Research Article

Geometric Asian Options Pricing under the Double Heston Stochastic Volatility Model with Stochastic Interest Rate

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This paper presents an extension of double Heston stochastic volatility model by incorporating stochastic interest rates and derives explicit solutions for the prices of the continuously monitored fixed and floating strike geometric Asian options. The discounted joint characteristic function of the log-asset price and its log-geometric mean value is computed by using the change of numeraire and the Fourier inversion transform technique. We also provide efficient approximated approach and analyze several effects on option prices under the proposed model. Numerical examples show that both stochastic volatility and stochastic interest rate have a significant impact on option values, particularly on the values of longer term options. The proposed model is suitable for modeling the longer time real-market changes and managing the credit risks.

1. Introduction

Asian option is a special type of option contract in which the payoff depends on the average of the underlying asset price over some predetermined time interval. The averaging feature allows Asian options to reduce the volatility inherent in the option. There are some advantages to trading Asian options in a financial market. One is that these decrease the risk of market manipulation of the financial derivative at expiry. Another is that Asian options have lower relative charge than European or American options. In general, the average considered can be a arithmetic or geometric one and it can be calculated either discretely, for which the average is taken over the underlying asset prices at discrete monitoring time points, or continuously, for which the average is calculated via the integration of the underlying asset price over the monitoring time period. Asian options can be differentiated into two main classes according to their payoff: fixed strike price options (sometimes called "average price") and floating strike price options (sometimes called "average strike"). All these details are specified by the contracts stipulated by two counterparts, as Asian options are traded actively on the OTC market among investors or traders for hedging the average price of a commodity. For a brief introduction to the development of Asian options, see Boyle and Boyle [1].

As the probability distribution of the average prices of the underlying asset generally does not have a simple analytical expression, it is difficult to obtain the analytical pricing formula for Asian option. Since the best-known closed-form pricing formula for the European vanilla option derived by Black and Scholes [2], many researchers have devoted themselves to developing the Asian options pricing based on the Black-Scholes assumptions; see, e.g., Kemna and Vorst (1990), Turnbull and Wakeman [3], Ritchken et al. [4], Geman and Yor [5], Rogers and Shi [6], Boyle et al. [7], Angus [8], Linetsky [9], Cui et al. [10], and the references therein. For a recent review, one can refer to Fusai and Roncoroni [11] and Sun et al. [12].

In practice, the Black-Scholes assumptions are hardly satisfied, especially the constant volatility and constant interest rate hypothesis. As the empirical behaviors of the implied volatility smile and heavy tailed in the distribution of log-returns are commonly observed in financial markets. For
this reason, stochastic volatility (hereafter SV) models have been proposed in finance (see Hull and White [13], Stein and Stein [14], Heston [15], and others). These models have been applied to value the Asian options (see, e.g., Wong and Cheung [16], Hubalek and Sgarra [17], Kim and Wee [18], and Shi and Yang [19]). In addition, interest rates are stochastic and stock returns are negatively correlated with interest rate changes, which have been examined in previous research.

Although these models mentioned above are able to account for the empirical behaviors, they are still based on a single-factor for volatility dynamics that is inconsistent with the long range memory characteristic of the volatility corrections and the stiff volatility skews. See Alizadeh et al. [20], Fiorentini et al. [21], Chernov et al. [22], Gourieroux [23], Christoffersen et al. [24], Romo [25], and Nagashima et al. [26] for the empirical results. To address this issue, multifactor SV models have recently generated attention in the option pricing literature. For instance, Duffie et al. [27] proposed multifactor affine stochastic volatility models. Based upon the Black-Scholes framework, Fouque et al. [28] introduced a multiscale SV model, in which the volatility processes are driven by two mean-reverting diffusion processes. Gourieroux [23] proposed a multivariate model in which the volatility-covolatility matrix follows a Wishart process.

On the basis of the findings of Christoffersen et al. [24], a double Heston (dbH) model, which consists of two independent variance processes, has recently been reported better than the plain Heston model in the performances of hedging (see Sun [29]) and has also been applied to arithmetic Asian option under discrete monitoring (Mehrdoust and Saber [30]) and forward starting option (Zhang and Sun [31]). However, its extension to continuously monitored geometric Asian option is yet to be considered. On the other hand, many of Asian options often have long-dated maturities since they are used as part of the structured notes which has a long maturity. The movement of interest rates becomes an issue in such cases and constant interest rate assumption should be replaced by an appropriate dynamic interest rate model. Several results are available on the Asian option in the stochastic interest rate framework; see, e.g., Nielsen and Sandmann [32, 33], Zhang et al. [34], Donnelly et al. [35], and He and Zhu [36]. In the above stochastic interest rate framework, the short-term interest rate is assumed to follow a specific parametric one-factor model (see, e.g., Cox et al. [37], Hull and White [13], and Vasicek [38]), which tends to oversimplify the true behavior of interest rate movement. However, empirical tests reported in Longstaff and Schwartz [39] and Pearson and Sun [40] show that the term structure for the interest rate should involve several sources of uncertainty, and introducing additional state variables (such as the rate of inflation, GDP, etc.) significantly improves the fit.

