Hindawi
Complexity
Volume 2019, Article ID 4675397, 22 pages
https://doi.org/10.1155/2019/4675397

Research Article

Finite-Time H_∞ Filtering for Discrete-Time Singular Markovian Jump Systems with Time Delay and Input Saturation

Wei Guan,[1] **Lei Fu**,[2] **and Yuechao Ma**[1,2]

[1] *School of Science, Yanshan University, Qinhuangdao Hebei, 066004, China*
[2] *School of Electrical Engineering, Yanshan University, Qinhuangdao Hebei, 066004, China*

Correspondence should be addressed to Lei Fu; leifuhappy@126.com

Received 29 November 2018; Revised 18 February 2019; Accepted 12 March 2019; Published 2 April 2019

Academic Editor: Xinzhi Liu

The paper is discussed with the problem of finite-time H_∞ filtering for discrete-time singular Markovian jump systems (SMJSs). The systems under consideration consist of time-varying delay, actuator saturation and partly unknown transition probabilities. We pay attention to the design of a H_∞ filtering which ensures the filtering error systems to be singular stochastic finite-time boundedness. By employing an adequate stochastic Lyapunov functional together with a class of linear matrix inequalities (LMIs), a sufficient condition is firstly established, which guarantees the systems to achieve our goal and satisfy a prescribed H_∞ attenuation level in the given finite-time interval. Considering the above conditions, a distinct presentation for the requested H_∞ filter is given. Finally, two numerical examples add to a dynamical Leontief model of economic systems are presented to illustrate the validity of the developed theoretical results.

1. Introduction

Over the past decades, signal estimation has received remarkable attention in the field of control, as an elementary problem in signal processing. As is known to all, the traditional Kalman filtering [1] is the most popular ways to deal with the signal estimation. However, the celebrated Kalman filtering scheme is no longer shaped while a priori information on the external noises are not accurately known. Some optimal estimation approaches are introduced to overcome this problem. One of them H_∞ filtering has a wider range of application value. Compared with the traditional kalman filtering, the main advantage of the H_∞ filtering is without assuming the statistics of the noise signal. Therefore, the H_∞ filtering problem has been investigated and widely introduced, see [2–8] as well as the references therein. In practice, to guarantee the error system to be asymptotically stable and satisfy a performance indicator is the main intention of H_∞ filtering design. Moreover, for the purpose of applying to the practical applications, it is also necessary to be bounded for the state of the error system.

As a special class of stochastic hybrid systems, Markovian jump systems have the advantage of better describing physical systems with sudden variations and put into use in manufacturing systems, communication systems [9–11], economic systems and networked control systems. In the past years, a mass of attention has been paid to the study of singular systems, which have been greatly attracted for the reason that singular systems have better than the state-space ones when to describe physical systems. However, many researchers of singular systems, often ignore the influence of Markovian jump, describing the actual systems with some limitations. With in-depth study, the researchers found that combined the singular systems with MJSs to form a new system, which not only can make up for the limitations of description, also can accurately describe more practical applications. This kind of combined singular systems and MJSs systems have been known as singular Markovian jump system (SMJSs) [12–14]. Meanwhile, many of the subsequent researches found that the actual environment mutation and singular system parameters or the internal structure change, all can use SMJSs description, so it is mostly important to study the SMJSs.

On the other side, as a precondition, complete knowledge of the mode transition rates was asked to analyse and synthesis MJSs in lots of the studies. Therefore, it is

necessary and significant to further study more universal jump systems together with partly unknown transition rates, rather than spending a lot of time to measure or estimate all the transitions rates. Many attractive problems have been solved and investigated for these systems, such as stability analysis, H_∞ filtering [15–17] and stabilization [18, 19].

It is well known that the Lyapunov stability describes the systems ranging on the dynamic behavior of the trajectories in infinite time space [20], which does not reflect the transient performance of the systems. To copie with this transient performance of control dynamics, short-time stability or finite-time stability were presented by some early researches in [21–24]. Varieties of significant results were provided for discrete-time or continuous-time by applying linear matrix inequality (LMIs) techniques or Lyapunov function approach, these field including singular systems [25, 26], network systems [27, 28], linear and nonlinear systems [29–33], switching systems [34–36], and so forth. In [37–41], the authors studied discrete finite-time stability, finite-time boundedness or finite-time H_∞ control. Compared with the above referred, we can find that the definition of finite-time stability have some difference in [42]. When performing the controller, what is noteworthy is that the discrete-time MJSs could be more significant than continuous-time in many practical applications. However, as far as we know, the problem about discrete finite-time H_∞ filtering for stochastic systems has less been researched, so that it promotes the main purpose of our study.

To the best of our knowledge, time-delay phenomena are usually the main causes for performance deterioration and instability of systems and happened in many practical systems, such as neural networks, economics, chemical processes and mechanics [43, 44]. On the other hand, if we do not consider the effect of actuator saturation when designing the controller, and it can lead to poor performance and unstable of closed-loop system. Consequently, the study of control systems with actuator saturation is challenging not only theory and research value, but also in practice has important research significance and broad application prospects. Recently, Ma and Chen [45] presented singular T-S fuzzy time-varying delay systems with actuator saturation for the problem of memory dissipative control. In [46], the author talked about passive control problem for singular time-delay systems with nonlinearity and actuator saturation. In the paper, we investigates the finite-time H_∞ filter design for discrete-time SMJSs with time-varying delay, partly unknown transition probabilities and actuator saturation. Our results are a little different from those previous ones which are in [47, 48] on finite-time H_∞ filtering have been presented for singular Markovian jump systems. Our primary purpose is designing a H_∞ filtering that can guarantee the filtering error discrete-time SMJSs to be stochastic finite-time stability and boundedness with a prescribed H_∞ performance index in the supplied finite-time interval. Enough conditions are supplied for the solvability of the problem, which can be worked out by making use of LMIs techniques. What is more, we consider to design filters for the systems with partly or completely known transition probabilities. Contributions from this paper are as follows

(i) A class of more general systems are investigated, where we take time-varying delay, input saturation, outside disturbance, Markov switching phenomena into consideration at the same time.

(ii) The singular filtering is used in this article, which makes the filtering problem more comprehensive and general.

(iii) The Markovian jump system takes into account the fact that the transfer matrix is not completely known.

(iv) The finite-time H_∞ filtering is investigated in this paper.

In this paper, the structure of the arrangement is as follows. In Section 2 we present the problem statement as well as preliminaries. Discussed the problem on stochastic finite-time H_∞ filtering for discrete-time SMJSs with time-varying delay, actuator saturation and partly known transition probabilities in Section 3. Moreover, the estimation of the largest domain of attraction is solved. Section 4 gives some examples to demonstrate the accuracy of the proposed approaches. Finally, we draw the conclusions and the direction of future research in Section 5.

Notations. In this paper, the following notations that will be used: $R^n, R^{n\times m}$ denote the sets of n-dimensional space, $n\times m$ real matrices. For any symmetric matrix P, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimum and maximum eigenvalues of matrix P, respectively. $\varepsilon\{.\}$ denotes the expectation operator in the case of some probability measure. We use T stands for matrix transposition or vector and $*$ stands for the transposed elements in the symmetric positions of a matrix.

2. Problem Statement and Preliminaries

Consider the following discrete-time singular Markovian jump systems (SMJSs) with time-varying delay and actuator saturation:

$$\begin{aligned} Ex(k+1) &= A(r_k)x(k) + A_d(r_k)x(k-d(k)) \\ &\quad + B(r_k)sat(u(k)) + B_w(r_k)w(k) \\ z(k) &= C(r_k)x(k) + C_d(r_k)x(k-d(k) \\ &\quad + D(r_k)w(k) \\ y(k) &= C_y(r_k)x(k) + D_y(r_k)w(k) \\ x(k) &= \phi(k), \quad k = -d_2, \ldots, -1, 0 \end{aligned} \tag{1}$$

where $x(k) \in R^n$ is the state variable, $z(k) \in R^q$ is the signal to be estimated, $u(k) \in R^p$ is the control input variable and $sat : R^p \longrightarrow R^p$ is the standard saturation function defined as follows:

$$\begin{aligned} &sat(u(k)) \\ &\quad = \left[sat(u_1(k)), sat(u_2(k)), \ldots, sat(u_p(k))\right]^T \end{aligned} \tag{2}$$

Where without loss of generality, $sat(u_i(k)) = \operatorname{sign}(u_i(k))\min\{1, |u_i(k)|\}$. Here the notation of $sat(.)$ is

abused to denote the scalar values and the variable valued saturation functions. $w(k) \in R^q$ is the disturbance input which belongs to

$$\varepsilon\left\{\sum_{k=0}^{\infty} w^T(k)\, w(k)\right\} \le d^2, \quad d \ge 0 \tag{3}$$

$y(k) \in R^l$ is the measured output variable, $\phi(k)$ is a given initial condition, $A(r_k), A_d(r_k), B(r_k), B_w(r_k), C(r_k), C_d(r_k), D(r_k)$ and $C_y(r_k)$ are real constant matrices with appropriate dimensions. $d(k)$ denotes the time-varying delay satisfying $1 \le d_1 \le d(k) \le d_2$, where d_1 and d_2 are known positive integers. The matrix $E \in R^{n\times n}$ may be singular and we assume that $\operatorname{rank}(E) = r \le n$. The process $\{r_k, k \ge 0\}$ is a discrete-time homogeneous Markovian chain taking values in a finite set $w = \{1, 2, \ldots, s\}$ with mode transition probabilities

$$P_r\left(r_{k+1} = j / r_k = i\right) = \pi_{ij} \tag{4}$$

where $\pi_{ij} \ge 0, \forall i, j \in w$, and $\sum_{j=1}^{s} \pi_{ij} = 1$ for $r_k = i, i \in w$.

In this paper, the transition probabilities of the Markovian chain are considered to be partially available, while some elements in matrix $\pi = \{\pi_{ij} \mid i, j \in w\}$ are unknown. For notation clarity, $\forall i \in w$, we denote $w = w_k^i + w_{uk}^i$ with

$$\begin{aligned} w_k^i &= \left\{j : \pi_{ij} \text{ is known}\right\} \\ w_{uk}^i &= \left\{j : \pi_{ij} \text{ is unknown}\right\} \end{aligned} \tag{5}$$

Moreover, if $w_k^i \ne \phi$, we denote w_k^i as the total number of elements in w_k^i and $w_k^i(m)$ as the mth element in w_k^i, obviously, $1 \le m \le w_k^i$, $1 \le w_k^i \le s$.

In order to estimate $z(k)$, we are interested in designing a filter of the following structure:

$$\begin{aligned} E\tilde{x}(k+1) &= A_f(r_k)\,\tilde{x}(k) + B_f(r_k)\, y_k \\ \tilde{z}(k) &= C_f(r_k)\,\tilde{x}(k) + D_f(r_k)\, y_k \\ u(k) &= K(r_k)\,\tilde{x}(k) \end{aligned} \tag{6}$$

where $\tilde{x}(k) \in R^n$ is the filter state variable and $\tilde{z}(k) \in R^q$ is the estimated signal, $A_f(r_k)$, $B_f(r_k)$, $C_f(r_k)$, $D_f(r_k)$ are appropriately dimensioned filter matrices to be determined.

For notational simplicity, in the sequel, a matrix $N(r_k)$ will be denoted by N_i, for instance, $A(r_k)$ will be denoted by A_i, $B_f(r_k)$ by B_{fi}, and so on.

Definition 1 (see [14]). For a matrix $H \in R^{p\times n}$ denote the lth row of H as h_{l_0}, and define $\hbar(H)$ as:

$$\hbar(H) = \left\{x(k) \in R^n : h_{l_0}^T x(k) \le 1,\ l_0 = 1, 2, \ldots, p\right\} \tag{7}$$

Let $P \in R^{n\times n}$ is a positive-symmetric matrix and $E^T PE \ge 0$, $\tau > 0$ be a scalar and denote

$$\aleph\left(E^T PE, \tau\right) = \left\{x(k) \in R^n : x^T(k)\, E^T PEx(k) \le \tau\right\} \tag{8}$$

Let D be the set of $p\times p$ diagonal matrices whose diagonal elements are either 1 or 0. Suppose each element of D is labeled as D_l and denote $D_l^- = 1 - D_l$, Clearly, if $D_l \in D$, then $D_l^- \in D$.

