Research Article

Nonlinear Dynamic Modeling and Model Reduction Strategy for Rotating Thin Cylindrical Shells

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Nonlinear dynamic modeling and model reduction strategy are studied in this paper for a rotating thin cylindrical shell. The nonlinear dynamic model is first established in terms of ordinary differential equations, in which the effects of Coriolis and centrifugal forces are considered, as well as the initial hoop tension due to rotation. This model describes both the in-plane vibrations and the flexural vibration and reflects the coupling effects of those deformations. Based on this original model, a novel model reduction strategy is proposed to reduce the degrees of freedom by neglecting vibration modes predominated by in-plane vibrations. Meanwhile, for the reduced-order model, the in-plane vibrations' contributions to the rotating shell's response are still preserved. To validate the dynamic model and the model reduction strategy, comparisons and simulations are carried out. Subsequently, nonlinear dynamic behaviors are investigated preliminarily by analyzing the rotating cylindrical shell's amplitude-frequency responses under different excitation levels.

1. Introduction

Rotating thin cylindrical shells have been widely used in engineering applications, such as rotors of aircraft engines, rotating satellite structures, and high-speed centrifugal separators. Their large vibration amplitudes can be easily excited in engineering. Hence the nonlinear shell theory instead of linear one should be applied, and it is of practical importance to carry out nonlinear dynamic studies on thin rotating cylindrical shells.

Saito et al. [1], Zohar et al. [2], and Huang et al. [3] conducted the early studies for the rotating cylinders. The travelling-modes phenomenon was reported in those researches. Also they systematically analyzed the effects of centrifugal and Coriolis forces. Subsequently, some researchers proposed several analytical and numerical methods to study the dynamic characteristics of rotating shells, such as Fourier series expansion approach [4], wave propagation method [5], discrete singular convolution method [6, 7], harmonic reproducing kernel particle approach [8],

Rayleigh–Ritz technique [9], and finite element method [10]. In recent years, due to the wide application of composite materials in lots of engineering fields, the dynamic behaviors of rotating composite shells are studied in numerous published articles [11, 12]. Mehrparvar et al. [13] and Malekzadeh et al. [14] studied functional graded rotating cylindrical shells' vibration behaviors. Liu et al. [15] analyzed the dynamic characteristics of rotating shells with ringers. Song et al. [16] carried out studies on rotating cross-ply laminated cylindrical shells. The knowledge of rotating cylindrical shells has been steadily improved by those researches.

A monograph has been published by Amabili et al. [17] to investigate both theoretical and experimental aspects of nonlinear vibrations of cylindrical shells. As summarized in this monograph, Chu [18] and Evensen [19] conducted early studies related to the cylindrical shell's large-amplitude vibration. After that, many extensive studies have been carried out which are about the classic thin cylindrical shell's general theories. And now the research interests of many scholars are developing theories and approaches to study the nonlinear

vibration of laminated cylindrical shells [20–25], nanotube-reinforced composite material [26], and functionally graded material [27–33]. Moreover, some studies have been carried out on modeling and simulation. In particular, Hasrati et al. [34] developed a numerical solution strategy (variational differential quadrature, VDQ) to solve nonlinear free and forced vibration problems of cylindrical shells. They also systematically studied the nonlinear dynamic characteristics of FG-CNTRC shells and plates with this method [35–39]. Based on the perturbation procedure and proper orthogonal decomposition, Goncalves et al. [40] proposed low-dimensional modes for the nonlinear vibration analysis of cylindrical shells.

Recently, the nonlinear vibration of rotating cylindrical shells has attracted the researchers' attentions. Wang et al. [41, 42] studied rotating cylindrical shells' nonlinear dynamic characteristics and the nonlinear vibrations of a rotating laminated composite cylindrical shell [43]. Liu et al. [44] analyzed the influences of damping, excitation, and nonlinearity on rotating shells' frequency responses. Han et al. [45] studied the thin cylinder's parametric instability whose rotating speed is periodically time-varying. Nevertheless, researches related to rotating cylindrical shells' nonlinear vibrations are still extremely limited. To the authors' knowledge, none of the previous studies have given the nonlinear dynamic model which could reflect the coupling of flexural vibration and the vibrations in plane (i.e., the longitudinal and torsional vibrations). In [46], the authors derived a nonlinear model containing those three vibrations of a rotating shell and investigated the nonlinear travelling wave vibrations. However, this model may have defects in some cases because the quadratic nonlinear terms are not contained in the dynamic equations. Moreover, in order to facilitate the investigations of rotating cylindrical shells' nonlinearities, most scholars neglected all inertia terms along in-plane directions to derive simplified governing equations, which means the contributions of in-plane vibrations to the dynamic response are discarded directly. However, it is a question whether the system described by this simplified governing equations can retain the characteristics of nonlinear vibration, especially the low-frequency characteristics.

In this paper, the nonlinear dynamic modeling and model reduction strategy are studied for a rotating cylindrical shell. Considering the quadratic nonlinear terms neglected in authors' previous study [46], the nonlinear dynamic model for the shell with simply supported conditions is first derived in terms of ordinary differential equations. This model can reflect the coupling of three deformation directions and describe both the flexural and the in-plane vibrations. The model derived is accurate because no simplification is applied during the modeling procedure. Then, based on the original model derived, a novel model reduction strategy is proposed to reduce the degrees of freedom (DOFs) by neglecting vibration modes predominated by in-plane vibrations, which is different from the previous studies neglecting in-plane vibrations directly. Finally, comparisons and simulations are conducted to validate the original nonlinear dynamic model and model reduction strategy, and nonlinear dynamic behaviors are investigated preliminarily.

FIGURE 1: Geometry and coordinates of a thin rotating cylindrical shell.

2. Nonlinear Dynamic Modeling

Figure 1 shows a simply supported thin cylindrical shell rotating on its symmetrical and horizontal axis with an angular velocity Ω . The shell's thickness, mean radius, and length are H , R , and L . u , v , and w represent the shell's deformations in x , θ , and z directions in the orthogonal coordinate system (x, θ, z) fixed on the shell's middle surface. The material of this shell is assumed to be isotropic and has mass density ρ , Young's modulus E , and Poisson's ratio μ .

2.1. Energy Equations. The kinetic energy of the rotating cylindrical shell is expressed as

$$T = \frac{\rho HL}{2} \iint_S (\mathbf{V} \cdot \mathbf{V}) \cdot R d\xi d\theta, \quad (1)$$

where $\xi = x/L$, and $\mathbf{V}$ is the velocity vector of an arbitrary point of the cylindrical shell that can be expressed as follows:

$$\mathbf{V} = \dot{\mathbf{r}} + (-\Omega \mathbf{i} \times \mathbf{r}). \quad (2)$$

The position vector $\mathbf{r}$ is given by

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad (3)$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in x , θ , and z directions. Considering (2) and (3), the rotating shell's kinetic energy can be expressed as follows:

$$T = \frac{\rho H}{2} \int_0^{2\pi} \int_0^1 \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 + 2\Omega(w\dot{v} - v\dot{w}) + \Omega^2(v^2 + w^2) \right] RL d\xi d\theta. \quad (4)$$

The material of the shell studied in this paper is elastic, isotropic, and homogeneous. The thickness of this shell is small. Therefore, Donnell's nonlinear shell theory which is suitable for thin shells is selected and used in the paper. According to Donnell's nonlinear theory [21], the strain-displacement relations are given by

$$\begin{Bmatrix} \varepsilon_\xi \\ \varepsilon_\theta \\ \varepsilon_{\xi\theta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_\xi^0 \\ \varepsilon_\theta^0 \\ \varepsilon_{\xi\theta}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_\xi^0 \\ \kappa_\theta^0 \\ \tau^0 \end{Bmatrix}. \quad (5)$$

where

$$\begin{aligned} \begin{Bmatrix} \varepsilon_{\xi}^0 \\ \varepsilon_{\theta}^0 \\ \varepsilon_{\xi\theta}^0 \end{Bmatrix} &= \begin{Bmatrix} \frac{\partial u}{L\partial\xi} + \frac{1}{2} \left(\frac{\partial w}{L\partial\xi} \right)^2 \\ \frac{\partial v}{R\partial\theta} + \frac{w}{R} + \frac{1}{2} \left(\frac{\partial w}{R\partial\theta} \right)^2 \\ \frac{\partial u}{R\partial\theta} + \frac{\partial v}{L\partial\xi} + \frac{\partial w}{L\partial\xi} \frac{\partial w}{R\partial\theta} \end{Bmatrix}, \\ \begin{Bmatrix} \kappa_{\xi}^0 \\ \kappa_{\theta}^0 \\ \tau^0 \end{Bmatrix} &= \begin{Bmatrix} -\frac{\partial^2 w}{L^2 \partial \xi^2} \\ -\frac{\partial^2 w}{R^2 \partial \theta^2} \\ -\frac{2\partial^2 w}{RL\partial\xi\partial\theta} \end{Bmatrix}. \end{aligned} \quad (6)$$

