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Research Article

Dynamical Analysis of a Stochastic Multispecies Turbidostat Model

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A stochastic turbidostat system in which the dilution rate is subject to white noise is investigated in this paper. First of all, sufficient conditions of the competitive exclusion among microorganisms are obtained by employing the techniques of stochastic analysis. Furthermore, the results demonstrate that the competition among microorganisms and stochastic disturbance will affect the dynamical behaviors of microorganisms. Finally, the theoretical results obtained in this contribution are illustrated by numerical simulations.

1. Introduction

The chemostat and turbidostat, two types of devices for continuous cultivation of microorganism, have been utilized to analyze population dynamics. Novick et al. [1] first proposed the mechanism of chemostat in 1950, and then an increasing number of researchers have devoted themselves to investigate chemostat systems [2–7]. However, there exist some drawbacks in chemostat model with a constant dilution rate, such as the waste of substrate and higher viscosity caused by the mass transfer efficiency. Models with the dilution rate related to the state of microorganism, which can be called turbidostat (see [8]), can overcome the above drawbacks and be helpful to design optimal strategies to improve the substrate utilization. Flegr [9] investigated turbidostat system with two species from numerical simulation and De Leenheer et al. [4] analyzed the system theoretically. Li [10] systematically investigated the turbidostat model and established sufficient conditions of coexistence of two species. Subsequently, many researchers investigated turbidostat and chemostat systems. Many valuable and interesting results were obtained [11–16].

According to May [17], the parameters of the system, such as birth rate, death rate, and the input concentration of nutrient, are inevitably disturbed by environmental factors. In recent years, more and more stochastic ecological models were chosen to describe the dynamics of populations. Grasmanet et al. [18] established a stochastic chemostat model with three trophic levels. Using singular perturbation methods, they obtained the expected breakdown time and analyzed the influence of stochastic factors on dynamic behaviors of the system in detail. Considering the stochastic factors in chemostat, Campillo et al. [19] proposed stochastic systems with different population scales, and they investigated these systems and derived the field of validity for different population scales. Collet et al. [5] pointed out that the population size in the device was determined not only by the concentration of the nutrient but also by the stochastic birth and death rates. By analyzing the long time behaviors of microbes, they derived the conditions for the global existence of the solutions of the system and the existence of quasi-stationary distribution. Meng et al. [6] constructed an impulsive stochastic chemostat model and investigated the extinction and permanence of the microorganisms. They also pointed out that small perturbation from white noise could cause the extinction of microorganisms. Xu et al. [7] investigated a chemostat model containing telegraph noise with the help of Markov chain and derived the break-even concentration which determines the persistence or extinction

of microorganisms. One can also find a large number of stochastic models on cultivation of microorganisms [14, 15, 20–23] as well as in other fields [24–28].

Competition is all around environment because of the limitation of natural resource. Ghoul et al. [29] and Kaye et al. [30] pointed out that microbes expressed many competitive behaviors with their neighbours or plants for scarce nutrients and limited spaces. Many practical results were obtained by investigating the corresponding competitive models. Butler et al. [16] established a system containing two competitors and a growth-limiting nutrient. They derived conditions for uniform persistence and local stability of the food web and also obtained the conditions for predator-mediated coexistence. Wolkowicz et al. [31] proposed a competitive chemostat model with distributed delay to describe the process of nutrient consumption. They proved that there existed only one survivor under any conditions and also pointed out that the theoretical results in their paper were valid for all systems with monotone growth response functions. Liu et al. [32] established a stochastic competitive model and obtained sufficient conditions of extinction and persistence (including weak persistence and strong persistence). They stated that only one species survived under certain stochastic noise perturbation. Xu et al. [33] analyzed a competitive chemostat model and derived the critical value of the noise. Zhao et al. [34] discussed on a stochastic competition system and derived sufficient conditions of persistence and stationary distribution for each population. Actually, a great number of stochastic competition models have been studied [35–40].

In this paper, considering the dilution rate of microorganisms related to the feedback control and the influence of stochastic factors from the environment, we establish the following stochastic turbidostat system based on Zhang et al. [21] who constructed a stochastic chemostat model and obtained the conditions of competitive exclusion.

$$\begin{aligned}
\mathrm{d}S(t) &= \left[\left(S^0 - S(t)\right)\left(d + \sum_{i=1}^{n} k_i x_i(t)\right)\right. \\
&\quad \left. - \sum_{i=1}^{n} \frac{a_i S(t)}{1 + b_i S(t)} x_i(t)\right] \mathrm{d}t + \sigma_1 S(t)\,\mathrm{d}B_1(t), \\
\mathrm{d}x_i(t) &= \left[-\left(d_i + \sum_{i=1}^{n} k_i x_i(t)\right) x_i(t)\right. \\
&\quad \left. + \frac{a_i S(t)}{1 + b_i S(t)} x_i(t)\right] \mathrm{d}t + \sigma_{i+1} x_i(t)\,\mathrm{d}B_{i+1}(t), \\
&\qquad i = 1, 2, \ldots, n,
\end{aligned} \tag{1}$$

where $S(t)$ and $x_i(t)$ $(i = 1, 2, \ldots, n)$ represent the concentration of nutrient and microorganisms at the time t, separately. S^0 expresses the input concentration of nutrition, $B_i(t)$ $(i = 1, 2, \ldots, n+1)$ is Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ $(\mathcal{F}_t = \sigma\{(S(t), x_1(t), x_2(t), \ldots, x_n(t));\ 0 \le t \le \tau_e\})$ and $\sigma_i > 0$ $(i = 1, 2, \ldots, n+1)$ stands for the density of the white noise, $d + \sum_{i=1}^{n} k_i x_i$ ($d > 0$ and $k_i > 0$ are constants) denotes the dilution rate of the turbidostat and $d_i + \sum_{i=1}^{n} k_i x_i$ ($d_i > 0$ is constant) is the sum of the dilution rate and death rate of the microorganisms in the system. $a_i S/(1 + b_i S)$ is the Holling 2 functional response function ($a_i > 0$ and $b_i \ge 0$ are constants).

This paper is organized as follows. In Section 2, the existence and uniqueness of the positive solution of system (1) are verified. Section 3 demonstrates the main results. A discussion is given and a numerical example is offered to verify our theoretical results in Section 4.

2. Existence and Uniqueness of the Positive Solution

We in this section demonstrate the existence and uniqueness of global positive solution of system (1).

Theorem 1. *If $d_i/k_i > S^0 + 1$ $(i = 1, 2, \ldots, n)$, then stochastic system (1) has a unique positive solution $(S(t), x_1(t), \ldots, x_n(t))$ almost surely with initial value $(S(0), x_1(0), \ldots, x_n(0)) \in R_+^{n+1}$.*

Proof. For $t \ge 0$, let

$$\begin{aligned}
u(t) &= \ln S(t), \\
v_i(t) &= \ln x_i(t), \quad (i = 1, 2, \ldots, n).
\end{aligned} \tag{2}$$

According to the Itô formula, we have

$$\begin{aligned}
\mathrm{d}u(t) &= \left[\frac{\left(S^0 - \mathrm{e}^{u(t)}\right)\left(d + \sum_{i=1}^{n} k_i \mathrm{e}^{v_i(t)}\right)}{\mathrm{e}^{u(t)}}\right. \\
&\quad \left. - \sum_{i=1}^{n} \frac{a_i \mathrm{e}^{v_i(t)}}{1 + b_i \mathrm{e}^{u(t)}} - \frac{\sigma_1^2}{2}\right] \mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\
\mathrm{d}v_i(t) &= \left[-\left(d_i + \sum_{i=1}^{n} k_i \mathrm{e}^{v_i(t)}\right) + \frac{a_i \mathrm{e}^{u(t)}}{1 + b_i \mathrm{e}^{u(t)}} - \frac{\sigma_{i+1}^2}{2}\right] \mathrm{d}t \\
&\quad + \sigma_{i+1} \mathrm{d}B_{i+1}(t), \\
&\qquad i = 1, 2, \ldots, n,
\end{aligned} \tag{3}$$

with initial value $(u(0), v_1(0), \ldots, v_n(0)) = (\ln S(0), \ln x_1(0), \ldots, \ln x_n(0))$. It is obvious that e^u and e^{v_i} are continuous and differentiable functions. Hence (3) satisfies local Lipschitz condition, which means that there exists a unique local solution $(u(t), v_1(t), \ldots, v_n(t))$ for $t \in [0, \tau_e)$. Then $(S(t), x_1(t), \ldots, x_n(t)) = (\mathrm{e}^{u(t)}, \mathrm{e}^{v_1(t)}, \ldots, \mathrm{e}^{v_n(t)})$ is the unique positive local solution for system (1) with the initial value $(S(0), x_1(0), \ldots, x_n(0)) \in R_+^{n+1}$. Next, we prove that the positive solution of system (1) is global; that is, $\tau_e = \infty$. We first define the filtration $\{\mathcal{F}_t\}$ as $\mathcal{F}_t = \{X_s; 0 \le s \le \tau_e\}$ and choose an enough large $k_0 > 0$ such that $(S(0), x_1(0), \ldots, x_n(0)) \in [1/k_0, k_0]$. Then for all $k \ge k_0$, we define a stopping time by

$$\begin{aligned}
\tau_k = \inf\Big\{&t \in [0, \tau_e) : \min\{S(t), x_1(t), \ldots, x_n(t)\} \\
&\le \frac{1}{k} \text{ or } \max\{S(t), x_1(t), \ldots, x_n(t)\} \ge k\Big\}.
\end{aligned} \tag{4}$$

Next, we further show $\tau_e \longrightarrow \infty$. According to the definition of the stopping time,

$$\begin{aligned} &\min\{S(t), x_1(t), \ldots, x_n(t)\} \le \frac{1}{k} \longrightarrow 0, \\ &\max\{S(t), x_1(t), \ldots, x_n(t)\} \ge k \longrightarrow \infty, \\ &\qquad\qquad\qquad\qquad \text{as } k \longrightarrow \infty. \end{aligned} \tag{5}$$

Therefore, τ_k is increasing as $k \longrightarrow \infty$. Set $\tau_\infty = \lim_{k\longrightarrow\infty} \tau_k$. Then $\tau_\infty \le \tau_e$. Now, it is sufficient for us to show $\tau_\infty = \infty$. Arguing by contradiction, we assume that $\tau_\infty \ne \infty$. Then there exist constants $T > 0$ and $\varepsilon \in (0,1)$ such that $P(\tau_\infty \le T) > \varepsilon$. Therefore, there is a constant $k_1 \ge k_0$ such that $P(\tau_k \le T) \ge \varepsilon$ for all $k \ge k_1$.