We denote $x(k, \phi)$ for the initial value of the state trajectory in (1), which the initial value is $x(k) = \phi(k) \in C_{n,d_2}[-d_2, 0]$, and the domain of attraction is followed as:

$$\begin{aligned} &\Gamma \\ &\triangleq \left\{\phi(k), k = -d_2, -d_2+1, \ldots, 0 : \lim_{k\longrightarrow\infty} x(k, \phi) = 0\right\} \end{aligned} \tag{9}$$

However, it is difficult to calculate accurately the domain of attraction in the actual production, so the estimates of the attraction domain is presented by the following set:

$$B_\nu = \left\{\phi(k) \in C_{n,d_2}[-d_2, 0] : \max_{[-d_2,0]} \|\phi(k)\| \le \nu\right\} \tag{10}$$

Lemma 2 (see [46]). *Let $K_i, H_i \in R^{p\times n}$, then for any $x(k) \in \hbar(H_i)$,*

$$sat\left(K_i x(k)\right) = \sum_{l=1}^{2^p} \alpha_l(k)\left(D_l K_i + D_l^- H_i\right) x(k) \tag{11}$$

where α_l for $l = 1, 2, \ldots, 2^p$ are some scalars which satisfy $0 \le \alpha_l \le 1$, $\sum_{l=1}^{2^p} \alpha_l = 1$

Define $\overline{x}^T(k) = [x^T(k) \quad x^T(k) - \tilde{x}^T(k)]$, $e(k) = z(k) - \tilde{z}(k)$, from Lemma 2, for any $\overline{x}(k) \in \hbar([H_i \quad -H_i])$, $\tilde{x}(k) \in \hbar(H_i)$, we have

$$\begin{aligned} sat\left(K_i \tilde{x}(k)\right) &= sat\left(\left[K_i \quad -K_i\right] \overline{x}(k)\right) \\ &= \sum_{l=1}^{2^p} \alpha_l(k)\left(D_l\left[K_i \quad -K_i\right] + D_l^-\left[H_i \quad -H_i\right]\right) \overline{x}(k) \end{aligned} \tag{12}$$

Then we can obtain the filtering error discrete-time SMJSs as follows:

$$\begin{aligned} \overline{E}\overline{x}(k+1) &= \sum_{l=1}^{2^p} \alpha_l(k)\, \overline{A}_{il}\overline{x}(k) + \overline{A}_{di}\overline{x}(k - d(k)) \\ &\quad + \overline{B}_{wi} w(k) \\ e(k) &= \overline{C}_i \overline{x}(k) + \overline{C}_{di}\overline{x}(k - d(k)) + \overline{D}_i w(k) \\ x(k) &= \phi(k), \quad k = -d_2, -d_2+1, \ldots, -1, 0 \end{aligned} \tag{13}$$

where

$$\begin{aligned}
\overline{E} &= \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \\
\overline{A}_{il} &= \begin{bmatrix} A_{il} & -B_{il} \\ A_i - A_{fi} - B_{fi}C_{yi} & A_{fi} \end{bmatrix}, \\
\overline{A}_{di} &= \begin{bmatrix} A_{di} & 0 \\ A_{di} & 0 \end{bmatrix}, \\
\overline{B}_{wi} &= \begin{bmatrix} B_{wi} & 0 \\ B_{wi} - B_{fi}D_{yi} & 0 \end{bmatrix}, \\
\overline{C}_i &= \left[C_i - D_{fi}C_{yi} - C_{fi} \quad C_{fi}\right], \\
\overline{C}_{di} &= \left[C_{di} \quad 0\right], \\
\overline{D}_i &= \left[D_i - D_{fi}D_{yi}\right]. \\
A_{il} &= A_i + B_i\left(D_lK_i + D_l^-H_i\right), \\
B_{il} &= B_i\left(D_lK_i + D_l^-H_i\right).
\end{aligned} \tag{14}$$

Definition 3 (see [16]). System $Ex(k+1) = A(r_k)x(k)$ (or pair $E, A(r_k)$) is said to be:

(1) regular if, $\det(zE - A(r_k))$ is not identically zero for any $r_k = i, i \in w$;

(2) causal if, degree $(\det(zE - A(r_k))) = \operatorname{rank}(E)$ for any $r_k = i, i \in w$.

Definition 4 (see [30], singular stochastic finite-time boundedness (SSFTB)). The discrete-time singular jumping system (20) is said to be SSFTB with respect to $(\theta, \delta, R_i, N, d)$ with $0 < \theta < \delta, R_i > 0$ if

$$\begin{aligned}
&\varepsilon\left\{x^T(0)E^TR_iEx(0)\right\} \le \theta^2 \Longrightarrow \varepsilon\left\{x^T(k)E^TR_iEx(k)\right\} \\
&\quad < \delta^2, \quad \forall i \in w,\ k \in \{1, 2, \ldots, N\}
\end{aligned} \tag{15}$$

Remark 5. In many papers, the authors often use the definition of singular stochastic finite-time stability. However, it should be pointed out that SSFTB is more close to practical application and used widely, and SSFTS can be regarded as a particular case of SSFTB by letting $w(t) = 0$.

Definition 6 (see [30], singular stochastic H_∞ filtering finite–time boundedness (SSH$_\infty$FTB)). The filtering error discrete-time MJS (20) is said to be SSH$_\infty$FTB with respect to $(\theta, \delta, R_i, N, d, \gamma)$, where $0 < \theta < \delta, R_i > 0, \gamma > 0$ and $N \in \mathbb{Z}_{k\ge 0}$, and under the zero-initial condition the filtering error $z(k)$ satisfies:

$$\begin{aligned}
&\varepsilon\left\{\sum_{k=0}^{N} z^T(k)z(k)\right\} < \gamma^2\varepsilon\left\{\sum_{k=0}^{N} w^T(k)w(k)\right\}, \\
&\qquad \forall k \in \{1, 2, \ldots, N\}
\end{aligned} \tag{16}$$

Lemma 7 (see [45]). *Given matrices X, Y, Z with appropriate dimensions and Y is symmetric positive definite then the following inequality holds:*

$$-X^TZ - Z^TX - Z^TYZ \le X^TY^{-1}X \tag{17}$$

Lemma 8 (see [22]). *Given matrices X, Y, Z with appropriate dimensions and Y is symmetric positive definite, then there exists scalar $\lambda \ge 0$, such that $\lambda I + Y > 0$ and*

$$-X^TZ - Z^TX - Z^TYZ \le X^T(\lambda I + Y)^{-1}X + \lambda Z^TZ \tag{18}$$

3. Main Results

This section provides stochastic finite-time H_∞ boundedness analysis for the filtering error discrete-time SMJSs (13) under the given filter gains (6). LMI conditions are established to show that the filtering error discrete-time SMJSs (13) is SSFTB and the filtering error $e(k)$ and disturbance $w(k)$ satisfies the constraint condition (16).

Theorem 9. *Let K_i be the state feedback controller gain matrix, the filtering error discrete-time SMJSs (13) is SSFTB with respect to $(\theta, \delta, N, R_i, d)$, if there exist a scalar $\mu \ge 1, \varepsilon_1 > 0, \varepsilon_2 > 0$, set of symmetric positive-definite matrix $P_i > 0, Q_i > 0, i = 1, 2, 3$ and nonsingular matrices G_i, S_i, such that*

$$\Pi_1 = \begin{bmatrix}
\Pi'_{11} & \Pi_{12} & 0 & 0 & \Pi_{15} & \overline{G}_i^T\overline{A}_{il}^T & I \\
* & \Pi'_{22} & 0 & 0 & \Pi_{25} & \overline{A}_{di}^T & \varepsilon_1 I \\
* & * & -Q_2 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_3 & 0 & 0 & 0 \\
* & * & * & * & \Pi'_{55} & \overline{B}_{wi}^T & \varepsilon_2 I \\
* & * & * & * & * & -\overline{P}_j^{-1} & 0 \\
* & * & * & * & * & * & -S_j
\end{bmatrix} < 0, \tag{19}$$

$$\forall j \in w_{uk}^i,\ l = 1, 2, \ldots, 2^p$$

$$\Pi_2 = \begin{bmatrix}
\Pi''_{11} & \Pi_{12} & 0 & 0 & \Pi_{15} & \overline{G}_i^T\overline{A}_{il}^T & \rho_k^i I \\
* & \Pi''_{22} & 0 & 0 & \Pi_{25} & \overline{A}_{di}^T & \rho_k^i\varepsilon_1 I \\
* & * & -\rho_k^iQ_2 & 0 & 0 & 0 & 0 \\
* & * & * & -\rho_k^iQ_3 & 0 & 0 & 0 \\
* & * & * & * & \Pi''_{55} & \overline{B}_{wi}^T & \rho_k^i\varepsilon_2 I \\
* & * & * & * & * & -\left(P_k^i\right)^{-1} & 0 \\
* & * & * & * & * & * & -S_k^i
\end{bmatrix} < 0, \quad \forall j \in w_k^i,\ l = 1, 2, \ldots, 2^p \tag{20}$$

$$\aleph\left(\overline{E}^T\overline{P}_i\overline{E},\gamma^2\mu\right)\subset\hbar\left(\overline{H}\right) \tag{21}$$

$$\begin{aligned}&\theta^2\sup_{i\in w}\{\lambda_{\max}(\widehat{p}_i)\}+\theta^2\left(d_2+\frac{(d_2+d_1-1)(d_2-d_1)}{2}\right)\\&\cdot\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_1\right)\right\}+\theta^2d_1\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_2\right)\right\}+\theta^2d_2\\&\cdot\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_3\right)\right\}+d^2\sup_{i\in w}\{\lambda_{\max}(Q_i)\}\le\delta^2\mu^{-N}\\&\cdot\inf_{i\in w}\left\{\lambda_{\min}\left(\widehat{P}_i\right)\right\}\end{aligned} \tag{22}$$

where

$$\begin{aligned}\Pi'_{11}&=\Pi_{11}+\overline{G}_i^T\overline{A}_{il}^TR^T+R\overline{A}_{il}\overline{G}_i,\\\Pi_{12}&=\varepsilon_1\overline{G}_i^T\overline{A}_{il}^TR^T+R\overline{A}_{di},\\\Pi''_{11}&=\rho_k^i\Pi_{11}+\overline{G}_i^T\overline{A}_{il}^TR^T+R\overline{A}_{il}\overline{G}_i\\\Pi_{11}&=(d_{12}+1)\overline{G}_i^TQ_1\overline{G}_i+\overline{G}_i^TQ_2\overline{G}_i+\overline{G}_i^TQ_3\overline{G}_i\\&\quad-\mu\overline{G}_i\overline{EP}_i\overline{EG}_i,\\\Pi'_{22}&=\varepsilon_1\overline{A}_{di}^TR^T+\varepsilon_1R\overline{A}_{di}-Q_1\\\Pi''_{22}&=\varepsilon_1\overline{A}_{di}^TR^T+\varepsilon_1R\overline{A}_{di}-\rho_k^iQ_1,\\\Pi_{15}&=\varepsilon_2\overline{G}_i^T\overline{A}_{il}^TR^T+R\overline{B}_{wi},\\\Pi_{25}&=\varepsilon_2\overline{A}_{di}^TR^T+\varepsilon_1R\overline{B}_{wi}\\\Pi''_{55}&=\varepsilon_2\overline{B}_{wi}^TR^T+\varepsilon_2R\overline{B}_{wi}-\rho_k^iQ_i,\\\Pi'_{55}&=\varepsilon_2\overline{B}_{wi}^TR^T+\varepsilon_2R\overline{B}_{wi}-Q_i,\\\widehat{P}_i&=R_i^{-1/2}\overline{P}_iR_i^{-1/2}\\\widehat{Q}_1&=R_i^{-1/2}Q_1R_i^{-1/2},\\\widehat{Q}_2&=R_i^{-1/2}Q_2R_i^{-1/2},\\\widehat{Q}_3&=R_i^{-1/2}Q_3R_i^{-1/2},\\d_{12}&=d_2-d_1\\\rho_k^i&=\sum_{j\in w_k^i}\pi_{ij},\\S_k^i&=\sum_{j\in w_k^i}\pi_{ij}S_j,\\P_k^i&=\sum_{j\in w_k^i}\pi_{ij}\overline{P}_j\end{aligned} \tag{23}$$

$\lambda_{\max}(.),\lambda_{\min}(.)$ respectively stand for the maximum and minimum eigenvalues of matrices, $R\in R^{2n\times 2n}$ is any constant matrix satisfying: $R\overline{E}=0$, $\operatorname{rank}(R)=2(n-r)$. Then system (13) is regular, causal and finite-time bounded inside $\aleph(\overline{E}^T\overline{P}_i\overline{E},\gamma^2\mu)$ with respect to (θ,δ,R,N,d). Moreover, the estimates of the attraction domain is given as the following form:

$$\nu_{\max}=\sqrt{\frac{\gamma^2\mu^{1-N}-\sup_{i\in w}\{\lambda_{\max}(Q_i)\}d^2}{\Upsilon}} \tag{24}$$

where

$$\begin{aligned}\Upsilon&=\sup_{i\in w}\left\{\lambda_{\max}\left(\overline{E}^T\overline{P}_i\overline{E}\right)\right\}+d_2\lambda_{\max}(Q_1)\\&\quad+\frac{(d_2+d_1-1)(d_2-d_1)}{2}\lambda_{\max}(Q_1)\\&\quad+d_1\lambda_{\max}(Q_2)+d_2\lambda_{\max}(Q_3)\end{aligned} \tag{25}$$

Proof. Here we consider the transition probability of the unknown situation, and use the same method we can get the transfer probability is known. Now, we prove that the system (13) is regular and causal within $\aleph(\overline{E}^T\overline{P}_i\overline{E},\gamma^2\mu)$. When $i\in w, j\in w_{uk}^i$, it follows from the condition (24), we can easily see that $\Pi'_{11}<0$, then $\overline{G}_i$ is nonsingular. Let $T=\operatorname{diag}\{\overline{G}_i,I,I,I,I,I,I\}$, then $T^T\Pi_1T<0$, which is equivalent to:

$$\overline{\Pi}_1=\begin{bmatrix}\overline{\Pi}'_{11}&\overline{\Pi}_{12}&0&0&\overline{\Pi}_{15}&\overline{G}_i^T\overline{A}_{il}^T&L_i^T\\*&\Pi'_{22}&0&0&\Pi_{25}&\overline{A}_{di}^T&\varepsilon_1I\\*&*&-Q_2&0&0&0&0\\*&*&*&-Q_3&0&0&0\\*&*&*&*&\Pi_{55}&\overline{B}_{wi}^T&\varepsilon_2I\\*&*&*&*&*&-\overline{P}_j^{-1}&0\\*&*&*&*&*&*&-S_j\end{bmatrix}<0,\quad l=1,2,\ldots,2^p \tag{26}$$

where

$$\begin{aligned}\overline{\Pi}'_{11}&=(1+d_{12})Q_1+Q_2+Q_3-\mu\overline{E}\,\overline{P}_i\overline{E}+\overline{A}_{il}^TR^TL_i\\&\quad+L_i^TR\overline{A}_{il}\\\overline{\Pi}_{12}&=\varepsilon_1\overline{A}_{il}^TR^T+L_i^TR\overline{A}_{di},\\\overline{\Pi}_{15}&=\varepsilon_2\overline{A}_{il}^TR^T+L_i^TR\overline{B}_{wi},\\L_i&=G_i^{-1}\end{aligned} \tag{27}$$

then for $\sum_{l=1}^{2^p}\alpha_l(k)=1$ the following is true:

$$\sum_{l=1}^{2^p}\alpha_l(k)\overline{\Pi}_1<0 \tag{28}$$

Applying Schur complement lemma. Eq.(28) is equivalent to

$$\Phi_1 = \begin{bmatrix} (1+d_{12})Q_1+Q_2+Q_3-\mu\overline{E}^T\overline{P}_i\overline{E} & 0 & 0 & 0 & 0 \\ * & -Q_1 & 0 & 0 & 0 \\ * & * & -Q_2 & 0 & 0 \\ * & * & * & -Q_3 & 0 \\ * & * & * & * & -Q_i \end{bmatrix} + \eta^T(k)\overline{P}_j\eta(k) + \eta^T(k)R^T\tau(k) + \tau^T(k)R\eta(k) + \tau^T(k)S_j^{-1}\tau(k) < 0 \quad (29)$$

where

$$\eta(k) = \left[\sum_{l=1}^{2^p}\alpha_l(k)\overline{A}_{il} \quad \sum_{l=1}^{2^p}\alpha_l(k)\overline{A}_{di} \quad 0 \quad 0 \quad \sum_{l=1}^{2^p}\alpha_l(k)\overline{B}_{wi}\right], \quad (30)$$

$$\tau(k) = \left[L_i \quad \varepsilon_1 I \quad 0 \quad 0 \quad \varepsilon_2 I\right]$$