In this paper, we study the pricing of the continuously monitored geometric Asian options under dbH stochastic volatility model with stochastic interest rate framework (hereafter, dbH-SI model). The contribution of the present paper is twofold. Firstly, this paper extends the dbH model by introducing stochastic interest rate, which is assumed to follow two-factor model with two state variables. Secondly, this paper provides a semieexplicit valuation formula for the geometric Asian options with fixed or floating strike price, which is extremely useful also for the arithmetic average option valuation via Monte Carlo methods with control variables.

The rest of the paper is organized as follows. Section 2 develops the underlying pricing model and describes the geometric Asian option. Section 3 derives the joint characteristic function of a log-return of the underlying asset and its geometric average. Section 4 obtains the analytic expressions for the prices of the fixed strike geometric Asian call option and the floating strike Asian call option under continuous monitoring. Section 5 provides some numerical examples for the proposed approach. Section 6 concludes the paper.

2. Model Formulation

We consider an arbitrage-free, frictionless financial market where only riskless asset and risky asset are traded continuously up to a fixed horizon date $T$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, where $Q$ is a risk-neutral probability measure. Suppose $W^j_t$ and $Z^j_t$ $(j = 1, 2)$ are all standard Brownian motions defined on the probability space, and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by these Brownian motions. Moreover, $dW^1_t dZ^1_t = \rho_1 dt$, $dW^1_t dZ^2_t = \rho_2 dt$, and any other Brownian motions are pairwise independent. Assume that the asset price process $S_t$, without paying any dividend, satisfies the following stochastic differential equation under $Q$:

$$\frac{dS_t}{S_t} = r_t dt + \sqrt{\nu_t} dW^1_t + \sqrt{\nu_2} dW^2_t,$$

$$d\nu_t = \kappa_1 (\theta_1 - \nu_t) dt + \sigma_1 \sqrt{\nu_t} dZ^1_t,$$

$$d\nu_2 = \kappa_2 (\theta_2 - \nu_2) dt + \sigma_2 \sqrt{\nu_2} dZ^2_t,$$

where $\kappa_j, \theta_j, \sigma_j$ $(j = 1, 2)$ are all nonnegative constants, which represent the mean-reverting rates, long-term mean levels, and volatilities of variance processes $\nu_t$, respectively. We suppose that $2\kappa / \theta \geq \sigma^2 / \theta$. The instantaneous interest rate, $r_t$, is assumed to be a linear combination of $\nu_t$ and $\nu_2$, i.e., $r_t = \nu_t + \nu_2$, which designates the interest rate as an affine function of two-factor economic variables $\nu_1$ and $\nu_2$ and offers the analytic tractability (see Duffie et al. [27]).

In financial market, there are four types of European style continuously monitoring geometric Asian options: fixed strike geometric Asian calls, fixed strike geometric Asian puts, floating strike geometric Asian calls, and floating strike geometric Asian puts. The payoffs at the expiration date $T$ for these options are as follows:
where \( K \) is a fixed strike price and \( G_{[0,T]} \) is the geometric average of the underlying asset price \( S_t \) until time \( T \); i.e., \( G_{[0,T]} = \exp((1/T) \int_0^T \ln S_u \, du) \). In the following, we consider only the pricing problem of the geometric Asian call options (hereafter, GAC), while the put options can be dealt with similarly.

For the instantaneous interest rate \( r_t = \nu_t + \nu_w \), one can express the price at time \( t \) of a zero-coupon bond with maturate \( T \) as follows (see Cox et al. [37]):

\[
P(t, T) = \exp \left\{ a(t, T) - b(t, T) \, r_t \right\},
\]

where

\[
a(t, T) = \sum_{j=1}^{2} \left[ \frac{\kappa_j \theta_j (\kappa_j - \gamma_j)(T-t)}{\sigma_j^2} \right] + \frac{2\kappa_j \theta_j \ln \gamma_j}{2\gamma_j + (\kappa_j - \gamma_j) (1 - e^{-\gamma_j/(T-t)})},
\]

\[
b(t, T) = \sum_{j=1}^{2} \left[ \frac{2(1 - e^{-\gamma_j(T-t)})}{2\gamma_j + (\kappa_j - \gamma_j) (1 - e^{-\gamma_j/(T-t)})} \right],
\]

\[
\gamma_j = \sqrt{k_j^2 + 2\sigma_j^2}, \quad \text{for } j = 1, 2.
\]

**3. The Joint Characteristic Function**

Given the dynamic of the underlying asset price, it is possible to obtain the discounted joint characteristic function for the log-asset value \( S_T \) and the log-geometric mean value of the asset price over a certain time period.