Define a C^2 function $V: R_+^{n+1} \longrightarrow \overline{R}_+$ by

$$\begin{aligned} V(S, x_1, x_2, \ldots, x_n) &= S - a - a\ln\frac{S}{a} \\ &\quad + \sum_{i=1}^{n}(x_i - 1 - \ln x_1), \end{aligned} \tag{6}$$

where $a = \min\{(d_i - (S^0+1)k_i)/(k_i + a_i)\}$ $(i = 1, 2, \ldots, n)$ is a positive constant. It is obvious that $u - 1 - \ln u \ge 0$, $\forall u \ge 0$. Hence $V(S, x_1, \ldots, x_n) \ge 0$. By the Itô formula, the following equation can be derived

$$\begin{aligned} \mathrm{d}V &= \left[\frac{S-a}{S}(S^0 - S)\left(d + \sum_{i=1}^{n} k_i x_i\right)\right. \\ &\quad - \frac{S-a}{S}\sum_{i=1}^{n}\frac{a_i S}{1+b_i S}x_i + \frac{a\sigma_1^2}{2} \\ &\quad - \sum_{i=1}^{n}\frac{x_i - 1}{x_i}\left(d_i + \sum_{i=1}^{n} k_i x_i\right)x_i + \sum_{i=1}^{n}\frac{x_i-1}{x_i}\frac{a_i S x_i}{1+b_i S} \\ &\quad \left. + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2}\right]\mathrm{d}t + (S-a)\sigma_1 \mathrm{d}B_1(t) \\ &\quad + \sum_{i=1}^{n}(x_i - 1)\sigma_{i+1}\mathrm{d}B_{i+1}(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} LV &= \frac{S-a}{S}(S^0 - S)\left(d + \sum_{i=1}^{n} k_i x_i\right) \\ &\quad - \frac{S-a}{S}\sum_{i=1}^{n}\frac{a_i S}{1+b_i S}x_i + \frac{a\sigma_1^2}{2} \\ &\quad - \sum_{i=1}^{n}\frac{x_i-1}{x_i}\left(d_i + \sum_{i=1}^{n} k_i x_i\right)x_i + \sum_{i=1}^{n}\frac{x_i-1}{x_i}\frac{a_i S x_i}{1+b_i S} \\ &\quad + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2} \\ &= (S^0 + a)\left(d + \sum_{i=1}^{n} k_i x_i\right) + \sum_{i=1}^{n}\frac{a a_i x_i}{1+b_i S} + \frac{a\sigma_1^2}{2} \\ &\quad + \sum_{i=1}^{n}\left(d_i + \sum_{i=1}^{n} k_i x_i\right) + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2} \\ &\quad - S\left(d + \sum_{i=1}^{n} k_i x_i\right) - \frac{aS^0}{S}\left(d + \sum_{i=1}^{n} k_i x_i\right) \\ &\quad - \sum_{i=1}^{n}\left(d_i + \sum_{i=1}^{n} k_i x_i\right)x_i - \sum_{i=1}^{n}\frac{a_i S}{1+b_i S}. \end{aligned} \tag{8}$$

Through calculations, the following inequality can be obtained

$$\begin{aligned} LV &\le (S^0 + a)\left(d + \sum_{i=1}^{n} k_i x_i\right) + \sum_{i=1}^{n}\left(d_i + \sum_{i=1}^{n} k_i x_i\right) \\ &\quad + \frac{a\sigma_1^2}{2} + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2} + \sum_{i=1}^{n} a a_i x_i - \sum_{i=1}^{n} d_i x_i \\ &\le (S^0 + a)d + \sum_{i=1}^{n} d_i + \frac{a\sigma_1^2}{2} + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2} \\ &\quad + \sum_{i=1}^{n}\left[(S^0 + a)k_i + k_i + a a_i - d_i\right]x_i \\ &= (S^0 + a)d + \sum_{i=1}^{n} d_i + \frac{a\sigma_1^2}{2} + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2} \\ &\quad + \sum_{i=1}^{n}(k_i + a_i)\left[a - \frac{d_i - (S^0+1)k_i}{k_i + a_i}\right]x_i. \end{aligned} \tag{9}$$

Defining $a = \min\{(d_i - (S^0+1)k_i)/(k_i + a_i)\}$ $(i = 1, 2, \ldots, n)$, we have

$$LV \le (S^0 + a)d + \sum_{i=1}^{n} d_i + \frac{a\sigma_1^2}{2} + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2}, \tag{10}$$

which yields

$$\begin{aligned} \mathrm{d}V &= LV\mathrm{d}t + (S^0 - a)\sigma_1 \mathrm{d}B_1(t) \\ &\quad + \sum_{i=1}^{n}\sigma_{i+1}(x_i - 1)\mathrm{d}B_{i+1}(t) \\ &\le (S^0 + a)d + \sum_{i=1}^{n} d_i + \frac{a\sigma_1^2}{2} + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2} \\ &\quad + (S^0 - a)\sigma_1 \mathrm{d}B_1(t) + \sum_{i=1}^{n}\sigma_{i+1}(x_i - 1)\mathrm{d}B_{i+1}(t). \end{aligned} \tag{11}$$

Integrating from 0 to $\tau_k \wedge T$ and taking expectation in both sides of inequality (11), one can obtain

$$\begin{aligned} &EV(S(\tau_k \wedge T), \ldots, x_n(\tau_k \wedge T)) \\ &\quad \le V(S(0), \ldots, x_n(0)) \\ &\qquad + \left((S^0 + a)d + \sum_{i=1}^{n} d_i + \frac{a\sigma_1^2}{2} + \sum_{i=1}^{n}\frac{\sigma_{i+1}^2}{2}\right)T. \end{aligned} \tag{12}$$

Let $\Omega_k = \{\tau_k \le T\}$ for all $k \ge k_1$, then $P(\Omega_k) \ge \varepsilon$. Based on the definition of the stopping time, we have $\mathrm{mi}\{S(t), x_1(t), \ldots, x_n(t)\} \le 1/k$ or $\mathrm{ma}\{S(t), x_1(t), \ldots, x_n(t)\} \ge k$ for all $\omega \in \Omega_k$. Hence there is a positive constant k such that $V(S(\tau_k \wedge T), \ldots, x_n(\tau_k \wedge T)) \ge k$. Furthermore,

$$\begin{aligned} &V(S(0), \ldots, x_n(0)) \\ &\quad + \left(\left(S^0 + a\right) d + \sum_{i=1}^{n} d_i + \frac{a\sigma_1^2}{2} + \sum_{i=1}^{n} \frac{\sigma_{i+1}^2}{2}\right) T \\ &\quad \ge EV\left(S(\tau_k \wedge T), \ldots, x_n(\tau_k \wedge T)\right) \\ &\quad \ge P(\Omega_k) V\left(S(\tau_k \wedge T), \ldots, x_n(\tau_k \wedge T)\right) \ge \varepsilon k. \end{aligned} \tag{13}$$

Setting $k \longrightarrow \infty$, we have $V(S(0), \ldots, x_n(0)) + ((S^0 + a)d + \sum_{i=1}^{n} d_i + a\sigma_1^2/2 + \sum_{i=1}^{n}(\sigma_{i+1}^2/2))T \ge \infty$, which is contradictory with $V(S(0), \ldots, x_n(0)) < \infty$. Hence $\tau_e = \infty$. This completes the proof. □

Remark 2. If we take $T = d_i/k_i - S^0$ $(i = 1, 2, \ldots, n)$ as a threshold value, then the microorganisms in system (1) will survive as $T > 1$ and be extinct as $T \le 1$.

3. The Principle of Competitive Exclusion

In this section, we prove that stochastic turbidostat system (1) satisfies the principle of competitive exclusion. For simplicity, we first define

$$\langle x(t) \rangle = \frac{1}{t} \int_0^t x(r)\, \mathrm{d}r. \tag{14}$$

Lemma 3. *The solution $(S(t), x_1(t), \ldots, x_n(t))$ of system (1) satisfies*

$$\limsup_{t \longrightarrow \infty} \frac{\ln\left[S(t) + \sum_{i=1}^{n} x_i(t)\right]}{\ln t} \le \frac{1}{\theta}, \quad a.s. \tag{15}$$

with any initial value $(S(0), x_1(0), \ldots, x_n(0)) \in R_+^{n+1}$. Here, θ is a positive constant satisfying

$$\begin{aligned} \lambda &= \min\{d, d_1, \ldots, d_n\} - \sum_{i=1}^{n} k_i S^0 \\ &\quad - \left(\frac{\theta - 1}{2} \vee 0\right)\left(\sigma_1^2 \vee \ldots \vee \sigma_n^2\right) > 0. \end{aligned} \tag{16}$$

Proof. Let

$$u(t) = S(t) + \sum_{i=1}^{n} x_i(t), \tag{17}$$

and

$$W(u) = (1 + u)^{\theta}, \tag{18}$$

where θ is a nonnegative constant decided later. By the Itô formula, we derive

$$\begin{aligned} \mathrm{d}u &= \left[\left(S^0 - S\right)\left(d + \sum_{i=1}^{n} k_i x_i\right) \right. \\ &\quad \left. - \sum_{i=1}^{n}\left(d_i + \sum_{i=1}^{n} k_i x_i\right) x_i\right] \mathrm{d}t + \sigma_1 S \mathrm{d}B_1(t) \\ &\quad + \sum_{i=1}^{n} \sigma_{i+1} x_i \mathrm{d}B_{i+1}(t). \end{aligned} \tag{19}$$

We can further obtain

$$\begin{aligned} \mathrm{d}W &= \theta(1 + u)^{\theta - 2} \left\{(1 + u)\left[\left(S^0 - S\right)\left(d + \sum_{i=1}^{n} k_i x_i\right) \right.\right. \\ &\quad \left. - \sum_{i=1}^{n}\left(d_i + \sum_{i=1}^{n} k_i x_i\right) x_i\right] + \frac{\theta - 1}{2}\left(\sigma_1^2 S^2 \right. \\ &\quad \left.\left. + \sum_{i=1}^{n} \sigma_{i+1}^2 x_i^2\right)\right\} \mathrm{d}t + \theta(1 + u)^{\theta - 1}\left[\sigma_1 S \mathrm{d}B_1(t) \right. \\ &\quad \left. + \sum_{i=1}^{n} \sigma_{i+1} x_i \mathrm{d}B_{i+1}(t)\right]. \end{aligned} \tag{20}$$

Denote

$$\begin{aligned} LW(u) &= \theta(1 + u)^{\theta - 2} \left\{(1 + u) \right. \\ &\quad \cdot \left[\left(S^0 - S\right)\left(d + \sum_{i=1}^{n} k_i x_i\right) - \sum_{i=1}^{n}\left(d_i + \sum_{i=1}^{n} k_i x_i\right) x_i\right] \\ &\quad \left. + \frac{\theta - 1}{2}\left(\sigma_1^2 S^2 + \sum_{i=1}^{n} \sigma_{i+1}^2 x_i^2\right)\right\} = \theta(1 + u)^{\theta - 2} \left\{(1 \right. \\ &\quad + u)\left[S^0\left(d + \sum_{i=1}^{n} k_i x_i\right) - S\left(d + \sum_{i=1}^{n} k_i x_i\right) \right. \\ &\quad \left. - \sum_{i=1}^{n}\left(d_i + \sum_{i=1}^{n} k_i x_i\right) x_i\right] + \frac{\theta - 1}{2}\left(\sigma_1^2 S^2 \right. \\ &\quad \left.\left. + \sum_{i=1}^{n} \sigma_{i+1}^2 x_i^2\right)\right\} \le \theta(1 + u)^{\theta - 2} \left\{(1 + u)\left[S^0 d \right.\right. \\ &\quad \left. + \left(S^0 \sum_{i=1}^{n} k_i - \min\{d, d_1, \ldots, d_n\}\right) u\right] \\ &\quad \left. + \frac{\theta - 1}{2}\left(\sigma_1^2 S^2 + \sum_{i=1}^{n} \sigma_{i+1}^2 x_i^2\right)\right\} \le \theta(1 + u)^{\theta - 2} \\ &\quad \cdot \left\{\left[\sum_{i=1}^{n} k_i S^0 - \min\{d, d_1, \ldots, d_n\} \right.\right. \\ &\quad \left.\left. + \left(\frac{\theta - 1}{2} \vee 0\right)\left(\sigma_1^2 \vee \ldots \vee \sigma_n^2\right)\right] u^2 + \left(S^0 d \right. \end{aligned}$$

$$+\sum_{i=1}^{n}k_iS^0-\min\{d,d_1,\ldots,d_n\}\Big)u+\Big(d+\sum_{i=1}^{n}k_i\Big)\cdot S^0\Big\}=\theta(1+u)^{\theta-2}\Big\{-\lambda u^2+\Big(S^0d+\sum_{i=1}^{n}k_iS^0-\min\{d,d_1,\ldots,d_n\}\Big)u+\Big(d+\sum_{i=1}^{n}k_i\Big)S^0\Big\}, \tag{21}$$

where

$$\lambda=\min\{d,d_1,\ldots,d_n\}-\sum_{i=1}^{n}k_iS^0-\Big(\frac{\theta-1}{2}\vee 0\Big)\big(\sigma_1^2\vee\ldots\vee\sigma_n^2\big)>0. \tag{22}$$

Let

$$A=\Big(d+\sum_{i=1}^{n}k_i\Big)S^0-\min\{d,d_1,\ldots,d_n\} \tag{23}$$

be a constant. Then

$$\mathrm{d}W(u)\le\theta(1+u)^{\theta-2}\Big\{-\lambda u^2+Au+\Big(d+\sum_{i=1}^{n}k_i\Big)S^0\Big\}\mathrm{d}t+\theta(1+u)^{\theta-1}\Big[\sigma_1S\mathrm{d}B_1(t)+\sum_{i=1}^{n}\sigma_{i+1}x_i\mathrm{d}B_{i+1}(t)\Big]. \tag{24}$$

