Finite-Time $H_\infty$ Filtering for Discrete-Time Singular Markovian Jump Systems with Time Delay and Input Saturation

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The paper is discussed with the problem of finite-time $H_\infty$ filtering for discrete-time singular Markovian jump systems (SMJSs). The systems under consideration consist of time-varying delay, actuator saturation and partly unknown transition probabilities. We pay attention to the design of a $H_\infty$ filtering which ensures the filtering error systems to be singular stochastic finite-time boundedness. By employing an adequate stochastic Lyapunov functional together with a class of linear matrix inequalities (LMIs), a sufficient condition is firstly established, which guarantees the systems to achieve our goal and satisfy a prescribed $H_\infty$ attenuation level in the given finite-time interval. Considering the above conditions, a distinct presentation for the requested $H_\infty$ filter is given. Finally, two numerical examples add to a dynamical Leontief model of economic systems are presented to illustrate the validity of the developed theoretical results.

1. Introduction

Over the past decades, signal estimation has received remarkable attention in the field of control, as an elementary problem in signal processing. As is known to all, the traditional Kalman filtering [1] is the most popular ways to deal with the signal estimation. However, the celebrated Kalman filtering scheme is no longer shaped while a priori information on the external noises are not accurately known. Some optimal estimation approaches are introduced to overcome this problem. One of them $H_\infty$ filtering has a wider range of application value. Compared with the traditional kalman filtering, the main advantage of the $H_\infty$ filtering is without assuming the statistics of the noise signal. Therefore, the $H_\infty$ filtering problem has been investigated and widely introduced, see [2–8] as well as the references therein. In practice, to guarantee the error system to be asymptotically stable and satisfy a performance indicator is the main intention of $H_\infty$ filtering design. Moreover, for the purpose of applying to the practical applications, it is also necessary to be bounded for the state of the error system.

As a special class of stochastic hybrid systems, Markovian jump systems have the advantage of better describing physical systems with sudden variations and put into use in manufacturing systems, communication systems [9–11], economic systems and networked control systems. In the past years, a mass of attention has been paid to the study of singular systems, which have been greatly attracted for the reason that singular systems have better than the state-space ones when to describe physical systems. However, many researchers of singular systems, often ignore the influence of Markovian jump, describing the actual systems with some limitations. With in-depth study, the researchers found that combined the singular systems with MJSs to form a new system, which not only can make up for the limitations of description, also can accurately describe more practical applications. This kind of combined singular systems and MJSs systems have been known as singular Markovian jump system (SMJSs) [12–14]. Meanwhile, many of the subsequent researches found that the actual environment mutation and singular system parameters or the internal structure change, all can use SMJSs description, so it is mostly important to study the SMJSs.

On the other side, as a precondition, complete knowledge of the mode transition rates was asked to analyse and synthesis MJSs in lots of the studies. Therefore, it is
necessary and significant to further study more universal jump systems together with partly unknown transition rates, rather than spending a lot of time to measure or estimate all the transitions rates. Many attractive problems have been solved and investigated for these systems, such as stability analysis, $H_{\infty}$ filtering [15–17] and stabilizat[18,19].

It is well known that the Lyapunov stability describes the systems ranging on the dynamic behavior of the trajectories in infinite time space [20], which does not reflect the transient performance of the systems. To copie with this transient performance of control dynamics, short-time stability or finite-time stability were presented by some early researches in [21–24]. Varieties of significant results were provided for discrete-time or continuous-time by applying linear matrix inequality (LMIs) techniques or Lyapunov function approach, these field including singular systems [25, 26], network systems [27, 28], linear and nonlinear systems [29–33], switching systems [34–36], and so forth. In [37–41], the authors studied discrete finite-time stability, finite-time boundedness or finite-time $H_{\infty}$ control. Compared with the above referred, we can find that the definition of finite-time stability have some difference in [42]. When performing the controller, what is noteworthy is that the discrete-time MJJs could be more significant than continuous-time in many practical applications. However, as far as we know, the problem about discrete finite-time $H_{\infty}$ filtering for stochastic systems has less been researched, so that it promotes the main purpose of our study.

To the best of our knowledge, time-delay phenomena are usually the main causes for performance deterioration and instability of systems and happened in many practical systems, such as neural networks, economics, chemical processes and mechanics [43, 44]. On the other hand, if we do not consider the effect of actuator saturation when designing the controller, and it can lead to poor performance and unstable of closed-loop system. Consequently, the study of control systems with actuator saturation is challenging not only theory and research value, but also in practice has important research significance and broad application prospects. Recently, Ma and Chen [45] presented singular T-S fuzzy time-varying delay systems with actuator saturation for the problem of memory dissipative control. In [46], the author talked about passive control problem for singular time-delay systems with nonlinearity and actuator saturation. In the paper, we investigates the finite-time $H_{\infty}$ filter design for discrete-time SMJSs with time-varying delay, partly unknown transition probabilities and actuator saturation. Our results are a little different from those previous ones which are in [47, 48] on finite-time $H_{\infty}$ filtering have been presented for singular Markovian jump systems. Our primary purpose is designing a $H_{\infty}$ filtering that can guarantee the filtering error discrete-time SMJSs to be stochastic finite-time stability and boundedness with a prescribed $H_{\infty}$ performance index in the supplied finite-time interval. Enough conditions are supplied for the solvability of the problem, which can be worked out by making use of LMIs techniques. What is more, we consider to design filters for the systems with partly or completely known transition probabilities. Contributions from this paper are as follows

(i) A class of more general systems are investigated, where we take time-varying delay, input saturation, outside disturbance, Markov switching phenomena into consideration at the same time.

(ii) The singular filtering is used in this article, which makes the filtering problem more comprehensive and general.

(iii) The Markovian jump system takes into account the fact that the transfer matrix is not completely known.

(iv) The finite-time $H_{\infty}$ filtering is investigated in this paper.

In this paper, the structure of the arrangement is as follows. In Section 2 we present the problem statement as well as preliminaries. Discussed the problem on stochastic finite-time $H_{\infty}$ filtering for discrete-time SMJSs with time-varying delay, actuator saturation and partly known transition probabilities in Section 3. Moreover, the estimation of the largest domain of attraction is solved. Section 4 gives some examples to demonstrate the accuracy of the proposed approaches. Finally, we draw the conclusions and the direction of future research in Section 5.

**Notations.** In this paper, the following notations that will be used: $R^n$, $R^{n \times m}$ denote the sets of n-dimensional space, $n \times m$ real matrices. For any symmetric matrix $P$, $\lambda_{\text{min}}(P)$ and $\lambda_{\text{max}}(P)$ denote the minimum and maximum eigenvalues of matrix $P$, respectively. $\epsilon_{\cdot}$ denotes the expectation operator in the case of some probability measure. We use $T$ stands for matrix transposition or vector and $*$ stands for the transposed elements in the symmetric positions of a matrix.

### 2. Problem Statement and Preliminaries

Consider the following discrete-time singular Markovian jump systems (SMJSs) with time-varying delay and actuator saturation:

\[
E x(k+1) = A(r_k)x(k) + A_d(r_k)x(k-d(k)) \\
+ B(r_k)\text{sat}(u(k)) + B_w(r_k)w(k) \\
z(k) = C(r_k)x(k) + C_d(r_k)x(k-d(k)) \\
+ D(r_k)w(k) \\
y(k) = C_f(r_k)x(k) + D_f(r_k)w(k) \\
x(k) = \phi(k), \quad k = -d_2, \ldots, -1, 0
\]

where $x(k) \in R^n$ is the state variable, $z(k) \in R^q$ is the signal to be estimated, $u(k) \in R^p$ is the control input variable and $\text{sat} : R^p \rightarrow R^p$ is the standard saturation function defined as follows:

\[
\text{sat}(u(k)) = [\text{sat}(u_1(k)), \text{sat}(u_2(k)), \ldots, \text{sat}(u_p(k))]^T
\]

Where without loss of generality, $\text{sat}(u_i(k)) = \text{sign}(u_i(k)) \min\{1, |u_i(k)|\}$. Here the notation of $\text{sat}(\cdot)$ is...
abused to denote the scalar values and the variable valued saturation functions. \( w(k) \in \mathbb{R}^d \) is the disturbance input which belongs to
\[
e \left\{ \sum_{k=0}^{\infty} w^T(k) w(k) \right\} \leq d^2, \quad d \geq 0 \quad (3)
\]
y(k) \in \mathbb{R}^d \) is the measured output variable, \( \psi(k) \) is a given initial condition, \( A(r_k), A_d(r_k), B(r_k), B_d(r_k), C(r_k), C_d(r_k), D(r_k) \) and \( C_r(r_k) \) are real constant matrices with appropriate dimensions. \( d(k) \) denotes the time-varying delay satisfying \( 1 \leq d_1 \leq d(k) \leq d_2 \), where \( d_1 \) and \( d_2 \) are known positive integers. The matrix \( E \in \mathbb{R}^{n \times n} \) may be singular and we assume that \( \text{rank}(E) = r \leq n \). The process \( \{ r_k, k \geq 0 \} \) is a discrete-time homogeneous Markovian chain taking values in a finite set \( w = \{1, 2, \ldots, s\} \) with mode transition probabilities
\[
P_i (r_{k+1} = j/r_k = i) = \pi_{ij} \quad (4)
\]
where \( \pi_{ij} \geq 0, \forall i, j \in w, \) and \( \sum_{j=1}^{s} \pi_{ij} = 1 \) for \( r_k = i, i \in w \).

In this paper, the transition probabilities of the Markovian chain are considered to be partially available, while some elements in matrix \( \pi = \{\pi_{ij} \mid i, j \in w\} \) are unknown. For notation clarity, \( \forall i \in w, \) we denote \( w = u_i^j + u_i^{jk} \) with
\[
w_i^j = \{j: \pi_{ij} \text{ is known}\}
\]
\[
w_i^{jk} = \{j: \pi_{ij} \text{ is unknown}\} \quad (5)
\]
Moreover, if \( u_i^j \neq \phi \), we denote \( u_i^j \) as the total number of elements in \( u_i^j \) and \( u_i^j \) as the \( m \)th element in \( u_i^k \), obviously, \( 1 \leq m \leq u_i^j, 1 \leq u_i^j \leq s \).

In order to estimate \( z(k) \), we are interested in designing a filter of the following structure:
\[
E \tilde{x}(k + 1) = A_f (r_k) \tilde{x}(k) + B_f (r_k) y_k
\]
\[
\bar{z}(k) = C_f (r_k) \tilde{x}(k) + D_f (r_k) y_k
\]
\[
u(k) = K(r_k) \tilde{x}(k)
\]
where \( \tilde{x}(k) \in \mathbb{R}^n \) is the filter state variable and \( \bar{z}(k) \in \mathbb{R}^d \) is the estimated signal, \( A_f (r_k), B_f (r_k), C_f (r_k), D_f (r_k) \) are appropriately dimensioned filter matrices to be determined.

