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Research Article

Maximin Optimization Problem Subject to Min-Product Fuzzy Relation Inequalities with Application in Supply and Demand Scheme

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In a supply chain system, the prices with which the suppliers supply its local commodity to the retailers should satisfy the requirements of the retailers and the consumers. The supply and demand scheme satisfying these requirements is reduced into fuzzy relation inequalities (FRIs) with min-product composition. Due to the difference between the min-product composition and the classical max-t-norm one, we first study the resolution of such min-product FRI system. For optimization management in the supply chain system, we further investigate a maximin programming problem subject to the min-product FRIs. An algorithm is proposed to obtain the optimal solution based on the quasi-maximal matrix and corresponding index set. To illustrate the efficiency of our proposed algorithm, we provide a simple numerical example. The obtained optimal solution reflects an optimal pricing scheme, which maximizes the minimum prices of the commodity from the suppliers.

1. Introduction

The classical fuzzy relational equation (FRE) system could be formulated as

$$A \circ x = b, \tag{1}$$

in which

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \tag{2}$$

All the elements in A, x, and b belong to $[0,1]$. The composition $\circ$ in the equation is the classical *max-min* $(\vee - \wedge)$. For the convenience of presentation, we always denote the index sets $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ from now on. Moreover, the components in the right hand vector b are assumed to be $b_m \leqslant b_{m-1} \leqslant \cdots \leqslant b_2 \leqslant b_1$ in this paper.

Concept of FRE was first introduced by E. Sanchez [1–3]. Since E. Sanchez investigated the max-min FREs with application in medical diagnosis, some fuzzy mathematical scholars became to focus on FRE and its resolution. Efficient method for finding out all its solutions was one of the most interesting and important research topics [4–10]. The operations in a classical linear equations system are addition

(+) and multiple (.). However, the operations in system (1) are max ($\vee$) and min ($\wedge$). Hence resolution of system (1) is much different from that of the classical linear equations. As known to everyone, solution set of the classical (+, .) is usually a convex set. However the solution set of system (1) (when it is consistent) is nonconvex in most cases. We call system (1) consistent (or solvable), iff $X^* \neq \emptyset$, where

$$X^* = \{x \in [0,1]^n \mid A \circ x = b\} \tag{3}$$

represents its complete solution set.

It is well known that the consistency of (1) could be checked by its potential maximum (or greatest) solution. The necessary and sufficient condition, with which system (1) is consistent, is there exists a (unique) maximum solution. When the system is consistent, then its complete solution set could be formed and generated by the unique maximum solution and all the minimal ones. In general situation, a consistent system of max-min FREs has a finite number of minimal solutions. Although the number of minimal solutions is finite, it is difficult to find out all of them. The set of all minimal solutions could be found theoretically. But it is still a challenging problem until now. Due to its important role in solving (1), many scholars tried their bests to develop novel and efficient approaches to obtain the minimal solution set [11–17].

In the FRI system (1), the max-min composition could be extended to other ones, such as max-product and addition-min [18–22]. Data transmission on the Peer-to-Peer (P2P) network system was investigated by Yang et al. FRIs with addition-min composition were introduced to describe the transmission mechanism in such system [18, 20, 21]. The author established some optimization management models to the P2P network system. Specific methods were proposed to search one of the optimal solution(s), which represent(s) an optimal quantity of flow.

In a fuzzy relational equations system, the composition operator is indispensable. As mentioned above, the most common and typical one is *max-min* ($\vee, \wedge$). However, for wide application, the composition was generalized to max-t-norm one. Here t is a continuous triangular norm. However we have to point out, the minimum operator, as a specific kind t-norm, is irreplaceable and most frequently adopted in many application fields. Moreover, resolution method for a max-t-norm FRE or FRI system is similar to that for system (1). The structures of their complete solution sets are also the same.

In many practical application fields, mathematical programming with FREs or FRIs constraint was established and investigated, for describing the corresponding optimization model. Resolution of such optimization problems is usually related to the properties and structure of the feasible domain (i.e., the solution set of a FREs or FRIs system), and also related to the characteristic of the objective function.

Wang [23] was the pioneer who investigated the fuzzy relation programming problem. In such optimization problem studied in [23], the target is the latticized linear function, and the constraint is formed by a group of FRIs composed by ($\vee, \wedge$). The constraint system was firstly discussed. To characterize the feasible domain, the authors hoped to find out all the minimal solutions to the constraint. Conservative path approach was the effective method that proposed in [23]. Characteristic matrix was defined for the constraint system, based on which all the conservative paths could be found. It was further proved that a conservative path always corresponds to a minimal solution. This sets up the relationship between the minimal solution and the conservative path (formed by some indices). Due to the monotonicity of the objective function, the optimal solution of the target problem should be one of the minimal solutions. Hence, after finding out all the minimal solutions by the conservative path approach, the optimal solution could be selected by simple comparison calculation. S.-C. Fang is another settler on fuzzy relation optimization problem. Together with G. Li [24] and J. Loetamonphong [25], he proposed and studied the minimization problem with FRE constraints. Different from the optimization problem presented in [23], Fang and Li focused on the classical linear function. They tried to minimize a linear function, i.e., $\sum_{j=1}^{n} c_j x_j$, with the constraint of a group of max-min FREs. Corresponding problem was written as

$$\begin{aligned} \text{minimize} \quad & z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \text{subject to} \quad & A \circ x = b, \\ & 0 \leqslant x_j \leqslant 1. \end{aligned} \tag{4}$$

In problem (4), the coefficients were classified into two kinds, negative and nonnegative. Correspondingly, each kind of coefficients formed a new subproblem. Two subproblems were generated according to these objective coefficients. One of the subproblems was constructed by the negative coefficients. This subproblem could be simply solved by the maximum solution of the feasible domain. But resolution of the other subproblem was much harder. It might take much more computation cost. The optimal solution of such nonnegative subproblem could be selected from the minimal solutions. However, for decreasing the computation cost, the authors [24] avoided to compute all the minimal solutions. Novel way was developed for searching the optimal solution. They defined some index sets, corresponding the so-called quasi-minimal solutions. Complete feasible domain was characterized by the index sets. The nonnegative subproblem was further converted into 0-1 integer programming. Hence, the optimal solution could be found step by step according the the typical branch-and-bound approach. In fact, the idea that dividing the main problem into a negative-coefficients subproblem and a nonnegative-coefficients one, was further deeply developed and widely used afterwards, in other relevant fuzzy relation optimization problems. It turns out to be the most important method for such kind of problems [26–31].

Optimizing a linear function, i.e., $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$, which is restricted to the a FREs or FRIs system, becomes attractive and interesting research topic. Since the feasible domain is usually nonconvex, classical optimization method seems to be useless. To deal with such problem, some researchers improved the existing methods, or proposed some novel resolution methods [27–33]. In recent years, some

researchers have turned their attentions to the fuzzy relation nonlinear optimization problems [34–41], especially to the fuzzy relation geometric programming problems [42–49].

