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Research Article

Geometric Asian Options Pricing under the Double Heston Stochastic Volatility Model with Stochastic Interest Rate

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This paper presents an extension of double Heston stochastic volatility model by incorporating stochastic interest rates and derives explicit solutions for the prices of the continuously monitored fixed and floating strike geometric Asian options. The discounted joint characteristic function of the log-asset price and its log-geometric mean value is computed by using the change of numeraire and the Fourier inversion transform technique. We also provide efficient approximated approach and analyze several effects on option prices under the proposed model. Numerical examples show that both stochastic volatility and stochastic interest rate have a significant impact on option values, particularly on the values of longer term options. The proposed model is suitable for modeling the longer time real-market changes and managing the credit risks.

1. Introduction

Asian option is a special type of option contract in which the payoff depends on the average of the underlying asset price over some predetermined time interval. The averaging feature allows Asian options to reduce the volatility inherent in the option. There are some advantages to trading Asian options in a financial market. One is that these decrease the risk of market manipulation of the financial derivative at expiry. Another is that Asian options have lower relative charge than European or American options. In general, the average considered can be a arithmetic or geometric one and it can be calculated either discretely, for which the average is taken over the underlying asset prices at discrete monitoring time points, or continuously, for which the average is calculated via the integration of the underlying asset price over the monitoring time period. Asian options can be differentiated into two main classes according to their payoff: fixed strike price options (sometimes called "average price") and floating strike price options (sometimes called "average strike"). All these details are specified by the contracts stipulated by two counterparts, as Asian options are traded actively on the OTC market among investors or traders for hedging the average price of a commodity. For a brief introduction to the development of Asian options, see Boyle and Boyle [1].

As the probability distribution of the average prices of the underlying asset generally does not have a simple analytical expression, it is difficult to obtain the analytical pricing formula for Asian option. Since the best-known closed-form pricing formula for the European vanilla option derived by Black and Scholes [2]), many researchers have devoted themselves to developing the Asian options pricing based on the Black-Scholes assumptions; see, e.g., Kemna and Vorst (1990), Turnbull and Wakeman [3], Ritchken et al. [4], Geman and Yor [5], Rogers and Shi [6], Boyle et al. [7], Angus [8], Linetsky [9], Cui et al. [10], and the references therein. For a recent review, one can refer to Fusai and Roncoroni [11] and Sun et al. [12].

In practice, the Black-Scholes assumptions are hardly satisfied, especially the constant volatility and constant interest rate hypothesis. As the empirical behaviors of the implied volatility smile and heavy tailed in the distribution of log-returns are commonly observed in financial markets. For this reason, stochastic volatility (hereafter SV) models have been proposed in finance (see Hull and White [13], Stein and Stein [14], Heston [15], and others). These models have been applied to value the Asian options (see, e.g., Wong and Cheung [16], Hubalek and Sgarra [17], Kim and Wee [18], and Shi and Yang [19]). In addition, interest rates are stochastic and stock returns are negatively correlated with interest rate changes, which have been examined in previous research.

Although these models mentioned above are able to account for the empirical behaviors, they are still based on a single-factor for volatility dynamics that is inconsistent with the long range memory characteristic of the volatility corrections and the stiff volatility skews. See Alizadeh et al. [20], Fiorentini et al. [21], Chernov et al. [22], Gourieroux [23], Christoffersen et al. [24], Romo [25], and Nagashima et al. [26] for the empirical results. To address this issue, multifactor SV models have recently generated attention in the option pricing literature. For instance, Duffie et al. [27] proposed multifactor affine stochastic volatility models. Based upon the Black-Scholes framework, Fouque et al. [28] introduced a multiscale SV model, in which the volatility processes are driven by two mean-reverting diffusion processes. Gourieroux [23] proposed a multivariate model in which the volatility-covolatility matrix follows a Wishart process.

On the basis of the findings of Christoffersen et al. [24], a double Heston (dbH) model, which consists of two independent variance processes, has recently been reported better than the plain Heston [15] model in the performances of hedging (see Sun [29]) and has also been applied to arithmetic Asian option under discrete monitoring (Mehrdoust and Saber [30]) and forward starting option (Zhang and Sun [31]). However, its extension to continuously monitored geometric Asian option is yet to be considered. On the other hand, many of Asian options often have long-dated maturities since they are used as part of the structured notes which has a long maturity. The movement of interest rates becomes an issue in such cases and constant interest rate assumption should be replaced by an appropriate dynamic interest rate model. Several results are available on the Asian option in the stochastic interest rate framework; see, e.g., Nielsen and Sandmann [32, 33], Zhang et al. [34], Donnelly et al. [35], and He and Zhu [36]. In the above stochastic interest rate framework, the short-term interest rate is assumed to follow a specific parametric one-factor model (see, e.g., Cox et al. [37], Hull and White [13], and Vasicek [38]), which tends to oversimplify the true behavior of interest rate movement. However, empirical tests reported in Lonstaff and Schwartz [39] and Pearson and Sun [40] show that the term structure for the interest rate should involve several sources of uncertainty, and introducing additional state variables (such as the rate of inflation, GDP, etc.) significantly improves the fit.

In this paper, we study the pricing of the continuously monitored geometric Asian options under dbH stochastic volatility model with stochastic interest rate framework (hereafter, dbH-SI model). The contribution of the present paper is twofold. Firstly, this paper extends the dbH model by introducing stochastic interest rate, which is assumed to follow two-factor model with two state variables. Secondly, this paper provides a semiexplicit valuation formula for the geometric Asian options with fixed or floating strike price, which is extremely useful also for the arithmetic average option valuation via Monte Carlo methods with control variables.

The rest of the paper is organized as follows. Section 2 develops the underlying pricing model and describes the geometric Asian option. Section 3 derives the joint characteristic function of a log-return of the underlying asset and its geometric average. Section 4 obtains the analytic expressions for the prices of the fixed strike geometric Asian call option and the floating strike Asian call option under continuous monitoring. Section 5 provides some numerical examples for the proposed approach. Section 6 concludes the paper.

2. Model Formulation

We consider an arbitrage-free, frictionless financial market where only riskless asset and risky asset are traded continuously up to a fixed horizon date T. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\le t\le T}, Q)$ be a complete probability space equipped with a filtration $\{\mathscr{F}_t\}$ satisfying the usual conditions, where Q is a risk-neutral probability measure. Suppose W_t^j and Z_t^j $(j = 1, 2)$ are all standard Brownian motions defined on the probability space, and the filtration $\{\mathscr{F}_t\}_{t\ge 0}$ is generated by these Brownian motions. Moreover, $dW_t^1 dZ_t^1 = \rho_1 dt$, $dW_t^2 dZ_t^2 = \rho_2 dt$, and any other Brownian motions are pairwise independent. Assume that the asset price process S_t, without paying any dividend, satisfies the following stochastic differential equation under Q:

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t dt + \sqrt{v_{1t}} dW_t^1 + \sqrt{v_{2t}} dW_t^2, \\ dv_{1t} &= \kappa_1 (\theta_1 - v_{1t}) dt + \sigma_1 \sqrt{v_{1t}} dZ_t^1, \\ dv_{2t} &= \kappa_2 (\theta_2 - v_{2t}) dt + \sigma_2 \sqrt{v_{2t}} dZ_t^2, \end{aligned} \tag{1}$$

where $\kappa_j, \theta_j, \sigma_j$ $(j = 1, 2)$ are all nonnegative constants, which represent the mean-reverting rates, long-term mean levels, and volatilities of variance processes v_{jt}, respectively. We suppose that $2\kappa_j\theta_j \ge \sigma_j^2$. The instantaneous interest rate, r_t, is assumed to be a linear combination of v_{1t} and v_{2t}, i.e., $r_t = v_{1t} + v_{2t}$, which designates the interest rate as an affine function of two-factor economic variables v_1 and v_2 and offers the analytic tractability (see Duffie et al. [27]).

