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Research Article

Subgame Perfect Equilibrium in the Rubinstein Bargaining Game with Loss Aversion

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Rubinstein bargaining game is extended to incorporate loss aversion, where the initial reference points are not zero. Under the assumption that the highest rejected proposal of the opponent last periods is regarded as the associated reference point, we investigate the effect of loss aversion and initial reference points on subgame perfect equilibrium. Firstly, a subgame perfect equilibrium is constructed. And its uniqueness is shown. Furthermore, we analyze this equilibrium with respect to initial reference points, loss aversion coefficients, and discount factor. It is shown that one benefits from his opponent's loss aversion coefficient and his own initial reference point and is hurt by loss aversion coefficient of himself and the opponent's initial reference point. Moreover, it is found that, for a player who has a higher level of loss aversion than the other, although this player has a higher initial reference point than the opponent, this player can(not) obtain a high share of the pie if the level of loss aversion of this player is sufficiently low (high). Finally, a relation with asymmetric Nash bargaining is established, where player's bargaining power is negatively related to his own loss aversion and the initial reference point of the other and positively related to loss aversion of the opponent and his own initial reference point.

1. Introduction

A large number of experimental literature pieces on bargaining explore the nature of agreements and disagreements and the dynamics of bargaining. There are two critical conclusions: firstly, in real bargaining problem, bargaining is a gradual process and the agreements can be reached after many periods. Secondly, there is a strictly positive probability of disagreement. For the classical bargaining problem of dividing a pie, whose size is one unit, between two bargainers, Rubinstein [1] assumed that preferences of bargainers are time dependent. In many bargaining situations, however, the assumption may be violated and the share finally obtained by a bargainer may depend on the history of alternating offers made so far. In particular, the phenomenon of loss aversion in bargaining problems is pointed out by Driesen et al. [2] as follows: a share of x% is evaluated less if a share of y% with $y > x$ has been within reach at an earlier stage of the game.

Kahneman and Tversky [3] first proposed loss aversion. As the most striking result of the investigation of reference-dependent utility functions, loss aversion is applied to lots of applications with fixed reference point [4, 5]. For the situation of loss aversion where the reference points are fixed, we can regard it as a special case of risk aversion. Roth [6] investigated the impact of risk aversion on the classical Rubinstein alternating offers bargaining model in the context of full rationality of bargainers. However, in many applications, it is likely that a loss depends on history of the bargaining. That is, the reference points are endogenous [7]. Shalev [8] considered objective discount and loss aversion and obtained the unique subgame perfect equilibrium (SPE) of the Rubinstein bargaining with the transformed discount factors. Compte and Jehiel [9] assumed that a new bargaining phase begins at a fixed cost if a breakdown of the bargaining occurs. In each new bargaining phase, the reference points can be adjusted and the highest offer received over the process of bargaining and the first mover is chosen from the two agents at random with probability 1/2. Li [10] assumed that a player would rather reject any share that is less than the highest offer in the past and found a unique subgame perfect equilibrium. Schwartz and Wen [11] assumed that a proposal of a bargainer made to the other cannot be less than a proposal made to that player. Hyndman [12] assumed that a bargainer with reference dependent preferences prefers his current reference point to impasse to consuming below their current reference points. Closer to this paper is Driesen et al. [2]; they investigated the impact of loss aversion on the Rubinstein bargaining game based on the assumption that the initial reference points are zero and one's reference point in the process of bargaining is regarded as the highest rejected offer of his opponent last rounds of bargaining. Although it is reasonable that the reference points at the beginning of bargaining are zero in a lot of instances, it may be essential that the reference points at the beginning of bargaining are not zero in others. For example, a player may transfer his expectations derived from previous opponents when he enters into a new bargaining situation with another player. Thus, how to investigate the impact of loss aversion and the initial reference points on the classical Rubinstein bargaining game is a valuable topic and also the objective of this paper.

We adopt the model of loss aversion proposed by Shalev [13]. In Shalev's model, a player's preference is modeled by the following elements: basic utility function of decision-maker, loss aversion coefficient, and reference point. The outcomes that are less than some reference point are regarded as losses. And the corresponding values of utilities are scaled down by loss aversion parameter. A number of applications are consistent with Shalev's model of loss version. The basic assumption of Shalev's model is that the loss aversion coefficient is regarded as a constant parameter, which makes the model be easily used. The loss aversion coefficient is constant in the following two different aspects: first, for the utility of an outcome that is less than the reference outcome, it can be obtained from the basic utility by subtracting a disutility, which is obtained from the size of the loss multiplied by a parameter — loss aversion coefficient. Second, the parameter is constant across different reference points; that is, the loss aversion coefficient does not depend on reference point [14].

In this paper, we extend the analysis of the Rubinstein bargaining game to incorporate loss aversion and reference dependence, where the initial reference points are not zero. We assume that a bargainer's reference point in the bargaining process is equal to the highest rejected offer of his opponent that is higher than his own initial reference point, since it can be regarded as the share that could have been obtained so far. A simple modification of the Rubinstein bargaining game can be transformed into a new bargaining game with loss aversion and reference dependence through changed reference points, which depends on the history of bargaining. In our model, subgames depend not only on the initial reference points, but also on the impact the history of bargaining has on preferences, which leads to much more complications to analyzing the characterization of SPE. On the other hand, for Rubinstein bargaining game with loss aversion and reference dependence, where the initial reference points are not zero and the discount factor is regarded as the probability of entering a new phase of bargaining after rejecting a proposal of a player, we construct the unique subgame perfect equilibrium (SPE), and its features are shown. Finally, we analyze the impact of loss aversion coefficients and the discounting factor (or the probability of continuation) on subgame perfect equilibrium.

The remainder of the present paper is organized as follows. After preliminaries in Section 2, we define the SPE in the Rubinstein bargaining model with loss aversion, construct a SPE, and concern uniqueness of the SPE in Section 3. In Section 4, we discuss convergence of the subgame perfect equilibrium for the probability of continuation. In Section 5, conclusion is given.