For notational simplicity, in the sequel, a matrix \( N(r_k) \) will be denoted by \( N_j \), for instance, \( A(r_k) \) will be denoted by \( A_j \) by \( B_j (r_k) \) by \( B_j \), and so on.

**Definition 1** (see [14]). For a matrix \( H \in \mathbb{R}^{p \times n} \) denote the \( l \)th row of \( H \) as \( h_l \), and define \( h(H) \) as:
\[
h(H) = \{ x(k) \in \mathbb{R}^n : h_l x(k) \leq 1, \quad l_0 = 1, 2, \ldots, p \} \quad (7)
\]

Let \( P \in \mathbb{R}^{n \times n} \) is a positive-symmetric matrix and \( E^T P E \geq 0, \quad \tau > 0 \) be a scalar and denote
\[
\mathcal{N} (E^T P E, \tau) = \{ x(k) \in \mathbb{R}^n : x^T(k) E^T P E x(k) \leq \tau \}\quad (8)
\]

Let \( D \) be the set of \( p \times p \) diagonal matrices whose diagonal elements are either 1 or 0. Suppose each element of \( D \) is labeled as \( D_l \) and denote \( D_l = I - D_l \), clearly, if \( D_l \in D \), then \( D_l \in D \).

We denote \( x(k, \psi) \) for the initial value of the state trajectory in \( (1) \), which the initial value is \( x(k) = \psi(k) \in C_{n, d}, [-d_2, 0] \), and the domain of attraction is followed as:
\[
\Gamma^* = \{ \psi(k), k = -d_2, -d_2 + 1, \ldots, 0 : \lim_{k \to \infty} x(k) = 0 \} \quad (9)
\]

However, it is difficult to calculate accurately the domain of attraction in the actual production, so the estimates of the attraction domain is presented by the following set:
\[
B_x = \{ \phi(k) \in C_{n, d} : \max_{[−d_2, 0]} \| \phi(k) \| \leq \gamma \} \quad (10)
\]

**Lemma 2** (see [46]). Let \( K_j, H_j \in \mathbb{R}^{p \times m} \), then for any \( x(k) \in h(H_j) \),
\[
sat (K_j x(k)) = \sum_{l=1}^{2^p} \alpha_l (k) (D_l K_j + D_l^T H_j) x(k) \quad (11)
\]

where \( \alpha_l \) for \( l = 1, 2, \ldots, 2^p \) are some scalars which satisfy \( 0 \leq \alpha_l \leq 1, \sum_{l=1}^{2^p} \alpha_l = 1 \).

Define \( \overline{K} \equiv [x^T(k) \ x^T(k) \ \bar{z}^T(k)] e(k) = z(k) - \overline{z}(k) \), from Lemma 2, for any \( \overline{K}(k) \in h([H_j - H_j]), \overline{z}(k) \in h(H_j) \), we have
\[
sat (K_j \overline{K}(k)) = sat \left( [K_j - K_j] \overline{K}(k) \right)
\]
\[
= \sum_{l=1}^{2^p} \alpha_l (k) (D_l K_j - K_j + D_l^T [H_j - H_j]) \overline{K}(k) \quad (12)
\]

Then we can obtain the filtering error discrete-time SMJSS as follows:
\[
\overline{E} \overline{x}(k+1) = \sum_{l=1}^{2^p} \alpha_l (k) \overline{A}_{x l} \overline{x}(k) + \overline{A}_{dl} \overline{x}(k - d(k)) + \overline{P} \omega(k)
\]
\[
n \overline{x}(k) = \phi(k) \quad , \quad k = -d_2, -d_2 + 1, \ldots, -1, 0\]
where

\[
\overline{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \\
\overline{A}_g = \begin{bmatrix} A_{i1} & -B_{i1} \\ A_{i} - A_{fi} - B_{fi}C_{yi} & A_{fi} \end{bmatrix}, \\
\overline{A}_{di} = \begin{bmatrix} A_{di} & 0 \\ A_{di} & 0 \end{bmatrix}, \\
\overline{B}_wi = \begin{bmatrix} B_{wi} & 0 \\ B_{wi} - B_{fi}D_{yi} & 0 \end{bmatrix}, \\
\overline{C}_i = [C_i - D_{fi}C_{yi} - C_{fi} C_{fi}], \\
\overline{C}_{di} = [C_{di} 0], \\
\overline{D}_i = [D_i - D_{fi}D_{yi}], \\
A_g = A_i + B_i (D_iK_i + D_i^rH_i), \\
B_g = B_i (D_iK_i + D_i^rH_i).
\]

**Definition 3** (see [16]). System \(Ex(k+1) = A(r_k)x(k)\) (or pair \(E,A(r_k)\)) is said to be:

1. regular if, \(\det(zE - A(r_k))\) is not identically zero for any \(r_k = i, i \in w\);
2. causal if, degree \((\det(zE - A(r_k))\) = rank \((E)\) for any \(r_k = i, i \in w\).

**Definition 4** (see [30], singular stochastic finite-time boundedness (SSFTB)). The discrete-time singular jumping system (20) is said to be SSFTB with respect to \((\theta, \delta, R_i, N, d)\) with \(0 < \theta < \delta, R_i > 0\) if

\[
e \left\{ x^T(0) E^T R_i E x(0) \right\} \leq \theta^2 \implies e \left\{ x^T(k) E^T R_i E x(k) \right\} < \delta^2, \quad \forall i \in w, \; k \in \{1, 2, \ldots, N\}
\]

**Remark 5.** In many papers, the authors often use the definition of singular stochastic finite-time stability. However, it should be pointed out that SSFTB is more close to practical application and used widely, and SSFTS can be regarded as a particular case of SSFTB by letting \(u(t) = 0\).

**Definition 6** (see [30], singular stochastic \(H_{\infty}\) filtering finite-time boundedness (SSH_{\infty}FTB)). The filtering error discrete-time MJS (20) is said to be SSH_{\infty}FTB with respect to \((\theta, \delta, R_i, N, d, \gamma)\), where \(0 < \theta < \delta, R_i > 0, \gamma > 0\) and \(N \in \mathbb{Z}_{k \geq 0}\), and under the zero-initial condition the filtering error \(z(k)\) satisfies:

\[
e \left\{ \sum_{k=0}^{N} z^T(k) z(k) \right\} < \gamma^2 e \left\{ \sum_{k=0}^{N} w^T(k) w(k) \right\}, \quad \forall k \in \{1, 2, \ldots, N\}
\]

**Lemma 7** (see [45]). Given matrices \(X, Y, Z\) with appropriate dimensions and \(Y\) is symmetric positive definite then the following inequality holds:

\[-X^T Z - Z^T X - Z^T Y Z \leq X^T Y^{-1} X\]

**Lemma 8** (see [22]). Given matrices \(X, Y, Z\) with appropriate dimensions and \(Y\) is symmetric positive definite, then there exists scalar \(\lambda \geq 0\), such that \(\lambda I + Y > 0\) and

\[-X^T Z - Z^T X - Z^T Y Z \leq X^T (\lambda I + Y)^{-1} X + \lambda Z^T Z\]

### 3. Main Results

This section provides stochastic finite-time \(H_{\infty}\), boundedness analysis for the filtering error discrete-time SMJSs (13) under the given filter gains (6). LMI conditions are established to show that the filtering error discrete-time SMJSs (13) is SSFTB and the filtering error \(e(k)\) and disturbance \(w(k)\) satisfies the constraint condition (16).

**Theorem 9.** Let \(K_i\) be the state feedback controller gain matrix, the filtering error discrete-time SMJSs (13) is SSFTB with respect to \((\theta, \delta, R_i, N, d)\), if there exist a scalar \(\mu \geq 1, \varepsilon_1 > 0, \varepsilon_2 > 0\), set of symmetric positive-definite matrices \(P_i > 0, Q_i > 0, i = 1, 2, 3\) and nonsingular matrices \(G_i, S_i\), such that

\[
\Pi_1 = \begin{bmatrix} \Pi_{11} & \Pi_{12} & 0 & 0 & 0 & S_1 \\ \Pi_{12}^T & A_{di}^T & \overline{A}_g & I \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_1 \end{bmatrix} < 0, \quad \forall j \in w_{ik}, \quad l = 1, 2, \ldots, 2^p
\]

\[
\Pi_2 = \begin{bmatrix} \Pi_{11}' & \Pi_{12}' & 0 & 0 & 0 & S_1' \\ \Pi_{12}' & A_{di}' & \overline{A}_g & \rho_k I \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_1 \end{bmatrix} < 0, \quad \forall j \in w_{ik}', \quad l = 1, 2, \ldots, 2^p
\]

\[
\Pi_1' = \begin{bmatrix} \Pi_{11}'' & \Pi_{12}'' & 0 & 0 & 0 & S_1'' \\ \Pi_{12}'' & A_{di}'' & \overline{A}_g & \rho_k I \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_1 \end{bmatrix} < 0, \quad \forall j \in w_{ik}'', \quad l = 1, 2, \ldots, 2^p
\]
\[ \mathcal{N}(\mathbf{E}^T \tilde{P} \mathbf{E}, \gamma^2 \mu) \subset h(\mathbf{H}) \]

\[ \theta^2 \sup_{i \in u} \{ \lambda_{\max} (\tilde{P}_i) \} + \theta^2 \left( d_1 + \frac{(d_2 + d_1 - 1)(d_2 - d_1)}{2} \right) \]

\[ + \sup_{i \in u} \{ \lambda_{\max} (\tilde{Q}_i) \} + \theta^2 \sigma \sup_{i \in u} \{ \lambda_{\max} (\tilde{Q}_i) \} + \theta^2 \sigma \]

\[ + \sup_{i \in w} \{ \lambda_{\max} (\tilde{Q}_i) \} + \sigma^2 \sup_{i \in w} \{ \lambda_{\max} (\tilde{Q}_i) \} \leq \delta^2 \mu^{-N} \]

\[ \inf_{i \in u} \{ \lambda_{\min} (\tilde{P}_i) \} \]

where

\[ \Pi_{11} = \Pi_{11} + \tilde{G}_i \tilde{A}_d R^T + R \tilde{A}_2 \mathbf{G}_i, \]

\[ \Pi_{12} = \Pi_{12} + \tilde{G}_i \tilde{A}_d R^T + R \tilde{A}_d \mathbf{G}_i, \]

\[ \Pi_{11}'' = \Pi_{11}'' + \tilde{G}_i \tilde{A}_2 R^T + R \tilde{A}_2 \mathbf{G}_i, \]