The stress-strain relationship can be expressed as follows:

$$\begin{Bmatrix} \sigma_{\xi} \\ \sigma_{\theta} \\ \tau_{\xi\theta} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{\xi} \\ \varepsilon_{\theta} \\ \varepsilon_{\xi\theta} \end{Bmatrix}, \quad (7)$$

where

$$\begin{aligned} Q_{11} &= Q_{22} = \frac{E}{1-\mu^2}, \\ Q_{12} &= \frac{\mu E}{1-\mu^2}, \\ G &= \frac{E}{2(1+\mu)}. \end{aligned} \quad (8)$$

Then the shell's strain energy due to stretching and bending can be written as

$$\begin{aligned} U_{\varepsilon} &= \frac{1}{2} \int_{-H/2}^{H/2} \int_0^{2\pi} \int_0^1 \left[\sigma_{\xi}^0 \varepsilon_{\xi}^0 \right. \\ &\quad \left. + \sigma_{\theta}^0 \varepsilon_{\theta}^0 + \tau_{\xi\theta}^0 \varepsilon_{\xi\theta}^0 + z \left(\sigma_{\xi}^0 \kappa_{\xi}^0 + \sigma_{\theta}^0 \kappa_{\theta}^0 + \tau_{\xi\theta}^0 \tau^0 \right) \right] \\ &\quad \cdot RL d\xi d\theta dz. \end{aligned} \quad (9)$$

And the potential energy of the rotating cylindrical shell due to the initial hoop tension induced by rotation is expressed as follows:

$$\begin{aligned} U_h &= \frac{1}{2} \int_0^{2\pi} \int_0^1 N_{\theta}^0 \left\{ \left(\frac{1}{R} \frac{\partial u}{\partial \theta} \right)^2 + \left[\frac{1}{R} \left(\frac{\partial v}{\partial \theta} + w \right) \right]^2 \right. \\ &\quad \left. + \left[\frac{1}{R} \left(-\frac{\partial w}{\partial \theta} + v \right) \right]^2 \right\} RL d\xi d\theta, \end{aligned} \quad (10)$$

where N_{θ}^0 is the initial hoop tension due to the centrifugal force that can be expressed as follows:

$$N_{\theta}^0 = \rho H \Omega^2 R^2. \quad (11)$$

2.2. Displacement Field. The cylindrical shell is assumed to be simply supported and corresponding boundary conditions are

$$v = 0, \quad \text{at } \xi = 0, 1. \quad (12)$$

$$w = 0, \quad \text{at } \xi = 0, 1. \quad (13)$$

$$N_{\xi} = \int_{-H/2}^{H/2} \sigma_{\xi} dz = 0, \quad \text{at } \xi = 0, 1. \quad (14)$$

$$M_{\xi} = \int_{-H/2}^{H/2} \sigma_{\xi} z dz, \quad \text{at } \xi = 0, 1. \quad (15)$$

Here, N_{ξ} and M_{ξ} are the boundary force and boundary moment, respectively.

The displacements u , v , and w , satisfying the simply supported boundary conditions, can be expressed in Fourier series

$$\begin{aligned} u(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^6 A_{mni} p_{mni}(t) \cos(n\theta + \omega_{mni}t) \cos(m\pi\xi) \\ v(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^6 B_{mni} p_{mni}(t) \sin(n\theta + \omega_{mni}t) \sin(m\pi\xi) \\ w(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{i=1}^6 C_{mni} p_{mni}(t) \cos(n\theta + \omega_{mni}t) \sin(m\pi\xi), \end{aligned} \quad (16)$$

where $p_{mni}(t)$ is the modal participation factor. The displacements (u , v , and w) are expressed by infinite series of (m, n) combinations. Here m and n are numbers of axial half-waves and circumferential waves, respectively.

Rewrite expression (16) as

$$\begin{aligned} u(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[\left(\sum_{i=1}^6 \cos \omega_{mni}t \cdot A_{mni} p_{mni}(t) \right) \cos n\theta \right. \\ &\quad \left. - \left(\sum_{i=1}^6 \sin \omega_{mni}t \cdot A_{mni} p_{mni}(t) \right) \sin n\theta \right] \cos(m\pi\xi) \\ v(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[\left(\sum_{i=1}^6 \cos \omega_{mni}t \cdot B_{mni} p_{mni}(t) \right) \sin n\theta \right. \\ &\quad \left. + \left(\sum_{i=1}^6 \sin \omega_{mni}t \cdot B_{mni} p_{mni}(t) \right) \cos n\theta \right] \sin(m\pi\xi) \end{aligned}$$

$$\begin{aligned}
w(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[\left(\sum_{i=1}^6 \cos \omega_{mni} t \cdot C_{mni} p_{mni}(t) \right) \cos n\theta \right. \\
&\quad \left. - \left(\sum_{i=1}^6 \sin \omega_{mni} t \cdot C_{mni} p_{mni}(t) \right) \sin n\theta \right] \sin(m\pi\xi). \quad (17)
\end{aligned}$$

Let

$$\begin{aligned}
\begin{Bmatrix} u_1^{mn}(t) \\ v_1^{mn}(t) \\ w_1^{mn}(t) \end{Bmatrix} &= \sum_{i=1}^6 (\cos \omega_{mni} t) \begin{Bmatrix} A_{mni} p_{mni}(t) \\ B_{mni} p_{mni}(t) \\ C_{mni} p_{mni}(t) \end{Bmatrix}, \\
\begin{Bmatrix} u_2^{mn}(t) \\ v_2^{mn}(t) \\ w_2^{mn}(t) \end{Bmatrix} &= \sum_{i=1}^6 (\sin \omega_{mni} t) \begin{Bmatrix} A_{mni} p_{mni}(t) \\ B_{mni} p_{mni}(t) \\ C_{mni} p_{mni}(t) \end{Bmatrix}. \quad (18)
\end{aligned}$$

Then, the displacements u , v , and w expressed in (17) are rewritten as

$$\begin{aligned}
u(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [u_1^{mn}(t) \cos n\theta - u_2^{mn}(t) \sin n\theta] \cos(m\pi\xi) \\
&= \mathbf{U}(\xi, \theta) \mathbf{u}(t) \\
v(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [v_1^{mn}(t) \sin n\theta + v_2^{mn}(t) \cos n\theta] \sin(m\pi\xi) \quad (19) \\
&= \mathbf{V}(\xi, \theta) \mathbf{v}(t) \\
w(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [w_1^{mn}(t) \cos n\theta - w_2^{mn}(t) \sin n\theta] \sin(m\pi\xi) \\
&= \mathbf{W}(\xi, \theta) \mathbf{w}(t).
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{U}(\xi, \theta) &= [\cdots U_1^{mn}(\xi, \theta) \ U_2^{mn}(\xi, \theta) \ \cdots], \\
\mathbf{u}(t) &= \{\cdots u_1^{mn}(t) \ u_2^{mn}(t) \ \cdots\}^T, \\
\mathbf{V}(\xi, \theta) &= [\cdots V_1^{mn}(\xi, \theta) \ V_2^{mn}(\xi, \theta) \ \cdots], \\
\mathbf{v}(t) &= \{\cdots v_1^{mn}(t) \ v_2^{mn}(t) \ \cdots\}^T, \\
\mathbf{W}(\xi, \theta) &= [\cdots W_1^{mn}(\xi, \theta) \ W_2^{mn}(\xi, \theta) \ \cdots], \\
\mathbf{w}(t) &= \{\cdots w_1^{mn}(t) \ w_2^{mn}(t) \ \cdots\}^T. \quad (20)
\end{aligned}$$

and

$$\begin{aligned}
U_1^{mn}(\xi, \theta) &= \cos n\theta \cos(m\pi\xi), \\
U_2^{mn}(\xi, \theta) &= -\sin n\theta \cos(m\pi\xi), \\
V_1^{mn}(\xi, \theta) &= \sin n\theta \sin(m\pi\xi), \\
V_2^{mn}(\xi, \theta) &= \cos n\theta \sin(m\pi\xi), \\
W_1^{mn}(\xi, \theta) &= \cos n\theta \sin(m\pi\xi), \\
W_2^{mn}(\xi, \theta) &= -\sin n\theta \sin(m\pi\xi). \quad (21)
\end{aligned}$$

2.3. Governing Equations of Original Model. The viscous-type damping force is used and expressed with the Rayleigh dissipation function

$$P(t) = \frac{1}{2} c \int_0^{2\pi} \int_0^1 (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) RL \, d\xi \, d\theta, \quad (22)$$

where the value of c is related to the mode expansion terms. Considering expression (19), expression (22) can be expanded as follows:

$$\begin{aligned}
P(t) &= \frac{RL}{2} \\
&\cdot \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ c_{u1}^{mn} \int_0^{2\pi} \int_0^1 [U_1^{mn}(\xi, \theta) \dot{u}_1^{mn}(t)]^2 \, d\xi \, d\theta \right. \\
&\quad + c_{v1}^{mn} \int_0^{2\pi} \int_0^1 [V_1^{mn}(\xi, \theta) \dot{v}_1^{mn}(t)]^2 \, d\xi \, d\theta \\
&\quad + c_{w1}^{mn} \int_0^{2\pi} \int_0^1 [W_1^{mn}(\xi, \theta) \dot{w}_1^{mn}(t)]^2 \, d\xi \, d\theta \Big\} + \frac{RL}{2} \\
&\cdot \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ c_{u2}^{mn} \int_0^{2\pi} \int_0^1 [U_2^{mn}(\xi, \theta) \dot{u}_2^{mn}(t)]^2 \, d\xi \, d\theta \right. \\
&\quad + c_{v2}^{mn} \int_0^{2\pi} \int_0^1 [V_2^{mn}(\xi, \theta) \dot{v}_2^{mn}(t)]^2 \, d\xi \, d\theta \\
&\quad + c_{w2}^{mn} \int_0^{2\pi} \int_0^1 [W_2^{mn}(\xi, \theta) \dot{w}_2^{mn}(t)]^2 \, d\xi \, d\theta \Big\}. \quad (23)
\end{aligned}$$