In financial market, there are four types of European style continuously monitoring geometric Asian options: fixed strike geometric Asian calls, fixed strike geometric Asian puts, floating strike geometric Asian calls, and floating strike geometric Asian puts. The payoffs at the expiration date T for these options are as follows:

$$h\left(S_T, G_{[0,T]}\right) = \begin{cases} \max\left\{G_{[0,T]} - K, 0\right\}, & \text{a fixed strike geometric Asian calls,} \\ \max\left\{K - G_{[0,T]}, 0\right\}, & \text{a fixed strike geometric Asian puts,} \\ \max\left\{G_{[0,T]} - S_T, 0\right\}, & \text{a floating strike geometric Asian calls,} \\ \max\left\{S_T - G_{[0,T]}, 0\right\}, & \text{a floating strike geometric Asian puts,} \end{cases} \tag{2}$$

where K is a fixed strike price and $G_{[0,T]}$ is the geometric average of the underlying asset price S_t until time T; i.e., $G_{[0,T]} = \exp((1/T)\int_0^T \ln S_u du)$. In the following, we consider only the pricing problem of the geometric Asian call options (hereafter, GAC), while the put options can be dealt with similarly.

For the instantaneous interest rate $r_t = v_{1t} + v_{2t}$, one can express the price at time t of a zero-coupon bond with maturate T as follows (see Cox et al. [37]):

$$P(t,T) = \exp\left\{a(t,T) - b(t,T) r_t\right\}, \tag{3}$$

where

$$a(t,T) = \sum_{j=1}^{2}\left[\frac{\kappa_j\theta_j\left(\kappa_j - \gamma_j\right)(T-t)}{\sigma_j^2} + \frac{2\kappa_j\theta_j}{\sigma_j^2}\ln\frac{2\gamma_j}{2\gamma_j + \left(\kappa_j - \gamma_j\right)\left(1 - e^{-\gamma_j(T-t)}\right)}\right],$$

$$b(t,T) = \sum_{j=1}^{2}\left[\frac{2\left(1 - e^{-\gamma_j(T-t)}\right)}{2\gamma_j + \left(\kappa_j - \gamma_j\right)\left(1 - e^{-\gamma_j(T-t)}\right)}\right],$$

$$\gamma_j = \sqrt{\kappa_j^2 + 2\sigma_j^2}, \quad \text{for } j = 1, 2. \tag{4}$$

3. The Joint Characteristic Function

Given the dynamic of the underlying asset price, it is possible to obtain the discounted joint characteristic function for the log-asset value S_T and the log-geometric mean value of the asset price over a certain time period.

Let $\psi_t(s,w) = E^Q[e^{-\int_t^T r_\tau d\tau + s\ln G_{[t,T]} + w\ln S_T} \mid \mathscr{F}_t]$ be the discounted joint characteristic function of two-dimensional random variable, $(\ln G_{[t,T]}, \ln S_T)$, conditioned on $\mathscr{F}_t$ under Q, where $G_{[t,T]} = \exp\{(1/T)\int_t^T \ln S_u du\}$ and $E^Q[\cdot \mid \mathscr{F}_t]$ is the conditional expectation under Q for $t \in [0,T]$. Denote $D = \{(s,w) \in \mathbb{C}^2 : \mathfrak{R}(s) \geq 0, \mathfrak{R}(w) \geq 0, 0 \leq \mathfrak{R}(s) + \mathfrak{R}(w) \leq 1\}$.

Proposition 1. *Suppose that S_t, v_{1t}, and v_{2t} follow the dynamics in (1). If $(s,w) \in D$ and $t \in [0,T]$, then $E^Q[|e^{-\int_t^T r_\tau d\tau + s\ln G_{[t,T]} + w\ln S_T}|] < \infty$ and*

$$\psi_t(s,w) = e^{g_0} E^Q\left[\exp\left\{\sum_{j=1}^{2}\left[g_{j1}\int_t^T (T-\tau)^2 v_{j\tau} d\tau + g_{j2}\int_t^T (T-\tau) v_{j\tau} d\tau + g_{j3}\int_t^T v_{j\tau} d\tau + g_{j4} v_{jT}\right]\right\} \mid v_{1t}, v_{2t}\right], \tag{5}$$

where

$$g_0 = g_0(s,w) = s\left[\frac{T-t}{T}\ln S_t - \frac{1}{T}\sum_{j=1}^{2}\left(\frac{\rho_j\kappa_j\theta_j}{2\sigma_j}(T-t)^2 + \frac{\rho_j(T-t)}{\sigma_j}v_{jt}\right)\right] + w\left[\ln S_t - \sum_{j=1}^{2}\left(\frac{\rho_j\kappa_j\theta_j}{\sigma_j}(T-t) + \frac{\rho_j}{\sigma_j}v_{jt}\right)\right],$$

$$g_{j1} = g_{j1}(s,w) = \frac{s^2\left(1-\rho_j^2\right)}{2T^2},$$

$$g_{j2} = g_{j2}(s,w) = \frac{s\left(2\rho_j\kappa_j + \sigma_j\right)}{2\sigma_j T} + \frac{sw\left(1-\rho_j^2\right)}{T},$$

$$g_{j3} = g_{j3}(s,w) = \frac{s\rho_j}{T\sigma_j} + \frac{w\left(2\rho_j\kappa_j + \sigma_j\right)}{2\sigma_j} + \frac{w^2\left(1-\rho_j^2\right)}{2} - 1,$$

$$g_{j4} = g_{j4}(s,w) = \frac{w\rho_j}{\sigma_j}. \tag{6}$$

Proof. (i) We first prove that the integrability condition guarantees the existence of the cumulant function $\psi_t(s,w)$ in D. If $(s,w) \in D$ and $t \in [0,T)$, then

$$\begin{aligned}
&E^{Q}\left[\left|e^{-\int_t^T r_\tau d\tau + s\ln G_{[t,T]} + w\ln S_T}\right|\right] \\
&= P(t,T)\, E^{Q_T}\left[\left|e^{s\ln G_{[t,T]} + w\ln S_T}\right|\right] \\
&\le P(t,T)\, E^{Q_T}\left[\frac{1}{T-t}\int_t^T S_u du + S_T\right] \\
&= \left[\frac{1}{T-t}\int_t^T \frac{P(t,T)}{P(t,u)} du + 1\right] S_t < \infty,
\end{aligned} \tag{7}$$

where Q_T is the $T-$ forward measure given by the Radon-Nikodym derivative: $(dQ_T/dQ)|\mathscr{F}_T = \exp\{-\int_0^T r_u du\}/P(0,T)$, and $P(t,T)$ is given above in (3). In the case of $t = T$, it is triviality.

(ii) In order to determine (5), we start from model (1) and develop $\ln S_T$ with the Brownian motions W_t^1 and W_t^2 expressed as $W_t^1 = \rho_1 Z_t^1 + \sqrt{1-\rho_1{}^2}\widetilde{Z}_t^1$ and $W_t^2 = \rho_1 Z_t^2 + \sqrt{1-\rho_2{}^2}\widetilde{Z}_t^2$, respectively, where $(Z_t^1, Z_t^2, \widetilde{Z}_t^1, \widetilde{Z}_t^2)$ are 4-dimensional Brownian motion defined on the probability space. For the process, $\ln S_T$, we have

$$\begin{aligned}
\ln S_T &= \ln S_t + \sum_{j=1}^{2}\left[-\frac{\rho_j\kappa_j\theta_j}{\sigma_j}(T-t) - \frac{\rho_j}{\sigma_j}\nu_{jt}\right. \\
&\quad + \left(\frac{\rho_j\kappa_j}{\sigma_j} + \frac{1}{2}\right)\int_t^T \nu_{j\tau} d\tau + \frac{\rho_j}{\sigma_j}\nu_{jT} \\
&\quad \left. + \sqrt{1-\rho_j^2}\int_t^T \sqrt{\nu_{j\tau}} d\widetilde{Z}_\tau^j\right] = \ln S_t + \sum_{j=1}^{2}\left[\frac{\theta_j}{2}(T-t)\right. \\
&\quad + \frac{\nu_{jt}}{2\kappa_j} - \frac{\nu_{jT}}{2\kappa_j} + \left(\rho_j + \frac{\sigma_j}{2\kappa_j}\right)\int_t^T \sqrt{\nu_{j\tau}} dZ_\tau^j \\
&\quad \left. + \sqrt{1-\rho_j^2}\int_t^T \sqrt{\nu_{j\tau}} d\widetilde{Z}_\tau^j\right].
\end{aligned} \tag{8}$$