Bingyuan Cao [42, 44, 45] was the one who proposed the so-called "fuzzy relation geometric programming" for the first time. Some special forms of the geometric function were first studied. In [42], Yang and Cao gave the mathematical formulae of the monomial geometric programming problem. In such optimization problem, the objective function was nonlinear. Corresponding mathematical model was described as follows:

$$\begin{aligned} &\textit{minimize} \quad z(x_1, x_2, \ldots, x_n) = c\prod_{j=1}^{n} x_j^{r_j} \\ &\textit{subject to} \quad A \circ x = b. \end{aligned} \tag{5}$$

In the above problem, the coefficient c and all the exponents $\{r_j \mid j \in J\}$ are constant numbers. Notice that the feasible domain $\{x \in [0,1]^n \mid A \circ x = b\}$ could be written as a union of finite close intervals, whose left endpoints are indeed the minimal solutions. There are finite number of closed intervals. The optimal solution of one of the subproblems could be chosen from the minimal solution set. One just need to compare all the function values of the minimal solutions. Similar problem was also studied by Shivanian and Khorram [46]. They further improved the resolution method in [42], for the following fuzzy relational optimization:

$$\begin{aligned} &\textit{minimize} \quad z(x_1, x_2, \ldots, x_n) = c x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ &\textit{subject to} \quad A \star x \geq d^1, \\ &\qquad\qquad\quad B \star x \leq d^2. \end{aligned} \tag{6}$$

In order to accelerate resolution procedures and decrease the computation, they deleted the components of x, who were redundant and useless for searching the optimal solution.

With similar constraint, the mathematical models

$$\begin{aligned} &\min \quad z = (c_1 \wedge x_1^{r_1}) \vee (c_2 \wedge x_2^{\alpha_2}) \vee \cdots \vee (c_n \wedge x_n^{\alpha_n}) \\ &s.t. \quad A \circ x = b, \\ &\qquad 0 \leq x_j \leq 1, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\min \quad z = (c_1 \cdot x_1^{r_1}) \vee (c_2 \cdot x_2^{r_2}) \vee \cdots \vee (c_n \cdot x_n^{r_n}) \\ &s.t. \quad A \circ x = b, \\ &\qquad 0 \leq x_j \leq 1, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{8}$$

were investigated by Y.-K. Wu [43] and X.-G. Zhou [50], respectively. In problems (7) and (8), the coefficients are all nonnegative, i.e., $c_j \in R$ and $c_j \geq 0$, $j = 1, 2, \ldots, n$. The authors did not consider the negative coefficients. To solve these problems, it was unnecessary to compute all the (potential) minimal solutions. Hence the computation was decreased in some degree. Polynomial algorithms were proposed for such problems [43, 50].

Recently, fuzzy relation inequality with min-product composition was defined and investigated [51–53]. It was first introduced to describe the pricing in the supply and demand scheme [53]. We view this supply and demand scheme as a simple situation of a supply chain system. In [53], the authors only considered the requirement of the retailers. In such case, the requirement of the suppliers might not be fulfilled and a group of min-product fuzzy relation inequalities was constructed. In order to satisfy both requirement of the retailers and the suppliers, Yang et al. [51, 52] further established and studied the min-product system with one more constraint $x \leq \overline{x}$. In [51, 52], checking method of consistency of a min-product system and structure of its solution set (when consistent) were investigated. Based on the specific structure of the complete solution set of min-product fuzzy relation inequalities system, Yang et al. [52] further studied a lexicographic optimization problem. Detailed resolution algorithm was proposed for obtaining the unique optimal solution of such problem. The optimal solution is indeed an optimal pricing scheme maximizing the profits of the suppliers in a fixed lexicographic order relation. Or in other words, the profits of the suppliers are maximized with fixed priority grade. Hence, the suppliers are treated distinguishingly. However in some cases, the suppliers should not be treated with fixed priority grade. They should be treated equally. Based on such equalitarianism consideration, we propose a new type of optimal model, i.e., the maximin optimization problem subject to min-product fuzzy relation inequalities, and investigate its resolution method in this work.

The structure of the rest part of our work is as follows. In Section 2 we introduce the min-product FRIs system. Relevant properties and results are presented. Matrix-based method is proposed to find out all the solutions of such system in Section 3. In Section 4, we study the maximin programming problem subject to min-product FRIs system. Resolution algorithm is developed to search its optimal solution, with illustrating example. Sections 5 and 6 are simple discussion and conclusion respectively.

2. Preliminaries

2.1. Pricing Relation and the Corresponding Mathematical Model in a Supply Chain. This subsection provides simple application background for our studied min-product FRIs, which is adopted from [51, 52].

As shown in Figure 1. We assume that the supply chain is composed by n suppliers and m retailers. The suppliers were denoted by $S_1, S_2, \ldots, S_n$, while the retailers were denoted by $R_1, R_2, \ldots, R_m$. Suppose the suppliers will supply a sort of commodities to the retailers. The trade is free. We consider the subjective factors whether the suppliers are willing to supply its local commodities. What influence the transaction is the price of the commodities. For convenience to express, we define two index sets, i.e.,

$$\begin{aligned} I &= \{1, 2, \ldots, m\}, \\ J &= \{1, 2, \ldots, n\}. \end{aligned} \tag{9}$$

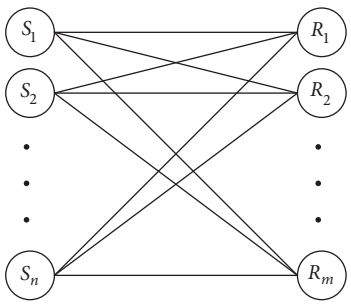

Figure 1: Supply chain system including n suppliers and m retailers.

Suppose the selling price of the commodities from S_j is x_j. Considering the transportation and other costs, when the commodities are received by R_i, the price should be bigger than x_j. We denote this received price by $a_{ij}x_j$, where a_{ij} is a real number bigger than 1. There is no doubt that the retailer will choose the suppler with the cheapest received price to stock the commodities. Assume that the price requirement of the commodity of retailer R_i is no bigger than b_i, $\forall i \in I$, then we have

$$a_{i1}x_1 \wedge a_{i2}x_2 \wedge \cdots \wedge a_{in}x_n \le b_i. \tag{10}$$

On the other hand, if the prime (cost) prices of the commodities for the suppliers $S_1, S_2, \ldots, S_n$ are $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$, respectively, then to ensure the supplier is profitable, it holds that

$$x_j \ge \underline{x}_j, \quad j \in J. \tag{11}$$

Combining these two aspects, the prices $x_1, x_2, \ldots, x_n$ should satisfy the following system:

$$\begin{aligned} &\bigwedge_{j\in J} a_{1j}x_j \le b_1, \\ &\bigwedge_{j\in J} a_{2j}x_j \le b_2, \\ &\cdots \\ &\bigwedge_{j\in J} a_{mj}x_j \le b_m, \\ &x_j \ge \underline{x}_j, \quad j \in J. \end{aligned} \tag{12}$$

In system (12), since the requirements of all the retailers are satisfies, this indicates all the retailers are supplied with the commodities by at least one supplier. However, this is not able to make all supplier to supply its local commodities to at least one retailer. That is to say, although some price scheme x satisfies system (12), there might exists some supplier who has not been chosen by the retailer. In order to make all suppliers able to supply their local commodities, a feasible price scheme x should also fulfill at least one of the inequalities below,

$$\begin{cases} a_{1j}x_j \le b_1, \\ a_{2j}x_j \le b_2, \\ \cdots \\ a_{mj}x_j \le b_m, \end{cases} \quad \text{i.e.} \quad \begin{cases} x_j \le \dfrac{b_1}{a_{1j}}, \\ x_j \le \dfrac{b_2}{a_{2j}}, \\ \cdots \\ x_j \le \dfrac{b_m}{a_{mj}}. \end{cases} \tag{13}$$

