in outdegree (or hub) centrality pinning algorithm, node with largest outdegree (or hub) centrality in each $\mathcal{S}_i$, $i = 1, \ldots, 4$, is pinned, and 2 other nodes with highest outdegree (or hub) centrality are pinned. In random algorithm, one node in each $\mathcal{S}_i$, $i = 1, \ldots, 4$, is randomly pinned, and the rest 2 nodes are pinned randomly in the network. Figure 4 plots the dynamics of $\mathrm{Err}(t)$, which shows that the pinned node set obtained from Algorithms 1 or 2 performs best, as compared to other pinning algorithms.

To further reveal the performance of different algorithms, Figures 5 and 6 plot the value of $\mathrm{Re}(\lambda_1(L + \kappa D))$ with respect to the pinning fraction and the pinning strength.

It can be seen from Figure 6 that, even for the case of medium pinning strength, our algorithms perform dominantly better than the existing methods.

6. Conclusion

We have discussed how to choose pinned node set to maximize the convergence rate of multiagent systems with digraph topologies in cases of sufficiently small and large pinning strength. We prove that when the pinning strength is sufficiently small, the left eigenvector(s) associated with eigenvalue 0 of the Laplacian matrix can be used to select the optimal pinned node set. In the case of sufficiently large pinning strength, nodes that increase the smallest eigenvalue of the Laplacian submatrix associated with unpinned nodes maximally should be pinned. Numerical simulations are given to illustrate the effectiveness of our theoretical results.

There are several interesting problems for future research. First, our study mainly focuses on the cases where the pinning strength is very small or very large. The extreme assumption is necessary for employing the perturbation approach to estimate the dominant eigenvalue of the pinned system. Nevertheless, it is important to study the optimization problem for the case of medium pinning strength, which is left as an open problem for future research. Second, for networks without spanning trees, we mainly consider the first-order approximation of the smallest eigenvalue of the perturbed Laplacian matrix and search for the pinned node set that maximizes the first-order term in the approximation. Sometimes, however, the set that maximizes the first-order term is not unique. Therefore, it is a significant future work to study the second-order approximation to obtain the optimal pinned node set. Third, it is interesting to extend our theoretical results to more general systems, for example, systems with time-varying delays and dynamical topology. However, this will make the analysis difficult because the dominant eigenvalue becomes time-dependent. Fourth, the present study does not consider the cost of applying feedback controllers. It is an interesting future work to study the optimization problem while considering the control cost.

Data Availability

The .mat data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors have no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is jointly supported by the National Natural Science Foundation of China [Grants nos. 61703271 and 61673119] and the Key Program of the National Natural Science Foundation of China [Grant no. 91630314].

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