\[ \Pi_{11} = (d_1 + 1) \mathbf{G}_i \tilde{Q}_i \mathbf{G}_i + \mathbf{G}_i^2 \mathbf{G}_i + \mathbf{G}_i^2 \mathbf{Q}_i \mathbf{G}_i - \mu \mathbf{G}_i \mathbf{E} \mathbf{F} \mathbf{E} \mathbf{G}_i, \]

\[ \Pi_{12}' = \epsilon_1 \tilde{A}_d R^T + \epsilon_1 R \tilde{A}_d - Q_i, \]

\[ \Pi_{12}'' = \epsilon_1 \tilde{A}_d R^T + \epsilon_1 R \tilde{A}_d - \rho_k^i \mathbf{Q}_i, \]

\[ \Pi_{15} = \epsilon_2 \tilde{G}_i \tilde{A}_2 R^T + R \tilde{B}_{wi}, \]

\[ \Pi_{25} = \epsilon_2 \tilde{A}_d R^T + \epsilon_2 R \tilde{B}_{wi}, \]

\[ \Pi_{15}' = \epsilon_2 \tilde{G}_i \tilde{A}_d R^T + \epsilon_2 R \tilde{B}_{wi} - \rho_k^i \mathbf{Q}_i, \]

\[ \Pi_{55} = \epsilon_2 \tilde{B}_{wi} R^T + \epsilon_2 R \tilde{B}_{wi}, \]

\[ \Pi_{55}' = \epsilon_2 \tilde{B}_{wi} R^T + \epsilon_2 R \tilde{B}_{wi} - \rho_k^i \mathbf{Q}_i, \]

\[ \mathbf{P}_i = R_{i}^{1/2} \mathbf{P} R_{i}^{-1/2}, \]

\[ \tilde{Q}_i = R_{i}^{1/2} \mathbf{Q}_i R_{i}^{-1/2}, \]

\[ \tilde{Q}_i = R_{i}^{1/2} \mathbf{Q}_i R_{i}^{-1/2}, \]

\[ d_{12} = d_2 - d_1 \]

\[ \rho_k^i = \sum_{j \in U_i} \pi_{ij}, \]

\[ S_k^i = \sum_{j \in U_k} \pi_{ij} S_j, \]

\[ P_k^j = \sum_{i \in U_k} \pi_{ij} \tilde{P}_j, \]

\[ \lambda_{\max} (\cdot), \lambda_{\min} (\cdot) \] respectively stand for the maximum and minimum eigenvalues of matrices, \( R \in \mathbb{R}^{2n \times 2n} \) is any constant matrix satisfying: \( R \mathbf{E} = 0 \), \( \text{rank}(R) = 2(n - r) \). Then system (13) is regular, causal and finite-time bounded inside \( \mathcal{N}(\mathbf{E}^T \tilde{P} \mathbf{E}, \gamma^2 \mu) \) with respect to \((\theta, \delta, R, N, d)\).

Moreover, the estimates of the attraction domain is given as the following form:

\[ \gamma_{\text{max}} = \sqrt{\gamma^2 \mu^{1-N} - \sup_{i \in u} \{ \lambda_{\max} (\tilde{Q}_i) \} d_i^2} \]

where

\[ Y = \sup_{i \in w} \{ \lambda_{\max} (\mathbf{E}^T \tilde{P} \mathbf{E}) \} + d_2 \lambda_{\max} (\tilde{Q}_i) \]

\[ + \frac{(d_2 + d_1 - 1)(d_2 - d_1)}{2} \lambda_{\max} (\tilde{Q}_i) + d_1 \lambda_{\max} (\tilde{Q}_2) + d_2 \lambda_{\max} (\tilde{Q}_3) \]

Proof. Here we consider the transition probability of the unknown situation, and use the same method we can get the transfer probability is known. Now, we prove that the system (13) is regular and causal within \( \mathcal{N}(\mathbf{E}^T \tilde{P} \mathbf{E}, \gamma^2 \mu) \). When \( i \in w, j \in \mathcal{U}_{k}', \) it follows from the condition (24), we can easily see that \( \Pi_{11}' < 0 \), then \( \mathbf{G}_i \) is nonsingular. Let \( T = \text{diag} \{ \mathbf{G}_i, I, I, I, I, I, I, I \} \), then \( T^T \Pi_i T < 0 \), which is equivalent to:

\[ \Pi_i = \begin{bmatrix} \Pi_{11} & \Pi_{12} & 0 & 0 & \Pi_{15} & \mathbf{G}_i & \tilde{A}_2^T & \tilde{A}_2^T \mathbf{L}_i^T \\ \Pi_{21} & \Pi_{22} & 0 & 0 & \Pi_{25} & \tilde{A}_d^T & \tilde{A}_d^T \mathbf{L}_i^T & \epsilon_1 \mathbf{I} \\ \* & \* & \* & \* & \* & \* & \* & \epsilon_2 \mathbf{I} \\ \* & \* & \* & \* & \* & \* & \* & \* \end{bmatrix} < 0, \]

where

\[ \Pi_{11} = (1 + d_{12}) \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 - \mu \mathbf{E} \mathbf{P} \mathbf{E} + \mathbf{A}_2^T \mathbf{P} R \mathbf{L}_i \]

\[ + L_i^T R \mathbf{A}_2 \]

\[ \Pi_{12} = \epsilon_1 \tilde{A}_2^T R^T + L_i^T R \tilde{A}_d, \]

\[ \Pi_{15} = \epsilon_2 \tilde{A}_d R^T + L_i^T R \tilde{B}_{wi}, \]

\[ L_i = \mathbf{G}_i^{-1} \]

then for \( \sum_{l=1}^{2^p} \alpha_l(k) = 1 \) the following is true:

\[ \sum_{l=1}^{2^p} \alpha_l(k) \Pi_i < 0 \]
Applying Schur complement lemma, Eq. (28) is equivalent to

$$\Phi_1 = \begin{bmatrix}
(1 + d_{12}) Q_1 + Q_2 + Q_3 - \mu E^T P_i E & 0 & 0 & 0 \\
* & -Q_1 & 0 & 0 \\
* & * & -Q_2 & 0 \\
* & * & * & -Q_3 \\
\end{bmatrix} + \eta^T (k) P_{i1} \eta(k) + \eta^T (k) R^T \tau(k) + \tau^T (k) R \eta(k)$$

$$+ \tau^T (k) S_j^{-1} \tau(k) < 0$$

where

$$\eta(k) = \sum_{l=1}^{2p} \alpha_l(k) A_{dl} - 2 \sum_{l=1}^{2p} \alpha_l(k) A_{dl} \tau(k) + \sum_{l=1}^{2p} \alpha_l(k) P_{wi}$$

$$\tau(k) = [L_1 \ e_1 I \ 0 \ 0 \ e_2 I]$$

From Lemma 7 and $S_j > 0$, we know that

$$-\eta^T (k) R^T S_j \eta(k) < 0$$

$$+ \eta^T (k) P_{i1} \eta(k) - \eta^T (k) R^T S_j R \eta(k) < 0$$

considering (29) and (31), it is obtained that

$$\Phi_1 = \begin{bmatrix}
(1 + d_{12}) Q_1 + Q_2 + Q_3 - \mu E^T P_i E & 0 & 0 & 0 \\
* & -Q_1 & 0 & 0 \\
* & * & -Q_2 & 0 \\
* & * & * & -Q_3 \\
\end{bmatrix} + \eta^T (k) P_{i1} \eta(k) - \eta^T (k) R^T S_j R \eta(k) < 0$$

using the similar method of the above, when $i \in w, j \in w_k$, we can obtain:

$$\Phi_2 = \begin{bmatrix}
(1 + d_{12}) Q_1 + Q_2 + Q_3 - \mu E^T P_i E \rho_i^T & 0 & 0 & 0 \\
* & -\rho_i^T Q_1 & 0 & 0 \\
* & * & -\rho_i^T Q_2 & 0 \\
* & * & * & -\rho_i^T Q_3 \\
\end{bmatrix} + \eta^T (k) P_{i1} \eta(k) - \eta^T (k) R^T S_j R \eta(k)$$

Then we take the unknown transition probability for an example to prove the system is regular and causal.

From (32), we can obtain that

$$(1 + d_{12}) Q_1 + Q_2 + Q_3 - \mu E^T P_i E$$

$$+ \sum_{l=1}^{2p} \alpha_l(k) A_{dl}^T (P_{j} - R^T S_j R) \sum_{l=1}^{2p} \alpha_l(k) A_{dl} < 0$$

since $Q_i > 0, i = 1, 2, 3$ and $P_i > 0$, then we have

$$\sum_{l=1}^{2p} \alpha_l(k) A_{dl}^T (P_{j} - R^T S_j R) \sum_{l=1}^{2p} \alpha_l(k) A_{dl} - \mu E^T P_i E$$

$$< 0$$

since $\Phi$ is singular and rank($\Phi$) = $r \leq n$, there exist two nonsingular matrices $M, N \in R^{nxn}$, such that

$$M^{\prime} N = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$M \sum_{l=1}^{2p} \alpha_l(k) A_{dl} N = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}$$

$$M^{-1} P_i M^{-1} = \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i3} & P_{i4} \end{bmatrix}$$

$$RM^{-1} = \begin{bmatrix} R_1 & R_2 \end{bmatrix}$$

Noting that $R \Phi = 0$ and rank($R$) = $2(n-r)$, we can obtain that $R_1 = 0$ and rank($R_2$) = $2(n-r), R_2 \in R^{2(n-r) \times 2(n-r)}$, that is

$$RM^{-1} = \begin{bmatrix} 0 & R_2 \end{bmatrix}$$

$$\begin{bmatrix} * & * \\ * & q_i \end{bmatrix} < 0$$
where * represents matrices that are not relevant in the following discussion, and

\[
\begin{align*}
\omega_i &= \mathbf{A}_{i_2}^{T} P_{ j_1} \mathbf{A}_{i_2} + \mathbf{A}_{i_4}^{T} P_{ j_3} \mathbf{A}_{i_2} + \mathbf{A}_{i_4}^{T} P_{ j_4} \mathbf{A}_{i_4} + \mathbf{A}_{i_4}^{T} P_{ j_5} \mathbf{A}_{i_4} \\
& \quad + \mathbf{A}_{i_4}^{T} R_{i_2} \mathbf{R}_{i_2} \mathbf{A}_{i_4} \tag{38}
\end{align*}
\]

from Eq.(37), it is obtained that

\[
\omega_i < 0 \tag{39}
\]

Now, we assume that the matrix $\mathbf{A}_{i_4}$ is singular for one $i \in w$, then there exists a vector $\zeta_i \in \mathbb{R}^{m_j}$ and $\zeta_i \neq 0$ such that $\mathbf{A}_{i_4} \zeta_i = 0$. Pre-multiplying and post-multiplying by $\zeta_i^T$ and $\zeta_i$ result in $\zeta_i^T \mathbf{A}_{i_2}^T P_{ j_1} \mathbf{A}_{i_2} \zeta_i < 0$ which contradicts $P_{ j_1} > 0$. Thus, $\mathbf{A}_{i_4}$ is nonsingular.