Satisfying at least one of the inequalities in system (13) is in fact equivalent to

$$x_j \le \bigvee_{i\in I} \frac{b_i}{a_{ij}}. \tag{14}$$

Combining both inequalities (12) and (14), we get

$$\begin{aligned} &\bigwedge_{j\in J} a_{ij}x_j \le b_i, \quad i \in I \\ &x_j \ge \underline{x}_j, \quad j \in J, \\ &x_j \le \bigvee_{i\in I} \frac{b_i}{a_{ij}}, \quad j \in J. \end{aligned} \tag{15}$$

After standardization of all the variables and parameters, system (15) becomes a system of min-product FRIs, with $a_{ij}, x_j, \underline{x}_j, b_i \in [0,1]$, $i \in I$, $j \in J$. Moreover, in this paper we always denote $\overline{x}_j = \bigvee_{i\in I}(b_i/a_{ij})$, $\forall j \in J$. Then the matrix form of (15) turns out to be

$$A \odot x \le b, \quad \underline{x} \le x \le \overline{x}. \tag{16}$$

In system (16), the entries of matrix $A_{m\times n}$ and vectors $x, \underline{x}, \overline{x}, b$ are $a_{ij}, x_j, \underline{x}_j, \overline{x}_j, b_i$ respectively.

2.2. Consistency Checking for System (16). In this subsection we provide some basic concepts and existing results of (16). The min-product FRI system has been studied in [52]. The authors investigated some basic properties of the system, including consistency checking and structure of its solution set. Besides, lexicographic maximum solution, as a specific solution, was defined and introduced to maximize the profits of the suppliers under some fixed priority grade [52].

In the rest we always denote $X = [0,1]^n$ for convenience. Moreover, the solution set of (16) is denoted by

$$X(A,b) = \{x \in X \mid A \odot x \le b,\ \underline{x} \le x \le \overline{x}\}, \tag{17}$$

in which $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n)$, $\overline{x} = (\bigvee_{i\in I}(b_i/a_{i1}), \ldots, \bigvee_{i\in I}(b_i/a_{in}))$. System (16) is called *consistent* unless $X(A,b) \ne \emptyset$. Otherwise, it is said to be *inconsistent*.

Definition 1 (see [51, 52]). The minimum (or minimal) element in $X(A,b)$ is called minimum (or minimal) solution of system (16). Analogously, the maximum solution and maximal solution could be defined in the same way.

The minimum solution and maximal solution are important for generating all the solutions of (16). Before computing all the solutions of (16), we present method for checking the system's consistency of (16).

Theorem 2 (see [51, 52]). *System (16) is consistent iff* $\underline{x} \in X(A,b)$.

It is clear that, if system (16) is consistent, then the vector $\underline{x} \in X(A,b)$ should be its minimum solution.

Define the matrix $D = (d_{ij})_{m\times n}$, where

$$d_{ij} = \begin{cases} \dfrac{b_i}{a_{ij}} \wedge 1, & \text{if } \underline{x}_j \le \dfrac{b_i}{a_{ij}}, \\ 0, & \text{if } \underline{x}_j > \dfrac{b_i}{a_{ij}}. \end{cases} \tag{18}$$

Theorem 3 (see [51, 52]). *System (16) is consistent if and only if there exists a nonzero element in every row and every column in the matrix D.*

Both Theorems 2 and 3 could be applied to check the consistency of (16).

When system (16) is consistent, i.e., its solution set is nonempty, then the special structure of this solution set is shown in the following Theorem 4.

Theorem 4 (see [51, 52]). *For the consistent system (16), suppose $X(A,b)$ represents its solution set. Then we have*

$$X(A,b) = \bigcup_{\hat{x}\in\widehat{X}(A,b)} [\check{x}, \hat{x}]. \tag{19}$$

Here $\check{x} = \underline{x}$ and $\widehat{X}(A,b)$ is the collection of all its maximal solutions.

Since $\underline{x}$ is self-evident, solving $X(A,b)$ is equal to computing $\widehat{X}(A,b)$. In next section we will provide matrix-based method for solving $X(A,b)$, i.e., the solution set of (16).

Illustrative example for the above-provided consistency-checking theorems could be found in References [51, 52].

3. Matrix-Based Resolution for System (16)

In this section we provide a novel method for obtaining the complete solution set $X(A,b)$ of system (16). The solution method is based on the below-defined concept of quasi-maximal Matrix. Thus we call it matrix-based method.

Theorem 5. *Take arbitrary $x \in X$. Suppose $\underline{x} \le x \le \overline{x}$. Then $x \in X(A,b)$ if and only if for any $i \in I$, there exists a corresponding index $j_i \in J$ such that $a_{ij_i}x_{j_i} \le b_i$.*

Proof. ($\Longrightarrow$) Since $x \in X(A,b)$, it holds for any $i \in I$ that

$$\bigwedge_{j\in J} a_{ij}x_j \le b_i. \tag{20}$$

Notice that J is a finite set. For any $i \in I$ there exists $j_i \in J$ such that

$$a_{ij_i}x_{j_i} = \bigwedge_{j\in J} a_{ij}x_j. \tag{21}$$

Thus we have $a_{ij_i}x_{j_i} \le b_i$.

($\Longleftarrow$) If for any $i \in I$,

$$\exists j_i \in J, \quad \text{s.t. } a_{ij_i}x_{j_i} \le b_i, \tag{22}$$

then

$$\bigwedge_{j\in J} a_{ij}x_j \le a_{ij_i}x_{j_i} \le b_i. \tag{23}$$

Combining $\underline{x} \le x \le \overline{x}$, it is clear that $x \in X(A,b)$. □

Based on the above-provided matrix $D = (d_{ij})$, we define concept of quasi-maximal matrix as follows.

Definition 6 (Quasi-maximal matrix). A matrix $q = (q_{ij})m \times n$ is called a quasi-maximal matrix, if it fulfills the following conditions:

(i) $q_{ij} \in \{0, d_{ij}\}$;

(ii) for any $i \in I$, $|J_i^q| = 1$, where $J_i^q = \{j \in J \mid q_{ij} > 0\}$.

It is shown in Definition 6 that the (i,j) element in a quasi-maximal matrix q is either $q_{ij} = 0$ or $q_{ij} = d_{ij}$. Moreover, in each row in the quasi-maximal matrix q, there exists and only exists one nonzero (or positive) element. By Theorem 3 and Definition 6 we quickly get the following Corollary 7.

Corollary 7. *If system (16) is consistent, then there exists at least a quasi-maximal matrix.*

For system (16), we denote the set of all quasi-maximal matrices by Q. Following Corollary 7, when (16) is consistent, it holds that $Q \ne \emptyset$.