2. Preliminaries

2.1. Rubinstein Bargaining Model. Player 1 and player 2 have to reach an agreement on how to divide one unit of a pie. The set of all possible partitions is denoted by

$$Z =: \left\{(z_1, z_2) \in R^2 \mid z_1 + z_2 = 1,\ z_1, z_2 \geq 0\right\}. \qquad (1)$$

Bargaining occurs at times $t \in T = \{1, 2, \ldots\}$. For simplicity, the set of odd moments is denoted by $T_{odd} \coloneqq \{1, 3, \ldots\ldots\}$ and the set of even moments is denoted by $T_{even} \coloneqq \{2, 4, \ldots\ldots\}$. At odd moments, player 1 offers $(z_1, z_2) \in Z$ and this proposal is accepted (Y) or rejected (N) by player 2. At even moments, for players 1 and 2, their roles are reversed. We assume that the history of bargaining is common knowledge; that is, one knows all previous offers at any moment $t \in T$, including his own and those of his opponent. If players accept the proposal (z_1, z_2), then player i $(i = 1, 2)$ obtains z_i and the bargaining ends. If a proposal offered by player i is rejected by his opponent, then the bargaining continues to a new phase with probability $0 < \delta < 1$ and ends in probability $1 - \delta$. For the latter case, it means that the bargaining game ends in disagreement; that is, players obtain the shares (r_1^0, r_2^0), where r_1^0 and r_2^0 represent the initial reference points of players.

For each odd moment, a strategy f played by player 1 specifies a proposal in Z that depends on the history of the bargaining so far; on the other hand, for each even moment, a decision Y or N is made, where this decision not only depends on the proposal at the current phase but also on the history of bargaining. Similarly, player 2 plays a strategy g, with the roles for moments $t \in T_{\text{odd}}$ and $t \in T_{\text{even}}$ are reversed.

At time $t \in T$, the history of the bargaining is denoted by h^t, which is defined as a vector of proposals of bargainers. Specifically, $h^t \coloneqq (z^1, z^2, \ldots, z^t)$, where $z^s \in Z$ for all $s \leq t$. Furthermore, at time $t \in T$, all possible histories h^t are denoted by H^t in the bargaining game. That is, $H^t \coloneqq \prod_{s=1}^{t} Z$, let $H^0 \coloneqq (h^0)$, where h^0 is an empty history.

Let F be strategy set of player 1, denoted by sequences of functions $(f^t)_{t\in T}$ where for $t = 1$: $f^t \in Z$, for $t > 1$ and $t \in T_{odd}$: $f^t\colon H^{t-1} \longrightarrow Z$, and for $t \in T_{even}$: $f^t\colon H^t \longrightarrow \{Y, N\}$, and let G be strategies of player 2, denoted by sequences of functions $(g^t)_{t\in T}$ where for $t \in T_{odd}$: $g^t\colon H^t \longrightarrow \{Y, N\}$ and for $t \in T_{even}$: $g^t\colon H^{t-1} \longrightarrow Z$. An agreement path is denoted by (h^t, a), which is a history $h^t \in H^t$ ending in agreement at time t. All time t agreement paths are denoted by $A^t \coloneqq \{(h^t, a) \mid h^t \in H^t\}$. The set $\mathrm{A} \coloneqq \bigcup_{t\in T} A^t$ contains all histories that end in agreement. Similarly, a disagreement path is denoted as (h^t, d), which means that a history $h^t \in H^t$ ends in disagreement at time t. $D^t := \{(h^t, d) \mid h^t \in H^t\}$ contains all time t disagreement paths. $D := \bigcup_{t\in T} D^t$ contains all histories ending in disagreement. All objects of (h^t, c) are denoted by the set C^t; that is, histories do not end at moment t. On the other hand, we define $H^{\infty} := \{(z^1, z^2, \ldots) \mid z^t \in Z \text{ for all } t \in T\}$. The elements of H^{∞} are defined as infinite paths. Therefore, the set that contains all paths of the bargaining game can be denoted by $\underline{H} := H^{\infty} \cup A \cup D$. Note that a set of paths in $\underline{H}$ is determined by a strategy profile $(f, g) \in F \times G$. In particular, if an agreement at time t is reached when players play (f, g), the set of paths associated with (f, g) not only contains $t - 1$ paths in the set D but also contains one in the set A. If an agreement is never reached when (f, g) is played, then the set only contains paths in D.

The function $\xi_i := \underline{H} \backslash H^{\infty} \longrightarrow [r_i^0, 1]$ is introduced, which is used to specify the share that player i $(i = 1, 2)$ obtains for each finite path in $\underline{H}$. Specifically, for all $h^t \in H^t$, $h^t = (z^1, z^2, \ldots, z^t)$, we have that $\xi_i(h^t, a) := z_i^t$ and $\xi_i(h^t, d) := r_i^0$, where r_i^0 is the initial reference point of player i.

2.2. Loss Aversion. Kahneman and Tversky [3] first proposed loss aversion; its central assumption is that gains are smaller than losses. For instance, the decrease in utility of loss of 10 dollars if one has 100 dollars is larger than the increase in utility of gain of 10 dollars if one has 90 dollars. Shalev [13] proposed another model of loss aversion to measure this. In Shalev's model, loss aversion of each player i $(i = 1, 2)$ is characterized by loss aversion coefficient λ_i, where λ_i is nonnegative. And let r_i be a reference point of player i. The utility function is given by the following transformation:

$$w_i(z_i, \lambda_i, r_i) = \begin{cases} z_i & \text{if } z_i \geq r_i \\ z_i - \lambda_i (r_i - z_i) & \text{if } z_i < r_i \end{cases} \tag{2}$$

or, equivalently,

$$w_i(z_i, \lambda_i, r_i) = (1 + \lambda_i) z_i - \lambda_i \max\{r_i, z_i\} \tag{3}$$

If the outcomes that are less than reference point are regarded as losses, then the values of utilities are scaled down by λ_i. If the values of the payoffs are higher than that of the reference point, then the payoffs are left unchanged.