For $0<q<\theta\lambda$, we have

$$\mathrm{d}\left[\mathrm{e}^{qt}W(u)\right]=L\left[\mathrm{e}^{qt}W(u)\right]\mathrm{d}t+\mathrm{e}^{qt}\theta(1+u)^{\theta-1}\cdot\Big[\sigma_1S\mathrm{d}B_1(t)+\sum_{i=1}^{n}\sigma_{i+1}x_i\mathrm{d}B_{i+1}(t)\Big], \tag{25}$$

where

$$L\left[\mathrm{e}^{qt}W(u)\right]=q\mathrm{e}^{qt}W(u)+\mathrm{e}^{qt}LW(u)\le q\mathrm{e}^{qt}(1+u)^{\theta}+\mathrm{e}^{qt}\theta(1+u)^{\theta-2}\Big\{-\lambda u^2+Au+\Big(d+\sum_{i=1}^{n}k_i\Big)S^0\Big\}=\mathrm{e}^{qt}\theta(1+u)^{\theta-2}\cdot\Big\{-\Big(\lambda-\frac{q}{\theta}\Big)u^2+\Big[A+\frac{2q}{\theta}\Big]u+S^0\Big(d+\sum_{i=1}^{n}k_i\Big)+\frac{q}{\theta}\Big\}. \tag{26}$$

Define a function $f(u)$ by

$$f(u)=(1+u)^{\theta-2}\Big\{-\Big(\lambda-\frac{q}{\theta}\Big)u^2+\Big[A+\frac{2q}{\theta}\Big]u+S^0\Big(d+\sum_{i=1}^{n}k_i\Big)+\frac{q}{\theta}\Big\},\quad u\in R^+ \tag{27}$$

and

$$F:=\sup_{u\in R_+}f(u)+1, \tag{28}$$

which gives

$$L\left[\mathrm{e}^{qt}W(u)\right]\le\theta\mathrm{e}^{qt}F. \tag{29}$$

Integrating from 0 to t and taking expectation for (25), one has

$$E\left[\mathrm{e}^{qt}W(u)\right]=W(u(0))+E\int_0^t L\left[\mathrm{e}^{qt}W(u)\right]\mathrm{d}t. \tag{30}$$

Then it is easy to obtain

$$E\left[\mathrm{e}^{qt}W(u)\right]\le(1+u(0))^{\theta}+E\int_0^t\mathrm{e}^{qt}\theta F\mathrm{d}t=(1+u(0))^{\theta}+\frac{\theta F}{q}\mathrm{e}^{qt}, \tag{31}$$

which leads to

$$\lim_{t\longrightarrow\infty}\sup E\left[(1+u(t))^{\theta}\right]\le\frac{\theta F}{q}:=F_0,\ a.s. \tag{32}$$

$u(t)$ is a continuous function, so there is a constant $M>0$ such that

$$E\left[(1+u(t))^{\theta}\right]\le M,\quad t\ge 0. \tag{33}$$

According to (24), for small enough $\delta>0$, $k=1,2,\ldots$, integrating from $k\delta$ to $(k+1)\delta$ and taking expectation, we can get

$$E\Big[\sup_{k\delta\le t\le(k+1)\delta}(1+u(t))^{\theta}\Big]\le E\left[(1+u(k\delta))^{\theta}\right]+I_1+I_2\le M+I_1+I_2, \tag{34}$$

where

$$\begin{aligned}I_1&=E\Big[\sup_{k\delta\le t\le(k+1)\delta}\Big|\int_{k\delta}^{t}\theta(1+u)^{\theta-2}\cdot\Big\{-\lambda u^2+Au+S^0\Big(d+\sum_{i=1}^{n}k_i\Big)\Big\}\mathrm{d}t\Big|\Big]\\&\le E\Big[\sup_{k\delta\le t\le(k+1)\delta}\Big|\int_{k\delta}^{t}\theta(1+u)^{\theta-2}\cdot\Big\{S^0\Big(d+\sum_{i=1}^{n}k_i\Big)u+S^0\Big(d+\sum_{i=1}^{n}k_i\Big)\Big\}\mathrm{d}t\Big|\Big]\\&=E\Big[\sup_{k\delta\le t\le(k+1)\delta}\Big|\int_{k\delta}^{t}\theta(1+u)^{\theta-2}\end{aligned}$$

$$\cdot \left\{ S^0 \left(d + \sum_{i=1}^{n} k_i \right) (1+u) \right\} \mathrm{d}t \Bigg| \Bigg] \leq S^0 \left(d + \sum_{i=1}^{n} k_i \right) \theta E \int_{k\delta}^{(k+1)\delta} (1+u)^{\theta} \, \mathrm{d}t \leq C_1 \delta E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^{\theta} \right], \tag{35}$$

here,

$$C_1 = S^0 \left(d + \sum_{i=1}^{n} k_i \right) \theta. \tag{36}$$

Applying Burkholder-Davis-Gundy inequality [41], we have

$$\begin{aligned} I_2 &= E \left[\sup_{k\delta \leq t \leq (k+1)\delta} \left| \int_{k\delta}^{t} \theta (1+u)^{\theta-1} \cdot \left\{ \sigma_1 S \mathrm{d}B_1(t) + \sum_{i=1}^{n} \sigma_{i+1} x_i \mathrm{d}B_{i+1}(t) \right\} \right| \right] \\ &\leq \sqrt{32} E \left[\int_{k\delta}^{(k+1)\delta} \theta^2 (1+u)^{2\theta-2} \left(\sigma_1^2 S^2 + \sum_{i=1}^{n} \sigma_{i+1}^2 x_i^2 \right) \mathrm{d}t \right]^{1/2} \leq \sqrt{32} E \left[\int_{k\delta}^{(k+1)\delta} \theta^2 (1+u)^{2\theta-2} \left(\sigma_1^2 u^2 + \sum_{i=1}^{n} \sigma_{i+1}^2 u^2 \right) \mathrm{d}t \right]^{1/2} \\ &\leq \sqrt{32} \theta \delta^{1/2} E \left[\sup_{k\delta \leq t \leq (k+1)\delta} \left(\sigma_1^2 \vee \ldots \vee \sigma_n^2 \right) (1+u)^{2\theta} \right]^{1/2} \leq \sqrt{32} \theta \delta^{1/2} \left(\sigma_1^2 \vee \ldots \vee \sigma_n^2 \right)^{1/2} \\ &\quad \cdot E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u)^{\theta} \right]. \end{aligned} \tag{37}$$

Hence, we have

$$\begin{aligned} E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u)^{\theta} \right] &\leq E \left[(1+u(k\delta))^{\theta} \right] \\ &\quad + \left[C_1 \delta + \sqrt{32} \theta \delta^{1/2} \left(\sigma_1^2 \vee \ldots \vee \sigma_n^2 \right)^{1/2} \right] \\ &\quad \times E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u)^{\theta} \right]. \end{aligned} \tag{38}$$

Particularly, one can choose $\delta > 0$ such that

$$C_1 \delta + \sqrt{32} \theta \delta^{1/2} \left(\sigma_1^2 \vee \ldots \vee \sigma_n^2 \right)^{1/2} \leq \frac{1}{2}, \tag{39}$$

which yields

$$E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u)^{\theta} \right] \leq 2E \left[(1+u(k\delta))^{\theta} \right] \leq 2M. \tag{40}$$

For any $\beta > 0$, the following inequality can be obtained by Chebyshev's inequality [41],

$$\begin{aligned} &P \left\{ \sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^{\theta} \geq (k\delta)^{1+\beta} \right\} \\ &\quad \leq \frac{E \left[\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^{\theta} \right]}{(k\delta)^{1+\beta}} \leq \frac{2M}{(k\delta)^{1+\beta}}. \end{aligned} \tag{41}$$

Using Borel-Cantelli Lemma [41], for all $\omega \in \Omega$ and sufficiently large k, we have

$$\sup_{k\delta \leq t \leq (k+1)\delta} (1+u(t))^{\theta} \leq (k\delta)^{1+\beta}. \tag{42}$$

Therefore, there is a $k_0(\omega)$, for all $\omega \in \Omega$ and $k \geq k_0$; the above equation holds. In addition, for all $\omega \in \Omega$, when $k \geq k_0$ and $k\delta \leq t \leq (k+1)\delta$,

$$\frac{\ln (1+u(t))^{\theta}}{\ln t} \leq \frac{\ln (k\delta)^{1+\beta}}{\ln t} \leq \frac{\ln (k\delta)^{1+\beta}}{\ln (k\delta)} = 1 + \beta, \tag{43}$$

which means

$$\lim_{t \longrightarrow \infty} \sup \frac{\ln (1+u(t))^{\theta}}{\ln t} \leq 1 + \beta, \ a.s. \tag{44}$$

When $\beta \longrightarrow 0$,

$$\lim_{t \longrightarrow \infty} \sup \frac{\ln (1+u(t))}{\ln t} \leq \frac{1}{\theta}, \ a.s. \tag{45}$$

where θ is determined by (22). Therefore,

$$\begin{aligned} \lim_{t \longrightarrow \infty} \sup \frac{\ln \left[S(t) + \sum_{i=1}^{n} x_i(t) \right]}{\ln t} &= \lim_{t \longrightarrow \infty} \sup \frac{\ln u(t)}{\ln t} \\ &\leq \lim_{t \longrightarrow \infty} \sup \frac{\ln (1+u(t))}{\ln t} \leq \frac{1}{\theta}, \ a.s. \end{aligned} \tag{46}$$

This completes the proof. □

Lemma 4. *If* $\min\{d, d_1, \ldots, d_n\} > \sum_{i=1}^{n} k_i S^0 + ((\theta - 1)/2 \vee 0)(\sigma_1^2 \vee \ldots \vee \sigma_n^2)$, *then the solution* $(S(t), x_1(t), \ldots, x_n(t))$ *of system (1) with initial value* $(S(0), x_1(0), \ldots, x_n(0)) \in R_+^{n+1}$ *satisfies*

$$\begin{aligned} &\lim_{t \longrightarrow \infty} \frac{\int_0^t (S(s) - \lambda_1) \, \mathrm{d}B_1(s)}{t} = 0, \ a.s. \\ &\lim_{t \longrightarrow \infty} \frac{\int_0^t (x_1(s) - x_1^*) \, \mathrm{d}B_2(s)}{t} = 0, \ a.s. \\ &\lim_{t \longrightarrow \infty} \frac{\int_0^t x_i(s) \, \mathrm{d}B_{i+1}(s)}{t} = 0, \ a.s. \\ &\qquad (i = 2, 3, \ldots, n), \end{aligned} \tag{47}$$

where λ_1 *is the break-even concentration of* $x_1(t)$ *and* x_1^* *is the equilibrium concentration of* $x_1(t)$.

Proof. Define

$$X(t) = \int_0^t (S(s) - \lambda_1)\, dB_1(s),$$
$$Y(t) = \int_0^t (x_1(s) - x_1^*)\, dB_2(s), \tag{48}$$
$$Z_i(t) = \int_0^t x_i(s)\, dB_{i+1}(s), \quad i = 2, 3, \ldots, n.$$

It follows from $\min\{d, d_1, \ldots, d_n\} > \sum_{i=1}^n k_i S^0 + ((\theta - 1)/2 \vee 0)(\sigma_1^2 \vee \ldots \vee \sigma_n^2)$, Lemma 3, and Burkholder-Davis-Gundy inequality [41] that

$$\begin{aligned} E\left[\sup_{0\le s\le t} |X(s)|^\theta\right] &\le C_\theta E\left[\int_0^t (S(s) - \lambda_1)^2\, ds\right]^{1/2} \\ &\le C_\theta t^{\theta/2} E\left[\sup_{0\le s\le t} (S(s) - \lambda_1)^\theta\right] \\ &\le C_\theta t^{\theta/2} E\left[\sup_{0\le s\le t} (1 + u(t))^\theta\right] \\ &\le 2MC_\theta t^{\theta/2}. \end{aligned} \tag{49}$$

By Doob's martingale inequality [41], for arbitrary $\alpha > 0$, it follows that

$$\begin{aligned} &P\left\{\omega : \sup_{k\delta\le t\le (k+1)\delta} |X(t)|^\theta > (k\delta)^{1+\alpha+\theta/2}\right\} \\ &\le \frac{E\left[|X((k+1)\delta)|^\theta\right]}{(k\delta)^{1+\alpha+\theta/2}} \le \frac{2MC_\theta}{(k\delta)^{1+\alpha}}\left(\frac{(k+1)\delta}{k\delta}\right)^{\theta/2} \\ &\le \frac{2^{1+\theta/2} MC_\theta}{(k\delta)^{1+\alpha}}. \end{aligned} \tag{50}$$