From Lemma 7 and $S_j > 0$, we know that

$$-\eta^T(k)R^TS_jR\eta(k) \le \eta^T(k)R^T\tau(k) + \tau^T(k)R\eta(k) + \tau^T(k)S_j^{-1}\tau(k) \quad (31)$$

considering (29) and (31), it is obtained that

$$\overline{\Phi}_1 = \begin{bmatrix} (1+d_{12})Q_1+Q_2+Q_3-\mu\overline{E}^T\overline{P}_i\overline{E} & 0 & 0 & 0 & 0 \\ * & -Q_1 & 0 & 0 & 0 \\ * & * & -Q_2 & 0 & 0 \\ * & * & * & -Q_3 & 0 \\ * & * & * & * & -Q_i \end{bmatrix} + \eta^T(k)\overline{P}_j\eta(k) - \eta^T(k)R^TS_jR\eta(k) < \Phi_1 < 0 \quad (32)$$

using the similar method of the above, when $i \in w$, $j \in w_k^i$, we can obtain:

$$\overline{\Phi}_2 = \begin{bmatrix} \left((1+d_{12})Q_1+Q_2+Q_3-\mu\overline{E}^T\overline{P}_i\overline{E}\right)\rho_k^i & 0 & 0 & 0 & 0 \\ * & -\rho_k^iQ_1 & 0 & 0 & 0 \\ * & * & -\rho_k^iQ_2 & 0 & 0 \\ * & * & * & -\rho_k^iQ_3 & 0 \\ * & * & * & * & -\rho_k^iQ_i \end{bmatrix} + \eta^T(k)P_k^i\eta(k) - \eta^T(k)R^TS_k^iR\eta(k) < 0 \quad (33)$$

Then we take the unknown transition probability for an example to prove the system is regular and causal.

From (32), we can obtain that

$$(1+d_{12})Q_1+Q_2+Q_3-\mu\overline{E}^T\overline{P}_i\overline{E} + \sum_{l=1}^{2^p}\alpha_l(k)\overline{A}_{il}^T\left(\overline{P}_j - R^TS_jR\right)\sum_{l=1}^{2^p}\alpha_l(k)\overline{A}_{il} < 0 \quad (34)$$

since $Q_i > 0, i = 1, 2, 3$ and $\overline{P}_i > 0$, then we have

$$\sum_{l=1}^{2^p}\alpha_l(k)\overline{A}_{il}^T\left(\overline{P}_j - R^TS_jR\right)\sum_{l=1}^{2^p}\alpha_l(k)\overline{A}_{il} - \mu\overline{E}^T\overline{P}_i\overline{E} < 0 \quad (35)$$

since $\overline{E}$ is singular and $\operatorname{rank}(\overline{E}) = r \le n$, there exist two nonsingular matrices $M, N \in R^{n\times n}$, such that

$$M\overline{E}N = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$
$$M\sum_{l=1}^{2^p}\alpha_l(k)\overline{A}_{il}N = \begin{bmatrix} \overline{A}_{i1} & \overline{A}_{i2} \\ \overline{A}_{i3} & \overline{A}_{i4} \end{bmatrix}$$
$$M^{-T}\overline{P}_iM^{-1} = \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i3} & P_{i4} \end{bmatrix}, \quad (36)$$
$$RM^{-1} = \left[R_1 \quad R_2\right]$$

Noting that $R\overline{E} = 0$ and $\operatorname{rank}(R) = 2(n-r)$, we can obtain that $R_1 = 0$ and $\operatorname{rank}(R_2) = 2(n-r)$, $R_2 \in R^{2n\times 2(n-r)}$, that is $RM^{-1} = \left[0 \quad R_2\right]$

$$\begin{bmatrix} * & * \\ * & \varpi_i \end{bmatrix} < 0 \quad (37)$$

where $*$ represents matrices that are not relevant in the following discussion, and

$$\begin{aligned}\varpi_i = {} & \overline{A}_{i2}^T\overline{P}_{j1}\overline{A}_{i2} + \overline{A}_{i4}^T\overline{P}_{j3}\overline{A}_{i2} + \overline{A}_{i2}^T\overline{P}_{j2}\overline{A}_{i4} + \overline{A}_{i4}^T\overline{P}_{j4}\overline{A}_{i4} \\ & + \overline{A}_{i4}^T R_2^T S_j R_2 \overline{A}_{i4}\end{aligned} \tag{38}$$

from Eq.(37), it is obtained that

$$\varpi_i < 0 \tag{39}$$

Now, we assume that the matrix $\overline{A}_{i4}$ is singular for one $i \in w$, then there exists a vector $\varsigma_i \in R^{n-r}$ and $\varsigma_i \neq 0$ such that $\overline{A}_{i4}\varsigma_i = 0$. Pre-multiplying and post-multiplying by ς^T and ς result in $\varsigma_i^T\overline{A}_{i2}^T\overline{P}_{j1}\overline{A}_{i2}\varsigma < 0$ which contradicts $\overline{P}_{j1} > 0$. Thus, $\overline{A}_{i4}$ is nonsingular.

On the other hand, for the case of $i \in w$, $j \in w_k^i$, by using the similar method of above, from (20), we also can obtain that $\overline{A}_{i4}$ is nonsingular. Therefore, by Definition 3, system (13) is regular and causal.

At this step we will show that system (13) is *SSFTB* with respect to $(\theta, \delta, N, R(r_k))$.

Consider the following Lyapunon-Krasovskii functional:

$$V(\overline{x}(k), r_k) = \sum_{i=1}^{3} V_i(\overline{x}(k), r_k) \tag{40}$$

where

$$\begin{aligned}V_1(\overline{x}(k), r_k) &= \overline{x}^T(k)\overline{E}^T\overline{P}_{r_k}\overline{E}\overline{x}(k) \\ V_2(\overline{x}(k), r_k) &= \sum_{m=k-d(k)}^{k-1}\overline{x}^T(m)Q_1\overline{x}(m) \\ &\quad + \sum_{\theta=-d_2+1}^{-d_1}\sum_{m=k+\theta}^{k-1}\overline{x}^T(m)Q_1\overline{x}(m) \\ V_3(\overline{x}(k), r_k) &= \sum_{m=k-d_1}^{k-1}\overline{x}^T(m)Q_2\overline{x}(m) \\ &\quad + \sum_{m=k-d_2}^{k-1}\overline{x}^T(m)Q_3\overline{x}(m)\end{aligned} \tag{41}$$

then

$$\begin{aligned}&\varepsilon\{V_1(\overline{x}(k+1), r_{k+1})\} - V_1(\overline{x}(k), r_k) \\ &= \overline{x}^T(k+1)\overline{E}^T\sum_{j\in w}\pi_{ij}\overline{P}_j\overline{E}\overline{x}(k+1) \\ &\quad - \overline{x}^T(k)\overline{E}^T\overline{P}_i\overline{E}\overline{x}(k) \\ &= \overline{x}^T(k+1)\overline{E}^T\left(\sum_{j\in w_k^i}\pi_{ij}\overline{P}_j + \sum_{j\in w_{uk}^i}\pi_{ij}\overline{P}_j\right)\overline{E}\overline{x}(k+1) \\ &\quad - \overline{x}^T(k)\overline{E}^T\left(\sum_{j\in w_k^i}\pi_{ij} + \sum_{j\in w_{uk}^i}\pi_{ij}\right)\overline{P}_i\overline{E}\overline{x}(k) \\ &= \overline{x}^T(k+1)\overline{E}^T\left(P_k^i + \sum_{j\in w_{uk}^i}\pi_{ij}\overline{P}_j\right)\overline{E}\overline{x}(k+1) \\ &\quad - \overline{x}^T(k)\overline{E}^T\left(\rho_k^i + \sum_{j\in w_{uk}^i}\pi_{ij}\right)\overline{P}_i\overline{E}\overline{x}(k)\end{aligned} \tag{42}$$

$$\begin{aligned}&\varepsilon\{V_2(\overline{x}(k+1), r_{k+1})\} - V_2(\overline{x}(k), r_k) \\ &\leq \overline{x}^T(k)Q_1\overline{x}(k) - \overline{x}^T(k-d(k))Q_1\overline{x}(k-d(k)) \\ &\quad + d_{12}\overline{x}^T(k)Q_1\overline{x}(k)\end{aligned} \tag{43}$$

$$\begin{aligned}&\varepsilon\{V_3(\overline{x}(k+1), r_{k+1})\} - V_3(\overline{x}(k), r_k) \\ &= \overline{x}^T(k)Q_2\overline{x}(k) + \overline{x}^T(k)Q_3\overline{x}(k) \\ &\quad - \overline{x}^T(k-d_1)Q_2\overline{x}(k-d_1) \\ &\quad - \overline{x}^T(k-d_2)Q_3\overline{x}(k-d_2)\end{aligned} \tag{44}$$

From $R\overline{E} = 0$, the following equation holds for any symmetric matrix S_i with appropriate dimensions and $r_k = i$.

$$\begin{aligned}0 &= -\sum_{j\in w}\pi_{ij}\overline{x}^T(k+1)\overline{E}^T R^T S_j R\overline{E}\overline{x}(k+1) \\ &= -\overline{x}^T(k+1)\overline{E}^T R^T\left(\sum_{j\in w_k^i}\pi_{ij} + \sum_{j\in w_{uk}^i}\pi_{ij}\right) \\ &\quad \cdot S_j R\overline{E}\overline{x}(k+1) = -\overline{x}^T(k+1)\overline{E}^T R^T S_k^i R\overline{E}\overline{x}(k+1) \\ &\quad - \sum_{j\in w_{uk}^i}\pi_{ij}\overline{x}^T(k+1)\overline{E}^T R^T S_j R\overline{E}\overline{x}(k+1)\end{aligned} \tag{45}$$

from (42)-(45), we have

$$\begin{aligned}
&\varepsilon\{V(k+1)\}-V(k)\le \xi^T(k)\\
&\cdot\left\{\begin{bmatrix}-(\overline{E}^T\overline{P}_i\overline{E}+(1+d_{12})Q_1 & & & & \\ +Q_2+Q_3)\rho_k^i & & & & \\ & -\rho_k^iQ_1 & & & \\ & & -\rho_k^iQ_2 & & \\ & & & -\rho_k^iQ_3 & \\ & & & & 0\end{bmatrix}\right.\\
&\left.+\eta^T(k)\left(P_k^i-R^TS_k^iR\right)\eta(k)\right\}\xi(k)+\sum_{j\in w_{uk}^i}\pi_{ij}\xi^T(k)\\
&\cdot\left\{\begin{bmatrix}-(\overline{E}^T\overline{P}_i\overline{E}+(1+d_{12})Q_1 & & & & \\ +Q_2+Q_3) & & & & \\ & -Q_1 & & & \\ & & -Q_2 & & \\ & & & -Q_3 & \\ & & & & 0\end{bmatrix}\right.\\
&\left.+\eta^T(k)\left(\overline{P}_j-R^TS_jR\right)\eta(k)\right\}\xi(k)=\xi^T(k)\\
&\cdot\left\{\begin{bmatrix}-(\overline{E}^T\overline{P}_i\overline{E}+(1+d_{12})Q_1 & & & & \\ +Q_2+Q_3) & & & & \\ & -Q_1 & & & \\ & & -Q_2 & & \\ & & & -Q_3 & \\ & & & & 0\end{bmatrix}\right.\\
&\left.+\eta^T(k)\left(\widehat{P}_i-R^T\widehat{S}_iR\right)\eta(k)\right\}\xi(k)
\end{aligned}\tag{46}$$

where

$$\begin{aligned}
&\xi(k)\\
&=\left[\overline{x}^T(k)\ \ \overline{x}^T(k-d(k))\ \ \overline{x}^T(k-d_1)\ \ \overline{x}^T(k-d_2)\ \ w^T(k)\right]^T\\
&\widehat{P}_i=\sum_{j\in w_k^i}\pi_{ij}\overline{P}_j+\sum_{j\in w_{uk}^i}\pi_{ij}\overline{P}_j=P_k^i+\sum_{j\in w_{uk}^i}\pi_{ij}\overline{P}_j,\\
&\widehat{S}_i=\sum_{j\in w_k^i}\pi_{ij}S_j+\sum_{j\in w_{uk}^i}\pi_{ij}S_j=S_k^i+\sum_{j\in w_{uk}^i}\pi_{ij}S_j
\end{aligned}\tag{47}$$

from Eq.(32) and (33), we have

$$\begin{aligned}
&\varepsilon\{V(k+1)\}-V(k)\\
&\quad<(\mu-1)x^T(k)\overline{E}^T\overline{P}_i\overline{E}x(k)+w^T(k)Q_iw(k)\\
&\quad\le(\mu-1)V(k)+\sup_{i\in w}\{\lambda_{\max}(Q_i)\}w^T(k)w(k)
\end{aligned}\tag{48}$$

therefore, we have

$$\begin{aligned}
&\varepsilon\{V(k+1)\}\le\mu\varepsilon\{V(k)\}\\
&\qquad+\sup_{i\in w}\{\lambda_{\max}(Q_i)\}\varepsilon\left\{w^T(k)w(k)\right\}
\end{aligned}\tag{49}$$

noting that $\mu\ge 1$, it follows from Eq.(49) that

$$\begin{aligned}
&\varepsilon\{V(k)\}\\
&\quad\le\mu^k\varepsilon\{V(0)\}\\
&\qquad+\sup_{i\in w}\{\lambda_{\max}(Q_i)\}\varepsilon\left\{\sum_{j=0}^{k-1}\mu^{k-j-1}w^T(k)w(k)\right\}\\
&\quad\le\mu^k\varepsilon\{V(0)\}+\sup_{i\in w}\{\lambda_{\max}(Q_i)\}\mu^kd^2
\end{aligned}\tag{50}$$