On the other hand, we have

$$\begin{aligned}
\ln G_{[t,T]} &= \frac{T-t}{T}\ln S_t + \frac{1}{T}\sum_{j=1}^{2}\left[\frac{\theta_j}{4}(T-t)^2\right. \\
&\quad + \frac{\nu_{jt}}{2\kappa_j}(T-t) - \frac{1}{2\kappa_j}\int_t^T \nu_{j\tau} d\tau \\
&\quad + \left(\rho_j + \frac{\sigma_j}{2\kappa_j}\right)\int_t^T (T-\tau)\sqrt{\nu_{j\tau}} dZ_\tau^j \\
&\quad \left. + \sqrt{1-\rho_j^2}\int_t^T (T-\tau)\sqrt{\nu_{j\tau}} d\widetilde{Z}_\tau^j\right].
\end{aligned} \tag{9}$$

Using the fact

$$\begin{aligned}
\int_t^T (T-\tau)\sqrt{\nu_{j\tau}} dZ_\tau^j &= \frac{1}{\sigma_j}\left[\kappa_j\int_t^T (T-\tau)\nu_{j\tau} d\tau\right. \\
&\quad \left. + \int_t^T \nu_{j\tau} d\tau - (T-t)\nu_{jt} - \frac{\kappa_j\theta_j (T-t)^2}{2}\right],
\end{aligned} \tag{10}$$

for $j = 1, 2$, then

$$\begin{aligned}
\ln G_{[t,T]} &= \frac{T-t}{T}\ln S_t + \frac{1}{T}\sum_{j=1}^{2}\left[-\frac{\rho_j\kappa_j\theta_j}{2\sigma_j}(T-t)^2\right. \\
&\quad - \frac{\rho_j\nu_{jt}}{\sigma_j}(T-t) + \frac{\rho_j}{\sigma_j}\int_t^T \nu_{j\tau} d\tau \\
&\quad + \left(\frac{\rho_j\kappa_j}{\sigma_j} + \frac{1}{2}\right)\int_t^T (T-\tau)\nu_{j\tau} d\tau \\
&\quad \left. + \sqrt{1-\rho_j^2}\int_t^T (T-\tau)\sqrt{\nu_{j\tau}} d\widetilde{Z}_\tau^j\right].
\end{aligned} \tag{11}$$

Let $\mathscr{G}$ be the $\sigma-$field generated by $\mathscr{F}_t$ and $\{(Z_u^1, Z_u^2) : t < u \le T\}$. By (8) and (11), for $(s,w) \in D$, we have

$$\begin{aligned}
\psi_t(s,w) &= E^{Q}\left[\exp\left\{-\int_t^T r_\tau d\tau + s\ln G_{[t,T]}\right.\right. \\
&\quad \left.\left. + w\ln S_T\right\} \mid \mathscr{F}_t\right] \\
&= E^{Q}\left[E^{Q}\left[\exp\left\{-\int_t^T r_\tau d\tau + s\ln G_{[t,T]} + w\ln S_T\right\} \mid\right.\right. \\
&\quad \left.\left.\mathscr{G}\right] \mid \mathscr{F}_t\right] = A_1 E^{Q}\left[A_2 E^{Q}\left[A_3 \mid \mathscr{G}\right] \mid \mathscr{F}_t\right],
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
A_1 &= \exp\left\{s\left[\frac{T-t}{T}\ln S_t - \frac{1}{T}\right.\right. \\
&\quad \left.\cdot \sum_{j=1}^{2}\left(\frac{\rho_j\kappa_j\theta_j}{2\sigma_j}(T-t)^2 + \frac{\rho_j (T-t)}{\sigma_j}\nu_{jt}\right)\right] \\
&\quad \left. + w\left[\ln S_t - \sum_{j=1}^{2}\left(\frac{\rho_j\kappa_j\theta_j}{\sigma_j}(T-t) + \frac{\rho_j}{\sigma_j}\nu_{jt}\right)\right]\right\}, \\
A_2 &= \exp\left\{\frac{s}{T}\sum_{j=1}^{2}\left(\frac{2\rho_j\kappa_j + \sigma_j}{2\sigma_j}\int_t^T (T-\tau)\nu_{j\tau} d\tau\right.\right. \\
&\quad \left.\left. + \frac{\rho_j}{\sigma_j}\int_t^T \nu_{j\tau} d\tau\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{2}\left[w\left(\frac{\rho_j v_{jT}}{\sigma_j}+\frac{2\rho_j\kappa_j+\sigma_j}{2\sigma_j}\int_t^T v_{j\tau}d\tau\right)\right.\\
&\left.\left.-\int_t^T v_{j\tau}d\tau\right]\right\},\\
&A_3=\exp\left\{\sum_{j=1}^{2}\sqrt{1-\rho_j^2}\int_t^T\left[\frac{s}{T}(T-\tau)+w\right]\right.\\
&\left.\cdot\sqrt{v_{j\tau}}d\widetilde{Z}_\tau^j\right\}.
\end{aligned}
\tag{13}
$$

Since

$$
\begin{aligned}
E^{Q}\left[A_3\mid\mathcal{G}\right]&=\exp\left\{\frac{1}{2}\sum_{j=1}^{2}\left(1-\rho_j^2\right)\right.\\
&\left.\cdot\int_t^T\left[\frac{s}{T}(T-\tau)+w\right]^2 v_{j\tau}d\tau\right\}\\
&=\exp\left\{\sum_{j=1}^{2}\frac{\left(1-\rho_j^2\right)}{2}\right.\\
&\left.\cdot\int_t^T\left[\frac{s^2}{T^2}(T-\tau)^2+\frac{2sw}{T}(T-\tau)+w^2\right]v_{j\tau}d\tau\right\},
\end{aligned}
\tag{14}
$$

substituting (14) into (12) and applying the Markov property of $\{v_{ju}:0\le u\le T\}$, $j=1,2$, lead to (5), which completes the proof. □

From Proposition 1, it is clear that we need to search for an exact formula for the discounted joint characteristic function of v_{1t} and v_{2t} and the three different integrals of v_{jt} ($j=1,2$) appearing in (5). We use the same approach introduced by Kim and Wee [18] to obtain an explicit formula for the discounted joint characteristic function $\psi_t(s,w)$. Therefore, two series of functions are introduced as follows. Define F_τ^j and $\widetilde{F}_\tau^j:\mathbb{C}^4\longrightarrow\mathbb{C}$ as

$$
F_\tau^j\left(g_{j1},g_{j2},g_{j3},g_{j4}\right)=\sum_{n=0}^{\infty}f_n^j, \tag{15}
$$

$$
\widetilde{F}_\tau^j\left(g_{j1},g_{j2},g_{j3},g_{j4}\right)=\sum_{n=1}^{\infty}\frac{n}{\tau}f_n^j, \tag{16}
$$

where $f_n^j, n=0,1,2,\cdots$, are functions of g_{j1},g_{j2},g_{j3}, and g_{j4} defined as

$$
\begin{aligned}
&f_{-2}^j=f_{-1}^j=0,\\
&f_0^j=1,\\
&f_1^j=\frac{\left(\kappa_j-g_{j4}\sigma_j^2\right)\tau}{2},\\
&f_n^j=-\frac{\sigma_j^2\tau^2}{2n(n-1)}\left(g_{j1}\tau^2 f_{n-4}^j+g_{j2}\tau f_{n-3}^j\right.\\
&\qquad\left.+\left(g_{j3}-\frac{\kappa_j^2}{2\sigma_j^2}\right)f_{n-2}^j\right),\quad n\ge 2,
\end{aligned}
\tag{17}
$$

for $j=1,2$.