For a given quasi-maximal matrix $q \in Q$, define

$$I_j^q = \left\{i \in I \mid q_{ij} > 0\right\}, \quad j \in J. \tag{24}$$

Then we are able to construct a corresponding vector, denoted by $x^q = (x_1^q, x_2^q, \ldots, x_n^q)$, as follows:

$$x_j^q = \begin{cases} \overline{x}_j, & \text{if } I_j^q = \emptyset, \\ \bigwedge_{i\in I_j^q} q_{ij}, & \text{if } I_j^q \ne \emptyset. \end{cases} \tag{25}$$

Theorem 8. *If $q \in Q$ is a quasi-maximal matrix of system (16), then the vector x^q defined by (25) is a solution of (16), i.e., $x^q \in X(A,b)$.*

Proof. (i) $x^q \ge \underline{x}$. Take arbitrary $j \in J$. If $I_j^q = \emptyset$, then $x_j^q = \overline{x}_j \ge \underline{x}_j$. If $I_j^q \ne \emptyset$, then for any $i \in I_j^q$, $q_{ij} > 0$. That is $q_{ij} = d_{ij} > 0$. Thus $\underline{x}_j \le b_i/a_{ij}$. On the other hand, it is clear that $\underline{x}_j \le 1$. Hence $\underline{x}_j \le (b_i/a_{ij}) \wedge 1$ and then

$$\underline{x}_j \le \bigwedge_{i\in I_j^q}\left(\frac{b_i}{a_{ij}} \wedge 1\right) = \bigwedge_{i\in I_j^q} d_{ij} = \bigwedge_{i\in I_j^q} q_{ij} = x_j^q. \tag{26}$$

(ii) $x^q \le \overline{x}$. Take arbitrary $j \in J$. If $I_j^q = \emptyset$, then $x_j^q = \overline{x}_j$. If $I_j^q \ne \emptyset$, then for any $i \in I_j^q$, similarly we get

$$q_{ij} = d_{ij} = \frac{b_i}{a_{ij}} \wedge 1 > 0. \tag{27}$$

Hence

$$x_j^q = \bigwedge_{i\in I_j^q} q_{ij} = \bigwedge_{i\in I_j^q} d_{ij} = \bigwedge_{i\in I_j^q}\left(\frac{b_i}{a_{ij}}\wedge 1\right) \le \bigwedge_{i\in I_j^q}\frac{b_i}{a_{ij}} \le \bigvee_{i\in I}\frac{b_i}{a_{ij}} = \overline{x}_j. \tag{28}$$

(iii) Since $q \in Q$ is a quasi-maximal matrix, it follows from Definition 6 that $J_i^q = \{j \in J \mid q_{ij} > 0\} \neq \emptyset$ and $|J_i^q| = 1$, for arbitrary $i \in I$. It is reasonable to assume that $J_i^q = \{j_i\}$, $\forall i \in I$. Then it holds that $q_{ij_i} = d_{ij_i} = (b_i/a_{ij_i}) \wedge 1 > 0$. Hence $I_{j_i}^q \neq \emptyset$ and $x_{j_i}^q = \bigwedge_{i\in I_{j_i}^q} q_{ij_i} \le q_{ij_i}$.

Notice that $a_{ij_i} > 0$. We have

$$a_{ij_i}x_{j_i}^q \le a_{ij_i}q_{ij_i} = a_{ij_i}\left(\frac{b_i}{a_{ij_i}}\wedge 1\right) \le a_{ij_i}\frac{b_i}{a_{ij_i}} = b_i. \tag{29}$$

According to Theorem 5, the above-proved (i), (ii) and (iii) contribute to $x^q \in X(A,b)$. □

Theorem 9. *Let $y \in X(A,b)$ be an arbitrary solution of system (16). There exists a quasi-maximal matrix $q \in Q$, such that $y \in [\underline{x}, x^q]$.*

Proof. Since $y \in X(A,b)$, $\underline{x} \le y \le \overline{x}$ is evident. According to Theorem 5, for each $i \in I$, there exists a $j_i \in J$ such that

$$a_{ij_i}y_{j_i} \le b_i, \quad \text{i.e. } y_{j_i} \le \frac{b_i}{a_{ij_i}}. \tag{30}$$

Until now we have found m indices $j_1, j_2, \ldots, j_m$. Inequality (30) indicates $\underline{x}_{j_i} \le y_{j_i} \le b_i/a_{ij_i}$. By (18), we have

$$d_{ij_i} = \frac{b_i}{a_{ij_i}} \wedge 1 > 0. \tag{31}$$

Define $q = (q_{ij})_{m\times n}$, in which

$$q_{ij} = \begin{cases} d_{ij_i}, & \text{if } j = j_i, \\ 0, & \text{if } j \neq j_i, \end{cases} \tag{32}$$

According to Definition 6, it is easy to check that $q \in Q$, i.e., q is a quasi-maximal matrix of system (16). To complete the proof, we just need to verify that $y \le x^q$. Take arbitrary $j \in J$. Next we check $y_j \le x_j^q$ in two cases.

(i) If $I_j^q \neq \emptyset$, then $x_j^q = \bigwedge_{i\in I_j^q} q_{ij}$. By (24), for any $i \in I_j^q$, it holds that $q_{ij} > 0$. By (32), it holds that

$$j = j_i, \quad \forall i \in I_j^q, \tag{33}$$

and

$$q_{ij_i} = d_{ij_i} = \frac{b_i}{a_{ij_i}} \wedge 1 > 0, \quad \forall i \in I_j^q. \tag{34}$$

It is obvious that $y_{j_i} \le 1, \forall i \in I$. Combining (30), we have

$$y_{j_i} \le \frac{b_i}{a_{ij_i}} \wedge 1, \quad \forall i \in I. \tag{35}$$

For $i \in I_j^q$, it holds that

$$y_j = y_{j_i} \le \frac{b_i}{a_{ij_i}} \wedge 1 = q_{ij_i} = q_{ij}. \tag{36}$$

Hence $y_j \le \bigwedge_{i\in I_j^q} q_{ij} = x_j^q$.

(ii) If $I_j^q = \emptyset$, then $x_j^q = \overline{x}_j$. It is obvious that $y_j \le \overline{x}_j = x_j^q$. □

Combining the above Theorems 8 and 9, the following Theorem 10 is self-evident.

Theorem 10. *Suppose (16) is consistent system, with minimum solution $\underline{x}$. Besides, the set of all its quasi-maximal matrices is denoted by Q. Then we have*

$$X(A,b) = \bigcup_{q\in Q}[\underline{x}, x^q]. \tag{37}$$

As shown in Theorem 8, for arbitrary $q \in Q$, the vector x^q defined by (25) is indeed a solution of (16). We call x^q the quasi-maximal solution corresponding to the q. Hence the set of all quasi-maximal solutions is

$$X^Q = \{x^q \mid q \in Q\}. \tag{38}$$

Based on the definition of quasi-maximal matrix, it is easily found that Q is a finite set. Consequently, X^Q is also a finite set. Moreover, the relation between X^Q and the maximal solution set $\widehat{X}(A,b)$ is as described below.

Proposition 11. *In system (16) it holds that $\widehat{X}(A,b) \subseteq X^Q$.*

To get the maximal solution set of system (16), denoted by $\widehat{X}(A,b)$ previously, we may delete all the quasi-maximal solutions which are not maximal from the set X^Q.

In fact, Theorem 10 indicates a resolution approach for obtaining the complete solution set of system (16). Based on the matrix D, we can compute all the quasi-maximal matrices. Then we further get the set of all quasi-maximal matrices, i.e., Q, and the set of all quasi-maximal solution, i.e., X^Q. At last we are able to find the solution set $X(A,b)$ by Theorem 10 or by Theorem 4.

4. Maximin Programming with Min-Product FRIs Constraint

In this section we study the maximin programming problem subject to system (16), i.e.,

$$\begin{aligned} \max \quad & z(x) = x_1 \wedge x_2 \wedge \cdots \wedge x_n \\ \text{s.t.} \quad & A \odot x^T \le b^T, \quad \underline{x} \le x \le \overline{x}. \end{aligned} \tag{39}$$

In problem (39), the constraint is system (16).