On the other hand, for the case of $i \in w, j \in w_{i_3}$, by using the similar method of above, from (20), we also can obtain that $\mathbf{A}_{i_4}$ is nonsingular. Therefore, by Definition 3, system (13) is regular and causal.

At this step we will show that system (13) is SSFTB with respect to $(\theta, \delta, N, R(r_k))$.

Consider the following Lyapunov-Krasovskii functional:

\[
V(\mathbf{x}(k), r_k) = \sum_{i=1}^{3} V_i(\mathbf{x}(k), r_k) \tag{40}
\]

where

\[
V_i(\mathbf{x}(k), r_k) = \begin{cases} 
\mathbf{x}^T(k) \bar{E}^T P_{i_1} \bar{E} \mathbf{x}(k) 
\end{cases} \tag{41}
\]

\[
V_2(\mathbf{x}(k), r_k) = \sum_{m=k-d(k)}^{k-1} \mathbf{x}^T(m) Q_1 \mathbf{x}(m) + \sum_{\theta = -d_1+1}^{-d_2} \sum_{m=\theta m+k \theta}^{k-1} \mathbf{x}^T(m) Q_1 \mathbf{x}(m) \tag{42}
\]

\[
V_3(\mathbf{x}(k), r_k) = \sum_{m=k-d_1}^{k-1} \mathbf{x}^T(m) Q_2 \mathbf{x}(m) + \sum_{m=k-d_2}^{k-1} \mathbf{x}^T(m) Q_3 \mathbf{x}(m) \tag{43}
\]

From $R_{\mathbf{E}} = 0$, the following equation holds for any symmetric matrix $S_j$ with appropriate dimensions and $r_k = i$.

\[
0 = \sum_{j \in w} \pi_{ij} \mathbf{x}^T(k+1) R^T S_j R \mathbf{E} \mathbf{x}(k+1) \\
= -\mathbf{x}^T(k+1) E^T R^T \left( \sum_{j \in w_{i_3}} \pi_{ij} + \sum_{j \in w_{i_3}} \pi_{ij} \right) \tag{44}
\]

\[
\cdot S_j R \mathbf{E} \mathbf{x}(k+1) = -\mathbf{x}^T(k+1) E^T R^T S_j R \mathbf{E} \mathbf{x}(k+1) - \sum_{j \in w_{i_3}} \pi_{ij} \mathbf{x}^T(k+1) E^T R^T S_j R \mathbf{E} \mathbf{x}(k+1) \tag{45}
\]
from (42)-(45), we have
\[ \varepsilon \{ V(k+1) - V(k) \} \leq \xi^T (k) \]
\[
\begin{bmatrix}
-\left( \bar{E} \bar{P} \bar{E} + (1 + d_{12}) Q_1 \\
+ Q_2 + Q_3 \right) \\
\vdots \\
-\rho_3 (Q_3) \\
-\rho_2 (Q_2) \\
-\rho_1 (Q_1) \\
0
\end{bmatrix}
\]
\[ + \eta^T (k) \left( \bar{P}_i - R^T \bar{S}_i R \right) \eta (k) \xi (k) = \xi^T (k) \]
\[ + \eta^T (k) \left( \bar{P}_i - R^T \bar{S}_i R \right) \eta (k) \xi (k) = \xi^T (k) \]

from Eq.(32) and (33), we have
\[ \varepsilon \{ V(k+1) - V(k) \} \]
\[ < (\mu - 1) x^T (k) \bar{E} \bar{P} \bar{E} x (k) + w^T (k) Q_i w (k) \]
\[ \leq (\mu - 1) V(k) + \sup_{i \in \omega} \lambda_{\max} (Q_i) w^T (k) w (k) \]

therefore, we have
\[ \varepsilon \{ V(k+1) \} \leq \mu \varepsilon \{ V(0) \} \]
\[ + \sup_{i \in \omega} \lambda_{\max} (Q_i) \varepsilon \left\{ w^T (k) w (k) \right\} \]

noting that \( \mu \geq 1 \), it follows from Eq.(49) that
\[ \varepsilon \{ V(k) \} \]
\[ \leq \mu^k \varepsilon \{ V(0) \} \]
\[ + \sup_{i \in \omega} \lambda_{\max} (Q_i) \mu^k d^2 \]

Letting \( \bar{P}_i = R_i^{1/2} \bar{P} R_i^{1/2}, \bar{Q}_1 = R_i^{1/2} Q_i R_i^{1/2}, \bar{Q}_2 = R_i^{1/2} \bar{Q}_1 R_i^{1/2}, \bar{Q}_3 = R_i^{1/2} \bar{Q}_1 R_i^{1/2} \) and noting that
\[ \varepsilon \{ x^T (0) \bar{E} \bar{P} \bar{E} x (0) \} \leq \theta^2, \]
we have
\[ \varepsilon \{ V(0) \} \]
\[ = \varepsilon \left\{ x^T (0) R_i^{1/2} \bar{P} R_i^{1/2} x (0) \right\} \]
\[ + \varepsilon \left\{ \sum_{m=-d(0)}^{-1} x^T (m) R_i^{1/2} \bar{Q}_1 R_i^{1/2} x (m) \right\} \]
\[ + \varepsilon \left\{ \sum_{m=-d_i+1}^{-1} \sum_{n=m+1}^{-1} x^T (m) R_i^{1/2} \bar{Q}_2 R_i^{1/2} x (m) \right\} \]
\[ + \varepsilon \left\{ \sum_{m=-d_i}^{-1} x^T (m) R_i^{1/2} \bar{Q}_3 R_i^{1/2} x (m) \right\} \]
\[ \leq \sup_{i \in \omega} \lambda_{\max} (\bar{P}_i) \theta^2 + d_2 \sup_{i \in \omega} \lambda_{\max} (\bar{Q}_1) \theta^2 \]
\[ + \frac{(d_2 + d_1 - 1)(d_2 - d_1)}{2} \sup_{i \in \omega} \lambda_{\max} (\bar{Q}_2) \theta^2 \]
\[ + d_4 \sup_{i \in \omega} \lambda_{\max} (\bar{Q}_3) \theta^2 \]

On the other hand, for all \( i \in \omega \), the following is true:
\[
\varepsilon \{V(k)\} \geq \varepsilon \{x^T(k) F_{\varepsilon} E_{\varepsilon} x(k)\} = \varepsilon \{x^T(k) F_{\varepsilon} P_{\varepsilon}^{1/2} P_{\varepsilon}^{1/2} E_{\varepsilon} x(k)\} \geq \inf_{i\in w} \{\lambda_{\min}(\ddot{P}_i)\} \varepsilon \{x^T(k) F_{\varepsilon} E_{\varepsilon} x(k)\}
\]

(52)

\[
\varepsilon \{x^T(k) F_{\varepsilon} E_{\varepsilon} x(k)\} \leq \frac{\mu^k \theta^2 \sup_{i\in w} \{\lambda_{\max}(\bar{P}_i)\}}{\inf_{i\in w} \{\lambda_{\min}(\ddot{P}_i)\}} + \frac{d_2 \mu^k \theta^2 \sup_{i\in w} \{\lambda_{\max}(\bar{Q}_2)\}}{\inf_{i\in w} \{\lambda_{\min}(\ddot{P}_i)\}} \sup_{i\in w} \{\lambda_{\max}(\bar{Q}_3)\} + \mu^k d^2 \sup_{i\in w} \{\lambda_{\max}(\bar{Q}_4)\}
\]

(53)

Noticing that condition (22), it follows from Eq. (53) that \(\varepsilon \{x^T(k) F_{\varepsilon} E_{\varepsilon} x(k)\} < \delta^2\). This completes the proof of this theorem.

Moreover, according to (50), we can obtain

\[
x^T(k) F_{\varepsilon} P_{\varepsilon} E_{\varepsilon} x(k) \leq V(k)
\]

\[
\leq \mu^k V(0) + \mu^k \sup_{i\in w} \{\lambda_{\max}(\bar{Q}_i)\} d^2 \leq \gamma^2 \mu
\]

(54)

\[
\|\phi(k)\|^2 \leq \frac{\gamma^2 \mu(1-N)}{\sup_{i\in w} \{\lambda_{\max}(\bar{Q}_i)\}} d^2 \leq \mathcal{F}(\phi)
\]

(55)

**Remark 10.** For all \(\phi \in \mathcal{F}(\phi)\), we have \(\mathcal{F}(\phi) \leq \gamma^2 \mu\) and from (54), we also obtain that for all \(\phi \in \mathcal{F}(\phi)\), we have \(x^T(k) F_{\varepsilon} P_{\varepsilon} E_{\varepsilon} x(k) \leq \gamma^2 \mu\). This situation shows that began in the trajectory of \(\mathcal{F}(\phi)\) state of \(x(k)\) would have been in attracting field.

**Remark 11.** Though the problem of \(H_{\infty}\) filtering for discrete-time SMJS subject to time-varying delay was considered in [16]. Compared with it, the finite-time and actuator saturation have been taken into the above Theorem, which is more practical and can guarantee the large values of states could not be accepted in the presence of saturations.

Base on Theorem 9, the filtering error system (13) can be proved that SSFTB, following we will propose a \(H_{\infty}\) finite-time sufficient condition for SMJS.