Letting $\widehat{P}_i=R_i^{-1/2}\overline{P}_iR_i^{-1/2}$, $\widehat{Q}_1=R_i^{-1/2}Q_1R_i^{-1/2}$, $\widehat{Q}_2=R_i^{-1/2}Q_2R_i^{-1/2}$, $\widehat{Q}_3=R_i^{-1/2}Q_3R_i^{-1/2}$ and noting that $\varepsilon\{x^T(0)\overline{E}^TR_i\overline{E}x(0)\}\le\theta^2$, we have

$$\begin{aligned}
&\varepsilon\{V(0)\}\\
&\quad=\varepsilon\left\{x^T(0)R_i^{1/2}\widehat{P}R_i^{1/2}x(0)\right\}\\
&\qquad+\varepsilon\left\{\sum_{m=-d(0)}^{-1}x^T(m)R_i^{1/2}\widehat{Q}_1R_i^{1/2}x(m)\right\}\\
&\qquad+\varepsilon\left\{\sum_{\theta=-d_2+1}^{-d_1}\sum_{m=\theta}^{-1}x^T(m)R_i^{1/2}\widehat{Q}_1R_i^{1/2}x(m)\right\}\\
&\qquad+\varepsilon\left\{\sum_{m=-d_1}^{-1}x^T(m)R_i^{1/2}\widehat{Q}_2R_i^{1/2}x(m)\right\}\\
&\qquad+\varepsilon\left\{\sum_{m=-d_2}^{-1}x^T(m)R_i^{1/2}\widehat{Q}_3R_i^{1/2}x(m)\right\}\\
&\quad\le\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{P}_i\right)\right\}\theta^2+d_2\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_1\right)\right\}\theta^2\\
&\qquad+\frac{(d_2+d_1-1)(d_2-d_1)}{2}\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_1\right)\right\}\theta^2\\
&\qquad+d_1\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_2\right)\right\}\theta^2\\
&\qquad+d_2\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_3\right)\right\}\theta^2
\end{aligned}\tag{51}$$

On the other hand, for all $i\in w$, the following is true:

$$\varepsilon\{V(k)\} \geq \varepsilon\left\{x^T(k)\overline{E}\,\overline{P}_i\overline{E}x(k)\right\} = \varepsilon\left\{x^T(k)\overline{E}R_i^{1/2}\widehat{P}_iR_i^{1/2}\overline{E}x(k)\right\} \geq \inf_{i\in w}\left\{\lambda_{\min}\left(\widehat{P}_i\right)\right\}\varepsilon\left\{x^T(k)\overline{E}R_i\overline{E}x(k)\right\} \tag{52}$$

$$\begin{aligned}\varepsilon\left\{x^T(k)\overline{E}R_i\overline{E}x(k)\right\} \leq{}& \frac{\mu^k\theta^2\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{P}_i\right)\right\} + \mu^k\theta^2\left(d_2 + \left(d_2 + d_1 - 1\right)\left(d_2 - d_1\right)/2\right)\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_1\right)\right\}}{\inf_{i\in w}\left\{\lambda_{\min}\left(\widehat{P}_i\right)\right\}} \\ &+ \frac{d_1\mu^k\theta^2\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_2\right)\right\} + d_2\mu^k\theta^2\sup_{i\in w}\left\{\lambda_{\max}\left(\widehat{Q}_3\right)\right\} + \mu^k d^2\sup_{i\in w}\left\{\lambda_{\max}\left(Q_i\right)\right\}}{\inf_{i\in w}\left\{\lambda_{\min}\left(\widehat{P}_i\right)\right\}}\end{aligned} \tag{53}$$

Noticing that condition (22), it follows from Eq. (53) that $\varepsilon\{x^T(k)\overline{E}^T R_i\overline{E}x(k)\} < \delta^2$. This completes the proof of this theorem. □

Moreover, according to (50), we can obtain

$$\begin{aligned}x^T(k)\overline{E}^T\overline{P}_i\overline{E}^T x(k) &\leq V(k) \\ &\leq \mu^k V(0) + \mu^k\sup_{i\in w}\left\{\lambda_{\max}\left(Q_i\right)\right\}d^2 \leq \gamma^2\mu\end{aligned} \tag{54}$$

$$\left\|\phi(k)\right\|^2 \leq \frac{\gamma^2\mu^{1-N} - \sup_{i\in w}\left(\lambda_{\max}\left(Q_i\right)\right)d^2}{\Upsilon} \triangleq \mathfrak{I}(\phi) \tag{55}$$

Remark 10. For all $\phi \in \mathfrak{I}(\phi)$, we have $\mathfrak{I}(\phi) \leq \gamma^2\mu$ and from (54), we also obtain that for all $\phi \in \mathfrak{I}(\phi)$, we have $x^T(k)\overline{E}^T\overline{P}_i\overline{E}^T x(k) \leq \gamma^2\mu$. This situation shows that began in the trajectory of $\mathfrak{I}(\phi)$ state of $x(k)$ would have been in attracting field.

Remark 11. Though the problem of $H\infty$ filtering for discrete-time SMJS subject to time-varying delay was considered in [16]. Compared with it, the finite-time and actuator saturation have been taken into the above Theorem, which is more practical and can guarantee the large values of states could not be accepted in the presence of saturations.

Base on Theorem 9, the filtering error system (13) can be proved that SSFTB, following we will propose a H_∞ finite-time sufficient condition for SMJS.

Theorem 12. *The filtering error system (13) is SSH$_\infty$FTB with respect to $(\theta, \delta, N, \gamma, R(r_k))$, if there exist scalars $\mu \geq 1, \varepsilon_1 > 0, \varepsilon_1 > 0, \delta > 0, \gamma > 0$, symmetric definite matrices $\overline{P}_i > 0, Q_1, Q_2, Q_3 > 0$, and nonsingular matrices $\overline{G}_i, S_i$,such that*

$$\overline{\Pi}_1 = \begin{bmatrix} \Pi'_{11} & \Pi_{12} & 0 & 0 & \Pi_{15} & \overline{G}_i^T\overline{A}_{il}^T & I & \overline{G}_i^T\overline{C}_i^T \\ * & \Pi'_{22} & 0 & 0 & \Pi_{25} & \overline{A}_{di}^T & \varepsilon_1 I & \overline{C}_{di}^T \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Q_3 & 0 & 0 & 0 & 0 \\ * & * & * & * & \overline{\Pi}'_{55} & \overline{B}_{wi}^T & \varepsilon_2 I & \overline{D}_i^T \\ * & * & * & * & * & -\overline{P}_j^{-1} & 0 & 0 \\ * & * & * & * & * & * & -S_j & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \quad \forall j \in w_{uk}^i \tag{56}$$

$$\overline{\Pi}_2 = \begin{bmatrix} \Pi''_{11} & \Pi_{12} & 0 & 0 & \Pi_{15} & \overline{G}_i^T\overline{A}_{il}^T W_i & \rho_k^i I & \rho_k^i\overline{G}_i^T\overline{C}_i^T \\ * & \Pi''_{22} & 0 & 0 & \Pi_{25} & \overline{A}_{di}^T W_i & \rho_k^i\varepsilon_1 I & \rho_k^i\overline{C}_{di}^T \\ * & * & -\rho_k^i Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\rho_k^i Q_3 & 0 & 0 & 0 & 0 \\ * & * & * & * & \overline{\Pi}''_{55} & \overline{B}_{wi}^T W_i & \rho_k^i\varepsilon_2 I & \rho_k^i D_i^T \\ * & * & * & * & * & -\overline{P}^{-1} & 0 & 0 \\ * & * & * & * & * & * & -S_k^i & 0 \\ * & * & * & * & * & * & * & -\rho_k^i I \end{bmatrix} < 0, \quad \forall j \in w_k^i \tag{57}$$

$$\theta^2 \sup_{i\in w}\{\lambda_{\max}(\widehat{P}_i)\} + \theta^2\left(d_2 + \frac{(d_2+d_1-1)(d_2-d_1)}{2}\right)\sup_{i\in w}\{\lambda_{\max}(\widehat{Q}_1)\} + \theta^2 d_1 \sup_{i\in w}\{\lambda_{\max}(\widehat{Q}_2)\} + \theta^2 d_2 \sup_{i\in w}\{\lambda_{\max}(\widehat{Q}_3)\} + \mu^{-N}\gamma^2 d^2 \le \delta^2\mu^{-N}\inf_{i\in w}\{\lambda_{\min}(\widehat{P}_i)\} \tag{58}$$

$$\aleph\left(\overline{E}^T\overline{P}_i\overline{E}, \gamma^2\mu\right) \subset \hbar\left(\overline{H}\right) \tag{59}$$

where

$$\begin{aligned}
\overline{\Pi}'_{55} &= -\gamma^2\mu^{-N}I + \varepsilon_2\overline{B}_{wi}^T R^T + \varepsilon_2 R\overline{B}_{wi},\\
\overline{\Pi}''_{55} &= -\rho_k^i\gamma^2\mu^{-N}I + \varepsilon_2\overline{B}_{wi}^T R^T + \varepsilon_2 R\overline{B}_{wi},\\
W_i &= \left[\sqrt{\pi_{i1}}I, \sqrt{\pi_{i2}}I, \ldots, \sqrt{\pi_{is}}I\right],\\
\overline{P}^{-1} &= \operatorname{diag}\left\{\overline{P}_1^{-1}, \overline{P}_2^{-1}, \ldots, \overline{P}_s^{-1}\right\}
\end{aligned} \tag{60}$$

Proof. Using Schur complement lemma, it follows from Eq.(56) that the following inequality holds:

$$\widehat{\Pi}_1 = \begin{bmatrix} \Pi'_{11} & \Pi_{12} & 0 & 0 & \Pi_{15} & \overline{G}_i^T\overline{A}_{il}^T & I\\ * & \Pi'_{22} & 0 & 0 & \Pi_{25} & \overline{A}_{di}^T & \varepsilon_1 I\\ * & * & -Q_2 & 0 & 0 & 0 & 0\\ * & * & * & -Q_3 & 0 & 0 & 0\\ * & * & * & * & \overline{\Pi}'_{55} & \overline{B}_{wi}^T & \varepsilon_2 I\\ * & * & * & * & * & -\overline{P}_j^{-1} & 0\\ * & * & * & * & * & * & -S_j \end{bmatrix} < 0, \quad \forall j \in w_{uk}^i,\ l = 1, 2, \ldots, 2^p \tag{61}$$

Let $Q_i = \mu^{-N}\gamma^2 I$, then conditions (58) and (61) can guarantee that the filtering error SMJS(13) is *SSFTB* with respect to $(\theta, \delta, N, R(r_k), d)$ according to Theorem 9. Thus, we only need to prove that the constraint relation (16) holds under zero-value initial condition. Applying Schur complement to (56) we have

$$\widehat{\Pi}'_1 = \begin{bmatrix} \Pi'_{11} + \overline{G}_i^T\overline{C}_i^T\overline{C}_i\overline{G}_i & \Pi_{12} + \overline{G}_i^T\overline{C}_i^T\overline{C}_{di} & 0 & 0 & \Pi_{15} + \overline{G}_i^T\overline{C}_i^T\overline{D}_i & \overline{G}_i^T\overline{A}_{il}^T & I\\ * & \Pi'_{22} + \overline{C}_{di}^T\overline{C}_{di} & 0 & 0 & \Pi_{25} + \overline{C}_{di}^T\overline{D}_i & \overline{A}_{di}^T & \varepsilon_1 I\\ * & * & -Q_2 & 0 & 0 & 0 & 0\\ * & * & * & -Q_3 & 0 & 0 & 0\\ * & * & * & * & \overline{\Pi}'_{55} + \overline{D}_i^T\overline{D}_i & \overline{B}_{wi}^T & \varepsilon_2 I\\ * & * & * & * & * & -\overline{P}_j^{-1} & 0\\ * & * & * & * & * & * & -S_j \end{bmatrix} < 0, \quad \forall j \in w_{uk}^i \tag{62}$$

Using the same Lyapunov-Krasovskii functional as Theorem 9, it follows that:

$$\varepsilon\{V(k+1)\} - \mu V(k) + e^T(k)e(k) - \gamma^2\mu^{-N}w^T(k)w(k) < 0 \tag{63}$$

then we have

$$\varepsilon\{V(k+1)\} \le \mu\varepsilon\{V(k)\} + \varepsilon\left\{e^T(k)e(k)\right\} - \gamma^2\mu^{-N}\varepsilon\left\{w^T(k)w(k)\right\} \tag{64}$$

according to Eq.(64), we can deduce

$$\varepsilon\{V(k)\} \le \mu^k\varepsilon\{V(0)\} + \sum_{j=0}^{k-1}\mu^{k-j-1}\varepsilon\left\{e^T(j)e(j)\right\} + \gamma^2\mu^{-N}\varepsilon\left\{\sum_{j=0}^{k-1}\mu^{k-j-1}w^T(j)w(j)\right\} \tag{65}$$

under the zero-value initial condition and noting that $V(k) \ge 0$ for all $k \in \{1, 2, \ldots, N\}$ we have

$$\sum_{j=0}^{k-1}\mu^{k-j-1}\varepsilon\left\{e^T(j)e(j)\right\} < \gamma^2\mu^{-N}\varepsilon\left\{\sum_{j=0}^{k-1}\mu^{k-j-1}w^T(j)w(j)\right\} \tag{66}$$

Notice that $\mu \geq 1$, we have

$$\begin{aligned}\varepsilon\left\{\sum_{k=0}^{N} e^{T}(k)\, e(k)\right\} &= \sum_{k=0}^{N} \varepsilon\left\{e^{T}(k)\, e(k)\right\} \\ &\leq \sum_{k=0}^{N} \varepsilon\left\{\mu^{N-k} e^{T}(k)\, e(k)\right\} \\ &< \gamma^{2}\mu^{-N}\varepsilon\left\{\sum_{k=0}^{N} \mu^{N-k} w^{T}(k)\, w(k)\right\} \\ &< \gamma^{2}\varepsilon\left\{\sum_{k=0}^{N} w^{T}(k)\, w(k)\right\}\end{aligned} \tag{67}$$

Therefore, condition (16) holds. In addition, taking into account (57) and using the similar method of the above, (16) can be worked out. This proof is completed. □

To solve Theorem 12, letting $\overline{P}_i = \operatorname{diag}\{P_i, P_i\}$, $R_i = \operatorname{diag}\{\overline{R}_i, \overline{R}_i\}$, the following theorem provides LMI conditions to ensure stochastic H_∞ finite-time boundedness of the filtering errors SMJS (13).