By Borel-Cantelli Lemma [41], for all $\omega \in \Omega$ and sufficient large k, we derive

$$\sup_{k\delta\le t\le (k+1)\delta} |X(t)|^\theta \le (k\delta)^{1+\alpha+\theta/2}. \tag{51}$$

Thus, there exists a positive constant $k_{X_0}(\omega)$ such that for all $k \ge k_{X_0}$ and $k\delta \le t \le (k+1)\delta$,

$$\frac{\ln |X(t)|^\theta}{\ln t} \le \frac{\ln (k\delta)^{1+\alpha+\theta/2}}{\ln t} \le \frac{\ln (k\delta)^{1+\alpha+\theta/2}}{\ln (k\delta)}, \tag{52}$$

which implies

$$\frac{\ln |X(t)|^\theta}{\ln t} \le 1 + \alpha + \frac{\theta}{2}. \tag{53}$$

Taking superior limit leads to

$$\lim_{t\to\infty} \sup \frac{\ln |X(t)|}{\ln t} \le \frac{1 + \alpha + \theta/2}{\theta}. \tag{54}$$

Sending $\alpha \longrightarrow 0$, we have

$$\lim_{t\to\infty} \sup \frac{\ln |X(t)|}{\ln t} \le \frac{1}{2} + \frac{1}{\theta}. \tag{55}$$

Then there exist a constant $T'(\omega) > 0$ and a set Ω_η for small enough $0 < \eta < 1/2 - 1/\theta$ such that $P(\Omega_\eta) \ge 1 - \eta$. Thus for $t \ge T'(\omega)$, $\omega \in \Omega_\eta$, we have

$$\frac{\ln |X(t)|}{\ln t} \le \frac{1}{2} + \frac{1}{\theta} + \eta, \tag{56}$$

which means

$$\lim_{t\to\infty} \sup \frac{|X(t)|}{t} \le \lim_{t\to\infty} \sup \frac{t^{1/2+1/\theta+\eta}}{t} = 0. \tag{57}$$

Together with

$$\begin{aligned} \lim_{t\to\infty} \inf \frac{|X(t)|}{t} &= \lim_{t\to\infty} \inf \frac{\left|\int_0^t (S(s) - \lambda_1)\, dB_1(s)\right|}{t} \\ &\ge 0, \end{aligned} \tag{58}$$

we derive

$$\lim_{t\to\infty} \frac{|X(t)|}{t} = 0, \ a.s. \tag{59}$$

and

$$\lim_{t\to\infty} \frac{\int_0^t (S(s) - \lambda_1)\, dB_1(s)}{t} = 0, \ a.s. \tag{60}$$

Applying the same methods, we can prove that

$$\begin{aligned} &\lim_{t\to\infty} \frac{\int_0^t (x_1(s) - x_1^*)\, dB_2(s)}{t} = 0, \ a.s. \\ &\lim_{t\to\infty} \frac{\int_0^t x_i(t)\, dB_{i+1}(s)}{t} = 0, \ a.s. \\ &\qquad i = 2, 3, \ldots, n. \end{aligned} \tag{61}$$

This completes the proof. □

Definition 5. Stochastic equation

$$\begin{aligned} d\phi(t) = &\left[\left(S^0 - S(t)\right)\left(d + \sum_{i=1}^n k_i x_i\right)\right] dt \\ &+ \sigma_1 \phi(t)\, dB_1(t) \end{aligned} \tag{62}$$

has a solution $\phi(t)$ weakly converging to the distribution ν. Here ν is a probability measure of R_+^1 such that $\int_0^\infty z\nu(dz) = S^0$. Particularly, ν has density $(A\sigma_1^2 z^2 p(z))^{-1}$, where A is a normal constant and

$$\begin{aligned} p(z) = &\exp\left[\frac{2\left(d + \sum_{i=1}^n k_i x_i\right) S^0}{\sigma_1^2 z}\right] \\ &\cdot \exp\left[\frac{-2\left(d + \sum_{i=1}^n k_i x_i\right) S^0}{\sigma_1^2}\right] z^{2(d+\sum_{i=1}^n k_i x_i)/\sigma_1^2}. \end{aligned} \tag{63}$$

Theorem 6. *$(S(t), x_1(t), \ldots, x_n(t))$ is the solution of system (1) with any initial value $(S(0), x_1(0), \ldots, x_n(0)) \in R_+^{n+1}$. Moreover, if*

$$\begin{aligned} &0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_n < S^0, \\ &\min\{d, d_1, \ldots, d_n\} \\ &\quad > \sum_{i=1}^{n} k_i S^0 + \left(\frac{\theta - 1}{2} \vee 0\right)\left(\sigma_1^2 \vee \ldots \vee \sigma_n^2\right), \\ &\frac{a_i \lambda_1}{1 + b_i \lambda_1} + \frac{a_i}{2\sqrt{b_i}\left(1 + b_i \lambda_1\right)}\left(\frac{H}{d + N\sum_{i=1}^{n} k_i}\right)^{1/2} \\ &\quad \le \left(d_i + N\sum_{i=1}^{n} k_i\right) + \frac{\sigma_{i+1}^2}{2} \quad (i = 2, 3, \ldots, n), \\ &\text{and } a_1 \int_0^\infty \frac{x\nu(\mathrm{d}x)}{1 + b_1 x} > \frac{\sigma_2^2}{2} + \left(d + M\sum_{i=1}^{n} k_i\right), \end{aligned} \tag{64}$$

then we have

$$\begin{aligned} &\lim_{t\longrightarrow\infty} \inf \langle x_1(t)\rangle > 0, \ a.s. \\ &\lim_{t\longrightarrow\infty} \sup \frac{\ln x_i(t)}{t} \le 0, \ a.s. \quad (i = 2, \ldots, n), \end{aligned} \tag{65}$$

where $M = \max\{x_1, \ldots, x_n\}$, $N = \min\{x_1, \ldots, x_n\}$, λ_i $(i = 1, 2, \ldots, n)$ is the break-even concentration of $x_i(t)$, H is determined later by (74), $(\lambda_1, x_1^*, 0, \ldots, 0)$ is the boundary equilibrium of the corresponding deterministic model, and ν is defined in Definition 5.

Proof. The deterministic system corresponding to (1) has an equilibrium $(\lambda_1, x_1^*, 0, \ldots, 0)$ under the condition of $\lambda_1 < S^0$, where λ_1 and x_1^* satisfy

$$\begin{aligned} \left(S^0 - \lambda_1\right)\left(d + k_1 x_1^*\right) &= \frac{a_1 \lambda_1 x_1^*}{1 + b_1 \lambda_1}, \\ d_1 + k_1 x_1^* &= \frac{a_1 \lambda_1}{1 + b_1 \lambda_1}. \end{aligned} \tag{66}$$

Define C^2 function $V : R_+^{n+1} \longrightarrow \overline{R}_+$ by

$$\begin{aligned} &V(S, x_1, \ldots, x_n) \\ &\quad = S - \lambda_1 - \lambda \ln \frac{S}{\lambda_1} \\ &\qquad + \left(1 + b_1 \lambda_1\right)\left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*}\right) \\ &\qquad + \sum_{i=2}^{n} \left(1 + b_i \lambda_1\right) x_i. \end{aligned} \tag{67}$$

Then by the Itô formula, we get

$$\begin{aligned} \mathrm{d}V &= \frac{S - \lambda_1}{S}\left\{\left[\left(S^0 - S\right)\left(d + \sum_{i=1}^{n} k_i x_i\right) - \sum_{i=1}^{n} \frac{a_i S}{1 + b_i S} x_i\right]\mathrm{d}t + \sigma_1 S \mathrm{d}B_1(t)\right\} + \frac{1}{2}\frac{\lambda_1}{S^2}\sigma_1^2 S^2 \mathrm{d}t \\ &\quad + \left[\left(1 + b_1\lambda_1\right) - \left(1 + b_1\lambda_1\right)\frac{x_1^*}{x_1}\right] \cdot \left\{\left[-\left(d_1 + \sum_{i=1}^{n} k_i x_i\right) x_1 + \frac{a_1 S}{1 + b_1 S} x_1\right]\mathrm{d}t \right. \\ &\quad \left. + \sigma_2 x_1 \mathrm{d}B_2(t)\right\} + \frac{1}{2}\frac{\left(1 + b_1\lambda_1\right) x_1^*}{x_1^2}\sigma_2^2 x_1^2 \mathrm{d}t + \sum_{i=2}^{n}\left(1 + b_i\lambda_1\right)\left\{\left[-\left(d_i + \sum_{i=1}^{n} k_i x_i\right) x_i + \frac{a_i S}{1 + b_i S} x_i\right]\mathrm{d}t \right. \\ &\quad \left. + \sigma_{i+1} x_i \mathrm{d}B_{i+1}(t)\right\} = LV\mathrm{d}t + \left(S - \lambda_1\right)\sigma_1 \mathrm{d}B_1(t) \\ &\quad + \left[\left(1 + b_1\lambda_1\right) - \left(1 + b_1\lambda_1\right)\frac{x_1^*}{x_1}\right]\sigma_2 x_1 \mathrm{d}B_2(t) \\ &\quad + \sum_{i=2}^{n}\left(1 + b_i\lambda_1\right)\sigma_{i+1} x_i \mathrm{d}B_{i+1}(t), \end{aligned} \tag{68}$$

where

$$\begin{aligned} LV &= \frac{S - \lambda_1}{S}\left[\left(S^0 - S\right)\left(d + \sum_{i=1}^{n} k_i x_i\right) - \sum_{i=1}^{n} \frac{a_i S}{1 + b_i S} x_i\right] + \frac{1}{2}\lambda_1 \sigma_1^2 + \frac{1}{2}\left(1 + b_1\lambda_1\right) x_1^* \sigma_2^2 \\ &\quad + \left[\left(1 + b_1\lambda_1\right) - \left(1 + b_1\lambda_1\right)\frac{x_1^*}{x_1}\right] \cdot \left[-\left(d_1 + \sum_{i=1}^{n} k_i x_i\right) x_1 + \frac{a_1 S}{1 + b_1 S} x_1\right] \\ &\quad + \sum_{i=2}^{n}\left(1 + b_i\lambda_1\right)\left[-\left(d_i + \sum_{i=1}^{n} k_i x_i\right) x_i + \frac{a_i S}{1 + b_i S} x_i\right]. \end{aligned} \tag{69}$$

According to (66), we have

$$\begin{aligned} S^0 &= \lambda_1 + \frac{a_1 \lambda_1 x_1^*}{\left(1 + b_1\lambda_1\right)\left(d + k_1 x_1^*\right)}, \\ d_1 &= \frac{a_1 \lambda_1}{1 + b_1 \lambda_1} - k_1 x_1^*. \end{aligned} \tag{70}$$

It follows from some calculations that

$$\begin{aligned} &LV \\ &\quad = -\frac{\left(d + \sum_{i=1}^{n} k_i x_i\right)\left(S - \lambda_1\right)^2}{S} \\ &\qquad + \frac{a_1 \lambda_1 x_1^*\left(S - \lambda_1\right)\sum_{i=2}^{n} k_i x_i}{S\left(1 + b_1\lambda_1\right)\left(d + k_1 x_1^*\right)} \\ &\qquad + \frac{a_1 k_1 x_1^*\left(S - \lambda_1\right)\left(x_1 - x_1^*\right)}{\left(1 + b_1 S\right)\left(d + k_1 x_1^*\right)} \end{aligned}$$

$$
\begin{aligned}
&- k_1 (1 + b_1\lambda_1)(x_1 - x_1^*)^2 \\
&- (1 + b_1\lambda_1)(x_1 - x_1^*)\sum_{i=2}^{n} k_i x_i \\
&+ \sum_{i=2}^{n} x_i \left[\frac{a_i\lambda_1}{1 + b_i\lambda_1} - \left(d_i + \sum_{i=1}^{n} k_i x_i\right)\right] + \frac{1}{2}\lambda_1\sigma_1^2 \\
&+ \frac{1}{2}(1 + b_1\lambda_1) x_1^*\sigma_2^2 \\
\le& -\frac{(d + \sum_{i=1}^{n} k_i x_i)(S - \lambda_1)^2}{S} \\
&+ \left[\frac{a_1\lambda_1 x_1^*}{(1 + b_1\lambda_1)(d + k_1 x_1^*)} + (1 + b_1\lambda_1) x_1^*\right]\sum_{i=2}^{n} k_i x_i \\
&+ \frac{a_1 k_1 x_1^* x_1}{d + k_1 x_1^*} + \frac{a_1 k_1\lambda_1 (x_1^*)^2}{d + k_1 x_1^*} \\
&+ \sum_{i=2}^{n} x_i \left[-\sum_{i=1}^{n} k_i (x_i - x_i^*)\right] + \frac{1}{2}\lambda_1\sigma_1^2 \\
&+ \frac{1}{2}(1 + b_1\lambda_1) x_1^*\sigma_2^2.
\end{aligned} \tag{71}
$$