For $j=1,2$, denote

$$
\begin{aligned}
D_{j\tau}=\Bigg\{&\left(g_{j1},g_{j2},g_{j3},g_{j4}\right)\in\mathbb{C}^4:\\
&E^{Q}\left[\exp\left\{\Re\left(g_{j1}\right)\int_0^\tau(\tau-t)^2 v_{jt}dt\right.\right.\\
&+\Re\left(g_{j2}\right)\int_0^\tau(\tau-t)v_{jt}dt+\Re\left(g_{j3}\right)\int_0^\tau v_{jt}dt\\
&\left.\left.+\Re\left(g_{j4}\right)v_{j\tau}\right\}\right]<\infty\Bigg\}.
\end{aligned}
\tag{18}
$$

Proposition 2. *(1) The argument of F_τ^j: $\arg F_\tau^j(g_{j1},g_{j2},g_{j3},g_{j4})\ne 0$ for every $(g_{j1},g_{j2},g_{j3},g_{j4})\in D_{j\tau}$ ($j=1,2$). In particular, $\arg F_\tau^j$ is continuous on $D_{j\tau}$, and $\arg F_\tau^j(g_{j1},g_{j2},g_{j3},g_{j4})=0$ for g_{j1},g_{j2},g_{j3}, and g_{j4} are all real numbers in $D_{j\tau}$ for $j=1,2$.*

(2) For $(g_{j1},g_{j2},g_{j3},g_{j4})\in D_{j\tau}$, $j=1,2$, we have

$$
\begin{aligned}
&E^{Q}\left[\exp\left\{\sum_{j=1}^{2}\left[g_{j1}\int_0^\tau(\tau-t)^2 v_{jt}dt+g_{j2}\int_0^\tau(\tau-t)v_{jt}dt+g_{j3}\int_0^\tau v_{jt}dt+g_{j4}v_{j\tau}\right]\right\}\right]\\
&=\exp\left\{\sum_{j=1}^{2}\left(\frac{\kappa_j v_{j0}+\kappa_j^2\theta_j\tau}{\sigma_j}-\frac{2v_{j0}}{\sigma_j^2}\frac{\widetilde{F}_\tau^j\left(g_{j1},g_{j2},g_{j3},g_{j4}\right)}{F_\tau^j\left(g_{j1},g_{j2},g_{j3},g_{j4}\right)}-\frac{2\kappa_j\theta_j}{\sigma_j^2}\ln F_\tau^j\left(g_{j1},g_{j2},g_{j3},g_{j4}\right)\right)\right\}.
\end{aligned}
\tag{19}
$$

The proof of Proposition 2 is similar to that of Kim and Wee [18].

Using Proposition 2 to Proposition 1 leads to the explicit expression of $\psi_t(s,w)$ in Proposition 3. To describe the simplicity of the result, we need to introduce new functions. Define $H^j_{t,T}$ and $\widetilde{H}^j_{t,T}$: $\mathbb{C}^2 \longrightarrow \mathbb{C}$ as

$$H^j_{t,T}(s,w) = F^j_{T-t}\left(g_{j1}(s,w), g_{j2}(s,w), g_{j3}(s,w), g_{j4}(s,w)\right), \tag{20}$$

$$\widetilde{H}^j_{t,T}(s,w) = \widetilde{F}^j_{T-t}\left(g_{j1}(s,w), g_{j2}(s,w), g_{j3}(s,w), g_{j4}(s,w)\right), \tag{21}$$

with $j = 1, 2$.

Proposition 3. *(1) If $(s,w) \in D$, then $H^j_{t,T}(s,w) \neq 0$ and $\arg H^j_{t,T}$ is continuous on D. In particular, $\arg H^j_{t,T}(s,w) = 0$ if s and w are real numbers for $j = 1, 2$.*

(2) For $\psi_t(s,w) \in D$ with $\arg H^j_{t,T}$ defined as above, then

$$\psi_t(s,w) = \exp\left\{\sum_{j=1}^{2}\left(-c_{j1}\frac{\widetilde{H}^j_{t,T}(s,w)}{H^j_{t,T}(s,w)} - c_{j2}\ln H^j_{t,T}(s,w) + c_{j3}s + c_{j4}w + c_{j5}\right)\right\}. \tag{22}$$

Here $c_{j1} = 2v_{jt}/\sigma_j^2$, $c_{j2} = 2\kappa_j\theta_j/\sigma_j^2$,

$$\begin{aligned} c_{j3} &= \frac{T-t}{2T}\ln S_t - \frac{\rho_j\kappa_j\theta_j(T-t)^2}{2\sigma_j T} - \frac{\rho_j(T-t)}{\sigma_j T}v_{jt}, \\ c_{j4} &= \frac{\ln S_t}{2} - \frac{\rho_j}{\sigma_j}v_{jt} - \frac{\rho_j\kappa_j\theta_j(T-t)}{\sigma_j}, \\ c_{j5} &= \frac{\kappa_j v_{jt} + \kappa_j^2\theta_j(T-t)}{\sigma_j^2} \end{aligned} \tag{23}$$

for $j = 1, 2$.

Proof. Assume that $(s,w) \in D$, using the definitions of $g_{jk} = g_{jk}(s,w)$ and $D_{j\tau}$ for $k = 1,2,3,4$ and $j = 1,2$. Note that $(g_{j1}, g_{j2}, g_{j3}, g_{j4}) \in D^j_\tau \subset D^j_{T-t}$ for every $0 \leq t \leq T$ and $j = 1,2$. Therefore, (19) remains valid for any $v_{j0} > 0$. The time homogeneous Markov property of v_{jt} implies that (22) holds when Propositions 1 and 2 are satisfied.

Now Substituting the expressions of $g_{j1}, g_{j2}, g_{j3}, g_{j4}$, $j = 1, 2$ into (15) and (16), $H^j_{t,T}$ and $\widetilde{H}^j_{t,T}$ are expressed as follows, i.e., for $(s,w) \in \mathbb{C}^2$,

$$H^j_{t,T}(s,w) = \sum_{n=0}^{\infty} h^j_n(s,w), \tag{24}$$

$$\widetilde{H}^j_{t,T}(s,w) = \sum_{n=1}^{\infty} \frac{n}{T-t} h^j_n(s,w), \tag{25}$$

where

$$\begin{aligned} &h^j_{-2}(s,w) = h^j_{-1}(s,w) = 0, \\ &h^j_0(s,w) = 1, \\ &h^j_1(s,w) = \frac{(T-t)\left(\kappa_j - w\rho_j\sigma_j\right)}{2}, \\ &h^j_n(s,w) = \frac{(T-t)^2}{4n(n-1)T^2}\left\{-s^2\sigma_j^2\left(1-\rho_j^2\right)(T-t)^2\right. \\ &\quad \cdot h^j_{n-4}(s,w) - \left[s\sigma_j T\left(\sigma_j + 2\rho_j\kappa_j\right)\right. \\ &\quad \left. + 2sw\sigma_j^2 T\left(1-\rho_j^2\right)\right](T-t)h^j_{n-3}(s,w) + T\left[\kappa_j^2 T\right. \\ &\quad - 2s\rho_j\sigma_j - w\left(2\rho_j\kappa_j + \sigma_j\right)\sigma_j T - w^2\left(1-\rho_j^2\right)\sigma_j^2 T \\ &\quad \left. \left. + 2\sigma_j^2 T\right] h^j_{n-2}(s,w)\right\}, \quad n \geq 2, \end{aligned} \tag{26}$$

for $j = 1, 2$. □

4. Pricing Geometric Asian Option

Once the discounted joint characteristic function is found, the European continuously monitored geometric Asian option can be valued using numeraire change technique and the inverse Fourier transform approach which are applied in many research works (see, e.g., Geman et al. [41] and Deng [42]). This section derives the pricing formulas for the continuously monitored fixed and floating strike geometric Asian call options.

Theorem 4. *Suppose that S_t, v_{1t}, and v_{2t} follow the dynamics in (1), then the price at time $t \in [0,T]$ of the continuously monitored fixed strike geometric Asian call option with maturity T and the strike price K is given by*

$$GAC_{fi}(t,S,v_1,v_2,K,T) = e^{(1/T)\int_0^t \ln S_u du}\psi_t(1,0)\Pi_1 - KP(t,T)\Pi_2, \tag{27}$$

where $P(t,T)$ is given above in (3) and

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty}\mathfrak{R}\left[\frac{e^{-iu\ln K_{t,T}}\phi_j(u)}{iu}\right]du, \quad j = 1, 2,$$

$$K_{t,T} = Ke^{-(1/T)\int_0^t \ln S_u du},$$
$$\phi_1(u) = \frac{\psi_t(iu+1,0)}{\psi_t(1,0)},$$
$$\phi_2(u) = \frac{\psi_t(iu,0)}{\psi_t(0,0)}, \tag{28}$$

where i is the imaginary unit ($i^2 = -1$).