Let \( \psi_t(s, w) = E^Q[ e^{-\int_0^T r_t \, du + \ln G_{[0,T]} + \ln S_T} \mid \mathcal{F}_t ] \) be the discounted joint characteristic function of two-dimensional random variable, \((\ln G_{[0,T]}, \ln S_T)\), conditioned on \( \mathcal{F}_t \) under \( Q \), where \( G_{[0,T]} = \exp((1/T) \int_0^T \ln S_u \, du) \) and \( E^Q[ \cdot \mid \mathcal{F}_t ] \) is the conditional expectation under \( Q \) for \( t \in [0, T] \). Denote \( D = \{(s, w) \in \mathbb{C}^2 : \Re(s) \geq 0, \Re(w) \geq 0, 0 \leq \Re(s) + \Re(w) \leq 1 \} \).

**Proposition 1.** Suppose that \( S_t, v_{1t} \), and \( v_{2t} \) follow the dynamics in (1). If \( (s, w) \in D \) and \( t \in [0, T] \), then \( E^Q[ e^{-\int_0^T r_t \, du + \ln G_{[0,T]} + \ln S_T} \mid \mathcal{F}_t ] < \infty \) and

\[
\psi_t(s, w) = e^{g_0(t, w)} E^Q \left[ \exp \left\{ \sum_{j=1}^{2} \left[ g_{j1} \int_0^T (T-\tau)^2 v_{j1} \, d\tau + g_{j2} \int_0^T v_{j1} \, d\tau + g_{j3} \int_0^T v_{j1} v_{j2} + g_{j4} v_{j2} \right] \right\} \mid \mathcal{F}_t \right],
\]

where

\[
g_0 = g_0^0(s, w) = s \left[ \frac{T-t}{T} \ln S_t \right.
\]

\[
- \frac{1}{T} \sum_{j=1}^{2} \left[ \frac{\rho_j \kappa_j \theta_j}{2\sigma_j} (T-t)^2 + \frac{\rho_j (T-t)}{\sigma_j} v_{j1} \right]
\]

\[
+ w \left[ \ln S_t + \frac{1}{2} \sum_{j=1}^{2} \left( \frac{\rho_j \kappa_j \theta_j}{2\sigma_j} (T-t) + \frac{\rho_j}{\sigma_j} v_{j1} \right) \right],
\]

\[
g_{j1} = g_{j1}^0(s, w) = \frac{s \left( 1 - \rho_j^2 \right)}{2T^2},
\]

\[
g_{j2} = g_{j2}^0(s, w) = \frac{sw \left( 1 - \rho_j^2 \right)}{2T},
\]

\[
g_{j3} = g_{j3}^0(s, w) = \frac{s \left( 2 \rho_j \kappa_j + \sigma_j \right)}{2T},
\]

\[
g_{j4} = g_{j4}^0(s, w) = \frac{sw \left( 2 \rho_j \kappa_j + \sigma_j \right)}{2T},
\]

\[
g_{j5} = g_{j5}^0(s, w) = \frac{sw \left( 2 \rho_j \kappa_j + \sigma_j \right)}{2T}.
\]

\[
g_{j6} = g_{j6}^0(s, w) = \frac{sw \left( 2 \rho_j \kappa_j + \sigma_j \right)}{2T}.
\]
Proof. (i) We first prove that the integrability condition guarantees the existence of the cumulant function $\psi_j(s, w)$ in $D$. If $(s, w) \in D$ and $t \in [0, T)$, then

$$\ln S_t = \ln S_0 + \sum_{j=1}^2 \left( \frac{\rho_j k_j}{\sigma_j} (T - t) - \frac{\rho_j}{\sigma_j} v_j \right) + \left( \frac{\rho_j k_j}{\sigma_j} + 1 \right) \int_t^T v_j(T - t) \sqrt{v_j(T - t)} \, dt + \frac{\rho_j}{\sigma_j} \int_t^T v_j(T - t) \sqrt{v_j(T - t)} \, dt$$

Using the fact

$$\int_t^T (T - t) \sqrt{v_j(T - t)} \, dt = \frac{1}{\sigma_j} \left[ k_j \int_t^T (T - t) v_j(T - t) \, dt + \int_t^T v_j(T - t) \sqrt{v_j(T - t)} \, dt - \frac{\kappa_j \beta_j}{2} \right],$$

for $j = 1, 2$, then

$$\ln G_{[t,T]} = \frac{T - t}{T} \ln S_t + \sum_{j=1}^2 \left( \frac{\rho_j k_j}{2 \sigma_j} (T - t)^2 - \frac{\rho_j}{\sigma_j} v_j \right) + \frac{\rho_j k_j}{\sigma_j} \int_t^T (T - t) \sqrt{v_j(T - t)} \, dt + \frac{\rho_j}{\sigma_j} \int_t^T v_j(T - t) \sqrt{v_j(T - t)} \, dt,$$