**Theorem 12.** The filtering error system (13) is \(SSH_{\infty}, FTB\) with respect to \((\theta, \delta, N, \gamma, R(\gamma))\), if there exist scalars \(\mu \geq 1, \epsilon_1 > 0, \epsilon_2 > 0, \theta > 0, \gamma > 0\), symmetric definite matrices \(F_{\varepsilon} > 0, Q_1, Q_2, Q_3 > 0\), and nonsingular matrices \(F_{\varepsilon}, S_{\varepsilon}\), such that

\[
\Pi_1 = \begin{bmatrix}
\Pi_{11}' & \Pi_{12} & 0 & 0 & \Pi_{15} & G_{\varepsilon} E_{\varepsilon}^T & I & C_{\varepsilon}^T & C_{\varepsilon}^T_I
\end{bmatrix}
\]

\[
* \Pi_{22}' & 0 & 0 & \Pi_{25} & A_{\varepsilon}^T & \epsilon_1 I & C_{\varepsilon}^T & C_{\varepsilon}^T_I
\]

\[
* * -Q_2 & 0 & 0 & 0 & 0 & 0
\]

\[
* * * -Q_3 & 0 & 0 & 0 & 0
\]

\[
* * * * \Pi_{55} & B_{\varepsilon}^T & \epsilon_2 I & D_{\varepsilon} & D_{\varepsilon}^T
\]

\[
* * * * -P_{\varepsilon}^{-1} & 0 & 0
\]

\[
* * * * * -S_{\varepsilon} & 0
\]

\[
* * * * * * -I
\]

\[
< 0, \quad \forall j \in w_{uk}^j
\]

(56)

\[
\Pi_2 = \begin{bmatrix}
\Pi_{11}' & \Pi_{12} & 0 & 0 & \Pi_{15} & G_{\varepsilon} A_{\varepsilon}^T W_{\varepsilon} & \rho_k I & \rho_k C_{\varepsilon}^T & C_{\varepsilon}^T_I
\end{bmatrix}
\]

\[
* \Pi_{22}' & 0 & 0 & \Pi_{25} & A_{\varepsilon}^T W_{\varepsilon} & \rho_k^2 \epsilon_1 I & \rho_k C_{\varepsilon}^T & \rho_k C_{\varepsilon}^T_I
\]

\[
* * -\rho_k Q_2 & 0 & 0 & 0 & 0 & 0
\]

\[
* * * -\rho_k Q_3 & 0 & 0 & 0 & 0 & 0
\]

\[
* * * * \Pi_{55} & B_{\varepsilon}^T W_{\varepsilon} & \rho_k^2 \epsilon_2 I & \rho_k D_{\varepsilon}^T & D_{\varepsilon}^T
\]

\[
* * * * * -P_{\varepsilon}^{-1} & 0 & 0
\]

\[
* * * * * * -S_{\varepsilon} & 0
\]

\[
* * * * * * -I
\]

\[
< 0, \quad \forall j \in w_k^j
\]

(57)
\[ \Theta \sup_{i \in w} \{ \lambda_{\text{max}} (\widehat{P}_i) \} + \Theta^2 \left( \frac{(d_2 + d_1 - 1)(d_2 - d_1)}{2} \right) \sup_{i \in w} \{ \lambda_{\text{max}} (\widehat{Q}_i) \} + \Theta^2 d_1 \sup_{i \in w} \{ \lambda_{\text{max}} (\widehat{Q}_2) \} \]

\[ + \Theta^2 d_1 \sup_{i \in w} \{ \lambda_{\text{max}} (\widehat{Q}_3) \} + \mu^{-N} \gamma^2 d^2 \leq \delta^2 \mu^{-N} \inf_{i \in w} \{ \lambda_{\text{min}} (\widehat{P}_i) \} \]

\[ \forall \left( E^T P_i E, \gamma^2 \mu \right) < h (H) \] (58)

where

\[ \Pi_{55}' = -\gamma^2 \mu^{-N} I + \varepsilon_2 R_{wi}^T R^T + \varepsilon_2 R_{wi}^T, \]

\[ \Pi_{55}' = -\rho_2 \gamma \mu^{-N} I + \varepsilon_2 R_{wi}^T R^T + \varepsilon_2 R_{wi}^T, \] (60)

\[ W_i = \left[ \sqrt{\pi_{i1}}, \sqrt{\pi_{i2}},\ldots, \sqrt{\pi_{ip}} \right], \]

\[ \overline{P}^{-1} = \text{diag} \left\{ \overline{P}_1^{-1}, \overline{P}_2^{-1}, \ldots, \overline{P}_s^{-1} \right\} \]

Proof. Using Schur complement lemma, it follows from Eq.(56) that the following inequality holds:

\[
\begin{bmatrix}
\Pi_{11}' + G_i C_i^T C_i G_i & \Pi_{12}' + G_i C_i^T C_i d_i & 0 & 0 & \Pi_{15}' + G_i C_i^T D_i G_i^T A_i^T & I \\
* & \Pi_{22}' + C_{di}^T C_{di} & 0 & 0 & \Pi_{25}' + C_{di}^T D_i & A_i^T e_i I \\
* & * & -Q_2 & 0 & 0 & 0 \\
* & * & * & -Q_3 & 0 & 0 \\
* & * & * & * & \Pi_{55}' + D_i D_i & \overline{P}_{wi}^{-1} & e_i I \\
* & * & * & * & * & -S_j \\
* & * & * & * & * & * & * & * & * & -S_j
\end{bmatrix} < 0, \quad \forall j \in w_{ik} \] (62)

using the same Lyapunov-Krasovskii functional as Theorem 9, it follows that:

\[ \varepsilon \{ V (k+1) \} - \mu V (k) + \varepsilon^T (k) e (k) \]

\[ - \gamma^2 \mu^{-N} w^T (k) w (k) < 0 \] (63)

then we have

\[ \varepsilon \{ V (k+1) \} \leq \mu \varepsilon \{ V (k) \} + \varepsilon \{ e^T (k) e (k) \} \]

\[ - \gamma^2 \mu^{-N} \varepsilon \{ w^T (k) w (k) \} \] (64)

According to Eq.(64), we can deduce

\[ \varepsilon \{ V (k) \} \leq \mu^k \varepsilon \{ V (0) \} + \sum_{j=0}^{k-1} \mu^{k-j-1} \varepsilon \{ e^T (j) e (j) \} \]

\[ + \gamma^2 \mu^{-N} \varepsilon \left\{ \sum_{j=0}^{k-1} \mu^{k-j-1} w^T (j) w (j) \right\} \] (65)

under the zero-value initial condition and noting that \( V(k) \geq 0 \) for all \( k \in \{1, 2, \ldots, N\} \) we have

\[ \sum_{j=0}^{k-1} \mu^{k-j-1} \varepsilon \{ e^T (j) e (j) \} \]

\[ < \gamma^2 \mu^{-N} \varepsilon \left\{ \sum_{j=0}^{k-1} \mu^{k-j-1} w^T (j) w (j) \right\} \] (66)
Notice that $\mu \geq 1$, we have

$$\epsilon \left\{ e^T(k) e(k) \right\} = e \left\{ \sum_{k=0}^{N} e^T(k) e(k) \right\} \leq \sum_{k=0}^{N} \epsilon \left\{ e^T(k) e(k) \right\} \leq \sum_{k=0}^{N} \epsilon \left\{ N^{-k} e^T(k) e(k) \right\}$$

$$< \gamma^2 \mu^{-N} \epsilon \left\{ \sum_{k=0}^{N} w^T(k) w(k) \right\}$$

Therefore, condition (16) holds. In addition, taking into account (57) and using the similar method of the above, (16) can be worked out. This proof is completed. \(\square\)

To solve Theorem 12, letting $P_i = \text{diag} \{ P_{i1}, P_{i2} \}$, $R_i = \text{diag} \{ R_{i1}, R_{i2} \}$, the following theorem provides LMI conditions to ensure stochastic $H_{\infty}$ finite-time boundedness of the filtering errors $SMJS(13)$.

**Theorem 13.** The filtering error discrete-time SMJS (13) with saturating actuators within $N(E, P_i, E, y^T \mu)$ is stochastically $H_{\infty}$ finite-time bounded with respect to $(\delta, \delta, N, \gamma, R_{i1}, d)$, if there exist scalars $\mu \geq 1, \gamma > 0, \gamma_1 > 0, \gamma_2 > 0, \rho > 0, \eta_1 > 0, \eta_2 > 0$, a set of symmetric positive-definite matrices $P_i > 0, Q_i, Q_j, Q_j > 0$, and sets of nonsingular matrices $G_j, S_j, Z_j$, sets of matrices $U_j, V_j, X_j, F_j, Y_j, M_j, T_j$, for all $i \in \omega$, such that

$$C_{\gamma} G_i = Z_i C_{\gamma_j}$$

(68)

$$\begin{bmatrix}
-\frac{1}{y^T \mu} & \tilde{R}_{01} G_i & -\tilde{R}_{02} G_i \\
* & E G_i + G_i^T E^T - 2I + P_i & 0 \\
* & * & E G_i + G_i^T E^T - 2I + P_i
\end{bmatrix} \leq 0, \quad b = 1, 2, \ldots, p$$

(69)

$$\Pi_l = \begin{bmatrix} \Omega_1 & \Omega_2 \\
* & \Omega_3 \end{bmatrix} < 0, \quad \forall j \in \omega, l = 1, 2, \ldots, 2^p$$

(70)

$$\Pi_l = \begin{bmatrix} \Omega_1 & \Omega_2 \\
* & \Omega_3 \end{bmatrix} < 0, \quad \forall j \in \omega, l = 1, 2, \ldots, 2^p$$

(71)

$$\tilde{R}_i < P_i < \tilde{R}_i$$

(72)

$$\eta_1 \tilde{R}_i < \tilde{Q}_i, \eta_2 \tilde{R}_i$$

(73)

$$\delta + \frac{(d_2 + d_1 - 1)(d_2 - d_1) + d_2 + 2d_1}{2} \eta_1 \delta^2 < \eta_2 \delta^2 \eta_1 \delta^2 < \eta_2 \delta^2$$

where

$$\begin{bmatrix}
\Pi_{11} & \Pi_{12} & 0 & 0 & \Pi_{13} & \Pi_{14} & I & \Pi_{15} \\
\Pi_{12}^T & 0 & 0 & \Pi_{13} & \Pi_{14} & e_1 I & \Pi_{25} \\
* & -\tilde{Q}_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\tilde{Q}_3 & 0 & 0 & 0 & 0 \\
* & * & * & \tilde{Q}_{56}^T & \Pi_{56} & e_2 I & \Pi_{57} \\
* & * & * & * & \Pi_{12} & 0 & 0 \\
* & * & * & * & * & \tilde{P}_3 & 0
\end{bmatrix}$$

$$\Omega_1 = \begin{bmatrix}
\Pi_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} & \Pi_{20} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\Omega_2 = \begin{bmatrix}
\Pi_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} & \Pi_{20} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
\[ \Omega_1' = \begin{bmatrix} \tilde{\Pi}_{11}'' & \tilde{\Pi}_{12}'' & 0 & 0 & \tilde{\Pi}_{13}'' & \tilde{\Pi}_{14}'' & \rho_k^i I & \rho_k^i \Pi_{15} \\ * & \tilde{\Pi}_{12}'' & 0 & 0 & \tilde{\Pi}_{23}'' & \tilde{\Pi}_{24}'' & \rho_k^i \epsilon_1 I & \rho_k^i \Pi_{25} \\ * & * & -\rho_k^i \tilde{Q}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{55}'' & \Pi_{56}'' & \rho_k^i \epsilon_2 I & \rho_k^i \Pi_{57} \\ * & * & * & * & \tilde{\Pi}_{11}'' & 0 & 0 \\ * & * & * & * & \tilde{\Pi}_2 & 0 \\ * & * & * & * & \tilde{\Pi}_3 & 0 \end{bmatrix}, \]