The damping coefficients $c_{\lambda 1}^{mn}$ and $c_{\lambda 2}^{mn}$ ($\lambda = u, v, w$) in this expression are relevant to the modal damping ratio ζ obtained from experiments. Simple calculations of $c_{\lambda 1}^{mn}$ and $c_{\lambda 2}^{mn}$ are given by

$$c_{\lambda 1}^{mn} = c_{\lambda 2}^{mn} \triangleq c_{\lambda 0}^{mn} = 2\zeta \rho H \omega_{\lambda 0}^{mn} \quad (\lambda = u, v, w). \quad (24)$$

Here $c_{\lambda 0}^{mn}$ and $\omega_{\lambda 0}^{mn}$ ($\lambda = u, v, w$) are the stationary cylindrical shell's damping coefficient and circular frequencies with mode (m, n) , respectively.

The transverse excitation imposed on the rotating cylindrical shell is written as

$$F(t) = F_0 \cos(\omega_d t) \delta(\xi - \xi_0) \delta(\theta - \theta_0), \quad (25)$$

where F_0 , ω_d , and (ξ_0, θ_0) denote the amplitude, frequency, and position of the excitation. Symbol δ represents the Dirac function.

Using Lagrange equations, the nonlinear governing equations of the rotating cylindrical shell are derived and expressed as

$$\mathbf{M} = \begin{bmatrix} \rho HRH_1 & 0 & 0 \\ 0 & \rho HRH_7 & 0 \\ 0 & 0 & \rho HRH_{13} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} Rc_u H_1 & 0 & 0 \\ 0 & Rc_v H_7 & 0 \\ 0 & 0 & Rc_w H_{13} \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\rho HR\Omega H_9 \\ 0 & -2\rho HR\Omega H_9^T & 0 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} RHQ_{11}H_2 + \left(\frac{N_\theta^0}{R} + \frac{HG}{R}\right)H_3 & HQ_{12}H_4 + HGH_5 & HQ_{12}H_6 \\ \left(\frac{HQ_{22}}{R} + \frac{N_\theta^0}{R}\right)H_{10} + RHGH_8 & \left(\frac{HQ_{22}}{R} + \frac{N_\theta^0}{R}\right)H_{11} - \frac{N_\theta^0}{R}H_{12} & \left(\frac{HQ_{22}}{R} + \frac{N_\theta^0}{R}\right)H_{11} - \frac{N_\theta^0}{R}H_{12} \\ sym. & \frac{HQ_{22}}{R}H_{13} + \frac{RH^3Q_{11}}{12}H_{14} + \frac{H^3Q_{12}}{12R}H_{15} + \frac{H^3Q_{22}}{12R^3}H_{16} + \frac{H^3G}{3R}H_{17} + \frac{N_\theta^0}{R}H_{18} \end{bmatrix}, \quad (27)$$

$$\mathbf{q} = \{\mathbf{u}^T(t) \ \mathbf{v}^T(t) \ \mathbf{w}^T(t)\}^T,$$

$$\mathbf{N}(\mathbf{q}) = \{\mathbf{N}_u^T \ \mathbf{N}_v^T \ \mathbf{N}_w^T\}^T,$$

$$\mathbf{F} = \{0 \ 0 \ F_0 \cos(\omega_d t) \mathbf{W}(\xi_0, \theta_0)\}^T.$$

Here, the expressions of symbols in (27)-(29) are shown in Appendixes A and B.

Equation (26) gives the nonlinear dynamic model of rotating cylindrical shells. The model derived is accurate, because no simplification is applied during the modeling procedure and both the quadratic and cubic nonlinear terms are contained. For every vibration mode (m, n) , $\mathbf{q}$ is a 6×1 vector. To guarantee good accuracy for the calculation of the dynamic response, enough vibration modes must be taken into account, and the scale of DOFs of the overall system increases dramatically. Therefore, it is necessary to derive a reduced model which can retain the main characteristics of the original model.

3. Model Reduction

3.1. Frequencies and Modes of the Corresponding Linear Cylindrical Shell. The frequencies and modes of the corresponding linear cylindrical shell are obtained first for the purpose of the model reduction in the following formulation. For the travelling wave vibration mode with number $(m, n, n \neq 0)$, the displacement field expressed in (16) can be rewritten in general form

$$u(\xi, \theta, t) = A_{mn} \cos(n\theta + \omega_{mn}t) \cos(m\pi\xi),$$

$$v(\xi, \theta, t) = B_{mn} \sin(n\theta + \omega_{mn}t) \sin(m\pi\xi),$$

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{N}(\mathbf{q}) = \mathbf{F}, \quad (26)$$

where

$$w(\xi, \theta, t) = C_{mn} \cos(n\theta + \omega_{mn}t) \sin(m\pi\xi), \quad (30)$$

where A_{mn} , B_{mn} , and C_{mn} are the displacement amplitudes in the x , θ , and z directions, respectively. ω_{mn} represents the natural frequency.

The energy function $\mathbb{R}$ of the rotating cylindrical shell is

$$\mathbb{R} = T_{\max} - (U_h + U_\varepsilon)_{\max}. \quad (31)$$

Substituting (5)-(8) into (9) and (10) (the expressions of strain energy U_h and U_ε), one can express U_h and U_ε with shell displacements u , v , and w . Subsequently, substituting the displacement expression (30) into the expressions of kinetic energy (T , Eq. (4)) and strain energy (U_h and U_ε), the energy function $\mathbb{R}$ can be written as

$$\begin{aligned} \mathbb{R} = & \frac{\pi EHL}{(1-\mu^2)} \left\{ \frac{1}{4R} \left[\bar{\omega}_{mn}^2 (A_{mn}^2 + B_{mn}^2 + C_{mn}^2) \right. \right. \\ & + 4\bar{\Omega}\bar{\omega}_{mn}B_{mn}C_{mn} + \bar{\Omega}^2 (B_{mn}^2 + C_{mn}^2) \left. \right] - \frac{1}{4R} \\ & \cdot \bar{\Omega}^2 \left[n^2 (A_{mn}^2 + B_{mn}^2 + C_{mn}^2) + B_{mn}^2 \right. \end{aligned}$$

$$\begin{aligned}
& + C_{mn}^2 + 4nB_{mn}C_{mn} \Big] - \left[\frac{RA_{mn}^2 m^2 \pi^2}{4L^2} \right. \\
& - \frac{\mu A_{mn} m \pi}{2L} (nB_{mn} + C_{mn}) + \frac{1}{4R} (nB_{mn} + C_{mn})^2 \\
& + \frac{(1-\mu)R}{8} \left(\frac{-nA_{mn}}{R} + \frac{B_{mn} m \pi}{L} \right)^2 \Big] \\
& - \left[\frac{H^2 RC_{mn}^2 m^4 \pi^4}{48L^4} + \frac{\mu H^2 RC_{mn}^2 m^2 n^2 \pi^2}{24R^2 L^2} \right. \\
& \cdot \left. + \frac{H^2 RC_{mn}^2 n^4}{48R^4} + \frac{H^2 R(1-\mu) c^2 n^2 m^2 \pi^2}{24R^2 L^2} \right] \Big\} + \mathbb{N}.
\end{aligned} \quad (32)$$

Here, $\mathbb{N}$ denotes the high order terms of A_{mn} , B_{mn} , and C_{mn} , which are related to the nonlinear vibration. $\bar{\omega}_{mn}$ and $\bar{\Omega}$ represent the dimensionless frequency and the rotating speed, and they are expressed as

$$\begin{aligned}
\bar{\omega}_{mn} &= \gamma \omega_{mn}, \\
\bar{\Omega} &= \gamma \Omega,
\end{aligned} \quad (33)$$

where $\gamma = R\sqrt{\rho(1-\mu^2)/E}$.