Let $\mathcal{F}$ be the $\sigma$-field generated by $\mathcal{F}_t$ and $\{(Z_{u}^1, Z_{u}^2) : t < u \leq T\}$. By (8) and (11), for $(s, w) \in D$, we have

$$\psi_j(s, w) = E^Q \left[ \exp \left\{ - \int_t^T r_s \, ds + s \ln G_{[t,T]} \right\} \right]$$

On the other hand, we have

$$\ln G_{[t,T]} = \frac{T - t}{T} \ln S_t + \sum_{j=1}^2 \left( \frac{\theta_j}{4} (T - t)^2 - \frac{\rho_j k_j}{2 \sigma_j} (T - t)^2 + \frac{\rho_j}{\sigma_j} v_j \right) + \left( \frac{\rho_j k_j}{\sigma_j} + 1 \right) \int_t^T v_j(T - t) \sqrt{v_j(T - t)} \, dt + \frac{\rho_j}{\sigma_j} \int_t^T v_j(T - t) \sqrt{v_j(T - t)} \, dt$$

where $A_1 = \exp \left\{ s \left[ \frac{T - t}{T} \ln S_t - \frac{1}{2} \left( \sum_{j=1}^2 \frac{\rho_j k_j}{\sigma_j} (T - t)^2 + \frac{\rho_j}{\sigma_j} v_j \right) \right] ight\}$

and $A_2 = \exp \left\{ s \left( \sum_{j=1}^2 \frac{2 \rho_j k_j + \sigma_j}{2 \sigma_j} \int_t^T (T - t) v_j(T - t) \, dt + \frac{\rho_j}{\sigma_j} \int_t^T v_j(T - t) \, dt \right) \right\}$. 

Proof. (ii) In order to determine (5), we start from model (1) and develop $\ln S_t$ with the Brownian motions $W_i^1$ and $W_i^2$ expressed as $W_i^1 = \rho_i Z_i^1 + \sqrt{1 - \rho_i^2} Z_i^2$ and $W_i^2 = \rho_i Z_i^2 + \sqrt{1 - \rho_i^2} Z_i^2$, respectively, where $(Z_1^1, Z_2^1, Z_1^2, Z_2^2)$ are 4-dimensional Brownian motions defined on the probability space. For the process $\ln S_t$, we have

$$\ln S_t = \ln S_0 + \sum_{j=1}^2 \left( \frac{\rho_j k_j}{\sigma_j} \left( \frac{T - t}{2} \right) - \frac{\rho_j}{\sigma_j} v_j \right) + \left( \frac{\rho_j k_j}{\sigma_j} + \frac{1}{2} \right) \int_t^T v_j(T - t) \sqrt{v_j(T - t)} \, dt + \frac{1}{\sqrt{1 - \rho_j^2}} \int_t^T \sqrt{v_j(T - t)} \, dZ_j^1 + \frac{\rho_j}{2 \sigma_j} \int_t^T \sqrt{v_j(T - t)} \, dZ_j^2.$$
two series of functions are introduced as follows. Define $F_t^j$ and $\tilde{F}_t^j : \mathbb{C}^4 \rightarrow \mathbb{C}$ as

$$F_t^j \left( g_{j1}, g_{j2}, g_{j3}, g_{j4} \right) = \sum_{n=0}^{\infty} f_n^j,$$

(15)

$$\tilde{F}_t^j \left( g_{j1}, g_{j2}, g_{j3}, g_{j4} \right) = \sum_{n=1}^{\infty} f_n^j,$$

(16)

where $f_n^j, n = 0, 1, 2, \cdots$, are functions of $g_{j1}, g_{j2}, g_{j3},$ and $g_{j4}$ defined as

$$f_{n-2}^j = f_{n-1}^j = 0,$$

$$f_0^j = 1,$$

$$f_1^j = \left( \frac{\kappa_j - g_{j4}^2 \sigma_j^2}{2} \right) \tau,$$

(17)

$$f_n^j = -\frac{\sigma_j^2 \tau^2}{2n(n-1)} \left( g_{j1} \tau^2 f_{n-4}^j + g_{j2} \tau f_{n-3}^j \right) + \left( g_{j3} - \frac{\kappa_j^2}{2 \sigma_j^2} \right) f_{n-2}^j, \quad n \geq 2,$$

for $j = 1, 2$.