Theorem 13. *The filtering error discrete-time SMJS (13) with saturating actuators within $\aleph(\overline{E}^T \overline{P}_i \overline{E}, \gamma^2\mu)$ is stochastically H_∞ finite-time bounded with respect to $(\theta, \delta, N, \gamma, R(r_k), d)$, if there exist scalars $\mu \geq 1, \gamma > 0, \varepsilon_1 > 0, \varepsilon_2 > 0, \delta_e > 0, \rho > 0, \eta_1 > 0, \eta_2 > 0$, a set of symmetric positive-definite matrices $P_i > 0, Q_1, Q_2, Q_3 > 0$, and sets of nonsingular matrices G_i, S_i, Z_i, sets of matrices $U_i, V_i, X_i, F_i, Y_i, M_i, T_i$, for all $i \in w$, such that*

$$C_{yi} G_i = Z_i C_{yi} \tag{68}$$

$$\begin{bmatrix} -\dfrac{1}{\gamma^2\mu} & \overline{h}_{l_0} G_i & -\overline{h}_{l_0} G_i \\ * & EG_i + G_i^T E^T - 2I + P_i & 0 \\ * & * & EG_i + G_i^T E^T - 2I + P_i \end{bmatrix} \leq 0, \quad l_0 = 1, 2, \ldots, p \tag{69}$$

$$\widetilde{\Pi}_1 = \begin{bmatrix} \Omega_1 & \Omega_2 \\ * & \Omega_3 \end{bmatrix} < 0, \quad \forall j \in w_{uk}^{i},\ l = 1, 2, \ldots, 2^p \tag{70}$$

$$\widetilde{\Pi}_2 = \begin{bmatrix} \Omega'_1 & \sqrt{\rho_k^i}\,\Omega_2 \\ * & \Omega_3 \end{bmatrix} < 0, \quad \forall j \in w_{k}^{i},\ l = 1, 2, \ldots, 2^p \tag{71}$$

$$\begin{aligned} &\overline{R}_i < P_i < \delta_e \overline{R}_i, \\ &\eta_1 \overline{R}_i < \widetilde{Q}_1, \\ &\widetilde{Q}_2, \widetilde{Q}_3 < \eta_2 \overline{R}_i \end{aligned} \tag{72}$$

$$\begin{aligned} &\left\{\delta_e + \left[\frac{(d_2 + d_1 - 1)(d_2 - d_1)}{2} + d_1 + 2d_2\right]\eta_2\right\}\theta^2 \\ &\quad + \mu^{-N}\gamma^2 d^2 \leq \delta^2 \mu^{-N} \end{aligned} \tag{73}$$

where

$$\Omega_1 = \begin{bmatrix} \widetilde{\Pi}'_{11} & \widetilde{\Pi}_{12} & 0 & 0 & \widetilde{\Pi}_{13} & \Pi'_{14} & I & \Pi_{15} \\ * & \widetilde{\Pi}'_{22} & 0 & 0 & \widetilde{\Pi}_{23} & \Pi'_{24} & \varepsilon_1 I & \Pi_{25} \\ * & * & -\widetilde{Q}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\widetilde{Q}_3 & 0 & 0 & 0 & 0 \\ * & * & * & * & \widetilde{\Pi}'_{55} & \Pi'_{56} & \varepsilon_2 I & \Pi_{57} \\ * & * & * & * & * & \widehat{\Pi}'_1 & 0 & 0 \\ * & * & * & * & * & * & \widehat{\Pi}_2 & 0 \\ * & * & * & * & * & * & * & \widehat{\Pi}_3 \end{bmatrix},$$

$$\Omega_2 = \begin{bmatrix} \Pi_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} & \Pi_{20} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Omega'_1 = \begin{bmatrix} \widetilde{\Pi}''_{11} & \widetilde{\Pi}_{12} & 0 & 0 & \widetilde{\Pi}_{13} & \Pi''_{14} & \rho_k^i I & \rho_k^i \Pi_{15} \\ * & \widetilde{\Pi}''_{22} & 0 & 0 & \widetilde{\Pi}_{23} & \Pi''_{24} & \rho_k^i \varepsilon_1 I & \rho_k^i \Pi_{25} \\ * & * & -\rho_k^i \widetilde{Q}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\rho_k^i \widetilde{Q}_3 & 0 & 0 & 0 & 0 \\ * & * & * & * & \widetilde{\Pi}''_{55} & \Pi''_{56} & \rho_k^i \varepsilon_2 I & \rho_k^i \Pi_{57} \\ * & * & * & * & * & \widehat{\Pi}''_1 & 0 & 0 \\ * & * & * & * & * & * & \widehat{\Pi}_2 & 0 \\ * & * & * & * & * & * & * & \widehat{\Pi}_3 \end{bmatrix},$$

$$\Omega_3 = \begin{bmatrix} \widehat{\Pi}_4 & 0 & 0 & 0 & 0 \\ * & \widehat{\Pi}_5 & 0 & 0 & 0 \\ * & * & \widehat{\Pi}_6 & 0 & 0 \\ * & * & * & \widehat{\Pi}_7 & 0 \\ * & * & * & * & \widehat{\Pi}_8 \end{bmatrix}$$

$$\widetilde{\Pi}''_{11} = \begin{bmatrix} \Lambda_1 + \rho_k^i \mu \left(G_i^T E^T + E G_i \right) & \Lambda_2 \\ * & \Lambda_3 + \mu \rho_k^i \left(G_i^T E^T + E G_i \right) \end{bmatrix},$$

$$\widetilde{\Pi}_{12} = \begin{bmatrix} \varepsilon_1 \Lambda_4 + \left(R_1 + R_2 \right) A_{di} & \varepsilon_1 \Lambda_5 \\ \Lambda_6 & \Lambda_7 \end{bmatrix}$$

$$\widetilde{\Pi}'_{11} = \begin{bmatrix} \Lambda_1 + \mu \left(G_i^T E^T + E G_i \right) & \Lambda_2 \\ * & \Lambda_3 + \mu \left(G_i^T E^T + E G_i \right) \end{bmatrix},$$

$$\widetilde{\Pi}_{13} = \begin{bmatrix} \Lambda_8 & \varepsilon_2 \Lambda_5 \\ \Lambda_{10} & \Lambda_{11} \end{bmatrix},$$

$$Pi''_{14} = \Pi'_{14} W_i$$

$$\Pi'_{14} = \begin{bmatrix} 2\overline{\Lambda}_1{}^T + \overline{\Lambda}_2{}^T - X_i^T - \left(F_i C_{yi} \right)^T & 2\overline{\Lambda}_1{}^T + \overline{\Lambda}_2{}^T - X_i^T - \left(F_i C_{yi} \right)^T \\ X_i^T - \overline{\Lambda}_2{}^T & X_i^T - \overline{\Lambda}_2{}^T \end{bmatrix},$$

$$\Pi_{57} = \begin{bmatrix} D_i^T - D_{yi}^T D_{fi}^T \\ 0 \end{bmatrix}$$

$$\Pi_{15} = \begin{bmatrix} \Lambda_{12} \\ -Y_i^T \end{bmatrix},$$

$$\Pi_{16} = \begin{bmatrix} \sqrt{(1 + d_{12})} G_i^T & 0 \\ 0 & \sqrt{(1 + d_{12})} G_i^T \end{bmatrix},$$

$$\widetilde{Q}_2 = \begin{bmatrix} Q_{21} & Q_{22} \\ * & Q_{23} \end{bmatrix}$$

$$\Pi_{17} = \Pi_{18} = \begin{bmatrix} G_i^T & 0 \\ 0 & G_i^T \end{bmatrix},$$

$$\Pi_{19} = \begin{bmatrix} \mu\rho G_i^T E^T & \mu I \\ 0 & 0 \end{bmatrix},$$

$$\widehat{\Pi}_7 = \widehat{\Pi}_8 = \begin{bmatrix} -\mu\rho I & 0 \\ 0 & -\mu\left(\rho I + P_i\right) \end{bmatrix}$$

$$\widetilde{\Pi}''_{22} = \begin{bmatrix} \Lambda_{13} - \rho_k^i Q_{11} & \Lambda_{14} - \rho_k^i Q_{12} \\ * & -\rho_k^i Q_{13} \end{bmatrix},$$

$$\widetilde{\Pi}'_{22} = \begin{bmatrix} \Lambda_{13} - Q_{11} & \Lambda_{14} - Q_{12} \\ * & -Q_{13} \end{bmatrix},$$

$$\Pi''_{24} = \Pi'_{24} W_i$$

$$\widetilde{\Pi}_{23} = \begin{bmatrix} \Lambda_{15} & \varepsilon_2 A_{di}^T \left(R_3^T + R_4^T\right) \\ \varepsilon_1 \left(R_3 B_{wi} + R_4 B_{wi} - R_4 N_i\right) & 0 \end{bmatrix},$$

$$\Pi'_{24} = \begin{bmatrix} A_{di}^T & A_{di}^T \\ 0 & 0 \end{bmatrix}$$

$$\Pi'_{56} = \begin{bmatrix} B_{wi}^T & B_{wi}^T - N_i^T \\ 0 & 0 \end{bmatrix},$$

$$\Pi''_{56} = \Pi'_{56} W_i,$$

$$\Pi_{25} = \begin{bmatrix} C_{di}^T \\ 0 \end{bmatrix},$$

$$\Pi_{20} = \begin{bmatrix} 0 & 0 \\ \mu\rho G_i^T E^T & \mu I \end{bmatrix}$$

$$\widetilde{Q}_3 = \begin{bmatrix} Q_{31} & Q_{32} \\ * & Q_{33} \end{bmatrix},$$

$$\widetilde{Q}_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{13} \end{bmatrix},$$

$$\widetilde{\Pi}''_{55} = \begin{bmatrix} \Lambda_{16} - \rho_k^i \gamma^2 \mu^{-N} I & \varepsilon_2 B_{wi}^T \left(R_3^T + R_4^T\right) - \varepsilon_2 N_i^T R_4^T \\ * & -\rho_k^i \gamma^2 \mu^{-N} I \end{bmatrix}$$

$$\widehat{\Pi}'_1 = \begin{bmatrix} -2I + P_j & 0 \\ 0 & -2I + P_j \end{bmatrix},$$

$$\widehat{\Pi}_2 = \begin{bmatrix} -S_k^i & 0 \\ 0 & -S_k^i \end{bmatrix},$$

$$\widehat{\Pi}_3 = \begin{bmatrix} -\rho_k^i I \\ 0 \end{bmatrix},$$

$$\widehat{\Pi}_4 = \begin{bmatrix} -2I + Q_{11} & Q_{12} \\ * & -2I + Q_{13} \end{bmatrix}$$

$$\widehat{\Pi}_5 = \begin{bmatrix} -2I + Q_{21} & Q_{22} \\ * & -2I + Q_{23} \end{bmatrix},$$

$$
\begin{aligned}
\widehat{\Pi}_6 &= \begin{bmatrix} -2I + Q_{31} & Q_{32} \\ * & -2I + Q_{33} \end{bmatrix}, \\
\widehat{\Pi}_1'' &= \begin{bmatrix} -2I + P & 0 \\ 0 & -2I + P \end{bmatrix} \\
\Lambda_1 &= \operatorname{sym}\left(R_1\overline{\Lambda}_1 + R_1\overline{\Lambda}_2 - R_2F_iC_{yi} + R_2\overline{\Lambda}_1 - R_2X_i\right), \\
\overline{\Lambda}_1 &= A_iG_i \\
\Lambda_2 &= \left(\overline{\Lambda}_1^T + \overline{\Lambda}_2^T\right)R_3^T - \left(R_4F_iC_{yi}\right)^T + R_2X_i - R_1\overline{\Lambda}_2 + \overline{\Lambda}_1^TR_4^T - X_i^TR_4^T \\
\overline{\Lambda}_2 &= B_i\left(D_lU_i + D_l^-V_i\right), \\
\Lambda_3 &= X_i^TR_4^T + R_4X_i - R_3\overline{\Lambda}_2 - \left(R_3\overline{\Lambda}_2\right)^T \\
\Lambda_4 &= \left(\overline{\Lambda}_1^T + \overline{\Lambda}_2^T\right)R_1^T - \left(F_iC_{yi}\right)^TR_2^T + \overline{\Lambda}_1^TR_2^T - X_i^TR_2^T \\
\Lambda_5 &= \left(\overline{\Lambda}_1^T + \overline{\Lambda}_2^T\right)R_3^T - \left(F_iC_{yi}\right)^TR_4^T + \overline{\Lambda}_1^TR_4^T - X_i^TR_4^T \\
\Lambda_6 &= \varepsilon_1X_i^TR_2^T + \left(R_3 + R_4\right)A_{di} - \left(R_1\overline{\Lambda}_2\right)^T \\
\Lambda_7 &= \varepsilon_1X_i^TR_4^T - \left(R_3\overline{\Lambda}_2\right)^T, \\
\Lambda_8 &= \varepsilon_2\Lambda_4 + \left(R_1 + R_2\right)B_{wi} - R_2N_i \\
\Lambda_{10} &= \varepsilon_2X_i^TR_2^T + \left(R_3 + R_4\right)B_{wi} - \left(R_1\overline{\Lambda}_2\right)^T - R_4N_i \\
\Lambda_{11} &= \varepsilon_2X_i^TR_4^T - \left(R_3\overline{\Lambda}_2\right)^T, \\
\Lambda_{12} &= \left(C_iG_i\right)^T - \left(M_iC_{yi}\right)^T - Y_i^T \\
\Lambda_{13} &= \varepsilon_1\operatorname{sym}\left(\left(R_1 + R_2\right)A_{di}\right), \\
\Lambda_{16} &= \operatorname{sym}\left(\varepsilon_2R_1B_{wi} + \varepsilon_2R_2B_{wi} - \varepsilon_2R_2N_2\right) \\
\Lambda_{14} &= \varepsilon_1A_{di}^T\left(R_3^T + R_4^T\right), \\
\Lambda_{15} &= \varepsilon_2A_{di}^T\left(R_1^T + R_2^T\right) + \varepsilon_1\left(R_1 + R_2\right)B_{wi} - \varepsilon_1R_iN_i
\end{aligned}
\tag{74}
$$