To obtain the eigenvalue equation associated with the cylindrical shell's linear vibration, high order terms denoted by $\mathbb{N}$ are neglected, and the following expressions can be derived by Rayleigh-Ritz procedure

$$\frac{\partial(\mathbb{R} - \mathbb{N})}{\partial A_{mn}} = \frac{\partial(\mathbb{R} - \mathbb{N})}{\partial B_{mn}} = \frac{\partial(\mathbb{R} - \mathbb{N})}{\partial C_{mn}} = 0. \quad (34)$$

Equation (34) leads to

$$\begin{bmatrix} \bar{\omega}_{mn}^2 + a_1 & a_2 & a_3 \\ a_2 & \bar{\omega}_{mn}^2 + a_4 & a_5 \bar{\omega}_{mn} + a_6 \\ a_3 & a_5 \bar{\omega}_{mn} + a_6 & \bar{\omega}_{mn}^2 + a_7 \end{bmatrix} \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{Bmatrix} = \mathbf{0}. \quad (35)$$

Here,

$$\begin{aligned}
a_1 &= -n^2 \bar{\Omega}^2 - \frac{m^2 \pi^2}{L_r^2} - \frac{(1-\mu)n^2}{2}, \\
a_2 &= \frac{\mu m n \pi}{L_r} + \frac{(1-\mu) m n \pi}{2L_r}, \\
a_3 &= \frac{\mu m \pi}{L_r}, \\
a_4 &= -n^2 \bar{\Omega}^2 - n^2 - \frac{(1-\mu)m^2 \pi^2}{2L_r^2}, \\
a_5 &= 2\bar{\Omega},
\end{aligned}$$

$$a_6 = -2n\bar{\Omega}^2 - n,$$

$$\begin{aligned}
a_7 &= -n^2 \bar{\Omega}^2 - 1 - \left(\frac{m^4 \pi^4 H_r^2}{12L_r^4} + \frac{\mu m^2 n^2 \pi^2 H_r^2}{6L_r^2} \right. \\
& \left. + \frac{n^4 H_r^2}{12} + \frac{(1-\mu)m^2 n^2 \pi^2 H_r^2}{6L_r^2} \right),
\end{aligned} \quad (36)$$

where $L_r = L/R$ and $H_r = H/R$.

According to the nonzero solution identification theorem of homogeneous linear equations, the nondimensional frequency parameters can be obtained by solving the equation expressed as

$$\begin{vmatrix} \bar{\omega}_{mn}^2 + a_1 & a_2 & a_3 \\ a_2 & \bar{\omega}_{mn}^2 + a_4 & a_5 \bar{\omega}_{mn} + a_6 \\ a_3 & a_5 \bar{\omega}_{mn} + a_6 & \bar{\omega}_{mn}^2 + a_7 \end{vmatrix} = 0, \quad (37)$$

which can be expanded as

$$\bar{\omega}_{mn}^6 + e_4 \bar{\omega}_{mn}^4 + e_3 \bar{\omega}_{mn}^3 + e_2 \bar{\omega}_{mn}^2 + e_1 \bar{\omega}_{mn} + e_0 = 0, \quad (38)$$

where

$$\begin{aligned}
e_0 &= -a_1 a_6^2 + 2a_2 a_3 a_6 - a_2^2 a_7 - a_3^2 a_4 + a_1 a_4 a_7, \\
e_1 &= -2a_1 a_5 a_6 + 2a_2 a_3 a_5 \\
e_2 &= -a_1 a_5^2 + a_1 a_4 + a_1 a_7 - a_2^2 - a_3^2 + a_4 a_7 - a_6^2, \\
e_3 &= -2a_5 a_6, \\
e_4 &= -a_5^2 + a_1 + a_4 + a_7.
\end{aligned} \quad (39)$$

Six distinct natural frequencies $\bar{\omega}_{mni}$ ($i = 1, 2, \dots, 6$) can be obtained from (38) and the corresponding natural modes are given by

$$\begin{aligned}
\alpha_{mni} &= \frac{A_{mni}}{C_{mni}} = \frac{\begin{vmatrix} -a_3 & a_2 \\ -(a_5 \bar{\omega}_{mni} + a_6) & \bar{\omega}_{mni}^2 + a_4 \end{vmatrix}}{\begin{vmatrix} \bar{\omega}_{mni}^2 + a_1 & a_2 \\ a_2 & \bar{\omega}_{mni}^2 + a_4 \end{vmatrix}}, \\
\beta_{mni} &= \frac{B_{mni}}{C_{mni}} = \frac{\begin{vmatrix} \bar{\omega}_{mni}^2 + a_1 & -a_3 \\ a_2 & -(a_5 \bar{\omega}_{mni} + a_6) \end{vmatrix}}{\begin{vmatrix} \bar{\omega}_{mni}^2 + a_1 & a_2 \\ a_2 & \bar{\omega}_{mni}^2 + a_4 \end{vmatrix}}.
\end{aligned} \quad (40)$$

For a stationary cylindrical shell, the lowest frequency is related to the flexural vibration [17]. When the shell rotates on its axis, rotation induces Coriolis accelerations. The Coriolis acceleration bifurcates the shells' natural frequencies. Therefore, there are two frequencies related to the flexural vibration for a rotating cylindrical shell, and they are the lowest two which can be identified. Therefore, the lowest two (i.e., $\bar{\omega}_{mn1}$ and $\bar{\omega}_{mn2}$) among those six frequencies $\bar{\omega}_{mni}$ ($i = 1, 2, \dots, 6$) are related to the shell's flexural vibrations, and the other four frequencies are associated with in-plane vibrations (i.e., the longitudinal and torsional vibrations). Moreover,

both the literature and the author's validation show that the displacement amplitude ratios satisfy

$$\begin{aligned}\alpha_{mn1} &\approx \alpha_{mn2} \triangleq \alpha_{mn} = \frac{A_{mn}}{C_{mn}}, \\ \beta_{mn1} &\approx \beta_{mn2} \triangleq \beta_{mn} = \frac{B_{mn}}{C_{mn}}.\end{aligned}\quad (41)$$

For the travelling wave vibration mode ($m, n=0$), the displacement field expressed in (16) can be rewritten in general form

$$\begin{aligned}u(\xi, \theta, t) &= A_{mn} \cos(m\pi\xi) \cos(\omega_{mn}t), \\ v(\xi, \theta, t) &= B_{mn} \cdot 0, \\ w(\xi, \theta, t) &= C_{mn} \sin(m\pi\xi) \cos(\omega_{mn}t),\end{aligned}\quad (42)$$

Similar to the formulations for the mode with circumferential wave number n ($n \neq 0$), one can obtain the characteristic equation as

$$\bar{\omega}_{mn}^4 + (b_1 + b_3)\bar{\omega}_{mn}^2 + (b_1b_3 - b_2^2) = 0, \quad (43)$$

where

$$\begin{aligned}b_1 &= -\frac{m^2\pi^2}{L_r^2}, \\ b_2 &= \frac{\mu m\pi}{L_r}, \\ b_3 &= -\bar{\Omega}^2 - 1 - \frac{m^4\pi^4 H_r^2}{12L_r^4}.\end{aligned}\quad (44)$$

From (43), four natural frequencies $\bar{\omega}_{mni}$ ($i = 1, 2, 3, 4$) are derived and the corresponding natural modes are given by

$$\begin{aligned}\alpha_{mni} &= \frac{A_{mni}}{C_{mni}} = \frac{-b_2}{\bar{\omega}_{mni} + b_1}, \\ \beta_{mni} &= \frac{B_{mni}}{C_{mni}} = 0.\end{aligned}\quad (45)$$

Among the four frequencies derived from (43), the lowest two (i.e., $\bar{\omega}_{mn1}$ and $\bar{\omega}_{mn2}$) are relevant to the shell's flexural vibrations. It should be noted that $\bar{\omega}_{mn1}$ equals $\bar{\omega}_{mn2}$ because of the axial symmetry of the mode ($n=0$), which is different from the modes with circumferential wave number n ($n \neq 0$). Correspondingly, the displacement amplitude ratios also satisfy (41).

3.2. Reduced-Order Model. The previous studies of Amabili [17] found that the flexural vibrations predominate the dynamic response of cylindrical shells, and, for every mode (m, n ; both $n = 0$ and $n \neq 0$), the frequencies associated with the flexural vibrations are much lower than the other frequencies. This characteristics show us a viable path to reduce the model by neglecting the contributions of the vibration mode predominated by in-plane vibrations which

are associated with the higher frequencies. Considering (40), (41), and (45), expression (18) can be written as

$$\begin{aligned}\begin{Bmatrix} u_1^{mn}(t) \\ v_1^{mn}(t) \\ w_1^{mn}(t) \end{Bmatrix} &= \sum_{i=1}^6 (\cos \omega_{mni}t) \begin{Bmatrix} A_{mni}P_{mni}(t) \\ B_{mni}P_{mni}(t) \\ C_{mni}P_{mni}(t) \end{Bmatrix} \\ &= \sum_{i=1}^2 (\cos \omega_{mni}t) \begin{Bmatrix} A_{mni}P_{mni}(t) \\ B_{mni}P_{mni}(t) \\ C_{mni}P_{mni}(t) \end{Bmatrix} \\ &= \eta_1^{mn}(t) \begin{Bmatrix} \alpha_{mn} \\ \beta_{mn} \\ 1 \end{Bmatrix}, \\ \begin{Bmatrix} u_2^{mn}(t) \\ v_2^{mn}(t) \\ w_2^{mn}(t) \end{Bmatrix} &= \sum_{i=1}^6 (\sin \omega_{mni}t) \begin{Bmatrix} A_{mni}P_{mni}(t) \\ B_{mni}P_{mni}(t) \\ C_{mni}P_{mni}(t) \end{Bmatrix} \\ &= \sum_{i=1}^2 (\sin \omega_{mni}t) \begin{Bmatrix} A_{mni}P_{mni}(t) \\ B_{mni}P_{mni}(t) \\ C_{mni}P_{mni}(t) \end{Bmatrix} \\ &= \eta_2^{mn}(t) \begin{Bmatrix} \alpha_{mn} \\ \beta_{mn} \\ 1 \end{Bmatrix}.\end{aligned}\quad (46)$$

where

$$\begin{aligned}C_{mn} \sum_{i=1}^2 (\cos \omega_{mni}t) P_{mni}(t) &= \eta_1^{mn}(t), \\ C_{mn} \sum_{i=1}^2 (\sin \omega_{mni}t) P_{mni}(t) &= \eta_2^{mn}(t).\end{aligned}\quad (47)$$