In a number of applications, the reference points are usually given exogenously, which sidesteps the important question of the significance of the reference points. Thus, the fact that players' reference points are endogenous is the motivation for this paper.

3. Equilibrium in the Rubinstein Bargaining Game with Loss Aversion

At time t, all the proposals made to a player by his opponent so far, possibly including the proposal at time t, specify all the shares that this player could have obtained up to the current time t. Thus, the maximum of those shares can be regarded as the reference point of this player, since the maximum of those shares represents what this player could have obtained: shares below this reference point represent losses and their utilities are evaluated by (3).

For real bargaining problems, a player may transfer his expectations derived from previous opponents when he enters into a new bargaining situation with another player. For such situations, it is more appropriate that the initial reference points of players are not equal to zero. However, if player i starts the bargaining by offering an equal split (1/2, 1/2) to his opponent, there is a risk that — if breakdown occurs and a new bargaining phrase starts — the reference point of his opponent switches to a new value that it is larger than the initial value 1/2 [9]. Thus, let r_1^0 and r_2^0 be the initial reference points of players 1 and 2, where $r_1^0, r_2^0 \in (0, 1/2)$. At any moment $t \geq 1$, the reference point of player 1 is $r_1^t = \max\{r_1^0, z_1^s \mid s = 2, 4, \ldots\ldots \leq t\}$ and the reference point of player 2 is $r_2^t = \max\{r_2^0, z_2^s \mid s = 1, 3, \ldots\ldots \leq t\}$, if $z^1, z^2, \ldots, z^t \in Z$ are the offers made up to time t. Thus, the Rubinstein bargaining game that is extended to incorporate loss aversion and reference dependence is not history independent anymore.

For player i, the utility functions for agreement paths and disagreement paths are defined as $u_i(h^t, a) = w_i(\xi_i(h^t, a), r_i(h^t), \lambda_i, r_i^0)$ and $u_i(h^t, d) = w_i(\xi_i(h^t, d), r_i(h^t), \lambda_i, r_i^0)$, respectively. In H^{∞}, the utility evaluation of player i is defined as $u_i = -\lambda_i$ for all $h \in H^{\infty}$, which means that the utility of perpetual disagreement is $-\lambda_i$.

Let $U_i : F \times G \longrightarrow R$ be the expected utility function and the strategy profile $(f, g) \in F \times G$ be played from the moment $t \in T$, where t is the moment up until the history is known. Then $(f \mid h^t, g \mid h^t)$ is played at moment $t + 1$ and $U_i(f \mid h^t, g \mid h^t)$ is defined as the expected utility of player i at time t if $(f, g) \in F \times G$ is played.

Definition 1. The strategy profile (f, g) is called a SPE if, for every $t \in T$ and every $h^t \in H^t$, it satisfies the following two conditions:

$$U_1\left(f \mid h^t, g \mid h^t\right) \geq U_1\left(\tilde{f} \mid h^t, g \mid h^t\right) \quad \forall \tilde{f} \tag{4}$$

and

$$U_2\left(f \mid h^t, g \mid h^t\right) \geq U_2\left(f \mid h^t, \tilde{g} \mid h^t\right) \quad \forall \tilde{g}. \tag{5}$$

3.1. Constructing Equilibrium. In this subsection, we construct a SPE for the Rubinstein bargaining game with loss aversion and reference dependence, where the initial reference points are not zero. Players' strategies in the SPE are stationary Markov strategies: both proposals and decisions of acceptance or rejection depend only on the initial reference points and the current reference points. The SPE in this paper still satisfy the following two characteristics that share with SPE in the classical bargaining game proposed by Rubinstein: (i) every proposal in equilibrium is immediately accepted; and (ii) for the decision of acceptance or rejection, players are always indifferent in equilibrium. In our model, a SPE is constructed based on the assumption that a player's proposal should make the other one indifferent between this proposal and his own proposal in the next phrase.

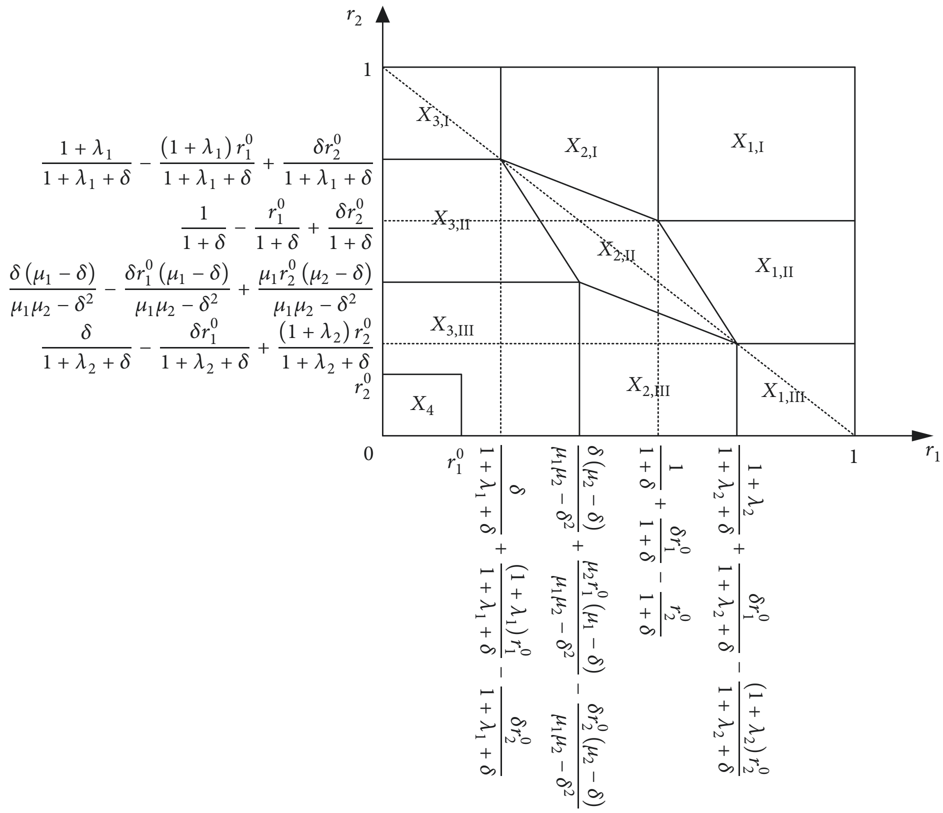

Figure 1: The partition $X_{1,\mathrm{I}}, \ldots\ldots, X_{3,\mathrm{III}}, X_4$, where X_4 represents the set of the initial reference points.