In this case, the desired filtering parameters and filtering controller gain can be chosen by:

$$
\begin{aligned}
K_i &= U_iG_i^{-1}, \\
H_i &= V_iG_i^{-1}, \\
A_{fi} &= X_iG_i^{-1}, \\
B_{fi} &= F_iZ_i^{-1}, \\
C_{fi} &= Y_iG_i^{-1}, \\
D_{fi} &= M_iT_i^{-1}
\end{aligned}
\tag{75}
$$

Proof. We firstly verify that conditions (68) and (70) are equivalent to condition (56). Letting

$$
\begin{aligned}
\overline{P}_i &= \begin{bmatrix} P_i & 0 \\ 0 & P_i \end{bmatrix}, \\
R_i &= \begin{bmatrix} \overline{R}_i & 0 \\ 0 & \overline{R}_i \end{bmatrix}, \\
\overline{G}_i &= \begin{bmatrix} G_i & 0 \\ 0 & G_i \end{bmatrix}, \\
R &= \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}, \\
Q_i &= \begin{bmatrix} Q_{i1} & Q_{i2} \\ * & Q_{i3} \end{bmatrix}
\end{aligned}
\tag{76}
$$

and R satisfies $R\overline{E} = 0$.

From Lemma 8, we know that:

$$\begin{aligned} -\mu G_i^T E^T P_i E G_i &\le \mu G_i^T E^T + \mu E G_i + \mu\left(\rho I + P_i\right)^{-1} \\ &\quad + \mu\rho G_i^T E^T E G_i \end{aligned} \tag{77}$$

$$\begin{aligned} -\mu\rho_k^i G_i^T E^T P_i E G_i &\le \mu\rho_k^i G_i^T E^T + \mu\rho_k^i E G_i \\ &\quad + \mu\rho_k^i\left(\rho I + P_i\right)^{-1} \\ &\quad + \mu\rho\rho_k^i G_i^T E^T E G_i \end{aligned} \tag{78}$$

noting that the fact,

$$\begin{aligned} -P_i^{-1} &\le -2I + P_i, \\ Q_i^{-1} &\le -2I + Q_i, \\ & i = 1, 2, 3 \end{aligned} \tag{79}$$

let

$$\begin{aligned} U_i &= K_i G_i, \\ V_i &= H_i G_i, \\ X_i &= A_{fi} G_i, \\ Y_i &= C_{fi} G_i, \\ B_{fi} C_{yi} G_i &= F_i C_{yi}, \\ F_i &= B_{fi} Z_i, \\ D_{fi} C_{yi} G_i &= M_i C_{yi}, \\ M_i &= D_{fi} Z_i, \\ N_i &= B_{fi} D_{yi} \end{aligned} \tag{80}$$

Applying (77), (79), (80) and by Schur complement, it is easily obtained that Eq.(68) and (70) is a sufficient condition for Eq.(56).

On the other hand, we denote $\widetilde{P}_i = \overline{R}^{-1/2} P_i \overline{R}^{-1/2}$, $\widetilde{Q}_i = \overline{R}^{-1/2} Q_i \overline{R}^{-1/2}$, $i = 1, 2, 3$. So condition (58) is equivalent to

$$\begin{aligned} &\theta^2 \left[\sup_{i\in w} \left\{\lambda_{\max}\left(\widetilde{p}_i\right)\right\} \right. \\ &\quad + \left(d_2 + \frac{\left(d_2 + d_1 - 1\right)\left(d_2 - d_1\right)}{2} \right) \\ &\quad \cdot \sup_{i\in w} \left\{\lambda_{\max}\left(\widetilde{Q}_1\right)\right\} + d_1 \\ &\quad \left. \cdot \sup_{i\in w} \left\{\lambda_{\max}\left(\widetilde{Q}_2\right)\right\} d_2 \sup_{i\in w} \left\{\lambda_{\max}\left(\widetilde{Q}_3\right)\right\} \right] + \mu^{-N}\gamma^2 d^2 \\ &\quad \le \delta^2 \mu^{-N} \inf_{i\in w} \left\{\lambda_{\min}\left(\widetilde{P}_i\right)\right\} \end{aligned} \tag{81}$$

Letting $\overline{R}_i < P_i < \delta_e \overline{R}_i, \eta_1 \overline{R}_i < Q_1, Q_2, Q_3 < \eta_2 \overline{R}_i$, it is easy to check that condition (58) can guarantee that condition (81) holds. When $j \in w_k^i$, we consider (78), (79), (80) and use the same method as above. Here, the proof is omitted. □

Remark 14. The condition (68) may be difficult to solve using the LMI toolbox of Matlab. To overcome this, we can replace the condition by the following one that may approximate this constraint:

$$\left[C_{yi} G_i - Z_i C_{yi}\right]^T \left[C_{yi} G_i - Z_i C_{yi}\right] < \alpha I \tag{82}$$

where α is a given sufficiently small positive scalar. Applying Schur complement, the constrained condition (82) is equivalent to the following LMI:

$$\begin{bmatrix} -\alpha I & * \\ C_{yi} G_i - Z_i C_{yi} & -I \end{bmatrix} < 0 \tag{83}$$

Remark 15. Using the same method in [46], condition $\aleph(\overline{E}^T \overline{P}_i \overline{E}, \gamma^2\mu) \subset \hbar(H_i)$ is equivalent to:

$$\begin{bmatrix} -\dfrac{1}{\gamma^2\mu} & \left[\overline{h}_j - \overline{h}_j\right] \overline{G}_i \\ * & -\overline{E}^T \overline{P}_i \overline{E} \end{bmatrix} \le 0 \tag{84}$$

$\overline{h}_j$ is the j-th row of H_i, pre-multiplying and post-multiplying inequality (84) by $\mathrm{diag}\{1, \overline{G}^T\}$ and $\mathrm{diag}\{1, \overline{G}\}$ respectively and then from Lemma 8 it is obtained:

$$\begin{bmatrix} -\dfrac{1}{\gamma^2\mu} & \overline{h}_j G_i & -\overline{h}_j G_i \\ * & E G_i + G_i^T E^T - 2I + P_i & 0 \\ * & * & E G_i + G_i^T E^T - 2I + P_i \end{bmatrix} \le 0 \tag{85}$$

Then the condition $\aleph(\overline{E}^T \overline{P}_i \overline{E}, \gamma^2\mu) \subset \hbar(H_i)$ holds.

Remark 16. According to the above discussion, we can obtain that the feasibility of conditions stated in Theorem 13 can be turned into the following LMI based on feasibility problem with a parameter μ, respectively:

$$\begin{aligned} \min \quad & \left(\delta^2 + \gamma^2\right) \\ & U_i, V_i, X_i, Y_i, F_i, M_i, N_i, \delta_e, P_i, S_i, G_i, \eta_1, \eta_2 \\ s.t. \quad & LMIs\,(69), (70).(71), (72), (73), (83) \ \ and \ \ (85) \end{aligned} \tag{86}$$

Remark 17. For all the ellipsoid invariant set of Theorem 9, we hope to get the biggest estimation of attraction domain. According to (86), the problem can be solved by the following method:

$$\begin{aligned} \min \quad & \kappa \\ s.t. \quad & \lambda_1 I - \overline{E}^T P_i \overline{E} \ge 0, \\ & \lambda_2 I - Q_1 \ge 0, \\ & \lambda_3 I - Q_2 \ge 0, \\ & \lambda_4 I - Q_3 \ge 0 \end{aligned} \tag{87}$$

where $\kappa = \lambda_1 + (d_2 + (d_2 + d_1 - 1)(d_2 - d_1)/2)\lambda_2 + d_1\lambda_3 + d_2\lambda_4$, then the biggest estimation of attraction domain as following:

$$\nu_{\max} = \sqrt{\frac{\gamma^2\mu^{1-N} - \gamma^2\mu^{-N}d^2}{\kappa_{\min}}} \tag{88}$$

4. Numerical Examples

Two numerical examples and a dynamical Leontief model of economic systems are shown to demonstrate the usefulness and flexibility of our theory in this section. The first example adopted to account for conservatism of the conditions shown less than [2]. The second example which is employed to apply the filter we designed is feasible and effective. Finally, as the third example, we use the dynamical Leontief model of economic systems to elucidatory the practical application in our life.

Example 1. In order to show the advantage of the method proposed, consider the systems shown in Example 1 of [2].

Mode 1

$$\begin{aligned} A_1 &= \begin{bmatrix} -2 & 0 \\ 1.5 & 1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ B_{w1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 1 & 1 \\ 0.5 & 1 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \\ C_{y1} &= \begin{bmatrix} 1 & 0 \\ 0.8 & 1 \end{bmatrix}. \end{aligned} \tag{89}$$

Mode 2

$$\begin{aligned} A_1 &= \begin{bmatrix} -2 & 0 \\ 1.5 & 1 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \\ B_{w2} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0.5 & 1 \\ 0.8 & 1 \end{bmatrix}, \\ D_2 &= \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.2 \end{bmatrix}, \\ C_{y2} &= \begin{bmatrix} 1.2 & 0 \\ 0.5 & 1 \end{bmatrix}. \\ R_1 &= R_2 = I_2, \\ \theta &= 1, \\ N &= 5, \\ \alpha &= 10^{-10}, \\ \mu &= 2.5, \\ \rho &= 1.6, \\ \varepsilon_1 &= 0.05, \\ \varepsilon_2 &= 0.05 \end{aligned} \tag{90}$$

In addition, the transition probability matrix is given by

$$P_r = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}. \tag{91}$$

Using the method in [2], the optimal values $\gamma = 54.8648$, while using the method in this paper, the optimal values $\gamma = 44.2536$. We can obtain that the method in this paper is better than it in [2]. The corresponding filter gains aregiven as follows:

$$\begin{aligned} A_{f1} &= \begin{bmatrix} -25.8409 & -1.0119 \\ -1.0856 & -20.1545 \end{bmatrix}, \\ A_{f2} &= \begin{bmatrix} -23.2328 & -2.5570 \\ -2.5630 & -24.0083 \end{bmatrix}, \\ B_{f1} &= \begin{bmatrix} -47.2815 & -1.9114 \\ -9.4202 & -39.5987 \end{bmatrix} \\ B_{f2} &= \begin{bmatrix} -39.5178 & -5.5231 \\ -2.3040 & -45.5163 \end{bmatrix}, \\ C_{f1} &= \begin{bmatrix} -37.3258 & -1.4616 \\ -1.5681 & -29.1121 \end{bmatrix}, \\ C_{f2} &= \begin{bmatrix} -32.5259 & -3.5798 \\ -3.5882 & -33.6117 \end{bmatrix} \\ D_{f1} &= \begin{bmatrix} -30.5939 & -1.2368 \\ -6.0954 & -25.6227 \end{bmatrix}, \\ D_{f2} &= \begin{bmatrix} -26.3452 & -3.6820 \\ -1.5360 & -30.3442 \end{bmatrix} \\ K_1 &= \begin{bmatrix} 1.7505 & 0.5009 \\ -0.7510 & -0.5002 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -0.5719 & -0.4784 \\ 0.1980 & -0.0094 \end{bmatrix} \end{aligned} \tag{92}$$

Then, we consider the systems shown in Example 2 of [2]. Without input feedback, we consider the two-mode MJS with the same matrices and transition matrix given in Example 1. The optimization result of this paper is $\gamma = 115.2842$, which is better than the value of $\gamma = 209.4661$ of [2]. The corresponding filter gains aregiven as follows:

$$\begin{aligned} A_{f1} &= \begin{bmatrix} -0.8287 & 0.1455 \\ 0.1641 & -0.6967 \end{bmatrix}, \\ A_{f2} &= \begin{bmatrix} -0.8435 & -0.1198 \\ -0.1159 & -0.8002 \end{bmatrix} \\ B_{f1} &= \begin{bmatrix} -0.1667 & 0.2685 \\ 0.2379 & -0.4011 \end{bmatrix}, \\ B_{f2} &= \begin{bmatrix} -0.4180 & 0.3611 \\ 0.3554 & -0.3213 \end{bmatrix} \\ C_{f1} &= \begin{bmatrix} -52.4816 & -2.9693 \\ -3.1335 & -47.9769 \end{bmatrix}, \\ C_{f2} &= \begin{bmatrix} -47.4428 & -6.0609 \\ -6.0976 & -47.0605 \end{bmatrix} \end{aligned} \tag{93}$$

Remark 18. The problem of finite-time H_∞ filtering for discrete-time Markovian jump systems were investigated in [2]. However, the time-delay and actuator saturation have not been considered, which have a great impact on the performance and stability of the system. Thus, we consider the singular Markovian jump systems with actuator saturation, time-varying delay and partly unknown transition probabilities. Moreover, the examples above have presented that our methods is feasible and less conservative than [2].