Then, the displacements u , v , and w are rewritten as

$$\begin{aligned}u(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \alpha_{mn} [\eta_1^{mn}(t) \cos n\theta - \eta_2^{mn}(t) \sin n\theta] \\ &\quad \cdot \cos(m\pi\xi) = \bar{\mathbf{U}}(\xi, \theta) \boldsymbol{\eta}(t) \\ v(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \beta_{mn} [\eta_1^{mn}(t) \sin n\theta + \eta_2^{mn}(t) \cos n\theta] \\ &\quad \cdot \sin(m\pi\xi) = \bar{\mathbf{V}}(\xi, \theta) \boldsymbol{\eta}(t) \\ w(\xi, \theta, t) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [\eta_1^{mn}(t) \cos n\theta - \eta_2^{mn}(t) \sin n\theta] \\ &\quad \cdot \sin(m\pi\xi) = \mathbf{W}(\xi, \theta) \boldsymbol{\eta}(t),\end{aligned}\quad (48)$$

where

$$\begin{aligned}\bar{\mathbf{U}}(\xi, \theta) &= \mathbf{U}(\xi, \theta) \boldsymbol{\alpha}_{mn}, \\ \bar{\mathbf{V}}(\xi, \theta) &= \mathbf{V}(\xi, \theta) \boldsymbol{\beta}_{mn},\end{aligned}\quad (49)$$

$$\begin{aligned}\boldsymbol{\alpha}_{mn} &= \begin{bmatrix} \ddots & & & \\ & \alpha_{mn} & & \\ & & \alpha_{mn} & \\ & & & \ddots \end{bmatrix}, \\ \boldsymbol{\beta}_{mn} &= \begin{bmatrix} \ddots & & & \\ & \beta_{mn} & & \\ & & \beta_{mn} & \\ & & & \ddots \end{bmatrix}.\end{aligned}\quad (50)$$

Similarly, using Lagrange equations, the nonlinear rotating cylindrical shell's reduced model can be obtained as follows:

$$\bar{\mathbf{M}}\ddot{\boldsymbol{\eta}} + \bar{\mathbf{C}}\dot{\boldsymbol{\eta}} + \bar{\mathbf{G}}\boldsymbol{\eta} + \bar{\mathbf{K}}\boldsymbol{\eta} + \bar{\mathbf{N}}(\boldsymbol{\eta}) = \bar{\mathbf{F}}, \quad (51)$$

where

$$\begin{aligned}\bar{\mathbf{M}} &= \rho H R (\boldsymbol{\alpha}_{mn}^2 \mathbf{H}_1 + \boldsymbol{\beta}_{mn}^2 \mathbf{H}_7 + \mathbf{H}_{13}), \\ \bar{\mathbf{C}} &= \boldsymbol{\alpha}_{mn}^2 R \mathbf{c}_u \mathbf{H}_1 + \boldsymbol{\beta}_{mn}^2 R \mathbf{c}_v \mathbf{H}_7 + R \mathbf{c}_w \mathbf{H}_{13},\end{aligned}\quad (52)$$

$$\begin{aligned}\bar{\mathbf{G}} &= 2 \boldsymbol{\beta}_{mn} \rho H R \Omega (\mathbf{H}_9 - \mathbf{H}_9^T), \\ \bar{\mathbf{K}} &= \boldsymbol{\alpha}_{mn}^2 R H Q_{11} \mathbf{H}_2 + \boldsymbol{\alpha}_{mn}^2 \left(\frac{N_\theta^0}{R} + \frac{H G}{R} \right) \mathbf{H}_3 \\ &\quad + 2 \boldsymbol{\alpha}_{mn} \boldsymbol{\beta}_{mn} (H Q_{12} \mathbf{H}_4 + H G \mathbf{H}_5) \\ &\quad + 2 \boldsymbol{\alpha}_{mn} H Q_{12} \mathbf{H}_6 \\ &\quad + \boldsymbol{\beta}_{mn}^2 \left(\frac{H Q_{22}}{R} + \frac{N_\theta^0}{R} \right) \mathbf{H}_{10} \\ &\quad + \boldsymbol{\beta}_{mn}^2 R H G \mathbf{H}_8 \\ &\quad + 2 \boldsymbol{\beta}_{mn} \left(\frac{H Q_{22}}{R} \mathbf{H}_{11} + \frac{N_\theta^0}{R} \mathbf{H}_{11} - \frac{N_\theta^0}{R} \mathbf{H}_{12} \right) \\ &\quad + \frac{H Q_{22}}{R} \mathbf{H}_{13} + \frac{R H^3 Q_{11}}{12} \mathbf{H}_{14} \\ &\quad + \frac{H^3 Q_{12}}{12 R} \mathbf{H}_{15} + \frac{H^3 Q_{22}}{12 R^3} \mathbf{H}_{16} + \frac{H^3 G}{3 R} \mathbf{H}_{17} \\ &\quad + \frac{N_\theta^0}{R} \mathbf{H}_{18},\end{aligned}\quad (53)$$

$$\boldsymbol{\eta} = \{\cdots \eta_1^{mn}(t) \eta_2^{mn}(t) \cdots\}^T,$$

$$\bar{\mathbf{N}}(\boldsymbol{\eta}) = \bar{\mathbf{N}}_u + \bar{\mathbf{N}}_v + \bar{\mathbf{N}}_w, \quad (54)$$

$$\bar{\mathbf{F}} = F_0 \cos(\omega_d t) \mathbf{W}^T(\xi_0, \theta_0).$$

Here, symbols in are shown in Appendix C.

From (51), one can obtain the nonlinear reduced model of the rotating cylindrical shell for vibration mode (m, n) . It is given by

$$\begin{aligned}\ddot{\eta}_1 + c\dot{\eta}_1 - g\dot{\eta}_2 + k\eta_1 + \tilde{k}\eta_1(\eta_1^2 + \eta_2^2) &= f_1, \\ \ddot{\eta}_2 + c\dot{\eta}_2 + g\dot{\eta}_1 + k\eta_2 + \tilde{k}\eta_2(\eta_1^2 + \eta_2^2) &= f_2.\end{aligned}\quad (55)$$

Here, the subscript mn and time variable t of $\eta_1^{mn}(t)$ and $\eta_2^{mn}(t)$ are neglected for brevity. And symbols used in (55) are expressed as

$$c = \frac{1}{\rho H (\alpha_{mn}^2 + \beta_{mn}^2 + 1)} (\alpha_{mn}^2 c_{u0}^{mn} + \beta_{mn}^2 c_{v0}^{mn} + c_{w0}^{mn}), \quad (56)$$

$$g = \frac{4\Omega\beta_{mn}}{\alpha_{mn}^2 + \beta_{mn}^2 + 1},$$

$$\begin{aligned}k &= \frac{1}{\gamma^2 (\alpha_{mn}^2 + \beta_{mn}^2 + 1)} \left\{ \frac{\alpha_{mn}^2 m^2 \pi^2}{L_r^2} \right. \\ &\quad + \left(\gamma^2 \Omega^2 + \frac{1-\mu}{2} \right) \alpha_{mn}^2 n^2 - \frac{2\mu\alpha_{mn}\beta_{mn}n\pi m}{L_r} \\ &\quad - \frac{\alpha_{mn}\beta_{mn}n\pi m(1-\mu)}{L_r} - \frac{2\mu\alpha_{mn}\pi m}{L_r} + (1 + \gamma^2 \Omega^2) \\ &\quad \cdot \beta_{mn}^2 n^2 + \frac{\beta_{mn}^2 m^2 \pi^2 (1-\mu)}{2L_r^2} + 2\beta_{mn}n \\ &\quad + 4\gamma^2 \Omega^2 \beta_{mn}n + 1 + \frac{H_r^2 m^4 \pi^4}{12L_r^4} + \frac{H_r^2 \mu m^2 \pi^2 n^2}{6L_r^2} \\ &\quad \left. + \frac{H_r^2 n^4}{12} + \frac{H_r^2 m^2 \pi^2 n^2 (1-\mu)}{24L_r^2} + \gamma^2 \Omega^2 n^2 \right\}\end{aligned}\quad (57)$$

$$\begin{aligned}\tilde{k} &= \frac{1}{\gamma^2 (\alpha_{mn}^2 + \beta_{mn}^2 + 1)} \left(\frac{9R^2 m^4 \pi^4}{32L^4} + \frac{9n^4}{32R^2} \right. \\ &\quad \left. + \frac{n^2 m^2 \pi^2}{16L^2} \right),\end{aligned}\quad (58)$$

$$\begin{aligned}f_1 &= \frac{2F_0}{(\alpha_{mn}^2 + \beta_{mn}^2 + 1) \rho \pi H L R} \cos(\omega_d t) \cos n\theta \\ &\quad \cdot \sin(m\pi\xi) \Big|_{\substack{\xi=\xi_0 \\ \theta=\theta_0}}, \\ f_2 &= \frac{-2F_0}{(\alpha_{mn}^2 + \beta_{mn}^2 + 1) \rho \pi H L R} \cos(\omega_d t) \sin n\theta \\ &\quad \cdot \sin(m\pi\xi) \Big|_{\substack{\xi=\xi_0 \\ \theta=\theta_0}},\end{aligned}\quad (59)$$

The reduced model given by (55) is actually a 2-DOF gyroscopic system with cubic nonlinearities. The information related to in-plane vibrations (u and v) is still included in the reduced model, because the present model reduction only

TABLE 1: Nondimensional frequency parameters of previous and present studies.