For $j = 1, 2$, denote

$$D_{jr} = \left\{ (g_{j1}, g_{j2}, g_{j3}, g_{j4}) \in \mathbb{C}^4 : \begin{array}{l}
\end{array} \right\}.$$

$$E^Q \left[ \exp \left\{ \Re \left( g_{j1} \right) \int_{0}^{\tau} (\tau - t)^2 v_{j1} dt + \Re \left( g_{j2} \right) \int_{0}^{\tau} (\tau - t) v_{j2} dt + \Re \left( g_{j3} \right) \int_{0}^{\tau} v_{j3} dt + \Re \left( g_{j4} v_{j4} \right) \right\} < \infty \right\}.$$

(18)

Proposition 2. (1) The argument of $F_t^j$: arg$F_t^j$($g_{j1}, g_{j2}, g_{j3}, g_{j4}$) $\neq$ 0 for every $(g_{j1}, g_{j2}, g_{j3}, g_{j4}) \in D_{jr}$ ($j = 1, 2$).

In particular, arg$F_t^j$ is continuous on $D_{jr}$, and

$$\arg F_t^j (g_{j1}, g_{j2}, g_{j3}, g_{j4}) = 0 \text{ for } g_{j1}, g_{j2}, g_{j3}, \text{ and } g_{j4} \text{ are all real numbers in } D_{jr} \text{ for } j = 1, 2.$$

(2) For $(g_{j1}, g_{j2}, g_{j3}, g_{j4}) \in D_{jr}, j = 1, 2$, we have

$$E^Q \left[ \exp \left\{ \sum_{j=1}^{2} \left[ g_{j1} \int_{0}^{\tau} (\tau - t)^2 v_{j1} dt + g_{j2} \int_{0}^{\tau} (\tau - t) v_{j2} dt + g_{j3} \int_{0}^{\tau} v_{j3} dt + g_{j4} v_{j4} \right] \right\} \right]$$

$$= \exp \left\{ \sum_{j=1}^{2} \left( \frac{\kappa_j v_{j1} + \kappa_j^2 \theta_j t}{\sigma_j} - \frac{2 v_{j1} F_t^j \left( g_{j1}, g_{j2}, g_{j3}, g_{j4} \right)}{\sigma_j^2 F_t^j \left( g_{j1}, g_{j2}, g_{j3}, g_{j4} \right)} - \frac{2 \kappa_j \theta_j}{\sigma_j^2} \ln F_t^j \left( g_{j1}, g_{j2}, g_{j3}, g_{j4} \right) \right) \right\}.$$
The proof of Proposition 2 is similar to that of Kim and Wee [18].

Using Proposition 2 to Proposition 1 leads to the explicit expression of \( \psi(s,w) \) in Proposition 3. To describe the simplicity of the result, we need to introduce new functions. Define \( H_{t,T}^j(s,w) \) and \( \tilde{H}_{t,T}^j(s,w) \) with

\[
H_{t,T}^j(s,w) = E^{-t} \left( g_{j_1}(s,w), g_{j_2}(s,w), g_{j_3}(s,w), g_{j_4}(s,w) \right),
\]

\[
\tilde{H}_{t,T}^j(s,w) = E^{-t} \left( g_{j_1}(s,w), g_{j_2}(s,w), g_{j_3}(s,w), g_{j_4}(s,w) \right),
\]

for \( j = 1, 2 \).

Proposition 3. (1) If \((s,w) \in D\), then \( H_{t,T}^j(s,w) \neq 0 \) and \( \arg H_{t,T}^j(s,w) \) is continuous on \( D \). In particular, \( \arg H_{t,T}^j(s,w) = 0 \) if \( s \) and \( w \) are real numbers for \( j = 1, 2 \).

(2) For \( \psi(s,w) \in D \) with \( \arg H_{t,T}^j \) defined as above, then

\[
\psi(s,w) = \exp \left\{ \sum_{j=1}^{2} \left( -c_{j1} \ln H_{t,T}^j(s,w) + c_{j2}s + c_{j3}w + c_{j4} \right) \right\}.
\]

Here \( c_{j1} = 2\nu_{j}/\sigma_j^2 \), \( c_{j2} = 2\kappa\theta_j/\sigma_j^3 \), \( c_{j3} = T-t/2T \ln S_t - \frac{\rho_j\kappa\theta_j(T-t)^2}{2\sigma_j^2} - \frac{\rho_j(T-t)}{\sigma_j T}v_{j\beta} \), \( c_{j4} = \frac{\ln S_t}{2} - \frac{\rho_j}{\sigma_j}v_{\beta} - \frac{\rho_j\kappa\theta_j(T-t)}{\sigma_j} \), \( c_{j5} = \frac{\theta_j(T-t)}{\sigma_j^2} \),

for \( j = 1, 2 \).