\[ \Omega_3 = \begin{bmatrix} \tilde{\Pi}_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \tilde{\Pi}_5 & 0 & 0 & 0 \\ * & * & \tilde{\Pi}_6 & 0 & 0 \\ * & * & * & \tilde{\Pi}_7 & 0 \\ * & * & * & * & \tilde{\Pi}_8 \
\end{bmatrix} \]

\[ \Pi_{11}'' = \begin{bmatrix} \Lambda_1 + \rho_k^i \mu (G_i^T E^T + E G_i) & \Lambda_2 \\ * & \Lambda_3 + \mu \rho_k^i (G_i^T E^T + E G_i) \end{bmatrix}, \]

\[ \Pi_{12} = \begin{bmatrix} \epsilon_1 \Lambda_4 + (R_1 + R_2) A_{di} & \epsilon_1 \Lambda_5 \\ \Lambda_6 & \Lambda_7 \end{bmatrix} \]

\[ \Pi_{11}' = \begin{bmatrix} \Lambda_1 + \mu (G_i^T E^T + E G_i) & \Lambda_2 \\ * & \Lambda_3 + \mu (G_i^T E^T + E G_i) \end{bmatrix}, \]

\[ \Pi_{13} = \begin{bmatrix} \Lambda_8 & \epsilon_2 \Lambda_5 \\ \Lambda_9 & \Lambda_11 \end{bmatrix}, \]

\[ p_{i14}'' = \Pi_{i14}' W_i \]

\[ \Pi_{14}' = \begin{bmatrix} 2 \overline{\Lambda}_1^T + \overline{\Lambda}_2^T - X_i^T - (F_j C_{ji})^T \\ 2 \overline{\Lambda}_1^T + \overline{\Lambda}_2^T - X_i^T - (F_j C_{ji})^T \end{bmatrix}, \]

\[ \Pi_{57} = \begin{bmatrix} D_i^T - D_j^T D_{ji}^T \\ 0 \end{bmatrix}, \]

\[ \Pi_{15} = \begin{bmatrix} \Lambda_{12} \\ -Y_i^T \end{bmatrix}, \]

\[ \Pi_{16} = \begin{bmatrix} \sqrt{(1 + d_{12}) G_i^T} & 0 \\ 0 & \sqrt{(1 + d_{12}) C_i^T} \end{bmatrix}, \]

\[ \tilde{Q}_2 = \begin{bmatrix} Q_{21} & Q_{22} \\ * & Q_{23} \end{bmatrix}, \]

\[ \Pi_{17} = \Pi_{18} = \begin{bmatrix} G_i^T & 0 \\ 0 & G_i^T \end{bmatrix}, \]
\[ \Pi_{19} = \begin{bmatrix} \mu \rho G^T E^T & \mu I \\ 0 & 0 \end{bmatrix}, \]
\[ \hat{\Pi}_7 = \hat{\Pi}_8 = \begin{bmatrix} -\mu \rho I & 0 \\ 0 & -\mu (\rho I + P_i) \end{bmatrix}, \]
\[ \Pi''_{22} = \begin{bmatrix} \Lambda_{13} - \rho_k^i Q_{11} & \Lambda_{14} - \rho_k^i Q_{12} \\ * & -\rho_k^i Q_{13} \end{bmatrix}, \]
\[ \Pi'_{22} = \begin{bmatrix} \Lambda_{13} - Q_{11} & \Lambda_{14} - Q_{12} \\ * & -Q_{13} \end{bmatrix}, \]
\[ \Pi''_{24} = \Pi'_{24} W_i, \]
\[ \Pi_{25} = \begin{bmatrix} C_{di}^T \\ 0 \end{bmatrix}, \]
\[ \Pi_{20} = \begin{bmatrix} 0 & 0 \\ \mu \rho G^T E^T & \mu I \end{bmatrix}, \]
\[ \bar{Q}_3 = \begin{bmatrix} Q_{31} & Q_{32} \\ * & Q_{33} \end{bmatrix}, \]
\[ \bar{Q}_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{13} \end{bmatrix}, \]
\[ \Pi''_{55} = \begin{bmatrix} \Lambda_{16} - \rho_k^i \gamma^2 \mu - N I & \epsilon_2 A^T_{\alpha \alpha} (R_3^T + R_4^T) - \epsilon_1 (R_3 B_{ui} + R_4 B_{ui} - R_4 N_i) \\ * & -\rho_k^i \gamma^2 \mu - N I \end{bmatrix}, \]
\[ \bar{\Pi}_1' = \begin{bmatrix} -2 I + P_j & 0 \\ 0 & -2 I + P_j \end{bmatrix}, \]
\[ \bar{\Pi}_2 = \begin{bmatrix} -S_{ki}^j & 0 \\ 0 & -S_{ki}^j \end{bmatrix}, \]
\[ \bar{\Pi}_3 = \begin{bmatrix} -\rho_k^i I \\ 0 \end{bmatrix}, \]
\[ \bar{\Pi}_4 = \begin{bmatrix} -2 I + Q_{11} & Q_{12} \\ * & -2 I + Q_{13} \end{bmatrix}, \]
\[ \bar{\Pi}_5 = \begin{bmatrix} -2 I + Q_{21} & Q_{22} \\ * & -2 I + Q_{23} \end{bmatrix}. \]
\[ \Omega_6 = \begin{bmatrix} -2I + Q_{31} & Q_{32} \\ * & -2I + Q_{33} \end{bmatrix}, \]
\[ \Omega_1'' = \begin{bmatrix} -2I + P & 0 \\ 0 & -2I + P \end{bmatrix} \]
\[ \Lambda_1 = \text{sym} \left( R_1 \bar{X}_1 + R_4 \bar{X}_2 - R_2 F_1 C_{yi} + R_2 \bar{X}_1 - R_2 X_i \right), \]
\[ \bar{X}_1 = A_i G_i \]
\[ \Lambda_2 = (\bar{X}_1^T + \bar{X}_2^T) R_4^T - (R_4 F_1 C_{yi})^T R_2^T + \bar{X}_1^T R_2^T - X_i^T R_4^T \]
\[ \bar{X}_2 = B_i (D_i U_i + D_i V_i), \]
\[ \Lambda_3 = X_i^T R_4^T + R_4 X_i - R_3 \bar{X}_2 - \left( R_3 \bar{X}_2 \right)^T \]
\[ \Lambda_4 = (\bar{X}_1^T + \bar{X}_2^T) R_4^T - (F_1 C_{yi})^T R_2^T + \bar{X}_1^T R_2^T - X_i^T R_4^T \]
\[ \Lambda_5 = (\bar{X}_1^T + \bar{X}_2^T) R_4^T - (F_1 C_{yi})^T R_4^T + \bar{X}_1^T R_4^T - X_i^T R_4^T \]
\[ \Lambda_6 = \epsilon_i X_i^T R_2 + (R_3 + R_4) A_{di} - \left( R_4 \bar{X}_2 \right)^T \]
\[ \Lambda_7 = \epsilon_i X_i^T R_4^T - \left( R_3 \bar{X}_2 \right)^T, \]
\[ \Lambda_8 = \epsilon_2 \Lambda_4 + (R_1 + R_2) B_{wi} - R_2 N_i \]
\[ \Lambda_{10} = \epsilon_2 X_i^T R_2^T + (R_3 + R_4) B_{wi} - \left( R_4 \bar{X}_2 \right)^T - R_4 N_i \]
\[ \Lambda_{11} = \epsilon_2 X_i^T R_4^T - \left( R_3 \bar{X}_2 \right)^T, \]
\[ \Lambda_{12} = (C_i G_i)^T - \left( M_1 C_{yi} \right)^T - Y_i^T \]
\[ \Lambda_{13} = \epsilon_i \text{sym} \left( (R_1 + R_2) A_{di} \right), \]
\[ \Lambda_{16} = \text{sym} \left( \epsilon_1 R_1 B_{wi} + \epsilon_2 R_2 B_{wi} - \epsilon_2 R_2 N_2 \right) \]
\[ \Lambda_{14} = \epsilon_i A_{di}^T \left( R_1^T + R_2^T \right), \]
\[ \Lambda_{15} = \epsilon_2 A_{di}^T \left( R_1^T + R_2^T \right) + \epsilon_1 (R_1 + R_2) B_{wi} - \epsilon_1 R_1 N_i \]

(74)

In this case, the desired filtering parameters and filtering controller gain can be chosen by:

\[ K_i = U_i G_i^{-1}, \]
\[ H_i = V_i G_i^{-1}, \]
\[ A_{fi} = X_i G_i^{-1}, \]
\[ B_{fi} = F_i Z_i^{-1}, \]
\[ C_{fi} = Y_i G_i^{-1}, \]
\[ D_{fi} = M_1 T_i^{-1} \]

(75)

**Proof.** We firstly verify that conditions (68) and (70) are equivalent to condition (56). Letting

\[ R = [R_1 \ R_2] \]
\[ Q_i = \begin{bmatrix} Q_{ii} & 0 \\ 0 & Q_{ii} \end{bmatrix} \]

and \( R \) satisfies \( RE = 0 \).
From Lemma 8, we know that:

$$-\mu G^T_i E^T P_i E G_i \leq \mu G^T_i E^T + \mu E G_i + \mu (p I + P_i)^{-1}$$

(77)

$$-\mu \beta_i G^T_i E^T P_i E G_i \leq \mu \beta_i G^T_i E^T + \mu \beta_i E G_i + \mu (p I + P_i)^{-1}$$

(78)

noting that the fact,

$$-P_i^{-1} \leq -2 I + P_i,$$

$$Q_i^{-1} \leq -2 I + Q_i,$$

(79)

$$i = 1, 2, 3$$

let

$$U_i = K_i G_i,$$

$$V_i = H_i G_i,$$

$$X_i = A_i G_i,$$

$$Y_i = C_i G_i,$$

$$B_j C_j G_i = F_j C_{ji},$$

(80)

$$F_i = B_j Z_i,$$

$$D_j C_j G_i = M_i C_{ji},$$

$$M_i = D_j Z_i,$$

$$N_i = B_j D_{ji}$$

Applying (77), (79), (80) and by Schur complement, it is easily obtained that Eq.(68) and (70) is a sufficient condition for Eq.(56).