Proof. Since

$$\begin{aligned} &GAC_{fi}(t,S,v_1,v_2,K,T) \\ &= E^Q\left[e^{-\int_t^T r_u du}(G_{[0,T]}-K)^+ \mid \mathcal{F}_t\right] \\ &= e^{(1/T)\int_0^t \ln S_u du} E^Q\left[e^{-\int_t^T r_u du} G_{[t,T]} 1_{\{\ln G_{[t,T]} \geq \ln K_{t,T}\}} \mid \right. \\ &\left.\mathcal{F}_t\right] - KE^Q\left[e^{-\int_t^T r_u du} 1_{\{\ln G_{[t,T]} \geq \ln K_{t,T}\}} \mid \mathcal{F}_t\right] \\ &= e^{(1/T)\int_0^t \ln S_u du}\psi_t(1,0) Q_1(\ln G_{[t,T]} \geq \ln K_{t,T}) \\ &- KP(t,T) Q_T(\ln G_{[t,T]} \geq \ln K_{t,T}), \end{aligned} \tag{29}$$

where Q_1 is defined by the following Radon-Nikodym derivative:

$$\left.\frac{dQ_1}{dQ}\right|_{\mathcal{F}_T} = e^{-\int_t^T r_u du}\frac{G_{[t,T]}}{E^Q\left[e^{-\int_t^T r_u du} G_{[t,T]} \mid \mathcal{F}_t\right]}, \tag{30}$$

and Q_T is the $T-$ forward measure given above, it is well known that the probability distribution functions can be calculated by using the Fourier inversion transform, and then the above two probabilities Q_1 and Q_T in (29) are given by

$$\begin{aligned} &Q_1(\ln G_{[t,T]} \geq \ln K_{t,T}) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \Re\left[\frac{e^{-iu\ln K_{t,T}}\phi_1(u)}{iu}\right] du := \Pi_1, \end{aligned} \tag{31}$$

$$\begin{aligned} &Q_T(\ln G_{[t,T]} \geq \ln K_{t,T}) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \Re\left[\frac{e^{-iu\ln K_{t,T}}\phi_2(u)}{iu}\right] du := \Pi_2, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \phi_1(u) &= E^{Q_1}\left[e^{iu\ln G_{[t,T]}} \mid \mathcal{F}_t\right] \\ &= \frac{E^Q\left[e^{-\int_t^T r_u du + iu\ln G_{[t,T]}} G_{[t,T]} \mid \mathcal{F}_t\right]}{E^Q\left[e^{-\int_t^T r_u du} G_{[t,T]} \mid \mathcal{F}_t\right]} \\ &= \frac{E^Q\left[e^{-\int_t^T r_u du + (iu+1)\ln G_{[t,T]}} \mid \mathcal{F}_t\right]}{E^Q\left[e^{-\int_t^T r_u du + \ln G_{[t,T]}} \mid \mathcal{F}_t\right]} \\ &= \frac{\psi_t(iu+1,0)}{\psi_t(1,0)}, \end{aligned} \tag{33}$$

$$\begin{aligned} \phi_2(u) &= E^{Q_T}\left[e^{iu\ln G_{[t,T]}} \mid \mathcal{F}_t\right] \\ &= \frac{E^Q\left[e^{-\int_t^T r_u du + iu\ln G_{[t,T]}} \mid \mathcal{F}_t\right]}{E^Q\left[e^{-\int_t^T r_u du} \mid \mathcal{F}_t\right]} = \frac{\psi_t(iu,0)}{P(t,T)}. \end{aligned} \tag{34}$$

From (29), (31), and (32), we can obtain the required Theorem 4. □

Theorem 5. *Suppose that S_t, v_{1t}, and v_{2t} follow the dynamics in (1), then the price at time $t \in [0,T]$ of the continuously monitored floating strike geometric Asian call option with maturity T is given by*

$$\begin{aligned} GAC_{fl}(t,S,v_1,v_2,T) &= S_t\widehat{\Pi}_1 \\ &- e^{(1/T)\int_0^t \ln S_u du}\psi_t(1,0)\widehat{\Pi}_2, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \widehat{\Pi}_1 &= \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} \Re\left[\frac{\psi_t(iu,1-iu)e^{(iu/T)\int_0^t \ln S_u du}}{iuS_t}\right] du, \\ \widehat{\Pi}_2 &= \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} \Re\left[\frac{\psi_t(iu+1,-iu)e^{(iu/T)\int_0^t \ln S_u du}}{iu\psi_t(1,0)}\right] du. \end{aligned} \tag{36}$$

Proof. Since

$$\begin{aligned} &GAC_{fl}(t,S,v_1,v_2,T) = E^Q\left[e^{-\int_t^T r_u du}(S_T - G_{[0,T]})^+ \mid \mathcal{F}_t\right] \\ &= e^{(1/T)\int_0^t \ln S_u du} E^Q\left[e^{-\int_t^T r_u du}\left(S_T e^{-(1/T)\int_0^t \ln S_u du} - G_{[t,T]}\right)^+ \mid \right. \\ &\left.\mathcal{F}_t\right] = E^Q\left[e^{-\int_t^T r_u du} S_T 1_{\{\ln(G_{[t,T]}/S_T) \leq -(1/T)\int_0^t \ln S_u du\}} \mid \mathcal{F}_t\right] \\ &- e^{(1/T)\int_0^t \ln S_u du} E^Q\left[e^{-\int_t^T r_u du} G_{[t,T]} 1_{\{\ln(G_{t,T}/S_T) \leq -(1/T)\int_0^t \ln S_u du\}} \mid \right. \end{aligned}$$

$$\left.\mathscr{F}_t\right] = S_t Q_2\left(\ln\left(\frac{G_{[t,T]}}{S_T}\right) \le -\frac{1}{T}\int_0^t \ln S_u du\right) - e^{(1/T)\int_0^t \ln S_u du}\psi_t(1,0)\, Q_1\left(\ln\left(\frac{G_{[t,T]}}{S_T}\right) \le -\frac{1}{T}\int_0^t \ln S_u du\right), \tag{37}$$

where Q_2 is defined by the following Radon-Nikodym derivative:

$$\left.\frac{dQ_2}{dQ}\right|_{\mathscr{F}_T} = e^{-\int_t^T r_u du}\frac{S_T}{S_t}, \tag{38}$$

under the two probability measures Q_1 and Q_2, the conditional characteristic functions of $\ln(G_{[t,T]}/S_T)$ are given by

$$\begin{aligned} E^{Q_1}&\left[e^{iu\ln(G_{[t,T]}/S_T)} \mid \mathscr{F}_t\right] \\ &= \frac{E^{Q}\left[e^{-\int_t^T r_u du + iu\ln(G_{[t,T]}/S_T)}G_{[t,T]} \mid \mathscr{F}_t\right]}{E^{Q}\left[e^{-\int_t^T r_u du}G_{[t,T]} \mid \mathscr{F}_t\right]} \\ &= \frac{\psi_t(iu+1,-iu)}{\psi_t(1,0)}, \end{aligned} \tag{39}$$

$$\begin{aligned} E^{Q_2}&\left[e^{iu\ln(G_{[t,T]}/S_T)} \mid \mathscr{F}_t\right] \\ &= \frac{E^{Q}\left[e^{-\int_t^T r_u du + iu\ln(G_{[t,T]}/S_T) + \ln S_T} \mid \mathscr{F}_t\right]}{S_t} \\ &= \frac{\psi_t(iu, 1-iu)}{S_t}. \end{aligned} \tag{40}$$

Therefore

$$\begin{aligned} Q_1&\left(\ln\left(\frac{G_{t,T}}{S_T}\right) \le -\frac{1}{T}\int_0^t \ln S_u du\right) \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty}\Re\left[\frac{\psi_t(iu+1,-iu)\,e^{(iu/T)\int_0^t \ln S_u du}}{iu\psi_t(1,0)}\right]du, \end{aligned} \tag{41}$$

$$\begin{aligned} Q_2&\left(\ln\left(\frac{G_{t,T}}{S_T}\right) \le -\frac{1}{T}\int_0^t \ln S_u du\right) \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty}\Re\left[\frac{\psi_t(iu,1-iu)\,e^{(iu/T)\int_0^t \ln S_u du}}{S_t \cdot iu}\right]du. \end{aligned} \tag{42}$$

Plugging (41) and (42) in (37) yields (35). □

In addition to option prices, one can compute derivatives to hedge against changes in the underlying asset price S and volatilities v_1 and v_2. We omit it due to its triviality.