Proof. Assume that \((s,w) \in D\), using the definitions of \( g_{j,k} = g_{j}(s,w) \) and \( D_{t} \) for \( k = 1, 2, 3, 4 \) and \( j = 1, 2 \). Note that \( (g_{j_1}, g_{j_2}, g_{j_3}, g_{j_4}) \in D_{t} \subset D_{t-\delta} \) for every \( 0 \leq t \leq T \) and \( j = 1, 2 \). Therefore, \((9)\) remains valid for any \( v_{j\beta} > 0 \). The time homogeneous Markov property of \( v_{j\beta} \) implies that \((22)\) holds when Propositions 1 and 2 are satisfied.

Now Substituting the expressions of \( g_{j_1}, g_{j_2}, g_{j_3}, g_{j_4}, j = 1, 2 \) into \((15)\) and \((16)\), \( H_{t,T}^j \) and \( \tilde{H}_{t,T}^j \) are expressed as follows, i.e., for \((s,w) \in C^2\),

\[
H_{t,T}^j(s,w) = \sum_{n=0}^{\infty} \hat{h}_{n,j}^j(s,w),
\]

\[
\tilde{H}_{t,T}^j(s,w) = \sum_{n=1}^{\infty} \frac{n}{T-t} \tilde{h}_{n,j}^j(s,w),
\]

where

\[
h_{n,j}^j(s,w) = h_{n,j}^j(s,w) = \frac{(T-t)(\kappa_j - wp_j\sigma_j)}{2},
\]

\[
h_{n,j}^j(s,w) = \frac{(T-t)^2}{4n(n-1)T^2} \left\{ -s^2\sigma_j^2(1-\rho_j^2)(T-t)^2 + h_{n-4}^j(s,w) - [s\sigma_jT(\sigma_j + 2\rho_j\kappa_j)] + 2sw\sigma_j^2T(1-\rho_j^2)(T-t)h_{n-3}^j(s,w) + T[\kappa_j^2T - 2\rho_j\sigma_j - w(2\rho_j\kappa_j + \sigma_j)]\sigma_jT - w^2(1-\rho_j^2)\sigma_j^2T + 2\sigma_j^2T]\tilde{h}_{n-3}^j(s,w) \right\}, \quad n \geq 2,
\]

for \( j = 1, 2 \).

\[\square\]

4. Pricing Geometric Asian Option

Once the discounted joint characteristic function is found, the European continuously monitored geometric Asian option can be valued using numeraire change technique and the inverse Fourier transform approach which are applied in many research works (see, e.g., Geman et al. [41] and Deng [42]). This section derives the pricing formulas for the continuously monitored fixed and floating strike geometric Asian call options.

Theorem 4. Suppose that \( S_t, \nu_{1,t}, \) and \( \nu_{2,t} \) follow the dynamics in \((1)\), then the price at time \( t \in [0,T] \) of the continuously monitored fixed strike geometric Asian call option with maturity \( T \) and the strike price \( K \) is given by

\[
GAC_{j_1}(t,S_t,\nu_{1,t},\nu_{2,t},K,T) = e^{(1/T)} \int_{0}^{\infty} \psi_{j_1}(1,0) \Pi_{1} - KP(t,T) \Pi_{2},
\]

where \( P(t,T) \) is given above in \((3)\) and

\[
\Pi_{j} = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} R e^{-i\ln K \tau} \psi_{j}(u) du,
\]

\[ j = 1, 2, \]
where $i$ is the imaginary unit ($i^2 = -1$).

**Proof.** Since

\[ \phi_1 (u) = \frac{\psi_t (iu+1,0)}{\psi_t (1,0)} , \]

\[ \phi_2 (u) = \frac{\psi_t (iu,0)}{\psi_t (0,0)} , \]

(28)

From (29), (31), and (32), we can obtain the required Theorem 4. □

**Theorem 5.** Suppose that $S_t$, $v_{1t}$, and $v_{2t}$ follow the dynamics in (1), then the price at time $t \in [0,T]$ of the continuously monitored floating strike geometric Asian call option with maturity $T$ is given by

\[ \text{GAC}_{f_1} (t, S, v_1, v_2, T) = S_t \Pi_1 - e^{(1/T) \int_0^T \ln S_{du}} \phi_1 (1,0) \Pi_2 , \]

(35)

where

\[ \Pi_1 = \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \Re \left[ \frac{\psi_t (iu+1-iu) e^{iu(T-t)} \int_0^T \ln S_{du}}{iuS_t} \right] du , \]

(36)

\[ \Pi_2 = \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \Re \left[ \frac{\psi_t (iu+1-iu) e^{iu(T-t)} \int_0^T \ln S_{du}}{iu\psi_t (1,0)} \right] du . \]

(37)