As pointed out previously, the $j$the supplier S_j supply its local commodity with price x_j. The bigger value x_j take, the higher profit S_j gain. To enhance the profit of the suppliers, we should maximize the prices $x_1, x_2, \ldots, x_n$. However, in most common cases, one is not able to maximize all the valuables $x_1, x_2, \ldots, x_n$ simultaneously. In this paper, based on the fairness consideration, we try to ensure the worst profit of all the suppliers. To reach such goal, we should maximize the minimum price, i.e., $x_1 \wedge x_2 \wedge \cdots \wedge x_n$. As a consequence, we construct the corresponding mathematical model as (39).

Next we investigate some properties of problem (39) and try to develop an algorithm to find out one of its optimal(s).

Theorem 12. *If system (16) is consistent, then problem (39) has at least one optimal solution. Furthermore, there exists a maximal solution $\hat{x} \in \widehat{X}(A,b)$, such that $\hat{x}$ is an optimal solution of (39).*

Proof. If system (16) is consistent, then X^{Q} is a finite set and $\widehat{X}(A,b) \subseteq X^{\mathrm{Q}}$. Obviously $\widehat{X}(A,b)$ is also a finite set. We may assume that

$$\widehat{X}(A,b) = \left\{\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^s\right\}. \tag{40}$$

Let

$$\hat{x}^t_{min} = \hat{x}^t_1 \wedge \hat{x}^t_2 \wedge \cdots \wedge \hat{x}^t_n, \tag{41}$$

$t = 1, 2, \ldots, s$, and

$$\hat{x}_{max} = \hat{x}^1_{min} \vee \hat{x}^2_{min} \vee \cdots \vee \hat{x}^s_{min}. \tag{42}$$

Obviously there exists $t^* \in \{1, 2, \ldots, s\}$ such that

$$\hat{x}^{t^*}_{min} = \hat{x}_{max}, \tag{43}$$

i.e.,

$$z\left(\hat{x}^{t^*}\right) = \hat{x}^{t^*}_1 \wedge \hat{x}^{t^*}_2 \wedge \cdots \wedge \hat{x}^{t^*}_n = \hat{x}_{max}. \tag{44}$$

Next we verify that $\hat{x}^{t^*}$ is an optimal solution of problem (39). Notice that $\hat{x}^{t^*} \in \widehat{X}(A,b) \subseteq X(A,b)$ is obviously a feasible solution of (39). It corresponding function value is $z(\hat{x}^{t^*}) = \hat{x}_{max}$. To finish the proof, we should further check that $z(y) \leq \hat{x}_{max}$ holds for any $y \in X(A,b)$.

Take arbitrary $y \in X(A,b)$. According to Theorem 4, there exists a maximal solution $\hat{x} \in \widehat{X}(A,b)$ such that $\underline{x} \leq y \leq \hat{x}$. Since $\hat{x} \in \widehat{X}(A,b)$, we have

$$\begin{aligned} z(y) &= y_1 \wedge y_2 \wedge \cdots \wedge y_n \leq \hat{x}_1 \wedge \hat{x}_2 \wedge \cdots \wedge \hat{x}_n \\ &= \hat{x}_{min} \leq \hat{x}^1_{min} \vee \hat{x}^2_{min} \vee \cdots \vee \hat{x}^s_{min} = \hat{x}_{max}. \end{aligned} \tag{45}$$

The proof is completed. □

Proposition 13. *If x^* is an optimal solution of problem (39), then any vector y fulfilling $\underline{x} \leq y \leq x^*$ is also an optimal solution of (39).*

Theorem 12 allows us to gain an optimal solution of problem (39) by comparing the function value of all the maximal solutions of system (16). However, it is difficult to obtain all the maximal solutions. Hence we usually do not recommend this resolution approach. Next we will provide another effective resolution algorithm for problem (39).

Based on the matrix D defined by (18), we further define the index sets $J^*_1, J^*_2, \ldots, J^*_m$ as follows. For each $i \in I$, let

$$d^*_i = d_{i1} \vee d_{i2} \vee \cdots \vee d_{in}, \tag{46}$$

and

$$J^*_i = \left\{j \in J \mid d_{ij} = d^*_i\right\}. \tag{47}$$

Suppose $J^*_1 \times J^*_2 \times \cdots \times J^*_m$ denotes the Cartesian product of $J^*_1, J^*_2, \ldots, J^*_m$.

For a given $(j^*_1, j^*_2, \ldots, j^*_m) \in J^*_1 \times J^*_2 \times \cdots \times J^*_m$, define matrix $q^* = (q^*_{ij})_{m\times n}$, in which

$$q^*_{ij} = \begin{cases} d_{ij} = d^*_i, & \text{if } j = j^*_i, \\ 0, & \text{if } j \neq j^*_i. \end{cases} \tag{48}$$

Proposition 14. *If system (16) is consistent, then $d^*_i > 0$ holds for all $i \in I$.*

Proof. If system (16) is consistent, it follows from Theorem 3 that each row in the matrix D has at least one nonzero element. The nonzero element should be positive. That is, for any $i \in I$, there exists some $d_{ij} > 0$. Hence $d^*_i = d_{i1} \vee d_{i2} \vee \cdots \vee d_{in} \geq d_{ij} > 0$. □

Proposition 15. *If system (16) is consistent, then for any $(j^*_1, j^*_2, \ldots, j^*_m) \in J^*_1 \times J^*_2 \times \cdots \times J^*_m$, the matrix q^* defined by (48) is a quasi-maximal matrix.*

Proof. The proof is trivial according to expression (48) and Proposition 14. □

Theorem 16. *Suppose system (16) is consistent, $(j^*_1, j^*_2, \ldots, j^*_m) \in J^*_1 \times J^*_2 \times \cdots \times J^*_m$ and q^* is defined by (48) based on $(j^*_1, j^*_2, \ldots, j^*_m)$. Then x^{q^*} is an optimal solution of problem (39).*

Proof. Denote $z^* = (d^*_1 \wedge d^*_2 \wedge \cdots \wedge d^*_m) \wedge (\overline{x}_1 \wedge \overline{x}_2 \wedge \cdots \wedge \overline{x}_n)$.

(i) According to Proposition 15 and Theorem 8, $x^{q^*} \in X(A,b)$ is a feasible solution of problem (39).

(ii) Take arbitrary $y \in X(A,b)$. For any $i \in I$, it follows from Theorem 5 that there exists $j_i \in J$ such that

$$a_{ij_i} y_{j_i} \leq b_i. \tag{49}$$

Thus $y_{j_i} \leq b_i/a_{ij_i}$. On the other hand, it is clear that $y_{j_i} \leq 1$. Then it holds that

$$y_{j_i} \leq \frac{b_i}{a_{ij_i}} \wedge 1 = d_{ij_i}. \tag{50}$$

Hence

$$\begin{aligned} z(y) &= y_1 \wedge \cdots \wedge y_n \leq y_{j_i} \leq d_{ij_i} \leq d_{i1} \wedge \cdots \wedge d_{in} \\ &= d^*_i. \end{aligned} \tag{51}$$

Due to the arbitrariness of i, we have $z(y) \le d_1^* \wedge d_2^* \wedge \cdots \wedge d_m^*$. On the other hand, $y \in X(A,b)$ indicates $y \le \overline{x}$. It is obvious that $z(y) = y_1 \wedge y_2 \wedge \cdots \wedge y_n \le \overline{x}_1 \wedge \overline{x}_2 \wedge \cdots \wedge \overline{x}_n$. So we get

$$\begin{aligned} z(y) &\le (d_1^* \wedge d_2^* \wedge \cdots \wedge d_m^*) \wedge (\overline{x}_1 \wedge \overline{x}_2 \wedge \cdots \wedge \overline{x}_n) \\ &= z^*. \end{aligned} \tag{52}$$

(iii) Take arbitrary $j \in J$.