Set

$$
\begin{aligned}
M &= \max\{x_1, \ldots, x_n\}, \\
N &= \min\{x_1, \ldots, x_n\},
\end{aligned} \tag{72}
$$

then

$$
\begin{aligned}
LV \le& -\frac{(d + N\sum_{i=1}^{n} k_i)(S - \lambda_1)^2}{S} \\
&+ \left[\frac{a_1\lambda_1 x_1^*}{(1 + b_1\lambda_1)(d + k_1 x_1^*)} + (1 + b_1\lambda_1) x_1^*\right] M\sum_{i=2}^{n} k_i \\
&+ \frac{a_1 k_1 x_1^* M}{d + k_1 x_1^*} + \frac{a_1 k_1\lambda_1 (x_1^*)^2}{d + k_1 x_1^*} + \sum_{i=2}^{n} M\sum_{i=1}^{n} k_i x_i^* + \frac{1}{2} \\
&\cdot \lambda_1\sigma_1^2 + \frac{1}{2}(1 + b_1\lambda_1) x_1^*\sigma_2^2 \\
=& -\frac{(d + N\sum_{i=1}^{n} k_i)(S - \lambda_1)^2}{S} + H,
\end{aligned} \tag{73}
$$

where

$$
\begin{aligned}
H &= \left[\frac{a_1\lambda_1 x_1^*}{(1 + b_1\lambda_1)(d + k_1 x_1^*)} + (1 + b_1\lambda_1) x_1^*\right] M\sum_{i=2}^{n} k_i \\
&+ \frac{a_1 k_1 x_1^* M}{d + k_1 x_1^*} + \frac{a_1 k_1\lambda_1 (x_1^*)^2}{d + k_1 x_1^*} \\
&+ (n - 1) M\sum_{i=1}^{n} k_i x_i^* + \frac{1}{2}\lambda_1\sigma_1^2 \\
&+ \frac{1}{2}(1 + b_1\lambda_1) x_1^*\sigma_2^2.
\end{aligned} \tag{74}
$$

Integrating (68) from 0 to t and dividing by t on both sides, one can obtain

$$
\begin{aligned}
&\frac{V(t) - V(0)}{t} \\
&\le -\frac{d + N\sum_{i=1}^{n} k_i}{t}\int_0^t \frac{(S(s) - \lambda_1)^2}{S(s)}\mathrm{d}s + H \\
&+ \sigma_1\frac{1}{t}\int_0^t (S(s) - \lambda_1)\,\mathrm{d}B_1(s) \\
&+ \sigma_2(1 + b_1\lambda_1)\frac{1}{t}\int_0^t (x_1(s) - x_1^*)\,\mathrm{d}B_2(t) \\
&+ \sum_{i=2}^{n}(1 + b_i\lambda_1)\sigma_{i+1}\int_0^t x_i(s)\,\mathrm{d}B_{i+1}(t).
\end{aligned} \tag{75}
$$

According to Lemma 4, it follows that

$$
\limsup_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \frac{(S(s) - \lambda_1)^2}{S(s)}\mathrm{d}s \le \frac{H}{d + N\sum_{i=1}^{n} k_i}. \tag{76}
$$

Define

$$
V_i(x_i(t)) = \ln x_i(t), \quad i = 2, 3, \ldots, n. \tag{77}
$$

Using the Itô formula, we have

$$
\begin{aligned}
\mathrm{d}V_i =& \left[-\left(d_i + \sum_{i=1}^{n} k_i x_i\right) + \frac{a_i S}{1 + b_i S} - \frac{\sigma_{i+1}^2}{2}\right]\mathrm{d}t \\
&+ \sigma_{i+1}\mathrm{d}B_{i+1}(t) \le \left[\frac{a_i(S - \lambda_1)}{(1 + b_i S)(1 + b_i\lambda_1)}\right. \\
&\left. - \left(d_i + N\sum_{i=1}^{n} k_i\right) + \frac{a_i\lambda_1}{1 + b_i\lambda_1} - \frac{\sigma_{i+1}^2}{2}\right]\mathrm{d}t \\
&+ \sigma_{i+1}\mathrm{d}B_{i+1}(t).
\end{aligned} \tag{78}
$$

Then integrating (78) from 0 to t on both sides and dividing all by t, we can get

$$
\begin{aligned}
&\frac{V_i(t) - V_i(0)}{t} \\
&\le \frac{a_i}{1 + b_i\lambda_1}\left\langle\frac{S(t) - \lambda_1}{1 + b_i S(t)}\right\rangle - \left(d_i + N\sum_{i=1}^{n} k_i\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a_i\lambda_1}{1+b_i\lambda_1}-\frac{\sigma_{i+1}^2}{2}+\frac{\sigma_{i+1}B_{i+1}(t)}{t}\\
&\le \frac{a_i}{2\sqrt{b_i}(1+b_i\lambda_1)}\left\langle\frac{|S(t)-\lambda_1|}{\sqrt{S(t)}}\right\rangle\\
&-\left(d_i+N\sum_{i=1}^{n}k_i\right)+\frac{a_i\lambda_1}{1+b_i\lambda_1}-\frac{\sigma_{i+1}^2}{2}\\
&+\frac{\sigma_{i+1}B_{i+1}(t)}{t}\\
&\le \frac{a_i}{2\sqrt{b_i}(1+b_i\lambda_1)}\left[\frac{1}{t}\int_0^t\frac{(S(t)-\lambda_1)^2}{S(t)}\mathrm{d}s\right]^{1/2}\\
&-\left(d_i+N\sum_{i=1}^{n}k_i\right)+\frac{a_i\lambda_1}{1+b_i\lambda_1}-\frac{\sigma_{i+1}^2}{2}\\
&+\frac{\sigma_{i+1}B_{i+1}(t)}{t},
\end{aligned}
\tag{79}
$$

which means

$$
\begin{aligned}
&\frac{\ln x_i(t)}{t}\\
&\le \frac{\ln x_i(0)}{t}\\
&+\frac{a_i}{2\sqrt{b_i}(1+b_i\lambda_1)}\left[\frac{1}{t}\int_0^t\frac{(S(t)-\lambda_1)^2}{S(t)}\mathrm{d}s\right]^{1/2}\\
&-\left(d_i+N\sum_{i=1}^{n}k_i\right)+\frac{a_i\lambda_1}{1+b_i\lambda_1}-\frac{\sigma_{i+1}^2}{2}\\
&+\frac{\sigma_{i+1}B_{i+1}(t)}{t}.
\end{aligned}
\tag{80}
$$

Based on (76), $\lim_{t\longrightarrow\infty}(B_{i+1}(t)/t)=0$ and $\lim_{t\longrightarrow\infty}(\ln x_i(0)/t)=0$, we know that

$$
\begin{aligned}
&\lim_{t\longrightarrow\infty}\sup\frac{\ln x_i(t)}{t}\\
&\le -\left(d_i+N\sum_{i=1}^{n}k_i\right)+\frac{a_i\lambda_1}{1+b_i\lambda_1}-\frac{\sigma_{i+1}^2}{2}\\
&+\frac{a_i}{2\sqrt{b_i}(1+b_i\lambda_1)}\left(\frac{H}{d+N\sum_{i=1}^{n}k_i}\right)^{1/2}\\
&\qquad (i=2,\ldots,n).
\end{aligned}
\tag{81}
$$

If

$$
\begin{aligned}
&\frac{a_i\lambda_1}{1+b_i\lambda_1}+\frac{a_i}{2\sqrt{b_i}(1+b_i\lambda_1)}\left(\frac{H}{d+N\sum_{i=1}^{n}k_i}\right)^{1/2}\\
&\le\left(d_i+N\sum_{i=1}^{n}k_i\right)+\frac{\sigma_{i+1}^2}{2},
\end{aligned}
\tag{82}
$$

then we have

$$
\lim_{t\longrightarrow\infty}\sup\frac{\ln x_i(t)}{t}\le 0,\ a.s.\quad (i=2,\ldots,n),
\tag{83}
$$

that is, $x_i(t)$ $(i=2,\ldots,n)$ tends to extinction exponentially almost surely.

On the basis of classical comparison theorem of stochastic differential equation, the solution $S(t)$ of system (1) and the solution $\phi(t)$ of system (62) satisfy $S(t)\le\phi(t)$. According to the Itô formula, it follows that

$$
\begin{aligned}
&\mathrm{d}\ln S(t)\\
&=\left[\frac{(S^0-S)(d+\sum_{i=1}^{n}k_ix_i)}{S}-\sum_{i=1}^{n}\frac{a_ix_i}{1+b_iS}-\frac{\sigma_1^2}{2}\right]\mathrm{d}t\\
&\quad+\sigma_1\mathrm{d}B_1(t),
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
&\mathrm{d}\ln\phi(t)\\
&=\left[\frac{(S^0-\phi)(d+\sum_{i=1}^{n})}{\phi}-\frac{\sigma_1^2}{2}\right]\mathrm{d}t+\sigma_1\mathrm{d}B_1(t).
\end{aligned}
$$

Integrating the above two equations from 0 to t on both sides and dividing them by t, we obtain

$$
\begin{aligned}
&\frac{\ln S(t)}{t}\\
&=\frac{\ln S(0)}{t}\\
&+\frac{1}{t}\int_0^t\left[\frac{(S^0-S)(d+\sum_{i=1}^{n}k_ix_i)}{S}-\sum_{i=1}^{n}\frac{a_ix_i}{1+b_iS}\right]\mathrm{d}t\\
&-\frac{\sigma_1^2}{2}+\frac{1}{t}\int_0^t\sigma_1\mathrm{d}B_1(t)\\
&=\left(d+\sum_{i=1}^{n}k_ix_i\right)S^0\left\langle\frac{1}{S}\right\rangle-a_1\left\langle\frac{x_1}{1+b_1S}\right\rangle\\
&-\sum_{i=2}^{n}a_i\left\langle\frac{x_i}{1+b_iS}\right\rangle-\left(d+\sum_{i=1}^{n}k_ix_i\right)-\frac{\sigma_1^2}{2}\\
&+\frac{\sigma_1B_1(t)}{t}+\frac{\ln S(0)}{t},
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
&\frac{\ln\phi(t)}{t}\\
&=\frac{\ln\phi(0)}{t}+\frac{1}{t}\int_0^t\frac{(S^0-\phi)(d+\sum_{i=1}^{n}k_ix_i)}{\phi}\mathrm{d}t-\frac{\sigma_1^2}{2}\\
&+\frac{\sigma_1B_1(t)}{t}\\
&=\left(d+\sum_{i=1}^{n}k_ix_i\right)S^0\left\langle\frac{1}{\phi}\right\rangle-\left(d+\sum_{i=1}^{n}k_ix_i\right)-\frac{\sigma_1^2}{2}\\
&+\frac{\sigma_1B_1(t)}{t}+\frac{\ln\phi(0)}{t}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}0 &\geq \frac{\ln S(t) - \ln \phi(t)}{t} \\ &= \left(d + \sum_{i=1}^{n} k_i x_i\right) S^0 \left\langle \frac{1}{S} - \frac{1}{\phi} \right\rangle - a_1 \left\langle \frac{x_1}{1 + b_1 S} \right\rangle \\ &\quad - \sum_{i=2}^{n} a_i \left\langle \frac{x_i}{1 + b_i S} \right\rangle \\ &\geq \left(d + N \sum_{i=1}^{n} k_i\right) S^0 \left\langle \frac{1}{S} - \frac{1}{\phi} \right\rangle - a_1 \langle x_1 \rangle \\ &\quad - \sum_{i=2}^{n} a_i \langle x_i \rangle ,\end{aligned} \tag{86}$$

which implies that

$$\begin{aligned}&\left\langle \frac{1}{S} - \frac{1}{\phi} \right\rangle \\ &\quad \leq \frac{1}{\left(d + N \sum_{i=1}^{n} k_i\right) S^0} \left(a_1 \langle x_1 \rangle + \sum_{i=2}^{n} a_i \langle x_i \rangle\right).\end{aligned} \tag{87}$$