Example 2. Consider a three-mode SMJSs (1) with the following parameters:

Mode 1

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 0.2 \\ 1.1 & 2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \\ A_{d1} &= \begin{bmatrix} 0.02 & -0.09 \\ 0.02 & -0.02 \end{bmatrix}, \\ B_{w1} &= \begin{bmatrix} 0.02 & 0 \\ -0.04 & 0.1 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} -0.12 & -0.2 \\ 1 & 0 \end{bmatrix} \\ C_{d1} &= \begin{bmatrix} 0.04 & 0.5 \\ 0.1 & 0.2 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0.3 & 1 \\ 1 & 0.1 \end{bmatrix}, \\ C_{y1} &= \begin{bmatrix} 1 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \\ D_{y1} &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \end{aligned} \tag{94}$$

Mode 2

$$\begin{aligned} A_2 &= \begin{bmatrix} -0.4 & 1 \\ -2 & 4 \end{bmatrix}, \\ A_{d2} &= \begin{bmatrix} 0.01 & 0 \\ -0.02 & -0.01 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \\ B_{w2} &= \begin{bmatrix} 0.1 & 0.02 \\ 0.1 & 0.3 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} -0.02 & -0.2 \\ 1 & 1 \end{bmatrix} \\ C_{d2} &= \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}, \\ D_2 &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \\ C_{y2} &= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \\ D_{y2} &= \begin{bmatrix} -0.01 & 0 \\ 0 & -0.01 \end{bmatrix} \end{aligned} \tag{95}$$

Mode 3

$$\begin{aligned} A_3 &= \begin{bmatrix} -0.3 & 1.5 \\ -3 & 4 \end{bmatrix}, \\ A_{d3} &= \begin{bmatrix} 0.02 & 0 \\ -0.03 & -0.01 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \\ B_{w3} &= \begin{bmatrix} 0.1 & 0.02 \\ 0.1 & 0.3 \end{bmatrix}, \end{aligned}$$

$$C_3 = \begin{bmatrix} -0.02 & -0.2 \\ 1 & 1 \end{bmatrix}$$

$$C_{d3} = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.2 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$C_{y3} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},$$

$$D_{y3} = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.01 \end{bmatrix} \tag{96}$$

In addition, the singular matrix and the transition matrix is given as follows:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Pr = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & ? & ? \\ ? & 0.3 & ? \end{bmatrix}. \tag{97}$$

Then, we choose $N = 3, d = 0.7, d_1 = 1, d_2 = 3, \varepsilon_1 = 0.05, \varepsilon_2 = 0.05, \rho = 2.65, \theta = 0.00001, \alpha = 10^{-10}$, and

$$R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}, \tag{98}$$

where

$$R_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1.7 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \tag{99}$$

$$R_4 = \begin{bmatrix} 0 & 0 \\ 0 & -0.2 \end{bmatrix},$$

$$\overline{R}_1 = \overline{R}_2 = \overline{R}_3 = I_2.$$

According to Theorem 13, the optimal bound with value of $\delta^2 + \gamma^2$ depend on the parameter μ.

When we choose $3.1 \le \mu \le 8.61$ that can find feasible solution. Figure 1 shows the optimal value with different value of μ. Moreover, while $\mu = 3.2$ we can obtain the optimal bound $\gamma = 8.0483, \delta = 8.0534$ and the controller gains:

$$H_1 = \begin{bmatrix} -2.9765 & -0.1328 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -2.9698 & 0.0705 \end{bmatrix},$$

$$A_{f1} = \begin{bmatrix} 2.2533 & 16.7914 \\ 0.7448 & 5.5915 \end{bmatrix}$$

$$B_{f1} = \begin{bmatrix} 2.1983 & 78.4629 \\ 4.3065 & 153.6901 \end{bmatrix},$$

$$C_{f1} = \begin{bmatrix} 0.3783 & 0.1237 \\ 0.1386 & -0.0826 \end{bmatrix},$$

$$D_{f1} = \begin{bmatrix} 0.0807 & -0.5132 \\ -0.9923 & 2.1256 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} -0.4575 & 1.4874 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.0445 & 0.1727 \end{bmatrix},$$

$$A_{f2} = \begin{bmatrix} -5.3360 & 17.1396 \\ 0.5732 & -1.8683 \end{bmatrix}$$

$$B_{f2} = \begin{bmatrix} -2.9259 & 18.3000 \\ -7.1910 & 44.7616 \end{bmatrix}, \tag{100}$$

$$C_{f2} = \begin{bmatrix} 0.0599 & -0.1984 \\ 0.0842 & 0.0842 \end{bmatrix},$$

$$D_{f2} = \begin{bmatrix} -0.0935 & 0.8072 \\ -1.1050 & 2.6129 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 1.8040 & -4.7583 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 1.6792 & -4.4952 \end{bmatrix},$$

$$A_{f3} = \begin{bmatrix} -4.4112 & 9.3815 \\ 3.0823 & -6.4983 \end{bmatrix}$$

$$B_{f3} = \begin{bmatrix} -0.9899 & 5.0880 \\ -16.5078 & 84.6279 \end{bmatrix},$$

$$C_{f3} = \begin{bmatrix} 0.0982 & -0.0111 \\ 0.0340 & 0.3148 \end{bmatrix},$$

$$D_{f3} = \begin{bmatrix} -0.4951 & 2.6592 \\ -1.4470 & 4.1678 \end{bmatrix}$$

Under the above optimal solution, chose the jumping mode as Figure 2., let the disturbance function $w(k) = 0.5e^{-k}$ and assume the given initial condition $x_1(0) = -0.2, x_2(0) = 0.4, \tilde{x}_1(0) = 0, \tilde{x}_2(0) = 0$, and the initial mode $r_0 = 3$, then, the error response of the resulting filtering error systems as

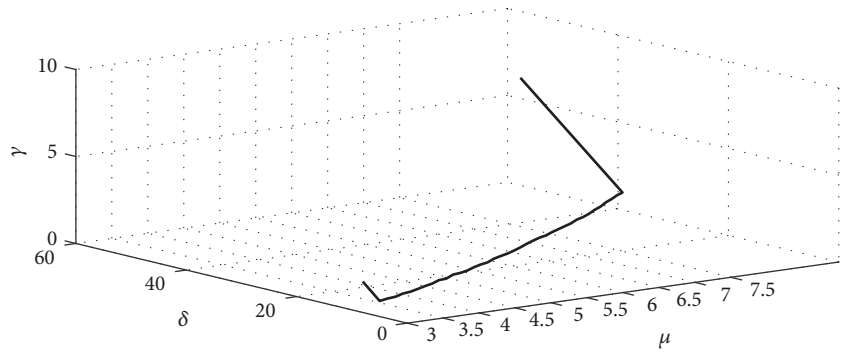

FIGURE 1: The local optimal bound of δ and γ.

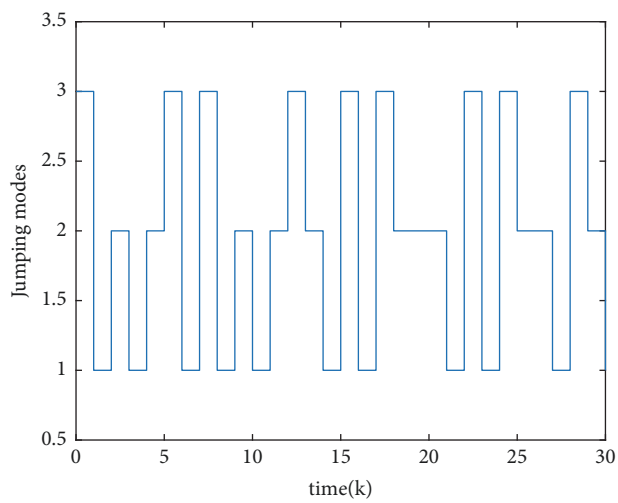

FIGURE 2: The jumping modes.

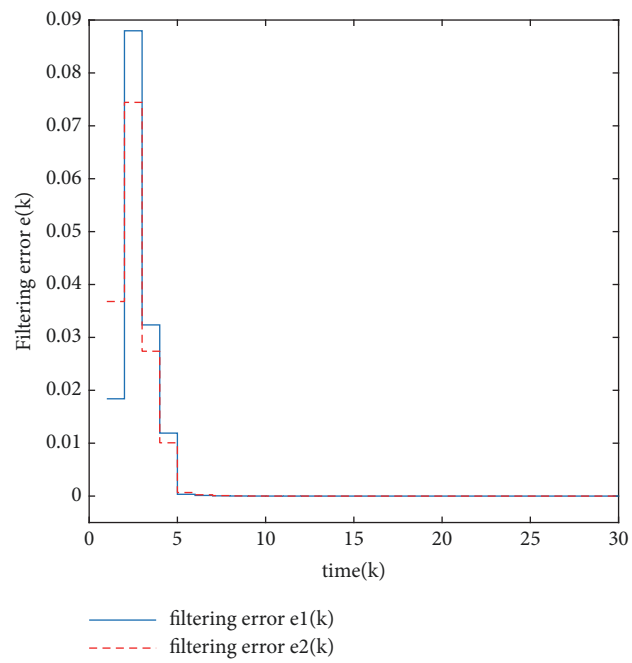

FIGURE 3: The filtering error $e(k)$.

depicted in Figure 3. From the figure, it is obvious that the filtering error system is finite-time bounded.

Remark 19. Using the same matrix variables, when $N = 3, d = 1, d_1 = 1, d_2 = 2, \varepsilon_1 = 0.05, \varepsilon_2 = 0.05, \rho = 1.5, \theta = 0.00001, \alpha = 10^{-10}, \mu = 2.8, \gamma = 7.3, \delta = 7.9$, according to (87), we can obtain the biggest estimation of attraction domain $\nu_{\max} = 1.3331$.

Example 3. Consider the dynamic Leontief model of economic systems, which explains the time pattern of production sectors shown by [49]

$$x(k) = Ax(k) + E[x(k+1) - x(k)] + v(k) \tag{101}$$

where $x(k)$ is the vector of production levels, A is the input-output matrix and $Ax(k)$ is the amount required as direct input for the current produce, E is the capital coefficient matrix, and we know $E[x(k+1) - x(k)]$ is the amount necessary for capacity expansion to be capable to produce $x(k+1)$ in the next period. $v(k)$ is the vector of final demands, according to [50], it is assumed that $v(k) = -B_i(sat(u(k))) - B_{wi}w(k) + H\theta(k)$, then (101) can be rewritten as

$$\begin{aligned} Ex(k+1) = (I - A + E)\,x(k) + B_i\,(sat\,(u\,(k))) \\ + B_{wi}w\,(k) - H\theta\,(k) \end{aligned} \tag{102}$$

It is assumed that the control law with fault model is given by

$$\theta^L(k) = L(\sigma(k))\left[\widehat{K}_1 x(k) + \widehat{K}_2 x(k+1)\right] \tag{103}$$

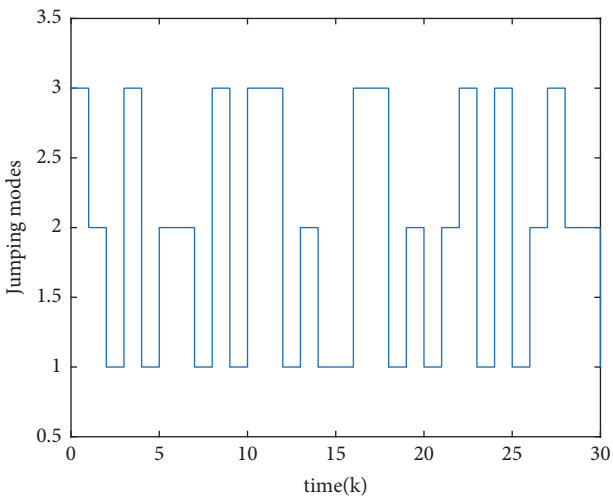

Figure 4: The jumping modes.

where $L(\sigma(k)) = \operatorname{diag}\{f_1\sigma(k), f_2\sigma(k), \ldots, f_p\sigma(k)\}, 0 \leq f_p\sigma(k) \leq 1, q = 1, 2, \ldots, p$ Then, consider a Leontief model described by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2.04 & 1 \\ 0.8 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} -1 \\ 3.05 \end{bmatrix} \tag{104}$$

Let $\widehat{K}_1 = [-0.01 \;\; 0.6]$, $\widehat{K}_2 = [0.1676 \;\; 0.117]$, $L_1 = 0.3$, $L_2 = 0.8$, $L_3 = 1$, then we can obtain the following parameters:

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.0430 & -0.8200 \\ -0.7909 & -0.5490 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -0.0480 & -0.5200 \\ -0.7756 & -1.4640 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -0.0500 & -0.4000 \\ -0.7695 & -1.8300 \end{bmatrix}, \\ A_{d1} &= \begin{bmatrix} 0.0503 & 0.0351 \\ -0.1534 & -0.1071 \end{bmatrix}, \\ A_{d2} &= \begin{bmatrix} 0.1341 & 0.0936 \\ -0.4089 & -0.2855 \end{bmatrix}, \\ A_{d3} &= \begin{bmatrix} -0.1676 & 0.1170 \\ -0.5112 & -0.3569 \end{bmatrix}. \end{aligned} \tag{105}$$

The other parameters are completely similar with Example 2. Then, by using the Matlab LMI Toolbox, the desired filter gains can be given as follows:

$$\begin{aligned} H_1 &= [1.4251 \;\; 5.8779], \\ H_2 &= [-2.1754 \;\; -12.4704], \\ H_3 &= [4.3387 \;\; 20.8669], \\ K_1 &= [1.0016 \;\; 0.4554], \\ K_2 &= [-1.0094 \;\; -5.7993], \\ K_3 &= [7.7551 \;\; 37.3305], \\ A_{f1} &= \begin{bmatrix} -1.3413 & -13.6819 \\ 0.3133 & 3.0341 \end{bmatrix}, \\ A_{f2} &= \begin{bmatrix} -12.8384 & -73.5209 \\ 1.8179 & 10.3895 \end{bmatrix}, \\ A_{f3} &= \begin{bmatrix} 56.1115 & 270.4071 \\ -7.5392 & -36.3306 \end{bmatrix}, \\ B_{f_1} &= \begin{bmatrix} -0.6284 & -30.7700 \\ -2.0133 & -98.6463 \end{bmatrix}, \\ B_{f_2} &= \begin{bmatrix} -6.2666 & -17.1297 \\ -17.4107 & -47.4759 \end{bmatrix}, \\ B_{f_3} &= \begin{bmatrix} 25.3906 & 46.1704 \\ 81.5071 & 148.2659 \end{bmatrix}, \\ C_{f_1} &= \begin{bmatrix} -0.1222 & -0.3708 \\ -0.0322 & -0.0553 \end{bmatrix}, \\ C_{f_2} &= \begin{bmatrix} 0.6806 & 3.8993 \\ 0.7904 & 4.5337 \end{bmatrix}, \\ C_{f_3} &= \begin{bmatrix} -5.1008 & -24.5798 \\ -5.6820 & -27.3749 \end{bmatrix}, \\ D_{f_1} &= \begin{bmatrix} 0.1606 & 4.2467 \\ -1.0143 & 2.0039 \end{bmatrix}, \\ D_{f_2} &= \begin{bmatrix} -1.7920 & -4.1510 \\ -2.7473 & -2.0614 \end{bmatrix}, \\ D_{f_3} &= \begin{bmatrix} 11.4045 & 21.6173 \\ 10.3752 & 23.5226 \end{bmatrix}. \end{aligned} \tag{106}$$

For simulation purposes, we assume that the initial conditions $x(0) = [0.2 \;\; -0.1]^T$, $\tilde{x}(0) = [0 \;\; 0]^T$, the exogenous disturbance signal $w(k)$ is the same as Example 2. Consider the jumping mode as Figure 4, and the trajectories of state are shown in Figure 5, from the picture, it is obvious that the filtering error systems is finite-time bounded.