5. Numerical Examples

In this section, we use the dbH-SI model to analyze the valuation of the continuously monitored fixed strike geometric Asian call option using some numerical examples. To implement the analytic formula given in (27) numerically, we first determine the number of terms taken for the computation of the infinite series expansions for $H_{t,T}^j$ and $\widetilde{H}_{t,T}^j$ for $j = 1,2$ in (24) and (25). Second, we investigate the accuracy and efficiency of the approximated analytic formula given in (27). In the end, we compare the option prices varying by S_0 and T under different models including the dbH-SI, H-SI (i.e., Heston stochastic volatility model with stochastic interest rate), dbH, and Heston models. Furthermore, we use this proposed model (1), i.e., the dbH-SI model, to examine the impacts of the model parameters on option prices. Here we implement the integral formulas given in (27) without truncation, modification, and approximation in Mathematica and Matlab software. Numerical integration was performed using the "NIntegrate()" or "quadgk()" commands which can handle infinite ranges, oscillatory integrands, and singularities in their default version, which employs automatic adaptive integration.

Model (1) parameters are set as follows: $\kappa_1 = 1.15$, $\kappa_2 = 2.2$, $\theta_1 = 0.2$, $\theta_2 = 0.15$, $\sigma_1 = 0.5$, $\sigma_2 = 0.8$, $\rho_1 = -0.4$, $\rho_2 = -0.64$, $v_{10} = 0.09$, $v_{20} = 0.04$, $S_0 = 100$, and $t = 0$. The results are shown in Table 1.

Table 1 investigates the influence of the number of terms n taken in (24) and (25). It is shown that the numerical values tend to be quite stable as the number of terms increases. In particular, the option prices stay unchanged after taking $n = 20$ and $n = 30$ terms, at most. In addition, Table 1 provides option prices with various maturities T, strike prices K, and the required CPU times (in seconds). From Table 1, we can see that some characteristics of the GAC_{fi} option are similar to the plain European call option. For example, the values of the GAC_{fi} option are decreasing functions of the strike price. In addition, under the same parameter values setting in dbH-SI model, longer maturating date T will result in higher option values.

It would be interesting to see the performance of our approximated analytic formula approach implemented to the fixed strike GAC option under dbH-SI model. Our numerical example uses the number of terms $n = 20$ taken in (24) and (25); other parameter values are similar to those settings of Table 1. We compute the 0.5-,1.5-, and 3-year maturity GAC options by our approximated analytic formula approach and compare the results with Monte Carlo simulation with 10000 sample paths and 100 points for time axis. The numerical results are displayed in Table 2.

Table 2 shows that the approximated analytic formula approach is considerably faster than those of the Monte Carlo simulation (MC). For a given set of parameters, the approximated analytic formula approach calculates the option prices for 5 different strikes and 3 different maturates in approximately 5 seconds. The Monte Carlo simulation takes around 110 seconds for each option price. Table 2 also compares their pricing accuracy. It can be seen that the relative error (RE) in prices is less than 0.4% for all cases. If

Table 1: GAC_{fi} option prices with different maturities, strike prices, and varying n.

T	K	n	Option price	CPU (sec.)	T	K	n	Option price	CPU (sec.)
0.5	90	5	17.8597	0.287	1.5	90	5	26.4805	0.291
		10	17.7859	0.291			10	25.2393	0.294
		15	17.7849	0.294			15	25.2254	0.299
		20	17.7849	0.296			20	25.2251	0.302
		25	17.7849	0.301			25	25.2251	0.307
		30	17.7849	0.312			30	25.2251	0.315
	100	5	9.7733	0.279		100	5	19.8642	0.276
		10	9.8624	0.288			10	19.1846	0.285
		15	9.8623	0.290			15	19.1758	0.290
		20	9.8623	0.295			20	19.1755	0.302
		25	9.8623	0.298			25	19.1755	0.309
		30	9.8623	0.302			30	19.1755	0.311
	110	5	4.2061	0.283		110	5	14.1405	0.283
		10	4.5395	0.291			10	14.0976	0.291
		15	4.5415	0.304			15	14.1003	0.297
		20	4.5415	0.306			20	14.1002	0.304
		25	4.5415	0.312			25	14.1002	0.310
		30	4.5415	0.315			30	14.1002	0.315
1	90	5	22.3192	0.268	3	90	5	30.5831	0.289
		10	22.0988	0.283			10	29.6250	0.293
		15	22.0938	0.287			15	29.2870	0.299
		20	22.0938	0.296			20	29.2835	0.305
		25	22.0938	0.313			25	29.2835	0.314
		30	22.0938	0.319			30	29.2835	0.317
	100	5	15.0971	0.281		100	5	26.7286	0.268
		10	15.1874	0.287			10	25.6231	0.284
		15	15.1850	0.294			15	25.3192	0.291
		20	15.1849	0.298			20	25.3151	0.295
		25	15.1849	0.301			25	25.3151	0.298
		30	15.1849	0.307			30	25.3151	0.307
	110	5	9.2791	0.275		110	5	22.5318	0.281
		10	9.7604	0.278			10	21.9684	0.284
		15	9.7650	0.281			15	21.7105	0.297
		20	9.7651	0.292			20	21.7064	0.301
		25	9.7651	0.294			25	21.7064	0.305
		30	9.7651	0.301			30	21.7064	0.310

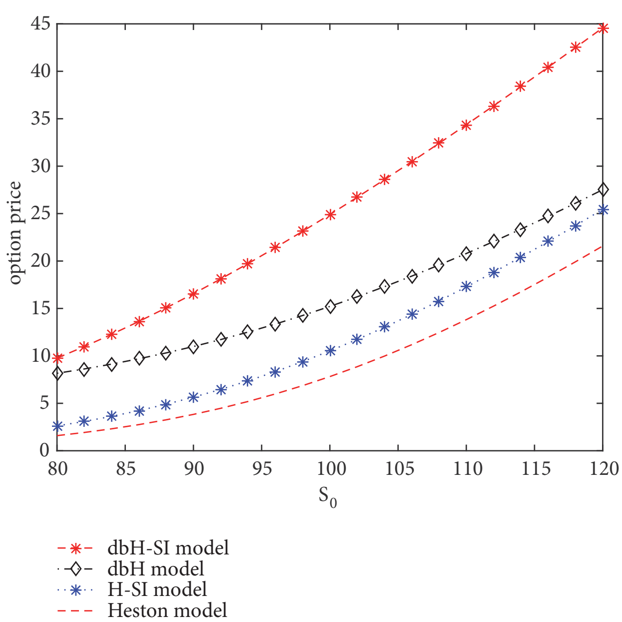
we regard the Monte Carlo price as the benchmark, then this numerical example confirms that the approximated analytic formula approach is accurate and efficient.

After examining accuracy and efficiency. We shall turn to investigate the comparison of the option prices under different models. Figure 1 compares the dbH-SI model with Heston, dbH, and H-SI models in pricing the fixed strike GAC option against the initial asset price S_0 and maturating date T with the strike price $K = 100$. From Figure 1, it is interesting to notice that the option prices of the fixed strike GAC option calculated from the dbH-SI model are much higher than those of the dbH model, H-SI model, and Heston model. One possible reason for this phenomena is that the volatilities are always bigger for the dbH-SI model than those for other models. This implies that the proposed model, i.e., the dbH-SI model, has a more significant influence than the dbH model, H-SI model, and Heston model on option prices. We can also clearly observe that the bigger the value of S_0(or T) is, the higher the GAC_{fi} option price is.