Parameters	(m, n)	Reference [5]		Present results	
		Forward wave $\bar{\omega}_{mn1}$	Backward wave $\bar{\omega}_{mn2}$	Forward wave $\bar{\omega}_{mn1}$	Backward wave $\bar{\omega}_{mn2}$
$\bar{\Omega} = 0.003$ $L_r = 20$ $H_r = 0.002$ $\mu = 0.3$	(1,1)	0.0131	0.0191	0.0131	0.0191
	(1,2)	0.0045	0.0093	0.0043	0.0091
	(1,3)	0.0074	0.0110	0.0073	0.0109
	(1,4)	0.0125	0.0153	0.0125	0.0154
	(1,5)	0.0185	0.0208	0.0186	0.0209

neglects vibration mode predominated by in-plane vibrations. However, the previous studies in the model simplification directly neglect in-plane vibrations (u and v) to obtain the 2-DOF system. Equation (55) preserves the rotating cylindrical shell's main vibration characteristics which could reflect the coupling of three deformation directions (u , v , and w). The approximate solution for (55), a gyroscopic system, can be derived by using some analytical approaches, such as L-P method, multiple scales method, harmonic balance method, and averaging method. Then, one can perform qualitative analysis to find the complex nonlinear dynamic phenomena on rotating thin cylindrical shell. Moreover, applying the model reduction procedure, the scale of DOFs of the overall system reduces to one-third of what it has been. Thus, the efficiency of numerical calculation for quantitative analysis can be improved as expected.

4. Numerical Results and Discussions

To validate the nonlinear dynamic model shown in (26), a comparison is conducted between the frequencies calculated by the corresponding linear system and those available in open literature, as shown in Table 1. Between the lowest two nondimensional frequencies (i.e., $\bar{\omega}_{mn1}$ and $\bar{\omega}_{mn2}$), (1) $\bar{\omega}_{mn1}$ is the lower one and corresponding to the forward wave frequency and (2) $\bar{\omega}_{mn2}$ is the higher one and corresponding to the backward wave frequency, as shown in literature [5]. It can be seen from Table 1 that good agreement is achieved.

Table 2 shows the simulation parameters for a rotating cylindrical shell considered in this paper. To validate the model reduction strategy, the first five nondimensional frequency parameters, computed by using the corresponding linear system of reduced-order model, are compared with those obtained from original model, as shown in Tables 3 and 4. Excellent agreement is shown in this comparison for a wide variety of modes with different rotating speeds. It may indicate that the model reduction strategy proposed in this paper is available to some extent.

Figures 2(a) and 2(b) illustrate the variation of the natural frequencies with respect to n with $m=1$ for different rotating speeds $\bar{\Omega}=0.005$ and $\bar{\Omega}=0.01$. It can be observed that, for a specific n , two frequencies exist, i.e., the forward and backward wave frequencies. For this two rotating speeds, the fundamental frequencies of them are both corresponding to the mode $(m=1, n=4)$, which is also shown in Table 3. However, the second order modes are mode $(m=1, n=5)$ and mode $(m=1, n=3)$ for $\bar{\Omega}=0.005$ and $\bar{\Omega}=0.01$, respectively.

TABLE 2: Geometric and material parameters of the rotating thin cylindrical shell.

Parameter	Value
Length, L	5 m
Radius, R	1 m
Thickness, H	0.002 m
Modulus of elasticity, E	70 GPa
Density of material, ρ	2600 kg/m ³
Poisson's ratio, μ	0.3

Figure 3 shows the variation of the natural frequencies with respect to nondimensional rotating speed for fundamental mode $(m=1, n=4)$. It can be observed that there is only one natural frequency at rotating speed $\bar{\Omega}=0$. As the rotating speed increases, this frequency bifurcates.

Without loss of generality, further investigation is carried out for the rotating speed $\bar{\Omega}=0.005$ by using Runge-Kutta method, and the cylindrical shell's first two modes at this rotating speed, i.e., mode $(m=1, n=4)$ and mode $(m=1, n=5)$, are considered.

Figures 4–6 give comparisons of the amplitude-frequency responses between original model and reduced model, under different levels of excitation. Here, the excitation is located at the center of shell, which is $(\xi_0 = 0.5, \theta_0 = 0)$. The damping ratio is $\zeta=0.01$. The symbol ω_0 is used to denote the corresponding stationary cylindrical shell's flexural vibration circular frequency. It can be observed that the low-frequency responses of original model can be preserved by the reduced model perfectly. It may indicate that the model reduction strategy is effective, and the reduced model is available to investigate the nonlinear dynamic behaviors of thin rotating cylindrical shells.

The amplitude-frequency responses for modes $(m=1, n=4)$ and $(m=1, n=5)$ under excitation level $F_0=500\text{N}$ are given by Figures 4 and 5, respectively. The system shows linear behavior, as depicted in the two figures. In each figure, two resonance peaks exist and are respectively corresponding to forward and backward waves. They are all offset upward from the one corresponding to a stationary shell ($\omega_d/\omega_0 = 1$). This phenomenon is induced by the effects of Coriolis and centrifugal forces, as well as the initial hoop tension due to rotation. The existing of two resonance peaks is consistent with the characteristics of rotating cylindrical shells, which can prove the models' correctness in some extent. It should be noted that two resonance peaks do not appear in the

FIGURE 2: Natural frequencies versus the circumferential wave number ($m=1$): (a) $\bar{\Omega}=0.005$, (b) $\bar{\Omega}=0.01$.

TABLE 3: Nondimensional frequency parameters of original model and reduced model ($\bar{\Omega} = 0.005$).

Frequency order	(m, n)	Original model		Reduced model	
		Forward wave $\bar{\omega}_{mn1}$	Backward wave $\bar{\omega}_{mn2}$	Forward wave $\bar{\omega}_{mn1}$	Backward wave $\bar{\omega}_{mn2}$
1	(1,4)	0.0274	0.0322	0.0274	0.0322
2	(1,5)	0.0288	0.0327	0.0288	0.0327
3	(1,6)	0.0349	0.0381	0.0349	0.0381
4	(1,3)	0.0371	0.0431	0.0371	0.0431
5	(1,7)	0.0430	0.0459	0.0430	0.0459

TABLE 4: Nondimensional frequency parameters of original model and reduced model ($\bar{\Omega} = 0.01$).

Frequency order	(m, n)	Original model		Reduced model	
		Forward wave $\bar{\omega}_{mn1}$	Backward wave $\bar{\omega}_{mn2}$	Forward wave $\bar{\omega}_{mn1}$	Backward wave $\bar{\omega}_{mn2}$
1	(1,4)	0.0380	0.0474	0.0380	0.0474
2	(1,3)	0.0391	0.0512	0.0391	0.0512
3	(1,5)	0.0466	0.0543	0.0466	0.0543
4	(1,6)	0.0580	0.0645	0.0580	0.0645
5	(1,2)	0.0689	0.0849	0.0689	0.0849

amplitude-frequency response curve of η_2 as expected. It may be led by the special location of excitation. One can see that, at the position ($\xi_0 = 0.5, \theta_0 = 0$) considered in the simulation, symbol f_2 in (55) equals 0.

As depicted in Figure 6, when the excitation amplitude increases to 1000N, the jump phenomenon appears, which is a typical characteristic of nonlinear system. As depicted in Figure 7, if the excitation amplitude increases furtherly, more complex nonlinear behaviors of the rotating shell will appear, such as the ultraharmonic resonance. Figures 8–11 give the close-up views of amplitude-frequency curves for the first mode ($m=1, n=4$) and the second mode ($m=1, n=5$) under different excitation levels. The variation of the

amplitude-frequency curves with respect to excitation level is well illustrated. The jump phenomenon and ultraharmonic resonance are clearly shown in those figures, especially in Figures 10 and 11. Further, one can conclude that the amplitude-frequency responses for the second mode ($m=1, n=5$) are more complex than those for the first mode ($m=1, n=4$).

5. Conclusions

Nonlinear dynamic modeling and model reduction strategy are studied in this research for a rotating cylindrical shell. Using Lagrange equations, the shell's nonlinear dynamic

FIGURE 3: Natural frequencies versus rotating speed (mode: $m = 1$, $n = 4$).