Case 1. If $I_j^{q^*} = \emptyset$, then $x_j^{q^*} = \overline{x}_j \ge z^*$.

Case 2. If $I_j^{q^*} \ne \emptyset$, then

$$x_j^{q^*} = \bigwedge_{i \in I_j^{q^*}} q_{ij}^*, \tag{53}$$

and

$$\overline{x}_j = \bigvee_{i \in I} d_{ij} \ge \bigwedge_{i \in I_j^{q^*}} d_{ij}. \tag{54}$$

Inequality (54) indicates $(\bigwedge_{i \in I_j^{q^*}} d_{ij}) \wedge \overline{x}_j = \bigwedge_{i \in I_j^{q^*}} d_{ij}$. Considering formulae (48), we further get

$$\begin{aligned} x_j^{q^*} &= \bigwedge_{i \in I_j^{q^*}} q_{ij}^* = \bigwedge_{i \in I_j^{q^*}} d_{ij} = \left(\bigwedge_{i \in I_j^{q^*}} d_{ij} \right) \wedge \overline{x}_j \\ &= \left(\bigwedge_{i \in I_j^{q^*}} d_i^* \right) \wedge \overline{x}_j \ge z^* \end{aligned} \tag{55}$$

Combining Cases 1 and 2, it holds for any $j \in J$ that $x_j^{q^*} \ge z^*$. So we have $z(x^{q^*}) = x_1^{q^*} \wedge x_2^{q^*} \wedge \cdots \wedge x_n^{q^*} \ge z^*$.

On the other hand, the above-proved point (ii) implies that $z(x^{q^*}) \le z^*$ since $x^{q^*} \in X(A,b)$ (see (i)). Hence it turns out to be $z(x^{q^*}) = z^*$.

Points (i), (ii) and (iii) contribute to the optimality of x^{q^*}. That is to say, x^{q^*} is an optimal solution of problem (39). □

To find an optimal solution of problem (39) without computing all the maximal solutions of (16), we now propose a novel resolution algorithm as follows.

Algorithm for Solving Problem (39)

Step 1. Compute the matrix D by (18).

Step 2. Check the feasibility of problem (39). If there exists at least a nonzero element in each row and also in each column in the matrix D, then system (16) is consistent and problem (39) is feasible, continue to Step 3. Otherwise, $X(A,b) = \emptyset$ and problem (39) has none optimal solution, stop.

Step 3. Compute $d_1^*, d_2^*, \ldots, d_m^*$ by (46).

Step 4. Compute $J_1^*, J_2^*, \ldots, J_m^*$ by (47).

Step 5. Construct the Cartesian product $J_1^* \times J_2^* \times \cdots \times J_m^*$ and take arbitrary index vector $(j_1^*, j_2^*, \ldots, j_m^*) \in J_1^* \times J_2^* \times \cdots \times J_m^*$.

Step 6. Based on the index vector $(j_1^*, j_2^*, \ldots, j_m^*)$, construct the quasi-maximal matrix q^* by (48).

Step 7. Compute the index sets $I_1^{q^*}, I_2^{q^*} \cdots, I_n^{q^*}$ by (24).

Step 8. Compute the quasi-maximal solution x^{q^*} corresponding to q^* by (25). Then x^{q^*} is an optimal solution of problem (39) according to Theorem 16.

Next we show the computational complexity of our proposed resolution algorithm.

(i) Computational Complexity

n – –Number of variables;

m – –Number of inequalities.

Computing the matrix D in Step 1 costs $4mn$ operations. In Step 2, checking whether the elements in each row and in each column of D are all zeroes costs $2mn$ operations. Computing $d_1^*, d_2^*, \ldots, d_m^*$ in Step 3 costs $m(n-1)$ operations, while computing $J_1^*, J_2^*, \ldots, J_m^*$ in Step 4 costs mn operations. In Steps 5 and 6, it costs $2mn$ operations for obtaining the quasi-maximal matrix q^*. Computing the index sets $I_1^{q^*}, I_2^{q^*} \cdots, I_n^{q^*}$ in Step 7 costs $2mn$ operations. At last, Step 8 costs $n + (m-1)n = mn$ operations for generating the optimal solution x^{q^*} of problem (39), based on the quasi-maximal matrix q^*. As a consequence, the above-proposed algorithm costs

$$\begin{aligned} &4mn + 2mn + m(n-1) + mn + 2mn + 2mn + mn \\ &\quad = 13mn - m \end{aligned} \tag{56}$$

operations in total. Hence the computational complexity of the algorithm is $O(mn)$. It also indicates our proposed algorithm could be achieved in polynomial time.

For illustrating the feasibility of the above-presented algorithm, we provide a numerical example as below.

Example 1. Let

$$A \odot x^T \le b^T, \quad \underline{x} \le x \le \overline{x}, \tag{57}$$

be the matrix form of a system of min-product FRIs. In system (57), the matrix A and the vectors $x, \underline{x}, b$ are as

$$A = \begin{bmatrix} 0.6 & 0.8 & 0.85 & 0.9 & 0.95 & 0.9 & 0.7 & 0.8 \\ 0.8 & 0.75 & 0.85 & 0.85 & 0.9 & 0.7 & 0.9 & 0.8 \\ 0.8 & 0.7 & 0.7 & 0.75 & 0.8 & 0.6 & 0.65 & 0.85 \\ 0.85 & 0.8 & 0.9 & 0.8 & 0.9 & 0.8 & 0.75 & 0.85 \\ 0.75 & 0.7 & 0.7 & 0.85 & 0.9 & 0.8 & 0.8 & 0.7 \\ 0.8 & 0.85 & 0.8 & 0.9 & 0.9 & 0.75 & 0.75 & 0.7 \end{bmatrix},$$

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_8), \\ \underline{x} &= (0.75, 0.8, 0.78, 0.72, 0.65, 0.7, 0.65, 0.75), \\ b &= (0.64, 0.60, 0.56, 0.60, 0.55, 0.63) \end{aligned} \tag{58}$$

Check whether system (57) is consistent. If it is consistent, find an optimal solution of the maximin programming subject to system (57), i.e.,

$$\begin{aligned} \max \quad & z(x) = x_1 \wedge x_2 \wedge \cdots \wedge x_n \\ \text{s.t.} \quad & A \odot x^T \le b^T, \quad \underline{x} \le x \le \overline{x}. \end{aligned} \tag{59}$$

Solution

Step 1-2. In fact, the matrix D is [51]

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.67 & 0.71 & 0.91 & 0.8 \\ 0 & 0 & 0 & 0 & 0.67 & 0.86 & 0.67 & 0 \\ 0 & 0.8 & 0.8 & 0.75 & 0.7 & 0.93 & 0.86 & 0 \\ 0 & 0 & 0 & 0.75 & 0.67 & 0.75 & 0.8 & 0 \\ 0 & 0 & 0.79 & 0 & 0 & 0 & 0.69 & 0.79 \\ 0.79 & 0 & 0.79 & 0 & 0.7 & 0.84 & 0.84 & 0.9 \end{bmatrix}. \tag{60}$$

Following Theorem 3, system (57) is consistent and problem (59) is feasible.