**Proof.** Since

\[ \text{GAC}_{f_1} (t, S, v_1, v_2, T) = E^Q \left[ e^{-\int_0^T r_{du} (S_t - G_{[T,T]})^t} | \mathcal{F}_t \right] \]

\[ = e^{(1/T) \int_0^T \ln S_{du}} E^Q \left[ e^{-\int_0^T r_{du} G_{[T,T]}^t} | \mathcal{F}_t \right] \]

\[ = \psi_t (iu+1,0) \psi_t (1,0) , \]

(33)

\[ \phi_2 (u) = E^Q \left[ e^{iu(T-t)} G_{[T,T]} | \mathcal{F}_t \right] \]

\[ = E^Q \left[ e^{-\int_0^T r_{du} G_{[T,T]}^t} | \mathcal{F}_t \right] \]

\[ = \psi_t (iu,0) P(t,T) , \]

(34)
\[
\mathcal{F}_t = S_t Q_2 \left( \ln \left( \frac{G_{0,T}}{S_T} \right) \right) - \frac{1}{T} \int_0^T \ln S_u \, du
\]

\[- \frac{1}{T} \int_0^T \ln S_u \, du \psi_t(1,0) Q_1 \left( \ln \left( \frac{G_{0,T}}{S_T} \right) \right) - \frac{1}{T} \int_0^T \ln S_u \, du \, ,
\]

(37)

where \( Q_2 \) is defined by the following Radon-Nikodym derivative:

\[
\frac{dQ_2}{dQ} \bigg|_{\mathcal{F}_t} = e^{-\int_0^T r_s dS_s / S_t}
\]

(38)

under the two probability measures \( Q_1 \) and \( Q_2 \), the conditional characteristic functions of \( \ln(G_{0,T}/S_T) \) are given by

\[
E^{Q_1} \left[ e^{i \ln(G_{0,T}/S_T)} \bigg| \mathcal{F}_t \right] = E^{Q_1} \left[ e^{-\int_t^T r_s dS_s + \ln(G_{0,T}/S_T)} G_{t,T} \bigg| \mathcal{F}_t \right]
\]

(39)

\[
= \psi_t(iu+1,-iu)
\]

\[
\psi_t(1,0) G_{t,T} e^{-\int_t^T r_s dS_s / S_t}
\]

(40)

Therefore

\[
Q_1 \left( \ln \left( \frac{G_{0,T}}{S_T} \right) \right) \leq - \frac{1}{T} \int_0^T \ln S_u \, du
\]

\[
= \frac{1}{2}
\]

(41)

\[- \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{\psi_t(iu+1,-iu) e^{iu/T}}{iu \psi_t(1,0)} \int_0^T \ln S_u \, du \right] \, du \, ,
\]

\[
Q_2 \left( \ln \left( \frac{G_{0,T}}{S_T} \right) \right) \leq - \frac{1}{T} \int_0^T \ln S_u \, du
\]

\[
= \frac{1}{2}
\]

(42)

\[- \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{\psi_t(iu,1-iu) e^{iu/T}}{S_t \cdot iu} \int_0^T \ln S_u \, du \right] \, du \, .
\]

Plugging (41) and (42) in (37) yields (35).

5. Numerical Examples

In this section, we use the dbH-SI model to analyze the valuation of the continuously monitored fixed strike geometric Asian call option using some numerical examples. To implement the analytic formula given in (27) numerically, we first determine the number of terms taken for the computation of the infinite series expansions for \( H_j^T \) and \( \tilde{H}_j^T \) for \( j = 1, 2 \) in (24) and (25). Second, we investigate the accuracy and efficiency of the approximated analytic formula given in (27). In the end, we compare the option prices varying by \( S_0 \) and \( T \) under different models including the dbH-SI, H-SI (i.e., Heston stochastic volatility model with stochastic interest rate), dbH, and Heston models. Furthermore, we use this proposed model (1), i.e., the dbH-SI model, to examine the impacts of the model parameters on option prices. Here we implement the integral formulas given in (27) without truncation, modification, and approximation in Mathematica and Matlab software. Numerical integration was performed using the \("NIntegrate()\) or \("quadlgk()\) commands which can handle infinite ranges, oscillatory integrands, and singularities in their default version, which employs automatic adaptive integration.

Model (1) parameters are set as follows: \( \kappa_1 = 1.15, \kappa_2 = 2.2, \theta_1 = 0.2, \theta_2 = 0.15, \sigma_1 = 0.5, \sigma_2 = 0.8, \rho_1 = -0.4, \rho_2 = -0.64, \nu_10 = 0.09, \nu_20 = 0.04, S_0 = 100, \) and \( T = 0. \) The results are shown in Table 1.