At $t \in T_{odd}$, player 1 offers $x \in Z$. We assume that player 2 offers $y \in Z$ at moment $t+1$ and this proposal will be accepted by his opponent if the proposal x is rejected. Let r_2^0 be the initial reference point of player 2 and let r_2 be a reference point of player 2 at time $t-1$. If x is accepted by player 2, then we have

$$\begin{aligned}&(1+\lambda_2)\left(x_2-r_2^0\right)-\lambda_2\max\left\{r_2-r_2^0, x_2-r_2^0\right\}\\&\quad\geq \delta\left(\left(1+\lambda_2\right)\left(y_2-r_2^0\right)\right.\\&\quad-\lambda_2\max\left\{y_2-r_2^0,\max\left\{r_2-r_2^0,x_2-r_2^0\right\}\right\}\left.\right)-(1\\&\quad-\delta)\lambda_2\max\left\{r_2-r_2^0,x_2-r_2^0\right\}\end{aligned} \tag{6}$$

which means that player 2 should estimate the proposal x at moment t at least as high as the proposal of himself y at the time $t+1$ after rejecting x. Similarly, we can give another inequality at even moments as follows:

$$\begin{aligned}&(1+\lambda_1)\left(y_1-r_1^0\right)-\lambda_1\max\left\{r_1-r_1^0, y_1-r_1^0\right\}\\&\quad\geq \delta\left(\left(1+\lambda_1\right)\left(x_1-r_1^0\right)\right.\\&\quad-\lambda_1\max\left\{x_1-r_1^0,\max\left\{r_1-r_1^0,y_1-r_1^0\right\}\right\}\left.\right)-(1\\&\quad-\delta)\lambda_1\max\left\{r_1-r_1^0,y_1-r_1^0\right\}\end{aligned} \tag{7}$$

The equilibrium can be constructed by assuming the inequalities (6) and (7) to be equalities. Let $\mu_i = 1+\lambda_i(1-\delta)$ for $i = 1, 2$. It follows from (6) with equality that we can obtain the following three cases:

(I) $r_2 > y_2 > x_2 : \delta(y_2 - r_2^0) = (x_2 - r_2^0)$.

(II) $y_2 \geq r_2 > x_2 : \delta(y_2 - r_2^0) = (1+\lambda_2)(x_2 - r_2^0) - \delta\lambda_2(r_2 - r_2^0)$.

(III) $y_2 > x_2 \geq r_2 : \delta(y_2 - r_2^0) = \mu_2(x_2 - r_2^0)$.

Similarly, we can obtain the following three cases from (7) with equality

(1) $r_1 > x_1 > y_1 : \delta(x_1 - r_1^0) = (y_1 - r_1^0)$.

(2) $x_1 \geq r_1 > y_1 : \delta(x_1 - r_1^0) = (1+\lambda_1)(y_1 - r_1^0) - \delta\lambda_1(r_1 - r_1^0)$.

(3) $x_1 > y_1 \geq r_1 : \delta(x_1 - r_1^0) = \mu_1(y_1 - r_1^0)$.

For reference points of players, we can obtain a partition of $[r_i^0, 1]$ $(i = 1, 2)$ of all possible pairs (r_1, r_2) into nine sets by combining these equations (see Figure 1). In Figure 1, these sets are denoted by $X_{1,\mathrm{I}}, \ldots, X_{3,\mathrm{III}}, X_4$, where X_4 represents the set of the initial reference points.

Therefore, the nine sets are formally described. All associated equilibrium proposals are given as follows.

① Region 1, I

$$\begin{aligned}X_{1,\mathrm{I}} = &\left\{\left(r_1, r_2\right), r_i \in \left[r_i^0, 1\right] \mid r_1 > \frac{1}{1+\delta} + \frac{\delta r_1^0}{1+\delta}\right.\\&\left.-\frac{r_2^0}{1+\delta}, r_2 > \frac{1}{1+\delta} - \frac{r_1^0}{1+\delta} + \frac{\delta r_2^0}{1+\delta}\right\}\end{aligned} \tag{8}$$

The equilibrium proposals in $X_{1,\mathrm{I}}$ are shown as follows:

$$x^{1,\mathrm{I}} = \left(\frac{1}{1+\delta} + \frac{\delta r_1^0}{1+\delta} - \frac{r_2^0}{1+\delta}, \frac{\delta}{1+\delta} - \frac{\delta r_1^0}{1+\delta} + \frac{r_2^0}{1+\delta}\right);$$
$$y^{1,\mathrm{I}} = \left(\frac{\delta}{1+\delta} + \frac{r_1^0}{1+\delta} - \frac{\delta r_2^0}{1+\delta}, \frac{1}{1+\delta} - \frac{r_1^0}{1+\delta} + \frac{\delta r_2^0}{1+\delta}\right) \quad (9)$$

② Region 1, III

$$X_{1,\mathrm{III}} = \left\{(r_1, r_2), r_i \in \left[r_i^0, 1\right] \mid r_1 > \frac{1+\lambda_2}{1+\lambda_2+\delta} + \frac{\delta r_1^0}{1+\lambda_2+\delta} - \frac{(1+\lambda_2) r_2^0}{1+\lambda_2+\delta}, r_2 \le \frac{\delta}{1+\lambda_2+\delta} - \frac{\delta r_1^0}{1+\lambda_2+\delta} + \frac{(1+\lambda_2) r_2^0}{1+\lambda_2+\delta}\right\} \quad (10)$$