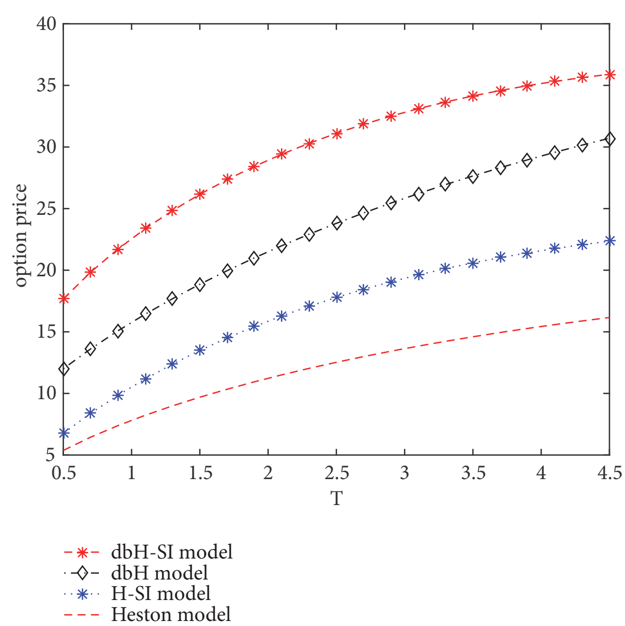
In Figure 2, we use the dbH-SI model to examine the effects of the mean-reverting rates (κ_1, κ_2) and volatilities (σ_1, σ_2) of variance processes (ν_{1t}, ν_{2t}) on the GAC_{fi} option prices (see Figures 2(a)–2(d)). We also examine the impacts of the correlation coefficients (ρ_1, ρ_2) (see Figures 2(e) and 2(f)), which determine both the correlation between the underlying asset and its volatilities, and the correlation between the underlying asset and the short interest rates.

Table 2: Comparison of the approximated approach and MC for GAC_{fi} options.

T	K	Approximated approach	CPU(sec.)	Monte Carlo	CPU(sec.)	RE(%)
0.5	90	17.7849	0.284	17.7892	110.302	-0.0242
	95	13.5305	0.291	13.5246	110.384	0.0436
	100	9.8623	0.278	9.8602	110.403	0.0213
	105	6.8598	0.296	6.8624	110.329	-0.3810
	110	4.5415	0.287	4.5407	110.376	0.0176
1.5	90	25.2251	0.265	25.2206	110.376	0.0178
	95	22.0851	0.281	22.0891	110.391	-0.0181
	100	19.1755	0.294	19.1787	110.417	-0.0167
	105	16.5109	0.312	16.5045	110.485	0.3888
	110	14.1002	0.293	14.1039	110.392	-0.0262
3	90	29.2835	0.271	29.2894	110.428	-0.0201
	95	27.2560	0.289	27.2513	110.397	0.0173
	100	25.3151	0.307	25.3106	110.405	0.0178
	105	23.4644	0.295	23.4595	110.471	0.0209
	110	21.7065	0.292	21.7128	110.396	-0.0290



(a) Option price against S_0



(b) Option price against T

Figure 1: Comparison of option prices under Heston, dbH, H-SI, and dbH-SI models.

From Figures 2(a) and 2(b), we can observe that the option prices increase as the mean-reverting rate increases. Particularly, it is quite remarkable that the mean-reverting rates have a significant effect on the longer term option values. The option price decreases as the volatilities of variance processes increase (see Figures 2(c) and 2(d)). The correlation coefficients (ρ_1, ρ_2) have several effects depending on the relation between the strike price and the initial asset price. A negative (ρ_1, ρ_2) tends to produce higher values for ITM (in-the-money) options and lower values for OTM (out-of-the-money) options (see Figure 2(e)). A similar result holds with respect to the expiration date: a negative (ρ_1, ρ_2) tends to produce higher value for the longer term options and lower values for the shorter options (see Figure 2(f)).

6. Conclusions

The proposed model (the dbH-SI model) incorporates several important features of the underlying asset returns variability. We derive the discounted joint characteristic function of the log-asset price and its log-geometric mean value over time period $[0, T]$ and obtain approximated analytic solutions to the continuously monitored fixed and floating strike geometric Asian call options using the change of numeraire

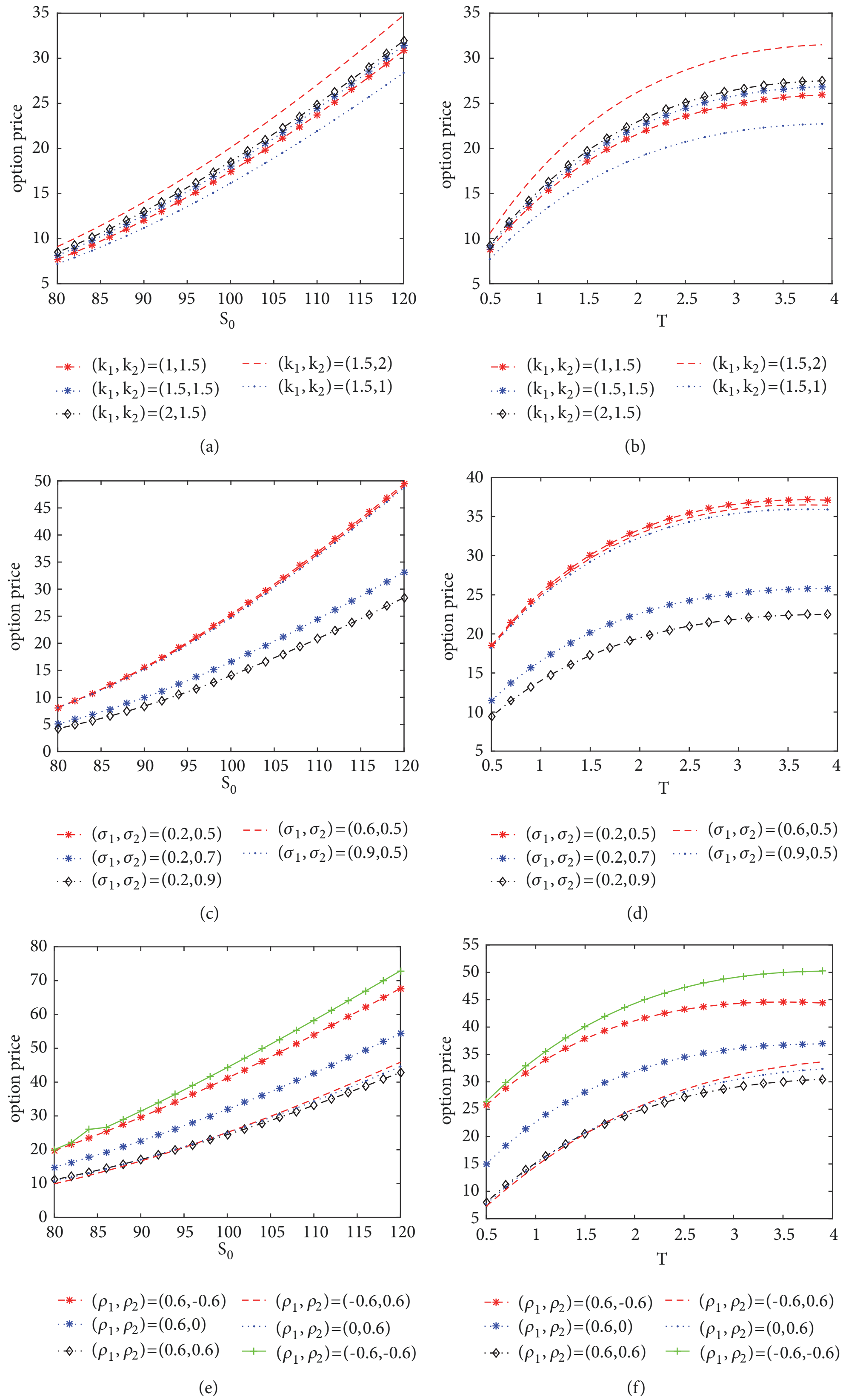


Figure 2: Option prices with respect to S_0 and T in the dbH-SI model.

technique and the Fourier inversion transform approach. Some numerical examples are provided to examine the effects of the proposed model, which reveals some additional features having a significant impact on option values, especially long-term options. The proposed model can be tested empirically by using the option price data from the option market.

Data Availability

All data used to support of the findings for this study are included within this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] P. Boyle and F. Boyle, *Derivatives: The Tools that Changed Finance*, Risk Books, London, UK, 2001.