On the other hand, we denote

$$\bar{P}_i = \bar{R}^{-1/2} P_i \bar{R}^{-1/2}, \bar{Q}_i = \bar{R}^{-1/2} Q_i \bar{R}^{-1/2}, i = 1, 2, 3.$$ So condition (58) is equivalent to

$$\theta^2 \left[ \sup_{i \in \omega} \lambda_{\max} (\bar{P}_i) \right]$$

$$+ \left( d_2 + \frac{(d_2 + d_1 - 1)(d_2 - d_1)}{2} \right)$$

$$\cdot \sup_{i \in \omega} \lambda_{\max} (\bar{Q}_i) + d_1$$

(81)

$$\cdot \sup_{i \in \omega} \lambda_{\max} (\bar{Q}_i) \cdot d_2 \sup_{i \in \omega} \lambda_{\max} (\bar{Q}_i) + \mu \gamma^2 \delta^2$$

$$\leq \delta^2 \mu^{-N} \inf_{i \in \omega} \left\{ \lambda_{\min} (\bar{P}_i) \right\}$$

holds. When $j \in \omega_k$, we consider (78), (79), (80) and use the same method as above. Here, the proof is omitted.

**Remark 14.** The condition (68) may be difficult to solve using the LMI toolbox of Matlab. To overcome this, we can replace the condition by the following one that may approximate this constraint:

$$[C_{ji} G_i - Z_{ji} C_{ji}]^T [C_{ji} G_i - Z_{ji} C_{ji}] < \alpha I$$

(82)

where $\alpha$ is a given sufficiently small positive scalar. Applying Schur complement, the constrained condition (82) is equivalent to the following LMI:

$$\begin{bmatrix}
-\alpha I & * \\
C_{ji} G_i - Z_{ji} C_{ji} & -I
\end{bmatrix} < 0$$

(83)

**Remark 15.** Using the same method in [46], condition $N(E^T P^* E, \gamma^2 \mu) < h(H_j)$ is equivalent to:

$$\begin{bmatrix}
\frac{1}{\gamma^2 \mu} [\bar{h}_j - \bar{h}_j] G_i \\
* & -E^T P^* E
\end{bmatrix} \leq 0$$

(84)

$\bar{h}_j$ is the $j$-th row of $H_j$, pre-multiplying and post-multiplying inequality (84) by diag$(1, G^T)$ and diag$(1, G)$ respectively and then from Lemma 8 it is obtained:

$$\begin{bmatrix}
\frac{1}{\gamma^2 \mu} \bar{h}_j G_i & -\bar{h}_j G_i \\
* & EG_i + G_i^T E^T - 2I + P_i
\end{bmatrix}$$

(85)

$$\leq 0$$

Then the condition $N(E^T P^* E, \gamma^2 \mu) < h(H_j)$ holds.

**Remark 16.** According to the above discussion, we can obtain that the feasibility of conditions stated in Theorem 13 can be turned into the following LMI based on feasibility problem with a parameter $\mu$, respectively:

$$\min \quad (\delta^2 + \gamma^2)$$

$$\begin{align*}
U_j, V_j, X_j, Y_j, F_i, M_i, N_i, \delta, P_i, S_i, G_i, \eta_1, \eta_2 \\
s.t. \quad \text{LMIs} (69), (70), (71), (72), (73), (83) \quad \text{and} \quad (85)
\end{align*}$$

(86)

**Remark 17.** For all the ellipsoid invariant set of Theorem 9, we hope to get the biggest estimation of attraction domain. According to (86), the problem can be solved by the following method:

$$\min \quad \kappa$$

$$\begin{align*}
\lambda_1 I - E^T P_i E & \geq 0, \\
\lambda_2 I - Q_i & \geq 0, \\
\lambda_3 I - Q_i & \geq 0, \\
\lambda_4 I - Q_i & \geq 0
\end{align*}$$

(87)
where \( \kappa = \lambda_1 + (d_2 + (d_2 + d_1 - 1)(d_2 - d_1)/2)\lambda_2 + d_1\lambda_3 + d_2\lambda_4 \),
then the biggest estimation of attraction domain as following:

\[
\gamma_{\max} = \frac{\sqrt{\gamma^2 \mu^{1-N} - \gamma^2 \mu^{1-N} d^2}}{\kappa_{\min}}
\]  (88)

4. Numerical Examples

Two numerical examples and a dynamical Leontief model of economic systems are shown to demonstrate the usefulness and flexibility of our theory in this section. The first example adopted to account for conservatism of the conditions shown less than [2]. The second example which is employed to apply the filter we designed is feasible and effective. Finally, as the third example, we use the dynamical Leontief model of economic systems to elucidatory the practical application in our life.

Example 1. In order to show the advantage of the method proposed, consider the systems shown in Example 1 of [2].

Mode 1

\[
A_1 = \begin{bmatrix} -2 & 0 \\ 1.5 & 1 \end{bmatrix},
B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},
B_{w1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
C_1 = \begin{bmatrix} 1 & 1 \\ 0.5 & 1 \end{bmatrix},
D_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix},
C_{y1} = \begin{bmatrix} 1 & 0 \\ 0.8 & 1 \end{bmatrix}.
\]  (89)

Mode 2

\[
A_1 = \begin{bmatrix} -2 & 0 \\ 1.5 & 1 \end{bmatrix},
B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},
B_{w2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},
C_2 = \begin{bmatrix} 0.5 & 1 \\ 0.8 & 1 \end{bmatrix},
D_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.2 \end{bmatrix},
\]

In addition, the transition probability matrix is given by

\[
P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}.
\]  (91)

Using the method in [2], the optimal values \( \gamma = 54.8648 \), while using the method in this paper, the optimal values \( \gamma = 44.2536 \). We can obtain that the method in this paper is better than it in [2]. The corresponding filter gains are given as follows:

\[
A_{f1} = \begin{bmatrix} -25.8409 & -0.10119 \\ -1.0856 & -20.1545 \end{bmatrix},
A_{f2} = \begin{bmatrix} -23.2328 & -2.5570 \\ -2.5630 & -24.0083 \end{bmatrix},
B_{f1} = \begin{bmatrix} -47.2815 & -1.9114 \\ -9.4202 & -39.5987 \end{bmatrix},
B_{f2} = \begin{bmatrix} -39.5178 & -5.5231 \\ -2.3040 & -45.5163 \end{bmatrix},
C_{f1} = \begin{bmatrix} -37.3258 & -1.4616 \\ -1.5681 & -29.1121 \end{bmatrix},
C_{f2} = \begin{bmatrix} -32.5259 & -3.5798 \\ -3.5882 & -33.6117 \end{bmatrix},
D_{f1} = \begin{bmatrix} -30.5939 & -5.0954 \\ -6.0954 & -25.6227 \end{bmatrix},
D_{f2} = \begin{bmatrix} -26.3452 & -3.6820 \\ -1.5360 & -30.3442 \end{bmatrix},
K_1 = \begin{bmatrix} 1.7505 & 0.5009 \\ -0.7510 & -0.5002 \end{bmatrix},
K_2 = \begin{bmatrix} -0.5719 & -0.4784 \\ 0.1980 & -0.0094 \end{bmatrix}.
\]  (92)
Then, we consider the systems shown in Example 2 of [2]. Without input feedback, we consider the two-mode MJJS with the same matrices and transition matrix given in Example 1. The optimization result of this paper is $\gamma = 115.2842$, which is better than the value of $\gamma = 209.4661$ of [2]. The corresponding filter gains are given as follows:

$$A_{f1} = \begin{bmatrix} -0.8287 & 0.1455 \\ 0.1641 & -0.6967 \end{bmatrix},$$
$$A_{f2} = \begin{bmatrix} -0.8435 & -0.1198 \\ -0.1159 & -0.8002 \end{bmatrix},$$
$$B_{f1} = \begin{bmatrix} -0.1667 & 0.2685 \\ 0.2379 & -0.4011 \end{bmatrix},$$
$$B_{f2} = \begin{bmatrix} -0.4180 & 0.3611 \\ 0.3554 & -0.3213 \end{bmatrix},$$
$$C_{f1} = \begin{bmatrix} -52.4816 & -2.9693 \\ -3.1335 & -47.9769 \end{bmatrix},$$
$$C_{f2} = \begin{bmatrix} -47.4428 & -6.0609 \\ -6.0976 & -47.0605 \end{bmatrix}.$$

**Remark 18.** The problem of finite-time $H_{\infty}$ filtering for discrete-time Markovian jump systems were investigated in [2]. However, the time-delay and actuator saturation have not been considered, which have a great impact on the performance and stability of the system. Thus, we consider the singular Markovian jump systems with actuator saturation, time-varying delay and partly unknown transition probabilities. Moreover, the examples above have presented that our methods is feasible and less conservative than [2].

**Example 2.** Consider a three-mode SMJSs (1) with the following parameters:

**Mode 1**

$$A_1 = \begin{bmatrix} 1 & 0.2 \\ 1.1 & 2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} 0.02 & -0.09 \\ 0.02 & -0.02 \end{bmatrix},$$
$$B_{w1} = \begin{bmatrix} 0.02 & 0 \\ -0.04 & 0.1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -0.12 & -0.2 \\ 1 & 0 \end{bmatrix}.$$

**Mode 2**

$$A_2 = \begin{bmatrix} -0.4 & 1 \\ 2 & 4 \end{bmatrix},$$
$$A_{d2} = \begin{bmatrix} 0.01 & 0 \\ -0.02 & -0.01 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix},$$
$$B_{w2} = \begin{bmatrix} 0.1 & 0.02 \\ 0.1 & 0.3 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} -0.02 & -0.2 \\ 1 & 1 \end{bmatrix},$$
$$C_{d2} = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.2 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$D_{y2} = \begin{bmatrix} 1 \end{bmatrix},$$
$$D_{y2} = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.01 \end{bmatrix}.$$

**Mode 3**

$$A_3 = \begin{bmatrix} -0.3 & 1.5 \\ -3 & 4 \end{bmatrix},$$
$$A_{d3} = \begin{bmatrix} 0.02 & 0 \\ -0.03 & -0.01 \end{bmatrix},$$
$$B_3 = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix},$$
$$B_{w3} = \begin{bmatrix} 0.1 & 0.02 \\ 0.1 & 0.3 \end{bmatrix}.$$
\[
C_3 = \begin{bmatrix} -0.02 & -0.2 \\ 1 & 1 \end{bmatrix},
\]
\[
C_{d3} = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.2 \end{bmatrix},
\]
\[
D_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},
\]
\[
C_{y3} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},
\]
\[
D_{y3} = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.01 \end{bmatrix}
\]
(96)

In addition, the singular matrix and the transition matrix is given as follows:

\[
E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\]
\[
Pr = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & ? & ? \\ ? & 0.3 & ? \end{bmatrix}
\]
(97)