Step 3. Compute $d_1^*, d_2^*, \ldots, d_6^*$ by (46).

$$\begin{aligned} d_1^* &= 1 \vee 0 \vee 0 \vee 0 \vee 0.67 \vee 0.71 \vee 0.91 \vee 0.8 = 1, \\ d_2^* &= 0 \vee 0 \vee 0 \vee 0 \vee 0.67 \vee 0.86 \vee 0.67 \vee 0 = 0.86, \\ d_3^* &= 0 \vee 0.8 \vee 0.8 \vee 0.75 \vee 0.7 \vee 0.93 \vee 0.86 \vee 0 \\ &= 0.93, \\ d_4^* &= 0 \vee 0 \vee 0 \vee 0.75 \vee 0.67 \vee 0.75 \vee 0.8 \vee 0 = 0.8, \\ d_5^* &= 0 \vee 0 \vee 0.79 \vee 0 \vee 0 \vee 0 \vee 0.69 \vee 0.79, \\ d_6^* &= 0.79 \vee 0 \vee 0.79 \vee 0 \vee 0.7 \vee 0.84 \vee 0.84 \vee 0.9 \\ &= 0.9. \end{aligned} \tag{61}$$

Step 4. Compute $J_1^*, J_2^*, \ldots, J_6^*$ by (47).

$$\begin{aligned} J_1^* &= \left\{ j \in J \mid d_{1j} = d_1^* \right\} = \left\{ j \in J \mid d_{1j} = 1 \right\} = \{1\}, \\ J_2^* &= \left\{ j \in J \mid d_{2j} = d_2^* \right\} = \left\{ j \in J \mid d_{2j} = 0.86 \right\} = \{6\}, \\ J_3^* &= \left\{ j \in J \mid d_{3j} = d_3^* \right\} = \left\{ j \in J \mid d_{3j} = 0.93 \right\} = \{6\}, \\ J_4^* &= \left\{ j \in J \mid d_{4j} = d_4^* \right\} = \left\{ j \in J \mid d_{4j} = 0.8 \right\} = \{7\}, \\ J_5^* &= \left\{ j \in J \mid d_{5j} = d_5^* \right\} = \left\{ j \in J \mid d_{5j} = 0.79 \right\} \\ &= \{3, 8\}, \\ J_6^* &= \left\{ j \in J \mid d_{6j} = d_6^* \right\} = \left\{ j \in J \mid d_{6j} = 0.9 \right\} = \{8\}. \end{aligned} \tag{62}$$

Step 5. Construct the Cartesian product $J_1^* \times J_2^* \times \cdots \times J_6^*$ and take arbitrary index vector $(j_1^*, j_2^*, \ldots, j_6^*) \in J_1^* \times J_2^* \times \cdots \times J_6^*$.

$$\begin{aligned} J_1^* \times J_2^* \times \cdots \times J_6^* = \{1\} \times \{6\} \times \{6\} \times \{7\} \times \{3, 8\} \\ \times \{8\}. \end{aligned} \tag{63}$$

We take the index vector $(j_1^*, j_2^*, \ldots, j_6^*) = (1, 6, 6, 7, 3, 8)$.

Step 6. Based on the index vector $(j_1^*, j_2^*, \ldots, j_6^*) = (1, 6, 6, 7, 3, 8)$, we are able to compute the elements in q^* by (48) as follows:

$$\begin{aligned} q_{11}^* &= d_{11} = 1, \\ q_{1j} &= 0 \quad \text{for any } j \ne 1, \ j \in \{1, 2, \ldots, 8\}, \\ q_{26}^* &= d_{26} = 0.86, \\ q_{2j} &= 0 \quad \text{for any } j \ne 6, \ j \in \{1, 2, \ldots, 8\}, \\ q_{36}^* &= d_{36} = 0.93, \\ q_{3j} &= 0 \quad \text{for any } j \ne 6, \ j \in \{1, 2, \ldots, 8\}, \\ q_{47}^* &= d_{47} = 0.8, \\ q_{4j} &= 0 \quad \text{for any } j \ne 7, \ j \in \{1, 2, \ldots, 8\}, \\ q_{53}^* &= d_{53} = 0.79 \\ q_{5j} &= 0 \quad \text{for any } j \ne 3, \ j \in \{1, 2, \ldots, 8\}, \\ q_{68}^* &= d_{68} = 0.9, \\ q_{6j} &= 0 \quad \text{for any } j \ne 8, \ j \in \{1, 2, \ldots, 8\}. \end{aligned} \tag{64}$$

The quasi-maximal matrix corresponding to $(1, 6, 6, 7, 3, 8)$ is

$$q^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.86 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.93 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0.79 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}. \tag{65}$$

Step 7. Compute the index sets $I_1^{q^*}, I_2^{q^*} \cdots, I_8^{q^*}$ by (24).

$$\begin{aligned} I_1^{q^*} &= \left\{ i \in I \mid q_{i1} > 0 \right\} = \{1\}, \\ I_2^{q^*} &= \emptyset, \\ I_3^{q^*} &= \left\{ i \in I \mid q_{i3} > 0 \right\} = \{5\}, \\ I_4^{q^*} &= \emptyset, \\ I_5^{q^*} &= \emptyset, \\ I_6^{q^*} &= \left\{ i \in I \mid q_{i6} > 0 \right\} = \{2, 3\}, \end{aligned}$$

$$I_7^{q^*} = \{i \in I \mid q_{i7} > 0\} = \{4\},$$
$$I_8^{q^*} = \{i \in I \mid q_{i8} > 0\} = \{6\}. \tag{66}$$

Step 8. Compute the quasi-maximal solution x^{q^*} corresponding to q^* by (25). Firstly,

$$\overline{x} = \left(\bigvee_{i\in I}\frac{b_i}{a_{i1}}, \bigvee_{i\in I}\frac{b_i}{a_{i2}}, \ldots, \bigvee_{i\in I}\frac{b_i}{a_{i8}}\right) = (1, 0.8, 0.8, 0.75, 0.7, 0.93, 0.91, 0.9). \tag{67}$$

Since $I_2^q = I_4^q = I_5^q = \emptyset$, we have

$$x_2^{q^*} = \overline{x}_2 = 0.8,$$
$$x_4^{q^*} = \overline{x}_4 = 0.75, \tag{68}$$
$$x_5^{q^*} = \overline{x}_5 = 0.7.$$

In addition,

$$\begin{aligned}
x_1^{q^*} &= \bigwedge_{i\in I_1^{q^*}} q_{i1}^* = q_{11}^* = 1,\\
x_3^{q^*} &= \bigwedge_{i\in I_3^{q^*}} q_{i3}^* = q_{53}^* = 0.79,\\
x_6^{q^*} &= \bigwedge_{i\in I_6^{q^*}} q_{i6}^* = q_{26}^* \wedge q_{36}^* = 0.86 \wedge 0.93 = 0.86,\\
x_7^{q^*} &= \bigwedge_{i\in I_7^{q^*}} q_{i7}^* = q_{47}^* = 0.8,\\
x_8^{q^*} &= \bigwedge_{i\in I_8^{q^*}} q_{i7}^* = q_{68}^* = 0.9.
\end{aligned} \tag{69}$$

Hence we find an optimal solution of problem (59) as

$$x^{q^*} = (1, 0.8, 0.79, 0.75, 0.7, 0.86, 0.8, 0.9). \tag{70}$$

5. Discussion

5.1. Optimal Solution Set of Problem (39). In Example 1, we have found an optimal solution of problem (59) as

$$x^{q^*} = (1, 0.8, 0.79, 0.75, 0.7, 0.86, 0.8, 0.9). \tag{71}$$

The corresponding optimal function value is

$$z\left(x^{q^*}\right) = 1 \wedge 0.8 \wedge 0.79 \wedge 0.75 \wedge 0.7 \wedge 0.86 \wedge 0.8 \wedge 0.9 = 0.7. \tag{72}$$

In Step 5, we take the index vector as $(j_1^{*\prime}, j_2^*, \ldots, j_6^*) = (1, 6, 6, 7, 8, 8)$. Following Steps 6–8, we get the vector $x^{q^{*\prime}} = (1, 0.8, 0.8, 0.75, 0.7, 0.86, 0.8, 0.79)$. It is easy to check that $x^{q^{*\prime}}$ satisfies the constraint in problem (59) and moreover, $z(x^{q^{*\prime}}) = 1 \wedge 0.8 \wedge 0.8 \wedge 0.75 \wedge 0.7 \wedge 0.86 \wedge 0.8 \wedge 0.79 = 0.7$. Therefore, $x^{q^{*\prime}}$ is also an optimal solution of problem (59).