Next, applying the Itô formula to $\ln x_1(t)$, one deduces

$$\begin{aligned}\mathrm{d} \ln x_1(t) &= \left[-\left(d_1 + \sum_{i=1}^{n} k_i x_i\right) + \frac{a_1 S}{1 + b_1 S} - \frac{\sigma_2^2}{2}\right] \mathrm{d}t \\ &\quad + \sigma_2 \mathrm{d}B_2(t) \\ &\geq \left[-\left(d_1 + M \sum_{i=1}^{n} k_i\right) + \frac{a_1 S}{1 + b_1 S} - \frac{\sigma_2^2}{2}\right] \mathrm{d}t \\ &\quad + \sigma_2 \mathrm{d}B_2(t).\end{aligned} \tag{88}$$

Integrating the above inequality from 0 to t on both sides and dividing all by t, we have

$$\begin{aligned}&\frac{\ln x_1(t)}{t} \\ &\quad \geq \frac{\ln x_1(0)}{t} - \frac{\sigma_2^2}{2} - \left(d + M \sum_{i=1}^{n} k_i\right) \\ &\qquad + a_1 \left\langle \frac{\phi}{1 + b_1 \phi} \right\rangle - a_1 \left\langle \frac{\phi - S}{(1 + b_1 \phi)(1 + b_1 S)} \right\rangle \\ &\qquad + \frac{\sigma_2 B_2(t)}{t} \\ &\quad \geq \frac{\ln x_1(0)}{t} - \frac{\sigma_2^2}{2} - \left(d + M \sum_{i=1}^{n} k_i\right) \\ &\qquad + a_1 \left\langle \frac{\phi}{1 + b_1 \phi} \right\rangle - \frac{a_1}{b_1^2} \left\langle \frac{1}{S} - \frac{1}{\phi} \right\rangle + \frac{\sigma_2 B_2(t)}{t} \\ &\quad \geq \frac{\ln x_1(0)}{t} - \frac{\sigma_2^2}{2} - \left(d + M \sum_{i=1}^{n} k_i\right) \\ &\qquad + a_1 \left\langle \frac{\phi}{1 + b_1 \phi} \right\rangle \\ &\qquad - \frac{a_1}{b_1^2 \left(d + N \sum_{i=1}^{n} k_i\right) S^0} \left(a_1 \langle x_1 \rangle + \sum_{i=2}^{n} a_i \langle x_i \rangle\right) \\ &\qquad + \frac{\sigma_2 B_2(t)}{t} \\ &\quad \geq \frac{\ln x_1(0)}{t} - \frac{\sigma_2^2}{2} - \left(d + M \sum_{i=1}^{n} k_i\right) \\ &\qquad + a_1 \left\langle \frac{\phi}{1 + b_1 \phi} \right\rangle - \frac{a_1^2 \langle x_1 \rangle}{b_1^2 \left(d + N \sum_{i=1}^{n} k_i\right) S^0} \\ &\qquad + \frac{\sigma_2 B_2(t)}{t}.\end{aligned} \tag{89}$$

According to Lemma 3, we have

$$\frac{\ln x_1(t)}{t} \leq \limsup_{t \longrightarrow \infty} \frac{\ln x_1(t)}{t} \leq 0, \tag{90}$$

which yields

$$\begin{aligned}\frac{a_1^2 \langle x_1 \rangle}{b_1^2 \left(d + N \sum_{i=1}^{n} k_i\right) S^0} &\geq \frac{\ln x_1(0)}{t} - \frac{\sigma_2^2}{2} \\ &\quad - \left(d + M \sum_{i=1}^{n} k_i\right) \\ &\quad + a_1 \left\langle \frac{\phi}{1 + b_1 \phi} \right\rangle + \frac{\sigma_2 B_2(t)}{t},\end{aligned} \tag{91}$$

that is,

$$\begin{aligned}\liminf_{t \longrightarrow \infty} \langle x_1(t) \rangle &\geq \frac{b_1^2 \left(d + N \sum_{i=1}^{n} k_i\right) S^0}{a_1^2} \left[-\frac{\sigma_2^2}{2}\right. \\ &\quad \left. - \left(d + M \sum_{i=1}^{n} k_i\right) + a_1 \int_0^{\infty} \frac{x \nu(\mathrm{d}x)}{1 + b_1 x}\right].\end{aligned} \tag{92}$$

Therefore, if

$$a_1 \int_0^{\infty} \frac{x \nu(\mathrm{d}x)}{1 + b_1 x} > \frac{\sigma_2^2}{2} + \left(d + M \sum_{i=1}^{n} k_i\right), \tag{93}$$

then

$$\liminf_{t \longrightarrow \infty} \langle x_1(t) \rangle > 0, \ a.s. \tag{94}$$

This completes the proof. □

Theorem 7. *System (1) with* $(S(0), x_1(0), \ldots, x_n(0)) \in R_+^{n+1}$ *admits the positive solution* $(S(t), x_1(t), \ldots, x_n(t))$. *Furthermore, if*

$$a_i \int_0^{\infty} \frac{x \nu(\mathrm{d}x)}{1 + b_i x} < d_i + N \sum_{i=1}^{n} k_i + \frac{\sigma_{i+1}^2}{2}, \quad i = 1, 2, \ldots, n, \tag{95}$$

then

$$\lim_{t\longrightarrow\infty}\frac{\int_0^t S(s)\,\mathrm{d}s}{t}=S^0,\ a.s.$$
$$\lim_{t\longrightarrow\infty}\sup\frac{\ln x_i(t)}{t}<0,\ a.s.\quad i=1,2,\ldots,n, \tag{96}$$

and $S(t)$ weakly converges to distribution v; here v is defined as in Definition 5, which means the biomass of all species in the vessel will be extinct exponentially.

Proof. It is obvious that $S(t)\le\phi(t)$ by the classical comparison theorem of stochastic differential equation, where $S(t)$ and $\phi(t)$ are the solutions of system (1) and (62), separately. Applying the Itô formula to $\ln(x_i(t))$ $(i=1,2,\ldots,n)$, we derive that

$$\begin{aligned}\mathrm{d}\ln x_i(t)&=\left(-d_i-\sum_{i=1}^n k_ix_i+\frac{a_iS}{1+b_iS}-\frac{\sigma_{i+1}^2}{2}\right)\mathrm{d}t\\&\quad+\sigma_{i+1}\mathrm{d}B_{i+1}(t)\\&\le\left(-d_i-N\sum_{i=1}^n k_i+\frac{a_i\phi}{1+b_i\phi}-\frac{\sigma_{i+1}^2}{2}\right)\mathrm{d}t\\&\quad+\sigma_{i+1}\mathrm{d}B_{i+1}(t).\end{aligned} \tag{97}$$

Since $a_iS/(1+b_iS)$ is a monotonous increasing function and $0<S(t)\le\phi(t)$,

$$\begin{aligned}\frac{\ln x_i(t)}{t}&\le\left(-d_i-N\sum_{i=1}^n k_i-\frac{\sigma_{i+1}^2}{2}\right)+a_i\left\langle\frac{\phi}{1+b_i\phi}\right\rangle\\&\quad+\frac{\sigma_{i+1}B_{i+1}(t)}{t}+\frac{\ln x_i(0)}{t}.\end{aligned} \tag{98}$$

Using the conditions that $\lim_{t\longrightarrow\infty}(B_{i+1}(t)/t)=0$, $\lim_{t\longrightarrow\infty}(\ln x_i(0)/t)=0$ and

$$a_i\int_0^\infty\frac{xv(\mathrm{d}x)}{1+b_ix}<d_i+N\sum_{i=1}^n k_i+\frac{\sigma_{i+1}^2}{2},\quad i=1,2,\ldots,n, \tag{99}$$

we know that

$$\lim_{t\longrightarrow\infty}\sup\frac{\ln x_i(t)}{t}\le0,\quad i=1,2,\ldots,n, \tag{100}$$

which means $x_i(t)$ $(i=1,2,\ldots,n)$ tends to zero exponentially almost surely.

Now we prove that (62) is stable in distribution. Set $Y(t)=\phi(t)-S^0$, then we have

$$\mathrm{d}Y=-\left(d+\sum_{i=1}^n k_ix_i\right)Y\mathrm{d}t+\sigma_1\left(S^0+Y\right)\mathrm{d}B_1(t). \tag{101}$$

According to Theorem 2.1 in [42], we know that diffusion process $Y(t)$ is stable in distribution when $t\longrightarrow\infty$. Therefore, $\phi(t)$ is also stable in distribution. We further prove that $S(t)$ is stable in distribution as $t\longrightarrow\infty$. We define the following stochastic process $S_\varepsilon(t)$ with the initial value $S_\varepsilon(0)=S(0)$ by

$$\begin{aligned}\mathrm{d}S_\varepsilon&=\left[\left(S^0-S_\varepsilon\right)\left(d+\sum_{i=1}^n k_ix_i\right)-\varepsilon S_\varepsilon\right]\mathrm{d}t\\&\quad+\sigma_1S_\varepsilon\mathrm{d}B_1(t).\end{aligned} \tag{102}$$

Then

$$\begin{aligned}\mathrm{d}\left(S-S_\varepsilon\right)&=\left[\left(\varepsilon-\sum_{i=1}^n\frac{a_ix_i}{1+b_iS}\right)S\right.\\&\quad\left.-\left(d+\varepsilon+\sum_{i=1}^n k_ix_i\right)\left(S-S_\varepsilon\right)\right]\mathrm{d}t+\sigma_1\left(S\right.\\&\quad\left.-S_\varepsilon\right)\mathrm{d}B_1(t).\end{aligned} \tag{103}$$

The solution of (103) is

$$\begin{aligned}&S(t)-S_\varepsilon(t)\\&\quad=\exp\left[\sigma_1B_1(t)-\left(d+\sum_{i=1}^n k_ix_i+\varepsilon+\frac{\sigma_1^2}{2}\right)t\right]\\&\quad\cdot\int_0^t\exp\left[\left(d+\sum_{i=1}^n k_ix_i+\varepsilon+\frac{\sigma_1^2}{2}\right)s-\sigma_1B_1(s)\right]\\&\quad\cdot\left(\varepsilon-\sum_{i=1}^n\frac{a_ix_i}{1+b_iS(s)}\right)S(s)\,\mathrm{d}s.\end{aligned} \tag{104}$$

According to Theorem 6, when $t\longrightarrow\infty$, $x_i\longrightarrow0$ $(i=2,3,\ldots,n)$, $x_1\longrightarrow x_1^*>0$ and $S\longrightarrow\lambda_1<S^0$. Therefore, for almost all $\omega\in\Omega$, $\exists T=T(\omega)$ such that

$$\varepsilon>\sum_{i=1}^n\frac{a_ix_i(t)}{1+b_iS(t)},\quad\forall t\ge T. \tag{105}$$