FIGURE 4: Comparison of the amplitude-frequency response between original model and reduced model ($m = 1$, $n = 4$; $\bar{\Omega}=0.005$; $F_0 = 500\text{N}$).

model is derived by considering the effects of Coriolis and centrifugal forces, as well as the initial hoop tension due to rotation. This model describes the flexural vibration and the in-plane vibrations. It can also reflect the coupling effects of those deformations. During the modeling procedure, no simplification is applied. Moreover, both the quadratic and cubic nonlinear terms are contained. So the model derived in this paper is accurate. Based on this model, a novel model reduction strategy has been proposed to reduce the DOFs

FIGURE 5: Comparison of the amplitude-frequency response between original model and reduced model ($m = 1$, $n = 5$; $\bar{\Omega}=0.005$; $F_0 = 500\text{N}$).

FIGURE 6: Comparison of the amplitude-frequency response between original model and reduced model ($m = 1$, $n = 4$; $\bar{\Omega} = 0.005$; $F_0 = 1000\text{N}$).

FIGURE 7: Comparison of the amplitude-frequency response between original model and reduced model ($m = 1$, $n = 4$; $\bar{\Omega} = 0.005$; $F_0 = 2000\text{N}$).

FIGURE 8: Amplitude-frequency curves related to η_1 for different excitation amplitudes ($m = 1$, $n = 4$; $\bar{\Omega} = 0.005$).

by neglecting vibration modes predominated by in-plane vibrations.

To validate the original nonlinear dynamic model and model reduction strategy, comparisons and simulations have been carried out, which show that the main characteristics of structural vibrations in all directions are still preserved in the reduced model. Meanwhile, nonlinear dynamic behaviors have been investigated preliminarily by analyzing the amplitude-frequency responses of rotating cylindrical shell

FIGURE 9: Amplitude-frequency curves related to η_2 for different excitation amplitudes ($m = 1$, $n = 4$; $\bar{\Omega} = 0.005$).

FIGURE 10: Amplitude-frequency curves related to η_1 for different excitation amplitudes. ($m = 1$, $n = 5$; $\bar{\Omega} = 0.005$).

under different excitation levels. Complex nonlinear behaviors are observed, such as ultraharmonic resonance and jump phenomenon.

The investigation in this paper is of great importance to make it possible and flexible to perform further studies on nonlinear behaviors of thin rotating cylindrical shells which have been widely used in practical engineering application. One can utilize the original model to obtain accuracy numerical results or utilize the reduced model to perform analytical analysis. It shows us a viable path to study the nonlinear dynamic characteristics of thin rotating cylindrical shells.

FIGURE 11: Amplitude-frequency curves related to η_2 for different excitation amplitudes ($m = 1, n = 5; \bar{\Omega} = 0.005$).

Appendix

A. Symbols in (26)-(28)

$\mathbf{c}_\lambda$ ($\lambda = u, v, w$) and $\mathbf{H}_j$ ($j = 1, 2, \dots, 18$) in (26)-(28) are

$$\mathbf{c}_\lambda = \text{diag}[\dots, c_{\lambda 0}^{mm}, c_{\lambda 0}^{mm}, \dots],$$

$$\mathbf{H}_1 = L \int_0^{2\pi} \int_0^1 \mathbf{U}^T(\xi, \theta) \mathbf{U}(\xi, \theta) d\xi d\theta,$$

$$\mathbf{H}_2 = \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} \right]^T \frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} d\xi d\theta,$$

$$\mathbf{H}_3 = L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \theta} \right]^T \frac{\partial \mathbf{U}(\xi, \theta)}{\partial \theta} d\xi d\theta,$$

$$\mathbf{H}_4 = \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} \right]^T \frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} d\xi d\theta,$$

$$\mathbf{H}_5 = \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \theta} \right]^T \frac{\partial \mathbf{V}(\xi, \theta)}{\partial \xi} d\xi d\theta,$$

$$\mathbf{H}_6 = \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} \right]^T \mathbf{W}(\xi, \theta) d\xi d\theta,$$

$$\mathbf{H}_7 = L \int_0^{2\pi} \int_0^1 \mathbf{V}^T(\xi, \theta) \mathbf{V}(\xi, \theta) d\xi d\theta,$$

$$\mathbf{H}_8 = \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \xi} \right]^T \frac{\partial \mathbf{V}(\xi, \theta)}{\partial \xi} d\xi d\theta,$$

$$\mathbf{H}_9 = L \int_0^{2\pi} \int_0^1 \mathbf{V}^T(\xi, \theta) \mathbf{W}(\xi, \theta) d\xi d\theta,$$

$$\mathbf{H}_{10} = L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} \right]^T \frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} d\xi d\theta,$$

$$\mathbf{H}_{11} = L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} \right]^T \mathbf{W}(\xi, \theta) d\xi d\theta,$$

$$\mathbf{H}_{12} = L \int_0^{2\pi} \int_0^1 \mathbf{V}^T(\xi, \theta) \frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} d\xi d\theta,$$

$$\mathbf{H}_{13} = L \int_0^{2\pi} \int_0^1 \mathbf{W}^T(\xi, \theta) \mathbf{W}(\xi, \theta) d\xi d\theta,$$

$$\mathbf{H}_{14} = \frac{1}{L^3} \int_0^{2\pi} \int_0^1 \left[\frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \xi^2} \right]^T \frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \xi^2} d\xi d\theta,$$

$$\mathbf{H}_{15}$$

$$= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left\{ \left[\frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \xi^2} \right]^T \frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \theta^2} + \left[\frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \theta^2} \right]^T \frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \xi^2} \right\} d\xi d\theta,$$

$$\mathbf{H}_{16} = L \int_0^{2\pi} \int_0^1 \left[\frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \theta^2} \right]^T \frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \theta^2} d\xi d\theta,$$

$$\mathbf{H}_{17} = \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \xi \partial \theta} \right]^T \frac{\partial^2 \mathbf{W}(\xi, \theta)}{\partial \xi \partial \theta} d\xi d\theta,$$

$$\mathbf{H}_{18} = L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T \frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} d\xi d\theta.$$

(A.1)

B. Symbols in (29)

$\mathbf{N}_\lambda$ ($\lambda = u, v, w$) in (29) are

$$\mathbf{N}_u = \frac{RH}{2} \mathbf{Q}_{11} \mathbf{N}_1 + \frac{H}{2R} \mathbf{Q}_{12} \mathbf{N}_2 + \frac{H}{R} \mathbf{GN}_3,$$

$$\mathbf{N}_v = \frac{H}{2} \mathbf{Q}_{12} \mathbf{N}_4 + \frac{H}{2R^2} \mathbf{Q}_{22} \mathbf{N}_5 + H \mathbf{GN}_6,$$

$$\mathbf{N}_w = \frac{RH \mathbf{Q}_{11}}{2} \mathbf{N}_7 + \frac{H \mathbf{Q}_{22}}{2R^3} \mathbf{N}_8 + \left(\frac{H \mathbf{Q}_{12}}{2R} + \frac{HG}{R} \right) \mathbf{N}_9$$

$$+ RH \mathbf{Q}_{11} \mathbf{N}_{10} + H \mathbf{Q}_{12} \mathbf{N}_{11} + \frac{H \mathbf{Q}_{12}}{2} \mathbf{N}_{12} \quad (\text{B.1})$$

$$+ H \mathbf{Q}_{12} \mathbf{N}_{13} + H \mathbf{Q}_{12} \frac{1}{R} \mathbf{N}_{14} + \frac{H}{R^2} \mathbf{Q}_{22} \mathbf{N}_{15}$$

$$+ \frac{H}{2R^2} \mathbf{Q}_{22} \mathbf{N}_{16} + \frac{H}{R^2} \mathbf{Q}_{22} \mathbf{N}_{17} + \frac{H}{R} \mathbf{GN}_{18}$$

$$+ H \mathbf{GN}_{19},$$

and

$$\mathbf{N}_1$$

$$= \frac{1}{L^2}$$

$$\cdot \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right]^2 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} \right]^T d\xi d\theta,$$

$$\mathbf{N}_2 = \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \right]^2 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} \right]^T d\xi d\theta,$$

$$\mathbf{N}_3$$

$$= \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right]$$

$$\cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \right] \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \theta} \right]^T d\xi d\theta,$$

$$\mathbf{N}_4$$

$$= \frac{1}{L}$$

$$\cdot \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right]^2 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} \right]^T d\xi d\theta,$$

$$\mathbf{N}_5$$

$$= L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \right]^2 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} \right]^T d\xi d\theta,$$

$\mathbf{N}_6$

$$= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right] \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \right] \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \xi} \right]^T d\xi d\theta,$$

 $\mathbf{N}_7$

$$= \frac{1}{L^3} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right]^3 \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right]^T d\xi d\theta,$$

 $\mathbf{N}_8$

$$= L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \right]^3 \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T d\xi d\theta,$$

 $\mathbf{N}_9$

$$= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left\{ \frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right]^2 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T + \frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \right]^2 \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right]^T \right\} d\xi d\theta,$$