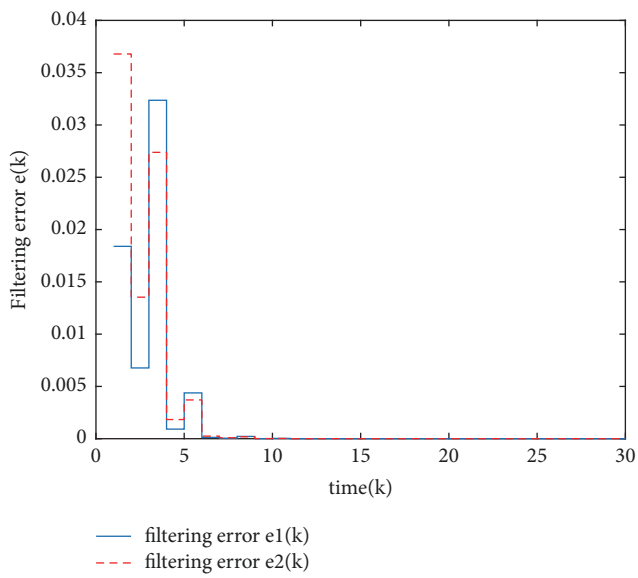

FIGURE 5: The filtering error $e(k)$.

5. Conclusion

The finite-time discrete SMJSsH_∞ filtering problems which are considering time-varying delay, partially unknown transition probabilities and actuator saturation. First, we realized our goal, a H_∞ filter has been designed while the filtering error system is stochastic boundedness in the finite time and satisfying a prescribed H_∞ attenuation level. Following, we enumerates three examples to prove the validity of the theory that we obtained. However, in practical application, the transition probabilities are often time-varying or difficult to available. Based on the above content, lots of attentions would focus on studying Markovian jump systems with transition probabilities unknown or time-varying under actuator saturation in the near future.

Data Availability

The underlying data related to my submission is available.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (No. 61273004), the Natural Science Foundation of Hebei province (No. F2018203099), and the Doctoral Foundation of Yanshan University (No. BL18046).

References

[1] D. S. Bernstein and W. M. Haddad, "Steady-state Kalman filtering with an H∞ error bound," *Systems & Control Letters*, vol. 12, no. 1, pp. 9–16, 1989.

[2] Y. Zhang, C. Liu, and Y. Song, "Finite-time H∞ filtering for discrete-time Markovian jump systems," *Journal of The Franklin Institute*, vol. 350, no. 6, pp. 1579–1595, 2013.

[3] S. Xu, T. Chen, and J. Lam, "Robust H∞ filtering for uncertain Markovian jump systems with mode-dependent time delays," *IEEE Transactions on Automatic Control*, vol. 48, no. 5, pp. 900–907, 2003.

[4] H. Gao, J. Lam, L. Xie, and C. Wang, "Newapproach to mixed H2 and H8 filtering for polytopic discrete-time systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 3183–3191, 2005.

[5] Z. Xiang, C. Qiao, and M. S. Mahmoud, "Robust H∞ filtering for switched stochastic systems under asynchronous switching," *Journal of The Franklin Institute*, vol. 349, no. 3, pp. 1213–1230, 2012.

[6] X. Su, P. Shi, L. Wu, and Y.-D. Song, "A novel approach to filter design for T-S fuzzy discrete-time systems with time-varying delay," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 6, pp. 1114–1129, 2012.

[7] R. Yang, P. Shi, and G. Liu, "Filtering for discrete-time networked nonlinear systems with mixed random delays and packet dropouts," *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2655–2660, 2011.

[8] J. Liu, Z. Gu, and S. Hu, "H∞ filtering for Markovian jump systems with time-varying delays," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 3, pp. 1299–1310, 2011.

[9] W. Chen, J. Xu, and Z. Guan, "Guaranteed cost control for uncertain Markovian jump systems with mode-dependent time-delay," *IEEE Trans. Autom. Control*, vol. 48, no. 12, pp. 2270–2277, 2003.

[10] L. Shen and U. Buscher, "Solving the serial batching problem in job shop manufacturing systems," *European Journal of Operational Research*, vol. 221, no. 1, pp. 14–26, 2012.

[11] M. Athans, "Command and control theory: a challenge to control science," *IEEE Transactions on Automatic Control*, vol. 32, no. 4, pp. 286–293, 1987.

[12] D. Zhang and Q. L. Zhang, "Sliding mode control for TCS fuzzy singular semi-Markovian jump system," *Nonlinear Analysis Hybrid Systems*, vol. 30, pp. 72–91, 2018.

[13] Z. G. Wu, J. H. Park, H. Y. Su, and J. Chu, "Delay-dependent passivity for singular Markov jump systems with time-delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 3, pp. 669–681, 2013.

[14] Y. Ma, X. Jia, and D. Liu, "Robust finite-time H∞ control for discrete-time singular Markovian jump systems with time-varying delay and actuator saturation," *Applied Mathematics and Computation*, vol. 286, pp. 213–227, 2016.

[15] G. Zong and D. Yang, "Robust resilient H8 control for stochastic systems with Markovian jump parameters under partially known transition probabilities," *Optimal Control Applications and Methods*, vol. 35, no. 5, pp. 539–558, 2014.

[16] J. Lin, S. Fei, and J. Shen, "Delay-dependent H∞ filtering for discrete-time singular Markovian jump systems with time-varying delay and partially unknown transition probabilities," *Signal Processing*, vol. 91, no. 2, pp. 277–289, 2011.

[17] G. Zong, D. Yang, L. Hou, and Q. Wang, "Robust finite-time H∞ control for Markovian jump systems with partially known transition probabilities," *Journal of The Franklin Institute*, vol. 350, no. 6, pp. 1562–1578, 2013.

[18] L. X. Zhang and E. Boukas, "Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 2, pp. 463–468, 2009.

[19] Y. Zhang, Y. He, M. Wu, and J. Zhang, "Stabilization for Markovian jump systems with partial information on transition probability based on free-connection weighting matrices," *Automatica*, vol. 47, no. 1, pp. 79–84, 2011.

[20] F. Amato, M. Ariola, and P. Dorato, "Finite-time H∞ control of linear systems subject to parametric uncertainties and disturbances," *Automatica*, vol. 37, no. 9, pp. 1459–1463, 2001.

[21] P. Dorato, "Short time stability in linear time-varying systems," in *Proceedings of IRE International Convention Record*, pp. 83–87, NewYork, NY, USA, 1961.

[22] Y. Ma, L. Fu, Y. Jing, and Q. Zhang, "Finite-time H∞ control for a class of discrete-time switched singular time-delay systems subject to actuator saturation," *Applied Mathematics and Computation*, vol. 261, no. 15, pp. 264–283, 2015.

[23] H. D'Angelo, *Linear Time-Varying Systems: Analysis and Synthesis*, Allyn and Bacon, Newton, 1970.

[24] H. J. Kushner, "Finite time stochastic stability and the analysis of tracking systems," *IEEE Transactions on Automatic Control*, vol. 11, no. 2, pp. 219–227, 1966.

[25] Y. Zhang, C. Liu, and X. Mu, "Robust finite-time H∞ control of singular stochastic systems via static output feedback," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5629–5640, 2012.

[26] Y. Zhang, C. Liu, and X. Mu, "Robust finite-time stabilization of uncertain singular Markovian jump systems," *Applied Mathematical Modelling*, vol. 36, no. 10, pp. 5109–5121, 2012.

[27] Y. Shen and C. Li, "LMI-based finite-time boundedness analysis of neural networks with parametric uncertainties," *Neurocomputing*, vol. 71, no. 4-6, pp. 502–507, 2008.

[28] S. He and F. Liu, "Stochastic finite-time boundedness of Markovian jumping neural network with uncertain transition probabilities," *Applied Mathematical Modelling*, vol. 35, no. 6, pp. 2631–2638, 2011.

[29] F. Amato, R. Ambrosino, M. Ariola, and C. Cosentino, "Finite-time stability of linear time-varying systems with jumps," *Automatica*, vol. 45, no. 5, pp. 1354–1358, 2009.

[30] Y. Ma and H. Chen, "Reliable finite-time H∞ filtering for discrete time-delay systems with Markovian jump and randomly occurring nonlinearities," *Applied Mathematics and Computation*, vol. 268, pp. 897–915, 2015.

[31] G. Garcia, S. Tarbouriech, and J. Bernussou, "Finite-time stabilization of linear time-varying continuous systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 364–369, 2009.

[32] Q. Y. Meng and Y. J. Shen, "Finite-time H∞ control for linear continuous system with norm-bounded disturbance," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 1043–1049, 2009.

[33] Y. Zhang, W. Cheng, and X. Mu, "Observer-based finitr-time H∞ control for singualr Markovian jump systems," *Journal of Applied Mathematics*, pp. 1–19, 2012.

[34] F. Amato, G. Carannante, and G. De Tommasi, "Finite-time stabilization of switching linear systems with uncertain resetting times," in *Proceedings of the 19th Mediterranean Conference on Control and Automation (MED)*, pp. 1361–1366, Corfu, Greece, 2011.

[35] L. Hou, G. Zong, and Y. Wu, "Finite-time control for switched delay systems via dynamic output feedback," *International Journal of Innovative Computing, Information and Control*, vol. 8, no. 7(A), pp. 4901–4913, 2012.

[36] H. Liu, Y. Shen, and X. Zhao, "Delay-dependent observer-based H∞ finite-time control for switched systems with time-varying delay," *Nonlinear Analysis: Hybrid Systems*, vol. 6, no. 3, pp. 885–898, 2012.

[37] F. Amato and M. Ariola, "Finite-time control of discrete-time linear systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 724–729, 2005.

[38] F. Amato, M. Ariola, and C. Cosentino, "Finite-time stabilization via dynamic output feedback," *Automatica*, vol. 42, no. 2, pp. 337–342, 2006.

[39] Y. Zang, C. Liu, and H. Sun, "Robust finite-time H∞ control for uncertain discrete jump systems with time delay," *Applied Mathematics and Computation*, vol. 219, no. 5, pp. 2465–2477, 2012.

[40] Z. Xiang, Y.-N. Sun, and M. S. Mahmoud, "Robust finite-time H∞ control for a class of uncertain switched neutral systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 4, pp. 1766–1778, 2012.

[41] Y. Zhang and C. Liu, "Observer-based finite-time H∞ control of discrete-time Markovian jump systems," *Applied Mathematical Modelling*, vol. 37, no. 6, pp. 3748–3760, 2013.

[42] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[43] E. Fridman and U. Shaked, "Delay-dependent stability and H∞ control: constant and time-varying delays," *International Journal of Control*, vol. 76, no. 1, pp. 48–60, 2003.

[44] B. Chen, X. Liu, K. Liu, P. Shi, and C. Lin, "Direct adaptive fuzzy control for nonlinear systems with time-varying delays," *Information Sciences*, vol. 180, no. 5, pp. 776–792, 2010.

[45] Y. C. Ma, M. H. Chen, and Q. L. Zhang, "Memory dissipative control for singular T-S fuzzy time-varying delay systems under actuator saturation," *Journal of The Franklin Institute*, vol. 352, no. 10, pp. 3947–3970, 2015.

[46] L. Fu and Y. Ma, "Passive control for singular time-delay system with actuator saturation," *Applied Mathematics and Computation*, vol. 289, pp. 181–193, 2016.

[47] X. Luan, F. Liu, and P. Shi, "Finite-time filtering for non-linear stochastic systems with partially known transition jump rates," *IET Control Theory & Applications*, vol. 4, no. 5, pp. 735–745, 2010.

[48] Z. G. Wu, P. Shi, H. Su, and J. Chu, "H2-H∞ filter design for discrete-time singular Markovian jump systems with time-varying delays," *Information Sciences*, vol. 181, no. 24, pp. 5534–5547, 2011.

[49] D. G. Luenberger, "Dynamic equations in descrptor form," *IEEE Transactions on Automatic Control*, vol. 22, no. 3, pp. 312–321, 1977.

[50] Z. Wu, P. Shi, H. Su, and J. Chu, "Delay-dependent stability analysis for discrete-time singular Markovian jump systems with time-varying delay," *International Journal of Systems Science*, vol. 43, no. 11, pp. 2095–2106, 2012.