In addition, for all $\omega\in\Omega$, if $t>T$, it follows that

$$\begin{aligned}S(t)-S_\varepsilon(t)&=\exp\left[\sigma_1B_1(t)-\left(d+\sum_{i=1}^n k_ix_i+\varepsilon\right.\right.\\&\quad\left.\left.+\frac{\sigma_1^2}{2}\right)t\right]\\&\quad\cdot\left\{\int_0^T\exp\left[\left(d+\sum_{i=1}^n k_ix_i+\varepsilon+\frac{\sigma_1^2}{2}\right)s-\sigma_1B_1(s)\right]\right.\\&\quad\cdot\left(\varepsilon-\sum_{i=1}^n\frac{a_ix_i}{1+b_iS(s)}\right)S(s)\,\mathrm{d}s\\&\quad+\int_T^t\exp\left[\left(d+\sum_{i=1}^n k_ix_i+\varepsilon+\frac{\sigma_1^2}{2}\right)s-\sigma_1B_1(s)\right]\end{aligned}$$

$$\cdot\left(\varepsilon-\sum_{i=1}^{n}\frac{a_i x_i}{1+b_i S(s)}\right)S(s)\,\mathrm{d}s\Bigg\}\geq\exp\left[\sigma_1 B_1(t)-\left(d+M\sum_{i=1}^{n}k_i+\varepsilon+\frac{\sigma_1^2}{2}\right)t\right]\cdot\int_0^T\exp\left[\left(d+N\sum_{i=1}^{n}k_i+\varepsilon+\frac{\sigma_1^2}{2}\right)s-\sigma_1 B_1(s)\right]\cdot\left(\varepsilon-\sum_{i=1}^{n}\frac{a_i x_i}{1+b_i S(s)}\right)S(s)\,\mathrm{d}s, \tag{106}$$

and when $t\longrightarrow\infty$,

$$\exp\left[\sigma_1 B_1(t)-\left(d+M\sum_{i=1}^{n}k_i+\varepsilon+\frac{\sigma_1^2}{2}\right)t\right]\longrightarrow 0,$$
$$\left|\int_0^T\exp\left[\left(d+N\sum_{i=1}^{n}k_i+\varepsilon+\frac{\sigma_1^2}{2}\right)s-\sigma_1 B_1(s)\right]\cdot\left(\varepsilon-\sum_{i=1}^{n}\frac{a_i x_i}{1+b_i S(s)}\right)S(s)\,\mathrm{d}s\right|<\infty, \tag{107}$$

which implies that

$$\lim_{t\longrightarrow\infty}\left(S(t)-S_\varepsilon(t)\right)\geq\lim_{t\longrightarrow\infty}\inf\left(S(t)-S_\varepsilon(t)\right)\geq 0,\ a.s. \tag{108}$$

Now, we consider

$$\begin{aligned}\mathrm{d}\left(\phi(t)-S_\varepsilon(t)\right)&=\left[\varepsilon S_\varepsilon(t)-\left(d+\sum_{i=1}^{n}k_i x_i\right)\left(\phi(t)-S_\varepsilon(t)\right)\right]\mathrm{d}t+\sigma_1\left(\phi(t)-S_\varepsilon(t)\right)\mathrm{d}B_1(t)\\&\leq\left[\varepsilon S_\varepsilon(t)-\left(d+N\sum_{i=1}^{n}k_i\right)\left(\phi(t)-S_\varepsilon(t)\right)\right]\mathrm{d}t+\sigma_1\left(\phi(t)-S_\varepsilon(t)\right)\mathrm{d}B_1(t).\end{aligned} \tag{109}$$

Integrating it from 0 to t, one can derive

$$\phi(t)-S_\varepsilon(t)\leq\int_0^t\left[\varepsilon S_\varepsilon(s)-\left(d+N\sum_{i=1}^{n}k_i\right)\left(\phi(s)-S_\varepsilon(s)\right)\right]\mathrm{d}s+\int_0^t\sigma_1\left(\phi(s)-S_\varepsilon(s)\right)\mathrm{d}B_1(s). \tag{110}$$

Taking expectation for both sides of the above inequality, we obtain

$$\begin{aligned}E\left|\phi(t)-S_\varepsilon(t)\right|&\leq E\left[\int_0^t\varepsilon S_\varepsilon(s)-\left(d+N\sum_{i=1}^{n}k_i\right)\left|\phi(s)-S_\varepsilon(s)\right|\mathrm{d}s\right]\\&\leq E\left[\int_0^t\varepsilon\phi(s)-\left(d+N\sum_{i=1}^{n}k_i\right)\left|\phi(s)-S_\varepsilon(s)\right|\mathrm{d}s\right],\end{aligned} \tag{111}$$

then

$$E\left|\phi(t)-S_\varepsilon(t)\right|\leq\varepsilon\sup_{0\leq s}E\phi(s), \tag{112}$$

which leads to

$$\lim_{\varepsilon\longrightarrow 0}\inf\lim_{t\longrightarrow\infty}E\left|\phi(t)-S_\varepsilon(t)\right|=0. \tag{113}$$

According to (108), (113), and $S(t)\leq\phi(t)$, we can get

$$\lim_{t\longrightarrow\infty}\left(\phi(t)-S(t)\right)=0. \tag{114}$$

Therefore, based on the fact that $\phi(t)$ is stable to distribution v as $t\longrightarrow\infty$, $S(t)$ weakly converges to distribution v. This completes the proof. □

4. Discussion and Numerical Simulation

In this paper, a turbidostat model subjected to competition and stochastic perturbation has been investigated. For system (1), we initially proved the existence and uniqueness of global positive solution in Section 2. We further analyzed the principle of competitive exclusion for system (1). Under the conditions of Theorem 6, the species x_i $(i=2,\ldots,n)$ in the vessel will be extinct and x_1 will survive due to the fact that x_i $(i=2,\ldots,n)$ is inadequate competitor [2] if $0<\lambda_1<\lambda_2\leq\cdots\leq\lambda_n<S^0$.

Besides the above assertion, we also obtain the condition for the extinction of system (1), which means that the principle of competitive exclusion becomes invalid if the conditions of Theorem 7 are satisfied. In addition, we have the following remark:

Remark 8. Any microorganism population $x_i(t)$ $(i=1,2,\ldots,n)$ in system (1) survives and the other microorganism populations in the vessel will be extinct if the following general condition holds:

$$0<\lambda_i<\lambda_j\leq\cdots\leq\lambda_k<S^0,\quad i,j,k=1,\ldots,n\text{ and }i\neq j\neq k. \tag{115}$$

For a better understanding to the results obtained in Theorem 6, we offer an example as follows.

Let $S^0=1.2$, $d=0.12$, $d_1=0.32$, $d_2=0.31$, $a_1=2$, $b_1=3$, $a_2=2$, $b_2=5$, $\sigma_1=0.1$, $\sigma_2=0.1$, $\sigma_3=0.2$, $k_1=0.007$, and $k_2=0.01$, then system (1) becomes

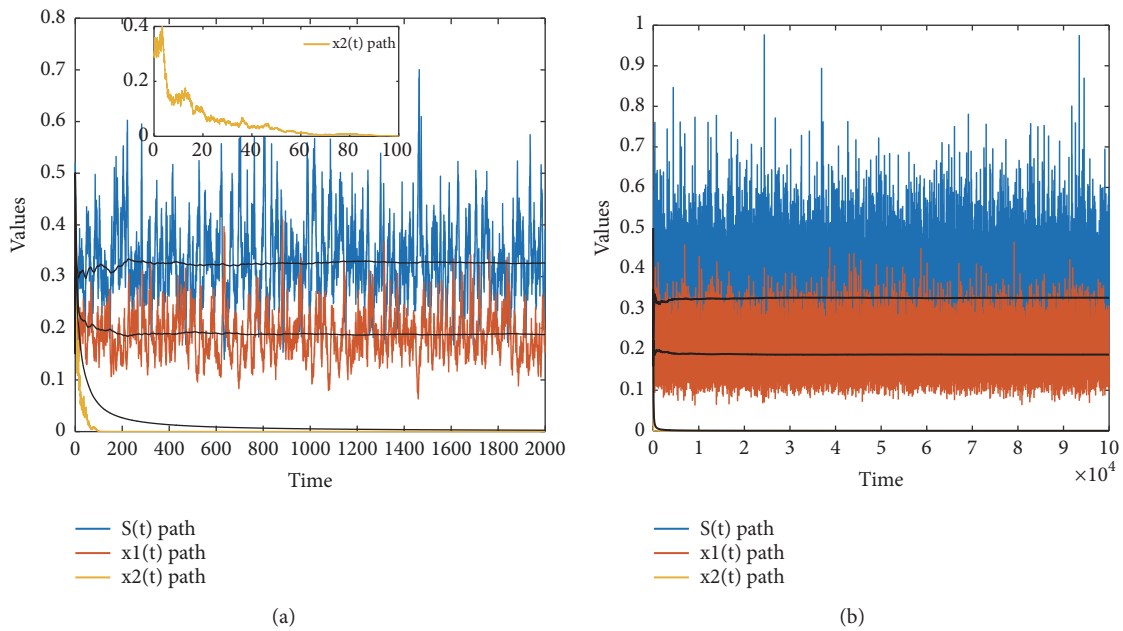

FIGURE 1: x_1 survive and x_2 will be extinct for system (116) with initial value $S(0) = 0.5$, $x_1(0) = x_2(0) = 0.3$. (a) and (b) are the time series of system with different time scale.

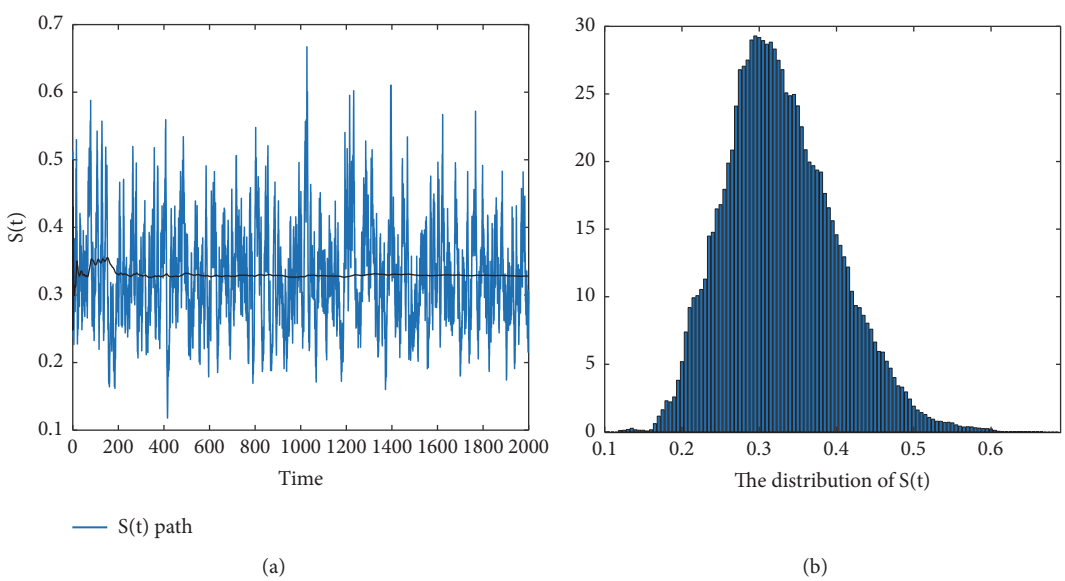

FIGURE 2: (a) is the time series of $S(t)$ with initial value $S(0) = 0.5$; (b) is the distribution interval of $S(t)$.

$$
\begin{aligned}
&\mathrm{d}S(t) \\
&\quad = \Big[(1.2 - S(t))\left(0.12 + 0.007x_1(t) + 0.01x_2(t)\right) \\
&\quad - \frac{2S(t)}{1+3S(t)}x_1(t) - \frac{2S(t)}{1+5S(t)}x_2(t)\Big]\,\mathrm{d}t \\
&\quad + 0.1S(t)\,\mathrm{d}B_1(t), \\
&\mathrm{d}x_1(t) = \Big[-\left(0.32 + 0.007x_1(t) + 0.01x_2(t)\right)x_1(t) \\
&\quad + \frac{2S(t)}{1+3S(t)}x_1(t)\Big]\,\mathrm{d}t + 0.1x_1(t)\,\mathrm{d}B_2(t), \\
&\mathrm{d}x_2(t) = \Big[-\left(0.31 + 0.007x_1(t) + 0.01x_2(t)\right)x_2(t) \\
&\quad + \frac{2S(t)}{1+5S(t)}x_2(t)\Big]\,\mathrm{d}t + 0.2x_2(t)\,\mathrm{d}B_3(t).
\end{aligned}
\tag{116}
$$

We derive $x_1^* = 0.1913$, $x_2^* = 0$, $\lambda_1 = 0.3102$, and $\lambda_2 = 0.7128$. Obviously, $\lambda_1 < \lambda_2 < S^0$. The concentration of

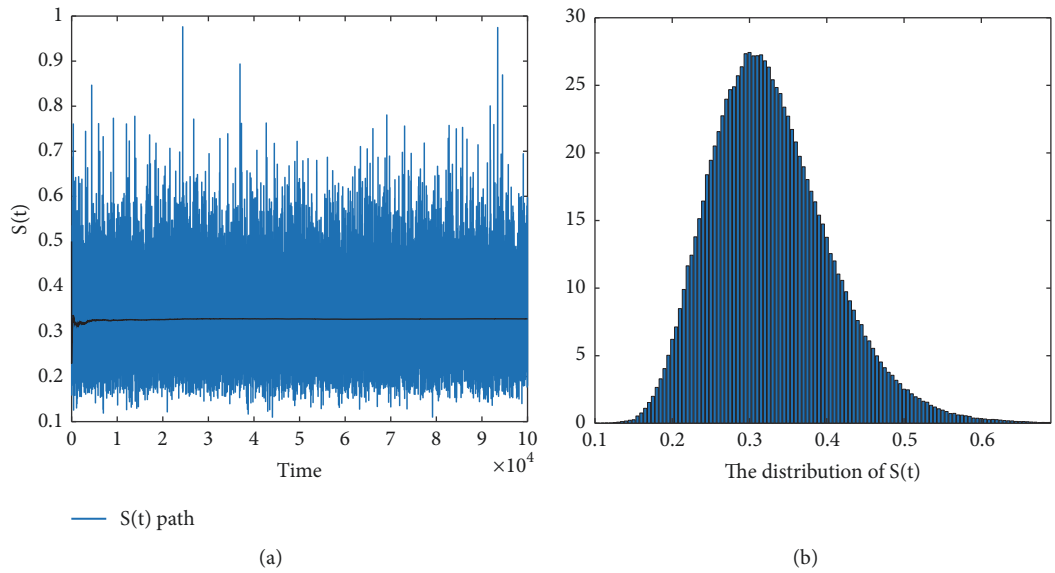

FIGURE 3: (a) is the time series of $S(t)$ with initial value $S(0) = 0.5$; (b) is the distribution interval of $S(t)$.