The equilibrium proposals in $X_{1,\mathrm{III}}$ are shown as follows:

$$x^{1,\mathrm{III}} = \left(\frac{1+\lambda_2}{1+\lambda_2+\delta} + \frac{\delta r_1^0}{1+\lambda_2+\delta} - \frac{(1+\lambda_2) r_2^0}{1+\lambda_2+\delta}, \frac{\delta}{1+\lambda_2+\delta} - \frac{\delta r_1^0}{1+\lambda_2+\delta} + \frac{(1+\lambda_2) r_2^0}{1+\lambda_2+\delta}\right);$$
$$y^{1,\mathrm{III}} = \left(\frac{\delta(1+\lambda_2)}{1+\lambda_2+\delta} + \frac{\mu_2 r_1^0}{1+\lambda_2+\delta} - \frac{\delta(1+\lambda_2) r_2^0}{1+\lambda_2+\delta}, \frac{1+\lambda_2(1-\delta)}{1+\lambda_2+\delta} - \frac{\mu_2 r_1^0}{1+\lambda_2+\delta} + \frac{\delta(1+\lambda_2) r_2^0}{1+\lambda_2+\delta}\right) \quad (11)$$

③ Region 3, I

$$X_{3,\mathrm{I}} = \left\{(r_1, r_2), r_i \in \left[r_i^0, 1\right] \mid r_1 \le \frac{\delta}{1+\lambda_1+\delta} + \frac{(1+\lambda_1) r_1^0}{1+\lambda_1+\delta} - \frac{\delta r_2^0}{1+\lambda_1+\delta}, r_2 > \frac{1+\lambda_1}{1+\lambda_1+\delta} - \frac{(1+\lambda_1) r_1^0}{1+\lambda_1+\delta} + \frac{\delta r_2^0}{1+\lambda_1+\delta}\right\} \quad (12)$$

The equilibrium proposals in $X_{3,\mathrm{I}}$ are shown as follows:

$$x^{3,\mathrm{I}} = \left(\frac{1+\lambda_1(1-\delta)}{1+\lambda_1+\delta} + \frac{\delta(1+\lambda_1) r_1^0}{1+\lambda_1+\delta} - \frac{(1+\lambda_1(1-\delta)) r_2^0}{1+\lambda_1+\delta}, \frac{\delta(1+\lambda_1)}{1+\lambda_1+\delta} - \frac{\delta(1+\lambda_1) r_1^0}{1+\lambda_1+\delta} + \frac{(1+\lambda_1(1-\delta)) r_2^0}{1+\lambda_1+\delta}\right);$$
$$y^{3,\mathrm{I}} = \left(\frac{\delta}{1+\lambda_1+\delta} + \frac{(1+\lambda_1) r_1^0}{1+\lambda_1+\delta} - \frac{\delta r_2^0}{1+\lambda_1+\delta}, \frac{1+\lambda_1}{1+\lambda_1+\delta} - \frac{(1+\lambda_1) r_1^0}{1+\lambda_1+\delta} + \frac{\delta r_2^0}{1+\lambda_1+\delta}\right) \quad (13)$$

④ Region 3, III

$$X_{3,\mathrm{III}} = \left\{(r_1, r_2), r_i \in \left[r_i^0, 1\right] \mid r_1 \le \frac{\delta(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\mu_2 r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\delta r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2}, r_2 \le \frac{\delta(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\delta r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\mu_1 r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2}\right\} \quad (14)$$

The equilibrium proposals in $X_{3,\mathrm{III}}$ are shown as follows:

$$x^{3,\mathrm{III}} = \left(\frac{\mu_1(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\delta r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\mu_1 r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2}, \frac{\delta(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\delta r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\mu_1 r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2}\right);$$
$$y^{3,\mathrm{III}} = \left(\frac{\delta(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\mu_2 r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\delta r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2}, \frac{\mu_2(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\mu_2 r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\delta r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2}\right) \quad (15)$$

⑤ Region 1, II

$$X_{1,\mathrm{II}} = \Bigg\{ (r_1, r_2), r_i \in \left[r_i^0, 1\right] \mid r_1 > \frac{(\mu_2 - \delta) + \delta\lambda_2 (1 - r_2)}{1 + \lambda_2 - \delta^2} + \frac{\delta (1 - \delta) r_1^0}{1 + \lambda_2 - \delta^2} - \frac{(1 + \lambda_2)(1 - \delta) r_2^0}{1 + \lambda_2 - \delta^2}, \frac{\delta}{1 + \lambda_2 + \delta} - \frac{\delta r_1^0}{1 + \lambda_2 + \delta} + \frac{(1 + \lambda_2) r_2^0}{1 + \lambda_2 + \delta} < r_2 \le \frac{1}{1 + \delta} - \frac{r_1^0}{1 + \delta} + \frac{\delta r_2^0}{1 + \delta} \Bigg\}. \quad (16)$$

The equilibrium proposals in $X_{1,\mathrm{II}}$ are shown as follows:

$$\begin{aligned} x^{1,\mathrm{II}} &= \Bigg(\frac{(\mu_2 - \delta) + \delta\lambda_2 (1 - r_2)}{1 + \lambda_2 - \delta^2} + \frac{\delta (1 - \delta) r_1^0}{1 + \lambda_2 - \delta^2} - \frac{(1 + \lambda_2)(1 - \delta) r_2^0}{1 + \lambda_2 - \delta^2}, \frac{\delta (1 - \delta) + \delta\lambda_2 r_2}{1 + \lambda_2 - \delta^2} - \frac{\delta (1 - \delta) r_1^0}{1 + \lambda_2 - \delta^2} + \frac{(1 + \lambda_2)(1 - \delta) r_2^0}{1 + \lambda_2 - \delta^2} \Bigg); \\ y^{1,\mathrm{II}} &= \Bigg(\frac{\delta (\mu_2 - \delta) + \delta^2 \lambda_2 (1 - r_2)}{1 + \lambda_2 - \delta^2} + \frac{(1 + \lambda_2)(1 - \delta) r_1^0}{1 + \lambda_2 - \delta^2} - \frac{\delta (1 + \lambda_2)(1 - \delta) r_2^0}{1 + \lambda_2 - \delta^2}, \frac{(\mu_2 - \delta) + \delta^2 \lambda_2 r_2}{1 + \lambda_2 - \delta^2} - \frac{(1 + \lambda_2)(1 - \delta) r_1^0}{1 + \lambda_2 - \delta^2} + \frac{\delta (1 + \lambda_2)(1 - \delta) r_2^0}{1 + \lambda_2 - \delta^2} \Bigg) \end{aligned} \quad (17)$$