 $\mathbf{N}_{10}$

$$= \frac{1}{L^2} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} \mathbf{u}(t) \right] \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right] \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right]^T d\xi d\theta,$$

 $\mathbf{N}_{11}$

$$= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} \mathbf{v}(t) \right] \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right]^T \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right] \mathbf{w}(t) d\xi d\theta,$$

$$\mathbf{N}_{12} = \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \mathbf{w}(t) \right]^2 \mathbf{W}^T(\xi, \theta) d\xi d\theta,$$

 $\mathbf{N}_{13}$

$$= \frac{1}{L} \int_0^{2\pi} \int_0^1 [\mathbf{W}(\xi, \theta) \cdot \mathbf{w}(t)] \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right]^T \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right] \cdot \mathbf{w}(t) d\xi d\theta,$$

 $\mathbf{N}_{14}$

$$= \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \xi} \mathbf{u}(t) \right] \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right] \mathbf{w}(t) d\xi d\theta,$$

 $\mathbf{N}_{15}$

$$= L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \theta} \mathbf{v}(t) \right] \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right] \mathbf{w}(t) d\xi d\theta,$$

$$\mathbf{N}_{16} = L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \mathbf{w}(t) \right]^2 \mathbf{W}^T(\xi, \theta) d\xi d\theta,$$

 $\mathbf{N}_{17}$

$$= L \int_0^{2\pi} \int_0^1 [\mathbf{W}(\xi, \theta) \cdot \mathbf{w}(t)] \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right] \cdot \mathbf{w}(t) d\xi d\theta,$$

 $\mathbf{N}_{18}$

$$= \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(\xi, \theta)}{\partial \theta} \mathbf{u}(t) \right] \left\{ \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right] + \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right]^T \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right] \right\} \cdot \mathbf{w}(t) d\xi d\theta,$$

 $\mathbf{N}_{19}$

$$= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(\xi, \theta)}{\partial \xi} \mathbf{v}(t) \right] \left\{ \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right]^T \cdot \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right] + \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \xi} \right]^T \left[\frac{\partial \mathbf{W}(\xi, \theta)}{\partial \theta} \right] \right\} \cdot \mathbf{w}(t) d\xi d\theta.$$

(B.2)

C. Symbols in (54)

$\tilde{\mathbf{N}}_\lambda$ ($\lambda = u, v, w$) in (54) are

$$\begin{aligned} \tilde{\mathbf{N}}_u &= \frac{RH}{2} Q_{11} \tilde{\mathbf{N}}_1 + \frac{H}{2R} Q_{12} \tilde{\mathbf{N}}_2 + \frac{H}{R} G \tilde{\mathbf{N}}_3, \\ \tilde{\mathbf{N}}_v &= \frac{H}{2} Q_{12} \tilde{\mathbf{N}}_4 + \frac{H}{2R^2} Q_{22} \tilde{\mathbf{N}}_5 + H G \tilde{\mathbf{N}}_6, \\ \tilde{\mathbf{N}}_w &= \frac{RHQ_{11}}{2} \tilde{\mathbf{N}}_7 + \frac{HQ_{22}}{2R^3} \tilde{\mathbf{N}}_8 + \left(\frac{HQ_{12}}{2R} + \frac{HG}{R} \right) \tilde{\mathbf{N}}_9 \\ &\quad + RHQ_{11} \tilde{\mathbf{N}}_{10} + HQ_{12} \tilde{\mathbf{N}}_{11} + \frac{HQ_{12}}{2} \tilde{\mathbf{N}}_{12} \\ &\quad + HQ_{12} \tilde{\mathbf{N}}_{13} + HQ_{12} \frac{1}{R} \tilde{\mathbf{N}}_{14} + \frac{H}{R^2} Q_{22} \tilde{\mathbf{N}}_{15} \\ &\quad + \frac{H}{2R^2} Q_{22} \tilde{\mathbf{N}}_{16} + \frac{H}{R^2} Q_{22} \tilde{\mathbf{N}}_{17} + \frac{H}{R} G \tilde{\mathbf{N}}_{18} \\ &\quad + H G \tilde{\mathbf{N}}_{19}, \end{aligned} \tag{C.1}$$

and

$$\begin{aligned}
\tilde{\mathbf{N}}_1 &= \frac{1}{L^2} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \right]^2 \left[\frac{\partial \mathbf{U}(x, \theta)}{\partial \xi} \right]^T \\
&\quad \cdot \boldsymbol{\alpha}_{mn} d\xi d\theta \\
\tilde{\mathbf{N}}_2 &= \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \right]^2 \left[\frac{\partial \mathbf{U}(x, \theta)}{\partial \xi} \right]^T \\
&\quad \cdot \boldsymbol{\alpha}_{mn} d\xi d\theta \\
\tilde{\mathbf{N}}_3 &= \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \right] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \right] \\
&\quad \cdot \left[\frac{\partial \mathbf{U}(x, \theta)}{\partial \theta} \right]^T \boldsymbol{\alpha}_{mn} d\xi d\theta \\
\tilde{\mathbf{N}}_4 &= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \right]^2 \left[\frac{\partial \mathbf{V}(x, \theta)}{\partial \theta} \right]^T \\
&\quad \cdot \boldsymbol{\beta}_{mn} d\xi d\theta \\
\tilde{\mathbf{N}}_5 &= L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \right]^2 \left[\frac{\partial \mathbf{V}(x, \theta)}{\partial \theta} \right]^T \\
&\quad \cdot \boldsymbol{\beta}_{mn} d\xi d\theta \\
\tilde{\mathbf{N}}_6 &= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \right] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \right] \\
&\quad \cdot \left[\frac{\partial \mathbf{V}(x, \theta)}{\partial \xi} \right]^T \boldsymbol{\beta}_{mn} d\xi d\theta \\
\tilde{\mathbf{N}}_7 &= \frac{1}{L^3} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \right]^3 \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right]^T d\xi d\theta \\
\tilde{\mathbf{N}}_8 &= L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \right]^3 \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right]^T d\xi d\theta \\
\tilde{\mathbf{N}}_9 &= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left\{ \frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \right]^2 \right. \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right]^T + \frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \right]^2 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right]^T \left. \right\} d\xi d\theta \\
\tilde{\mathbf{N}}_{10} &= \frac{1}{L^2} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(x, \theta)}{\partial \xi} \boldsymbol{\alpha}_{mn} \boldsymbol{\eta}(t) \right] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right. \\
&\quad \cdot \boldsymbol{\eta}(t) \left. \right] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right]^T d\xi d\theta
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{N}}_{11} &= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(x, \theta)}{\partial \theta} \boldsymbol{\beta}_{mn} \boldsymbol{\eta}(t) \right] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right]^T \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right] \boldsymbol{\eta}(t) d\xi d\theta \\
\tilde{\mathbf{N}}_{12} &= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \boldsymbol{\eta}(t) \right]^2 \mathbf{W}^T(x, \theta) d\xi d\theta \\
\tilde{\mathbf{N}}_{13} &= \frac{1}{L} \int_0^{2\pi} \int_0^1 [\mathbf{W}(x, \theta) \boldsymbol{\eta}(t)] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right]^T \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right] \boldsymbol{\eta}(t) d\xi d\theta \\
\tilde{\mathbf{N}}_{14} &= \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(x, \theta)}{\partial \xi} \boldsymbol{\alpha}_{mn} \boldsymbol{\eta}(t) \right] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right]^T \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right] \boldsymbol{\eta}(t) d\xi d\theta \\
\tilde{\mathbf{N}}_{15} &= L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(x, \theta)}{\partial \theta} \boldsymbol{\beta}_{mn} \boldsymbol{\eta}(t) \right] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right]^T \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right] \boldsymbol{\eta}(t) d\xi d\theta \\
\tilde{\mathbf{N}}_{16} &= L \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \boldsymbol{\eta}(t) \right]^2 \mathbf{W}^T(x, \theta) d\xi d\theta \\
\tilde{\mathbf{N}}_{17} &= L \int_0^{2\pi} \int_0^1 [\mathbf{W}(x, \theta) \boldsymbol{\eta}(t)] \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right]^T \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right] \boldsymbol{\eta}(t) d\xi d\theta \\
\tilde{\mathbf{N}}_{18} &= \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{U}(x, \theta)}{\partial \theta} \boldsymbol{\alpha}_{mn} \boldsymbol{\eta}(t) \right] \left\{ \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right]^T \right. \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right] + \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right]^T \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right] \left. \right\} \\
&\quad \cdot \boldsymbol{\eta}(t) d\xi d\theta \\
\tilde{\mathbf{N}}_{19} &= \frac{1}{L} \int_0^{2\pi} \int_0^1 \left[\frac{\partial \mathbf{V}(x, \theta)}{\partial \xi} \boldsymbol{\beta}_{mn} \boldsymbol{\eta}(t) \right] \\
&\quad \cdot \left\{ \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right]^T \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right] + \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \xi} \right]^T \right. \\
&\quad \cdot \left[\frac{\partial \mathbf{W}(x, \theta)}{\partial \theta} \right] \left. \right\} \boldsymbol{\eta}(t) d\xi d\theta
\end{aligned} \tag{C.2}$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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