Then, we choose \( N = 3, \) \( d = 0.7, \) \( d_1 = 1, \) \( d_2 = 3, \) \( \varepsilon_1 = 0.05, \) \( \varepsilon_2 = 0.05, \) \( \rho = 2.65, \) \( \theta = 0.00001, \) \( \alpha = 10^{-10}, \) and

\[
R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}
\]
(98)

where

\[
R_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]
\[
R_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1.7 \end{bmatrix},
\]
\[
R_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},
\]
\[
R_4 = \begin{bmatrix} 0 & 0 \\ 0 & -0.2 \end{bmatrix},
\]
\[
\overline{R}_1 = \overline{R}_2 = \overline{R}_3 = I_2.
\]
(99)

According to Theorem 13, the optimal bound with value of \( \delta^2 + \gamma^2 \) depend on the parameter \( \mu. \)

When we choose \( 3.1 \leq \mu \leq 8.61 \) that can find feasible solution. Figure 1 shows the optimal value with different value of \( \mu. \) Moreover, while \( \mu = 3.2 \) we can obtain the optimal bound \( \gamma = 8.0483, \) \( \delta = 8.0534 \) and the controller gains:

\[
H_1 = \begin{bmatrix} -2.9765 & -0.1328 \end{bmatrix},
\]
\[
K_1 = \begin{bmatrix} -2.9698 & 0.0705 \end{bmatrix},
\]
\[
A_{f1} = \begin{bmatrix} 2.2533 & 16.7914 \\ 0.7448 & 5.5915 \end{bmatrix},
\]
\[
B_{f1} = \begin{bmatrix} 2.1983 & 78.4629 \\ 4.3065 & 153.6901 \end{bmatrix},
\]
\[
C_{f1} = \begin{bmatrix} 0.3783 & 0.1237 \\ 0.1386 & -0.0826 \end{bmatrix},
\]
\[
D_{f1} = \begin{bmatrix} 0.0807 & -0.5132 \\ -0.9923 & 2.1256 \end{bmatrix},
\]
\[
H_2 = \begin{bmatrix} -0.4575 & 1.4874 \end{bmatrix},
\]
\[
K_2 = \begin{bmatrix} -0.0445 & 0.1727 \end{bmatrix},
\]
\[
A_{f2} = \begin{bmatrix} -5.3360 & 17.1396 \\ 0.5732 & -1.8683 \end{bmatrix},
\]
\[
B_{f2} = \begin{bmatrix} -2.9259 & 18.3000 \\ -7.1910 & 4.7616 \end{bmatrix},
\]
\[
C_{f2} = \begin{bmatrix} 0.0599 & -0.1984 \\ 0.0842 & 0.0842 \end{bmatrix},
\]
\[
D_{f2} = \begin{bmatrix} -0.0935 & 0.8072 \\ -1.1050 & 2.6129 \end{bmatrix},
\]
\[
H_3 = \begin{bmatrix} 1.8040 & -4.7583 \end{bmatrix},
\]
\[
K_3 = \begin{bmatrix} 1.6792 & -4.4952 \end{bmatrix},
\]
\[
A_{f3} = \begin{bmatrix} -4.4112 & 9.3815 \\ 3.0823 & -6.4983 \end{bmatrix},
\]
\[
B_{f3} = \begin{bmatrix} -0.9899 & 5.0880 \\ -16.5078 & 84.6279 \end{bmatrix},
\]
\[
C_{f3} = \begin{bmatrix} 0.0982 & -0.0111 \\ 0.0340 & 0.3148 \end{bmatrix},
\]
\[
D_{f3} = \begin{bmatrix} -0.4951 & 2.6592 \\ -1.4470 & 4.1678 \end{bmatrix}
\]
(100)

Under the above optimal solution, chose the jumping mode as Figure 2., let the disturbance function \( w(k) = 0.5e^{-k} \) and assume the given initial condition \( x_1(0) = -0.2, x_2(0) = 0.4, \) \( \overline{x}_1(0) = 0, \) \( \overline{x}_2(0) = 0, \) and the initial mode \( r_0 = 3, \) then, the error response of the resulting filtering error systems as
depicted in Figure 3. From the figure, it is obvious that the filtering error system is finite-time bounded.

**Remark 19.** Using the same matrix variables, when \( N = 3, d = 1, d_1 = 1, d_2 = 2, e_1 = 0.05, e_2 = 0.05, \rho = 1.5, \theta = 0.00001, \alpha = 10^{-10}, \mu = 2.8, \gamma = 7.3, \delta = 7.9 \), according to (87), we can obtain the biggest estimation of attraction domain \( \gamma_{\text{max}} = 1.3331 \).

**Example 3.** Consider the dynamic Leontief model of economic systems, which explains the time pattern of production sectors shown by [49]

\[
\dot{x}(k) = Ax(k) + E[x(k+1) - x(k)] + v(k) \tag{101}
\]

where \( x(k) \) is the vector of production levels, \( A \) is the input-output matrix and \( Ax(k) \) is the amount required as direct input for the current produce, \( E \) is the capital coefficient matrix, and we know \( E[x(k+1) - x(k)] \) is the amount necessary for capacity expansion to be capable to produce \( x(k+1) \) in the next period. \( v(k) \) is the vector of final demands, according to [50], it is assumed that \( v(k) = -B_i(sat(u(k))) - B_wi w(k) + H \theta(k) \), then (101) can be rewritten as

\[
Ex(k+1) = (I - A + E)x(k) + B_i(sat(u(k))) + B_wi w(k) - H \theta(k) \tag{102}
\]

It is assumed that the control law with fault model is given by

\[
\theta^f(k) = L(\sigma(k))[\bar{K}_1 x(k) + \bar{K}_2 x(k+1)] \tag{103}
\]
where $L(\sigma(k)) = \text{diag}\{f_1\sigma(k), f_2\sigma(k), \ldots, f_p\sigma(k)\}$, $0 \leq f_p\sigma(k) \leq 1$, $q = 1, 2, \ldots, p$. Then, consider a Leontief model described by

\begin{equation}
E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
A = \begin{bmatrix} 2.04 & 1 \\ 0.8 & 1 \end{bmatrix},
H = \begin{bmatrix} -1 \\ 3.05 \end{bmatrix}
\end{equation}

(104)

Let $\tilde{K}_1 = [-0.01 \ 0.6]$, $\tilde{K}_2 = [0.1676 \ 0.117]$, $L_1 = 0.3$, $L_2 = 0.8$, $L_3 = 1$, then we can obtain the following parameters:

\begin{align*}
A_1 &= \begin{bmatrix} -0.0430 & -0.8200 \\ -0.7909 & -0.5490 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -0.0480 & -0.5200 \\ -0.7756 & -1.4640 \end{bmatrix}, \\
A_3 &= \begin{bmatrix} -0.0500 & -0.4000 \\ -0.7695 & -1.8300 \end{bmatrix}, \\
A_{d1} &= \begin{bmatrix} 0.0503 & 0.0351 \\ -0.1534 & -0.1071 \end{bmatrix}, \\
A_{d2} &= \begin{bmatrix} 0.1341 & 0.0936 \\ -0.4089 & -0.2855 \end{bmatrix}, \\
A_{d3} &= \begin{bmatrix} -0.1676 & 0.1170 \\ -0.5112 & -0.3569 \end{bmatrix}.
\end{align*}

(105)

The other parameters are completely similar with Example 2. Then, by using the Matlab LMI Toolbox, the desired filter gains can be given as follows:

\begin{align*}
H_1 &= [1.4251 \ 5.8779], \\
H_2 &= [-2.1754 \ -12.4704], \\
H_3 &= [4.3387 \ 20.8669], \\
K_1 &= [1.0016 \ 0.4554], \\
K_2 &= [-1.0094 \ -5.7993], \\
K_3 &= [7.7551 \ 37.3305], \\
A_{f1} &= \begin{bmatrix} -1.3413 & -13.6819 \\ 0.3133 & 3.0341 \end{bmatrix}, \\
A_{f2} &= \begin{bmatrix} -12.8384 & -73.5209 \\ 1.8179 & 10.3895 \end{bmatrix}, \\
A_{f3} &= \begin{bmatrix} 56.1115 & 270.4071 \\ -7.5392 & -36.3306 \end{bmatrix}, \\
B_{f1} &= \begin{bmatrix} -0.6284 & -30.7700 \\ -2.0133 & -98.6463 \end{bmatrix}, \\
B_{f2} &= \begin{bmatrix} -6.2666 & -17.1297 \\ -17.4107 & -47.4759 \end{bmatrix}, \\
B_{f3} &= \begin{bmatrix} 25.3906 & 46.1704 \\ 81.5071 & 148.2659 \end{bmatrix}, \\
C_{f1} &= \begin{bmatrix} -0.1222 & -0.3708 \\ -0.0322 & -0.0553 \end{bmatrix}, \\
C_{f2} &= \begin{bmatrix} 0.6806 & 3.8993 \\ 0.7904 & 4.5337 \end{bmatrix}, \\
C_{f3} &= \begin{bmatrix} -5.1008 & -24.5798 \\ -5.6820 & -27.3749 \end{bmatrix}, \\
D_{f1} &= \begin{bmatrix} 0.1606 & 4.2467 \\ -1.0143 & 2.0039 \end{bmatrix}, \\
D_{f2} &= \begin{bmatrix} -1.7920 & -4.1510 \\ -2.7473 & -2.0614 \end{bmatrix}, \\
D_{f3} &= \begin{bmatrix} 11.4045 & 21.6173 \\ 10.3752 & 23.5226 \end{bmatrix}.
\end{align*}

(106)

For simulation purposes, we assume that the initial conditions $x(0) = [0.2 \ -0.1]^T$, $\bar{x}(0) = [0 \ 0]^T$, the exogenous disturbance signal $w(k)$ is the same as Example 2. Consider the jumping mode as Figure 4, and the trajectories of state are shown in Figure 5, from the picture, it is obvious that the filtering error systems is finite-time bounded.
5. Conclusion
The finite-time discrete SMJS $H_\infty$ filtering problems which are considering time-varying delay, partially unknown transition probabilities and actuator saturation. First, we realized our goal, a $H_\infty$ filter has been designed while the filtering error system is stochastic boundedness in the finite time and satisfying a prescribed $H_\infty$ attenuation level. Following, we enumerates three examples to prove the validity of the theory that we obtained. However, in practical application, the transition probabilities are often time-varying or difficult to available. Based on the above content, lots of attentions would focus on studying Markovian jump systems with transition probabilities unknown or time-varying under actuator saturation in the near future.

Data Availability
The underlying data related to my submission is available.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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