⑥ Region 3, II

$$X_{3,\mathrm{II}} = \Bigg\{ (r_1, r_2), r_i \in \left[r_i^0, 1\right] \mid r_1 \le \frac{\delta (\mu_2 - \delta + \lambda_2 \delta (1 - r_2))}{\mu_1 (1 + \lambda_2) - \delta^2} + \frac{(1 + \lambda_2)(\mu_1 - \delta) r_1^0}{\mu_1 (1 + \lambda_2) - \delta^2} - \frac{\delta (\mu_2 - \delta) r_2^0}{\mu_1 (1 + \lambda_2) - \delta^2}, \frac{\delta (\mu_1 - \delta)}{\mu_1 \mu_2 - \delta^2} - \frac{\delta r_1^0 (\mu_1 - \delta)}{\mu_1 \mu_2 - \delta^2} + \frac{\mu_1 r_2^0 (\mu_2 - \delta)}{\mu_1 \mu_2 - \delta^2} < r_2 < \frac{1 + \lambda_1}{1 + \lambda_1 + \delta} - \frac{(1 + \lambda_1) r_1^0}{1 + \lambda_1 + \delta} + \frac{\delta r_2^0}{1 + \lambda_1 + \delta} \Bigg\}. \quad (18)$$

The equilibrium proposals in $X_{3,\mathrm{II}}$ are shown as follows:

$$\begin{aligned} x^{3,\mathrm{II}} &= \Bigg(\frac{\mu_1 (\mu_2 - \delta + \lambda_2 \delta (1 - r_2))}{\mu_1 (1 + \lambda_2) - \delta^2} + \frac{\delta (\mu_1 - \delta) r_1^0}{\mu_1 (1 + \lambda_2) - \delta^2} - \frac{\mu_1 (\mu_2 - \delta) r_2^0}{\mu_1 (1 + \lambda_2) - \delta^2}, \frac{\delta (\mu_1 - \delta + \lambda_2 r_2 \mu_1)}{\mu_1 (1 + \lambda_2) - \delta^2} - \frac{\delta (\mu_1 - \delta) r_1^0}{\mu_1 (1 + \lambda_2) - \delta^2} + \frac{\mu_1 (\mu_2 - \delta) r_2^0}{\mu_1 (1 + \lambda_2) - \delta^2} \Bigg) \\ y^{3,\mathrm{II}} &= \Bigg(\frac{\delta (\mu_2 - \delta + \lambda_2 \delta (1 - r_2))}{\mu_1 (1 + \lambda_2) - \delta^2} + \frac{(1 + \lambda_2)(\mu_1 - \delta) r_1^0}{\mu_1 (1 + \lambda_2) - \delta^2} - \frac{\delta (\mu_2 - \delta) r_2^0}{\mu_1 (1 + \lambda_2) - \delta^2}, \frac{(1 + \lambda_2)(\mu_1 - \delta) + \lambda_2 \delta^2 r_2}{\mu_1 (1 + \lambda_2) - \delta^2} - \frac{(1 + \lambda_2)(\mu_1 - \delta) r_1^0}{\mu_1 (1 + \lambda_2) - \delta^2} + \frac{\delta (\mu_2 - \delta) r_2^0}{\mu_1 (1 + \lambda_2) - \delta^2} \Bigg) \end{aligned} \quad (19)$$

⑦ Region 2, I

$$X_{2,\mathrm{I}} = \Bigg\{ (r_1, r_2), r_i \in \left[r_i^0, 1\right] \mid \frac{\delta}{1 + \lambda_1 + \delta} + \frac{(1 + \lambda_1) r_1^0}{1 + \lambda_1 + \delta} - \frac{\delta r_2^0}{1 + \lambda_1 + \delta} < r_1 \le \frac{1}{1 + \delta} + \frac{\delta r_1^0}{1 + \delta} - \frac{r_2^0}{1 + \delta}, r_2 > \frac{(\mu_1 - \delta) + \delta\lambda_1 (1 - r_1)}{1 + \lambda_1 - \delta^2} - \frac{(1 - \delta)(1 + \lambda_1) r_1^0}{1 + \lambda_1 - \delta^2} - \frac{\delta (1 - \delta) r_2^0}{1 + \lambda_1 - \delta^2} \Bigg\}. \quad (20)$$

The equilibrium proposals in $X_{2,\mathrm{I}}$ are shown as follows:

$$\begin{aligned} x^{2,\mathrm{I}} &= \Bigg(\frac{(\mu_1 - \delta) + \delta^2 \lambda_1 r_1}{1 + \lambda_1 - \delta^2} + \frac{\delta (1 - \delta)(1 + \lambda_1) r_1^0}{1 + \lambda_1 - \delta^2} - \frac{(1 + \lambda_1)(1 - \delta) r_2^0}{1 + \lambda_1 - \delta^2}, \frac{\delta (\mu_1 - \delta) + \delta^2 \lambda_1 (1 - r_1)}{1 + \lambda_1 - \delta^2} - \frac{\delta (1 - \delta)(1 + \lambda_1) r_1^0}{1 + \lambda_1 - \delta^2} + \frac{(1 + \lambda_1)(1 - \delta) r_2^0}{1 + \lambda_1 - \delta^2} \Bigg); \\ y^{2,\mathrm{I}} &= \Bigg(\frac{\delta (1 - \delta) + \delta\lambda_1 r_1}{1 + \lambda_1 - \delta^2} + \frac{(1 - \delta)(1 + \lambda_1) r_1^0}{1 + \lambda_1 - \delta^2} - \frac{\delta (1 - \delta) r_2^0}{1 + \lambda_1 - \delta^2}, \frac{(\mu_1 - \delta) + \delta\lambda_1 (1 - r_1)}{1 + \lambda_1 - \delta^2} - \frac{(1 - \delta)(1 + \lambda_1) r_1^0}{1 + \lambda_1 - \delta^2} + \frac{\delta (1 - \delta) r_2^0}{1 + \lambda_1 - \delta^2} \Bigg). \end{aligned} \quad (21)$$