Table 1 investigates the influence of the number of terms \( n \) taken in (24) and (25). It is shown that the numerical values tend to be quite stable as the number of terms increases. In particular, the option prices stay unchanged after taking \( n = 20 \) and \( n = 30 \) terms, at most. In addition, Table 1 provides option prices with various maturities \( T \), strike prices \( K \), and the required CPU times (in seconds). From Table 1, we can see that some characteristics of the GAC \( P \) option are similar to the plain European call option. For example, the values of the GAC \( P \) option are decreasing functions of the strike price. In addition, under the same parameter values setting in dbH-SI model, longer maturing date \( T \) will result in higher option values.

It would be interesting to see the performance of our approximated analytic formula approach implemented to the fixed strike GAC option under dbH-SI model. Our numerical example uses the number of terms \( n = 20 \) taken in (24) and (25); other parameter values are similar to those settings of Table 1. We compute the 0.5-, 1.5-, and 3-year maturity GAC options by our approximated analytic formula approach and compare the results with Monte Carlo simulation with 10000 sample paths and 100 points for time axis. The numerical results are displayed in Table 2.

Table 2 shows that the approximated analytic formula approach is considerably faster than those of the Monte Carlo simulation (MC). For a given set of parameters, the approximated analytic formula approach calculates the option prices for 5 different strikes and 3 different maturates in approximately 5 seconds. The Monte Carlo simulation takes around 110 seconds for each option price. Table 2 also compares their pricing accuracy. It can be seen that the relative error (RE) in prices is less than 0.4% for all cases. If
we regard the Monte Carlo price as the benchmark, then this numerical example confirms that the approximated analytic formula approach is accurate and efficient.

After examining accuracy and efficiency. We shall turn to investigate the comparison of the option prices under different models. Figure 1 compares the dbH-SI model with Heston, dbH, and H-SI models in pricing the fixed strike GAC option against the initial asset price \(S_0\) and maturing date \(T\) with the strike price \(K = 100\). From Figure 1, it is interesting to notice that the option prices of the fixed strike GAC option calculated from the dbH-SI model are much higher than those of the dbH model, H-SI model, and Heston model. One possible reason for this phenomena is that the volatilities are always bigger for the dbH-SI model than those for other models. This implies that the proposed model, i.e., the dbH-SI model, has a more significant influence than the dbH model, H-SI model, and Heston model on option prices. We can also clearly observe that the bigger the value of \(S_0\) (or \(T\)) is, the higher the GAC option price is.

In Figure 2, we use the dbH-SI model to examine the effects of the mean-reverting rates \((\kappa_1, \kappa_2)\) and volatilities \((\sigma_1, \sigma_2)\) of variance processes \((v_1t, v_2t)\) on the GAC option prices (see Figures 2(a)–2(d)). We also examine the impacts of the correlation coefficients \((\rho_1, \rho_2)\) (see Figures 2(e) and 2(f)), which determine both the correlation between the underlying asset and its volatilities, and the correlation between the underlying asset and the short interest rates.
Table 2: Comparison of the approximated approach and MC for GAC$_{f_i}$ options.

<table>
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<tr>
<th>$T$</th>
<th>$K$</th>
<th>Approximated approach</th>
<th>CPU(sec.)</th>
<th>Monte Carlo</th>
<th>CPU(sec.)</th>
<th>RE(%)</th>
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<tr>
<td>0.5</td>
<td>90</td>
<td>17.7849</td>
<td>0.284</td>
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<td>6.8624</td>
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<tr>
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<td>4.5407</td>
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<tr>
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Figure 1: Comparison of option prices under Heston, dbH, H-SI, and dbH-SI models.

From Figures 2(a) and 2(b), we can observe that the option prices increase as the mean-reverting rate increases. Particularly, it is quite remarkable that the mean-reverting rates have a significant effect on the longer term option values. The option price decreases as the volatilities of variance processes increase (see Figures 2(c) and 2(d)). The correlation coefficients $(\rho_1, \rho_2)$ have several effects depending on the relation between the strike price and the initial asset price. A negative $(\rho_1, \rho_2)$ tends to produce higher values for ITM (in-the-money) options and lower values for OTM (out-of-the-money) options (see Figure 2(e)). A similar result holds with respect to the expiration date: a negative $(\rho_1, \rho_2)$ tends to produce higher value for the longer term options and lower values for the shorter options (see Figure 2(f)).

6. Conclusions

The proposed model (the dbH-SI model) incorporates several important features of the underlying asset returns variability. We derive the discounted joint characteristic function of the log-asset price and its log-geometric mean value over time period $[0, T]$ and obtain approximated analytic solutions to the continuously monitored fixed and floating strike geometric Asian call options using the change of numeraire
Figure 2: Option prices with respect to $S_0$ and $T$ in the dBH-SI model.
Some numerical examples are provided to examine the effects of the proposed model, which reveals some additional features having a significant impact on option values, especially long-term options. The proposed model can be tested empirically by using the option price data from the option market.

Data Availability

All data used to support of the findings for this study are included within this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References