It has shown above that the optimal solution of problem (59) is not unique. Hence we further discuss how to obtain all the optimal solutions of our proposed maximin optimization problem in this subsection. The following Theorem 2 gives exactly description of the set of all optimal solutions to problem (39), according to one of the optimal solutions obtained by our proposed algorithm in Section 4.

Theorem 2. *Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be an optimal solution to problem (39), with corresponding optimal value $z^* = z(x^*) = x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*$. Then the set of all optimal solutions to problem (39) is the solution set of the following inequalities,*

$$\begin{aligned}
&A \odot x \le b,\\
&\underline{x}_j \vee z^* \le x_j \le \overline{x}_j, \quad \forall j \in J.
\end{aligned} \tag{73}$$

Proof. We complete the proof from two aspects as follows:

(i) Take arbitrary $x^{*\prime} = (x_1^{*\prime}, x_2^{*\prime}, \ldots, x_n^{*\prime})$ satisfying system (73). It holds that $A \odot x^{*\prime} \le b$ and $\underline{x}_j \le \underline{x}_j \vee z^* \le x_j^{*\prime} \le \overline{x}_j, \forall j \in J$. Hence $x^{*\prime} \in X(A, b)$ is a solution of system (16), i.e., a feasible solution of problem (39).

Since $x^{*\prime}$ is a feasible solution of problem (39), while x^* is an optimal solution with objective function value z^*, we have

$$z\left(x^{*\prime}\right) = x_1^{*\prime} \wedge x_2^{*\prime} \wedge \cdots \wedge x_n^{*\prime} \le z^*. \tag{74}$$

On the other hand, since $x^{*\prime}$ satisfies system (73), we have $x_j^{*\prime} \ge \underline{x}_j \vee z^* \ge z^*, \forall j \in J$.

$$\begin{aligned}
z\left(x^{*\prime}\right) &= x_1^{*\prime} \wedge x_2^{*\prime} \wedge \cdots \wedge x_n^{*\prime} \ge z^* \wedge z^* \wedge \cdots \wedge z^*\\
&= z^*.
\end{aligned} \tag{75}$$

Inequalities (74) and (75) contribute to $z(x^{*\prime}) = z^*$.

Hence $x^{*\prime}$ is an optimal solution of problem (39).

(ii) Suppose $x^{*\prime}$ is an optimal solution of problem (39). Next we have to check that $x^{*\prime}$ satisfies system (73). It is obvious that $x^{*\prime}$ satisfies the constraint of (39), i.e., system (16). On the other hand, due to the optimality of $x^{*\prime}$, we have

$$z\left(x^{*\prime}\right) = x_1^{*\prime} \wedge x_2^{*\prime} \wedge \cdots \wedge x_n^{*\prime} = z^*. \tag{76}$$

This indicates $x_j^{*\prime} \ge x_1^{*\prime} \wedge x_2^{*\prime} \wedge \cdots \wedge x_n^{*\prime} = z^*$ for all $j \in J$. Hence $x_j^{*\prime} \ge z^* \vee \underline{x}_j, \forall j \in J$, and $x^{*\prime}$ is a solution of system (73). □

It is clear that system (73) is indeed a system of min-product fuzzy relation inequalities. In this sense, the set of all optimal solutions to problem (39) (when not unique), is exactly the solution set of a min-product fuzzy relation inequalities system. According to Theorem 10, there exist a unique minimum optimal solution and a finite number of maximal optimal solutions.

5.2. Comparison to the Existing Works. In this subsection we make some comparison between the results presented in this paper and those in the existing works [51–53].

In [53], only requirement of the retailers was considered. Both solutions and strong solutions were studied in [53]. The authors focused on the strong solutions. Structure and resolution method of the strong solution set are the major part of this work. A path-based method was proposed to find out all the strong solutions of a system of min-product fuzzy relation inequalities. No optimization problem was considered in this paper.

In [51], although both requirements of the retailers and the suppliers were considered, the authors only discussed the method of consistency checking of the min-product fuzzy relation inequalities system. In such system, a new constraint, i.e., $x \leq \overline{x}$, was added. The consistency of such system could be checked by its potential minimum solution $\underline{x}$, or by its distinguishing matrix D. Besides, structure of the solution set to such system was simply presented. No optimization problem was considered in this paper.

In [52] optimization problem under lexicographic order was considered. The constraint of the problem was exactly the min-product system proposed in [51]. In such optimization problem, the target was to maximize the profits of the suppliers. Under a specific lexicographic order, the suppliers were treated with fixed priority grade. This is conflict with the equalitarianism. The authors proposed a cyclic algorithm to obtain the unique optimal solution. There were n procedures in a cycle. In each procedure, a new vector y^k was generated and its feasibility was checked. Moreover, n procedures produced n components of the optimal solutions.

Both of the studied problem and the resolution method are different from those in the above-mentioned existing works. In this paper, we mainly study the maximin problem subject to min-product fuzzy relation inequalities, considering both requirements of the retailers and the suppliers. Quasi-maximal matrix is defined for investigating the structure of the complete solution set of the proposed min-product system. Based on the special structure of the solution set, a novel approach based on the quasi-maximal matrix and its corresponding index sets is proposed to find an optimal solution of the maximin optimization problem. The motivation of our proposed maximin optimization problem has been stated in Section 1.

6. Conclusion

FRIs with min-product composition could be applied to describe the price conditions in a supply chain. Consistency checking for the min-product system (16) has been discussed in [52]. In this work we further investigate the resolution method of (16). In fact, each solution of (16) represents a feasible pricing scheme of the suppliers in the supply chain. Consistency and solution-set characteristic of a system of min-product fuzzy relation inequalities are discussed in detail. The consistency could be checked in two ways. When consistent, the complete solution set of a system of min-product fuzzy relation inequalities is fully determined by its unique minimum solution and finite maximal (or quasi-maximal) solutions. Matrix-based method is proposed to compute all the solutions of (16). However, in practical application, it is no need to compute all the solutions in some cases. Optimal solution might be more meaningful, considering some specific optimization management objectives. In this paper, maximin programming is established for improving the minimal profit of the suppliers. In such maximin optimization problem, the objective is to maximize the maximin function, i.e., $x_1 \vee x_2 \vee \cdots \vee x_n$. We propose a novel algorithm to search an optimal solution, in case that the feasible domain is nonempty. The algorithm is developed step by step and illustrated by a detailed numerical example.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Hai-Tao Lin and Xiao-Bin Yang contributed equally to this work.

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