⑧ Region 2, III

$$X_{2,\mathrm{III}} = \Bigg\{ (r_1, r_2), r_i \in [r_i^0, 1] \mid \frac{\delta(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\mu_2 r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\delta r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} < r_1 \le \frac{1+\lambda_2}{1+\lambda_2+\delta} + \frac{\delta r_1^0}{1+\lambda_2+\delta} - \frac{(1+\lambda_2) r_2^0}{1+\lambda_2+\delta}, r_2 \le \frac{\delta(\mu_1-\delta+\lambda_1\delta(1-r_1))}{\mu_2(1+\lambda_1)-\delta^2} - \frac{\delta(\mu_1-\delta) r_1^0}{\mu_2(1+\lambda_1)-\delta^2} + \frac{(\mu_2-\delta)(1+\lambda_1) r_2^0}{\mu_2(1+\lambda_1)-\delta^2} \Bigg\}. \quad (22)$$

The equilibrium proposals in $X_{2,\mathrm{III}}$ are shown as follows:

$$x^{2,\mathrm{III}} = \Bigg(\frac{(\mu_2-\delta)(1+\lambda_1)+\delta^2\lambda_1 r_1}{\mu_2(1+\lambda_1)-\delta^2} + \frac{\delta(\mu_1-\delta) r_1^0}{\mu_2(1+\lambda_1)-\delta^2} - \frac{(\mu_2-\delta)(1+\lambda_1) r_2^0}{\mu_2(1+\lambda_1)-\delta^2}, \frac{\delta(\mu_1-\delta+\lambda_1\delta(1-r_1))}{\mu_2(1+\lambda_1)-\delta^2} - \frac{\delta(\mu_1-\delta) r_1^0}{\mu_2(1+\lambda_1)-\delta^2} + \frac{(\mu_2-\delta)(1+\lambda_1) r_2^0}{\mu_2(1+\lambda_1)-\delta^2} \Bigg);$$

$$y^{2,\mathrm{III}} = \Bigg(\frac{\delta(\mu_2-\delta+\lambda_1 r_1\mu_2)}{\mu_2(1+\lambda_1)-\delta^2} + \frac{\mu_2(\mu_1-\delta) r_1^0}{\mu_2(1+\lambda_1)-\delta^2} - \frac{\delta(\mu_2-\delta) r_2^0}{\mu_2(1+\lambda_1)-\delta^2}, \frac{\mu_2(\mu_1-\delta+\lambda_1\delta(1-r_1))}{\mu_2(1+\lambda_1)-\delta^2} - \frac{\mu_2(\mu_1-\delta) r_1^0}{\mu_2(1+\lambda_1)-\delta^2} + \frac{\delta(\mu_2-\delta) r_2^0}{\mu_2(1+\lambda_1)-\delta^2} \Bigg) \quad (23)$$

⑨ Region 2, II

For the set $X_{2,\mathrm{II}}$, its boundaries are described by the neighboring sets' boundaries. The equilibrium proposals in $X_{2,\mathrm{II}}$ are shown as follows:

$$x^{2,\mathrm{II}} = \Bigg(\frac{(1+\lambda_1)(\mu_2-\delta+\delta\lambda_2(1-r_2))+\delta^2\lambda_1 r_1}{(1+\lambda_1)(1+\lambda_2)-\delta^2} + \frac{\delta(1-\delta)(1+\lambda_1) r_1^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2} - \frac{(1-\delta)(1+\lambda_1)(1+\lambda_2) r_2^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2}, \frac{\delta(\mu_1-\delta+\lambda_1\delta(1-r_1)+\lambda_2(1+\lambda_1) r_2)}{(1+\lambda_1)(1+\lambda_2)-\delta^2} - \frac{\delta(1-\delta)(1+\lambda_1) r_1^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2} + \frac{(1-\delta)(1+\lambda_1)(1+\lambda_2) r_2^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2} \Bigg);$$

$$y^{2,\mathrm{II}} = \Bigg(\frac{\delta(\mu_2-\delta+\lambda_2\delta(1-r_2)+\lambda_1(1+\lambda_2) r_1)}{(1+\lambda_1)(1+\lambda_2)-\delta^2} + \frac{(1-\delta)(1+\lambda_1)(1+\lambda_2) r_1^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2} - \frac{\delta(1-\delta)(1+\lambda_2) r_2^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2}, \frac{(1+\lambda_2)(\mu_1-\delta+\lambda_1\delta(1-r_1))+\delta^2\lambda_2 r_2}{(1+\lambda_1)(1+\lambda_2)-\delta^2} - \frac{(1-\delta)(1+\lambda_1)(1+\lambda_2) r_1^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2} + \frac{\delta(1-\delta)(1+\lambda_2) r_2^0}{(1+\lambda_1)(1+\lambda_2)-\delta^2} \Bigg). \quad (24)$$

In the set $X_{1,\mathrm{I}}$, the equilibrium proposals are independent of loss aversion coefficients. If $r_1^0 = r_2^0 = 0$, we can obtain the classical Rubinstein equilibrium proposals $x = (1/(1+\delta), \delta/(1+\delta))$; $y = (\delta/(1+\delta), 1/(1+\delta))$. Moreover, if $r_1^0 = r_2^0 = 0$, we can obtain the equilibrium proposals of Driesen et al. [2] in the sets $X_{1,\mathrm{I}}, \ldots, X_{3,\mathrm{III}}$, respectively.