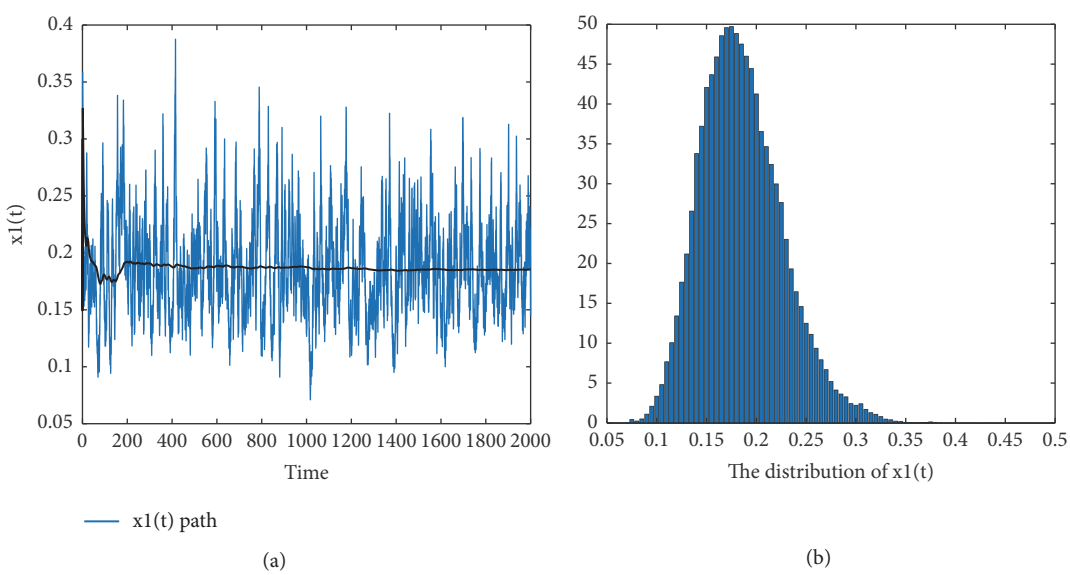

FIGURE 4: (a) is the time series of $x_1(t)$ with initial value $x_1(0) = 0.3$; (b) is the distribution interval of $x_1(t)$.

nutrient at equilibrium is $S^* = \lambda_1 = 0.3102$. Hence, the species besides $x_1(t)$ will be extinct eventually as Theorem 6 shows (see Figure 1). From the biological viewpoint, the natural resource for population will be fluctuated because of all kinds of reasons such as excessive development or natural disasters. This phenomenon also occurs in microorganisms cultivation. If the temperature or lightness is various (the density of white noise σ_i represents this phenomenon in system (1)), the dynamical behaviors, such as digestion ability for microorganisms, will be various at different points. That is, the stochastic factors will cause the variations of dynamical behaviors and this phenomenon is interacted between nutrient $S(t)$ and microorganisms $x_i(t)$ in the turbidostat. The stochastic factors always cause the fluctuation of $S(t)$ and $x_i(t)$ (see Figures 1–5).

Although the concentration of nutrients varies significantly in the system, it always fluctuates around $\lambda_1 = 0.3102$ (the black line in Figures 1(b), 2(a), and 3(a)) and has a stationary distribution in the end (see Figures 2(b) and 3(b)). The concentration of $x_1(t)$ fluctuates around $x_1^* =$

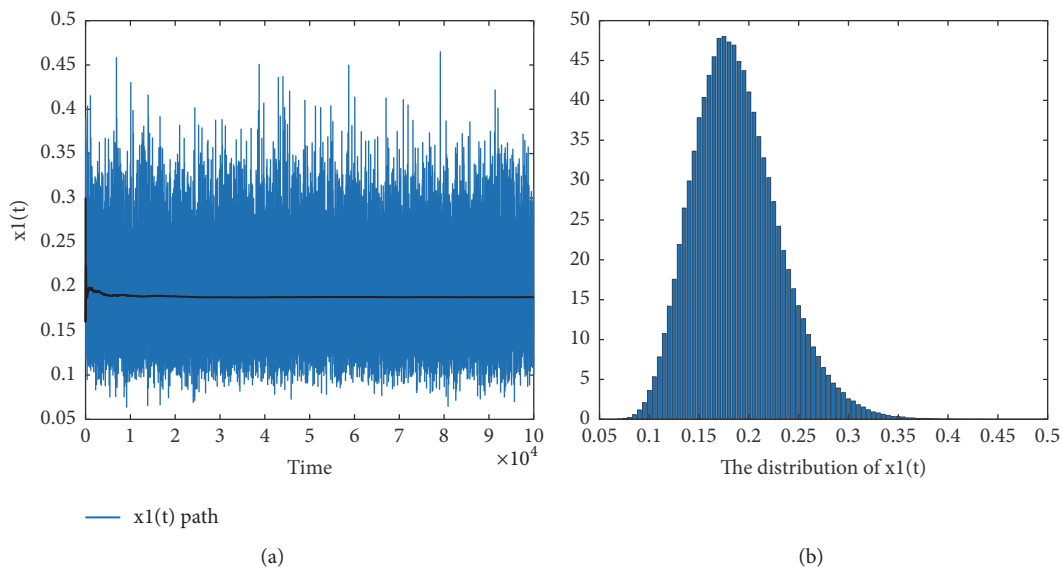

Figure 5: (a) is the time series of $x_1(t)$ with initial value $x_1(0) = 0.3$; (b) is the distribution interval of $x_1(t)$.

0.1913 (the black line in Figure 1(b)), and there exists a stationary distribution (see Figures 4(b) and 5(b)). This fluctuation phenomenon basically comes from the random factors which leads to variation of concentration for $S(t)$ and $x_i(t)$. The species $x_2(t)$ will be extinct since the concentration of nutrient $S^* = \lambda_1 = 0.3102$ is less than its break-even concentration $\lambda_2 = 0.7128$ (see Figure 1).

To further confirm the results obtained in Theorem 7, we give the following example.

Set $S^0 = 1.2$, $d = 0.8$, $d_1 = 0.9$, $d_2 = 0.9$, $a_1 = 1$, $b_1 = 0.4$, $a_2 = 1$, $b_2 = 0.6$, $\sigma_1 = 0.2$, $\sigma_2 = 0.8$, $\sigma_3 = 0.8$, $k_1 = 0.2$, and $k_2 = 0.2$, then system (1) arrives at

$$\begin{aligned}
\mathrm{d}S(t) &= \Big[(1.2 - S(t))\left(0.8 + 0.2x_1(t) + 0.2x_2(t)\right) \\
&\quad - \frac{S(t)}{1 + 0.4S(t)}x_1(t) - \frac{S(t)}{1 + 0.6S(\mathrm{t})}x_2(t)\Big]\mathrm{d}t \\
&\quad + 0.2S(t)\,\mathrm{d}B_1(t), \\
\mathrm{d}x_1(t) &= \Big[-\left(0.9 + 0.2x_1(t) + 0.2x_2(t)\right)x_1(t) \\
&\quad + \frac{S(t)}{1 + 0.4S(t)}x_1(t)\Big]\mathrm{d}t + 0.8x_1(t)\,\mathrm{d}B_2(t), \\
\mathrm{d}x_2(t) &= \Big[-\left(0.9 + 0.2x_1(t) + 0.2x_2(t)\right)x_2(t) \\
&\quad + \frac{S(t)}{1 + 0.6S(t)}x_2(t)\Big]\mathrm{d}t + 0.8x_2(t)\,\mathrm{d}B_3(t).
\end{aligned} \tag{117}$$

It follows from simple calculations to obtain $(1/a_i)[d_i + N(k_1 + k_2) + \sigma_{i+1}^2/2] \approx 1.22 > S^0$, which satisfies the conditions of Theorem 7. Hence, $x_1(t)$ and $x_2(t)$ tend to zero exponentially, which implies $x_1^* = 0$, $x_2^* = 0$ (see Figure 6).

In fact, if the density of white noise σ_{i+1} increased (such as extremely high temperature) so that $(1/a_i)[d_i + N(k_1 + k_2) + \sigma_{i+1}^2/2] > S^0$; that is, the environment is not suitable for microbial survival, both adequate and inadequate competitors will be extinct in the end (see Figure 6).

To sum up, the competition from interspecies and stochastic factors from environment may affect the dynamical behaviors of species. Inevitably, some species become adequate competitors and survive after a long time, and the other species in the system become inadequate competitors and die out in the end. However, adequate and inadequate competitors may be extinct under strong enough stochastic disturbance.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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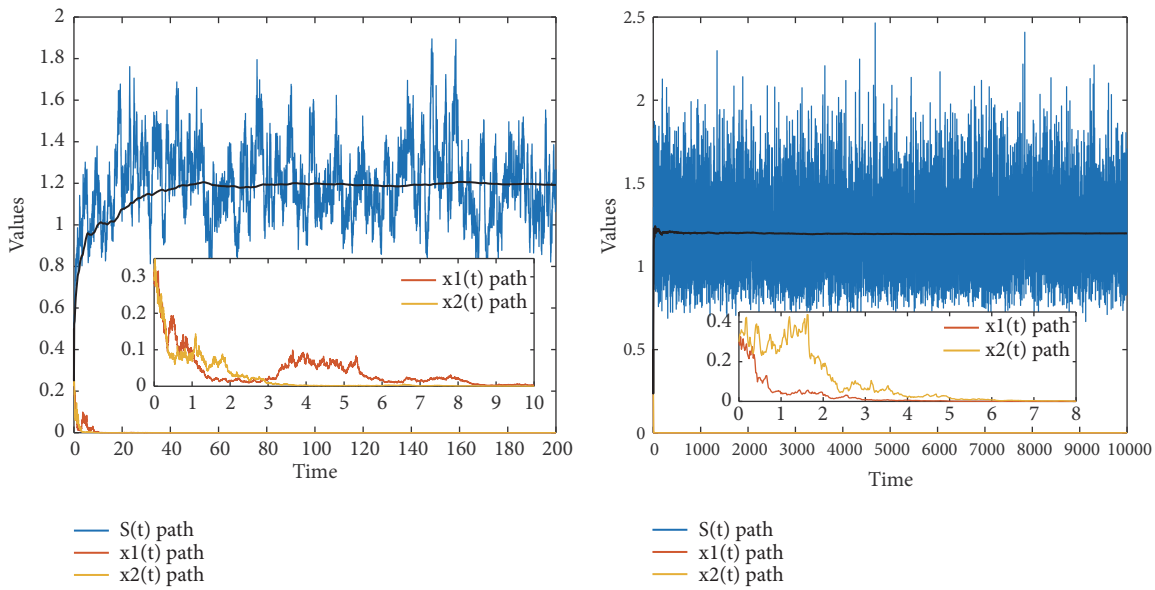

FIGURE 6: x_1 and x_2 die out for system (117) with initial value $S(0) = 0.5$, $x_1(0) = x_2(0) = 0.3$. (a) and (b) are the time series of system with different time scale.

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