[2] F. Black and M. Scholes, "The pricing of options corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–659, 1973.

[3] S. M. Turnbull and M. D. Wakeman, "A Quick Algorithm for Pricing European Average Options," *Journal of Financial and Quantitative Analysis*, vol. 26, no. 3, pp. 377–389, 1991.

[4] P. Ritchken, L. Sankarasubramanian, and A. M. Vijh, "The Valuation of Path Dependent Contracts on the Average," *Management Science*, vol. 39, no. 10, pp. 1202–1213, 1993.

[5] H. Geman and M. Yor, "Bessel processes, Asian options and perpetuities," *Mathematical Finance*, vol. 3, no. 4, pp. 349–375, 1993.

[6] L. C. Rogers and Z. Shi, "The value of an Asian option," *Journal of Applied Probability*, vol. 32, no. 4, pp. 1077–1088, 1995.

[7] P. Boyle, M. Broadie, and P. Glasserman, "Monte Carlo methods for security pricing," *Journal of Economic Dynamics & Control*, vol. 21, no. 8-9, pp. 1267–1321, 1997.

[8] J. Angus, "A note on pricing derivatives with continuous geometric averaging," *Journal of Future Markets*, vol. 19, pp. 823–839, 1999.

[9] V. Linetsky, "Spectral expansions for Asian (average price) options," *Operations Research*, vol. 52, no. 6, pp. 856–867, 2004.

[10] Z. Cui, C. Lee, and Y. Liu, "Single-transform formulas for pricing Asian options in a general approximation framework under Markov processes," *European Journal of Operational Research*, vol. 266, no. 3, pp. 1134–1139, 2018.

[11] G. Fusai and A. Roncoroni, *Asian Options: An Average Problem, in Problem Solving in Quantitative Finance: A Case-Study Approach*, Springer, 2008.

[12] J. Sun, L. Chen, and S. Li, "A Quasi-analytical pricing model for arithmetic Asian options," *Journal of Futures Markets*, vol. 33, no. 12, pp. 1143–1166, 2013.

[13] J. C. Hull and A. White, "The pricing of options on assets with stochastic volatilities," *The Journal of Finance*, vol. 42, no. 2, pp. 281–300, 1987.

[14] E. M. Stein and J. C. Stein, "Stock price distribution with stochastic volatility: an analytic approach," *Review of Financial Studies*, vol. 4, pp. 727–752, 1991.

[15] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The Review of Financial Studies*, vol. 6, no. 2, pp. 327–343, 1993.

[16] H. Y. Wong and Y. L. Cheung, "Geometric asian option: valuation and calibration with stochastic volatility," *Quantitative Finance*, vol. 4, pp. 301–314, 2004.

[17] F. Hubalek and C. Sgarra, "On the explicit evaluation of the geometric Asian options in stochastic volatility models with jumps," *Journal of Computational and Applied Mathematics*, vol. 235, no. 11, pp. 3355–3365, 2011.

[18] B. Kim and I.-S. Wee, "Pricing of geometric Asian options under Heston's stochastic volatility model," *Quantitative Finance*, vol. 14, no. 10, pp. 1795–1809, 2014.

[19] Q. Shi and X. Yang, "Pricing Asian options in a stochastic volatility model with jumps," *Applied Mathematics and Computation*, vol. 228, pp. 411–422, 2014.

[20] S. Alizadeh, M. Brandt, and F. Diebold, "Range-based estimation of stochastic volatility models," *Journal of Finance*, vol. 57, no. 3, pp. 1047–1091, 2002.

[21] G. Fiorentini, A. León, and G. Rubio, "Estimation and empirical performance of Heston's stochastic volatility model: the case of thinly traded market," *Journal of Empirical Finance*, vol. 9, pp. 225–255, 2002.

[22] M. Chernov, A. R. Gallant, E. Ghysels, and G. Tauchen, "Alternative models for stock price dynamics," *Journal of Econometrics*, vol. 116, no. 1-2, pp. 225–257, 2003.

[23] C. Gourieroux, "Continuous time Wishart process for stochastic risk," *Econometric Reviews*, vol. 25, no. 2-3, pp. 177–217, 2006.

[24] P. Christoffersen, S. Heston, and K. Jacobs, "The shape and term structure of the index option smirk: why multifactor stochastic volatility models work so well," *Management Science*, vol. 55, no. 12, pp. 1914–1932, 2009.

[25] J. M. Romo, "Pricing Forward Skew Dependent Derivatives. Multifactor Versus Single-Factor Stochastic Volatility Models," *Journal of Futures Markets*, vol. 34, no. 2, pp. 124–144, 2014.

[26] K. Nagashima, T.-K. Chung, and K. Tanaka, "Asymptotic Expansion Formula of Option Price Under Multifactor Heston Model," *Asia-Pacific Financial Markets*, vol. 21, no. 4, pp. 351–396, 2014.

[27] D. Duffie, J. Pan, and K. Singleton, "Transform analysis and asset pricing for affine jump-diffusions," *Econometrica*, vol. 68, pp. 1343–1376, 2000.

[28] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna, "Multiscale stochastic volatility asymptotics," *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal*, vol. 2, no. 1, pp. 22–42, 2003.

[29] Y. Sun, "Efficient pricing and hedging under the double Heston stochastic volatility jump-diffusion model," *International Journal of Computer Mathematics*, vol. 92, no. 12, pp. 2551–2574, 2015.

[30] F. Mehrdoust and N. Saber, "Pricing arithmetic Asian option under a two-factor stochastic volatility model with jumps,"

Journal of Statistical Computation and Simulation, vol. 85, no. 18, pp. 3811–3819, 2015.

[31] S. Zhang and Y. Sun, "Forward starting options pricing with dounble stochastic volatility, stochastic interest rate and double jumps," *Journal of Computational and Applied Mathematics*, vol. 325, pp. 34–41, 2017.

[32] J. A. Nielsen and K. Sandmann, "The pricing of Asian options under stochastic interest rates," *Applied Mathematical Finance*, vol. 3, no. 3, pp. 209–236, 1996.

[33] J. A. Nielsen and K. Sandmann, "Pricing of Asian exchange rate options under stochastic interest rates as a sum of options," *Finance and Stochastics*, vol. 6, no. 3, pp. 355–370, 2002.

[34] S. Zhang, S. Yuan, and L. Wang, "Prices of Asian options under stochastic interest rates," *Applied Mathematics: A Journal of Chinese Universities*, vol. 21, no. 2, pp. 135–142, 2006.

[35] R. Donnelly, S. Jaimungal, and D. H. Rubisov, "Valuing guaranteed withdrawal benefits with stochastic interest rates and volatility," *Quantitative Finance*, vol. 14, no. 2, pp. 369–382, 2014.

[36] X.-J. He and S.-P. Zhu, "A closed-form pricing formula for European options under the Heston model with stochastic interest rate," *Journal of Computational and Applied Mathematics*, vol. 335, pp. 323–333, 2018.

[37] J. C. Cox, J. E. Ingersoll, and S. Ross, "A theory of the term structure of interest rates," *Econometrica*, vol. 53, pp. 373–384, 1985.

[38] O. Vasicek, "An equilibrium characterization of the term structure," *Journal of Financial Economics*, vol. 5, no. 2, pp. 177–188, 1977.

[39] F. A. Longstaff and E. S. Schwartz, "Interest rate volatility and the term structure: a two-factor general equilibrium model," *The Journal of Finance*, vol. 47, no. 4, pp. 1259–1282, 1992.

[40] N. D. Pearson and T. Sun, "Exploiting the conditional density in estimating the term structure: an application to the Cox, Ingersoll and Ross model," *The Journal of Finance*, vol. 49, no. 4, pp. 1279–1304, 1994.

[41] H. Geman, N. Karoui, and J. C. Rochet, "Changes of numeraire, changes of probability measure and option pricing," *Journal of Applied Probability*, vol. 32, pp. 443–458, 1995.

[42] G. Deng, "Pricing European option in a double exponential jump-diffusion model with two market structure risks and its comparisons," *Applied Mathematics: A Journal of Chinese Universities*, vol. 22, no. 2, pp. 127–137, 2007.

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