The equilibrium proposals in the sets $X_{1,\mathrm{III}}$, $X_{3,\mathrm{I}}$, and $X_{3,\mathrm{III}}$ depend on the initial reference points but not on r_1 and r_2. The equilibrium proposals in the sets $X_{1,\mathrm{II}}$ and $X_{3,\mathrm{II}}$ depend on the initial reference points and the referent points r_2 but not on the referent points r_1. In the sets $X_{2,\mathrm{I}}$ and $X_{2,\mathrm{III}}$, the associated equilibrium proposals depend on the initial reference points and player 1's referent points r_1 but not on player 2's referent points r_2. In the set $X_{2,\mathrm{II}}$, the associated equilibrium proposals not only depend on the initial reference points but also the referent points r_1 and r_2.

3.2. Subgame Perfect Equilibrium and Its Uniqueness. To find a SPE, the strategies $\hat{f}$ and $\hat{g}$, which are the strategies of players 1 and 2, are defined according to the sets X_ω and the proposals x^ω and y^ω, where $\omega \in \{1, \mathrm{I}, \ldots, 3, \mathrm{III}\}$. At any time $t \in T_{odd}$, for player 1, take the (unique) X_ω containing reference point (r_1, r_2) for any (r_1, r_2) with $r_1 \ge r_1^0$, $r_2 \ge r_2^0$: then the corresponding proposal x^ω is made by player 1. At any time $t \in T_{even}$ and for any (r_1, r_2) with $r_1 \ge r_1^0$, $r_2 \ge r_2^0$, take again the relevant set X_ω: then a proposal z is accepted by player 1 if and only if $z_1 \ge y_1^\omega$. Similarly, the strategy $\hat{g}$ for player 2 can be defined.

Theorem 2. *The strategy profile* $(\hat{f}, \hat{g})$ *is a SPE; the outcome of the game is*

$$x^{3,\text{III}} = \left(\frac{\mu_1 (\mu_2 - \delta)}{\mu_1\mu_2 - \delta^2} + \frac{\delta r_1^0 (\mu_1 - \delta)}{\mu_1\mu_2 - \delta^2} - \frac{\mu_1 r_2^0 (\mu_2 - \delta)}{\mu_1\mu_2 - \delta^2}, \frac{\delta (\mu_1 - \delta)}{\mu_1\mu_2 - \delta^2} - \frac{\delta r_1^0 (\mu_1 - \delta)}{\mu_1\mu_2 - \delta^2} + \frac{\mu_1 r_2^0 (\mu_2 - \delta)}{\mu_1\mu_2 - \delta^2} \right). \quad (25)$$

Proof. See Appendix A. □

Obviously, the outcome in Theorem 2 depends on the initial reference points of players but not on the reference points r_1 and r_2. It is interesting to note that it is a SPE introduced by Driesen et al. [2] if $r_1^0 = r_2^0 = 0$; i.e., $(\mu_1(\mu_2 - \delta)/(\mu_1\mu_2 - \delta^2), \delta(\mu_1 - \delta)/(\mu_1\mu_2 - \delta^2))$ is the SPE in Driesen et al.'s [2] result, which is independent of the reference points.

For the situation where players have the same level of loss aversion, i.e., $\lambda := \lambda_1 = \lambda_2 > 0$, another interesting observation is that there exist the following cases.

(i) The initial reference points are equal but not zero, i.e., $r := r_1^0 = r_2^0 > 0$. In this case, the outcome is

$$\left(\frac{(\mu - (\mu - \delta) r)}{(\mu + \delta)}, \frac{(\delta + (\mu - \delta) r)}{(\mu + \delta)} \right), \quad \text{where } \mu := \mu_1 = \mu_2. \quad (26)$$

Since $\mu - \delta = (1 + \lambda)(1 - \delta) > 0$, we have $\delta/(\mu + \delta) + (\mu - \delta)r/(\mu + \delta) > \delta/(\mu + \delta)$, and $\delta/(\mu + \delta)$ is player 2's share in Driesen et al.'s [2] outcome. Thus, player 2 (the player who does not start proposing) benefits from the initial reference points.

Similarly, since $\mu/(\mu+\delta) - (\mu-\delta)r/(\mu+\delta) < \mu/(\mu+\delta)$, and $\mu/(\mu + \delta)$ is payoff of player 1 in Driesen et al.'s [2] outcome, player 1 suffers loss because of existing initial reference points.

(ii) The initial reference points of players are not equal; there exist the following three different cases.

If $\delta r_1^0 = \mu r_2^0$, then the outcome of the game is $(\mu/(\mu + \delta), \delta/(\mu + \delta))$, which is the outcome of Driesen et al.'s [2] game. This implies that players do not benefit from the initial reference points compared to Driesen et al.'s [2] case.

If $\delta r_1^0 < \mu r_2^0$, then the outcome is $((\mu + \delta r_1^0 - \mu r_2^0)/(\mu + \delta), (\delta - \delta r_1^0 + \mu r_2^0)/(\mu + \delta))$, which means that $\mu/(\mu + \delta) > (\mu + \delta r_1^0 - \mu r_2^0)/(\mu + \delta)$; $(\delta - \delta r_1^0 + \mu r_2^0)/(\mu + \delta) > \delta/(\mu + \delta)$. Compared to Driesen et al.'s [2] case, player 1, who starts proposing, has a disadvantage and player 2, who does not start proposing, has an advantage. In other words, player 2 (the player who does not start proposing) benefits from the initial reference points and player 1, who starts proposing, suffers loss because of existing initial reference points.

If $\delta r_1^0 > \mu r_2^0$, then $\mu/(\mu + \delta) < (\mu + \delta r_1^0 - \mu r_2^0)/(\mu + \delta)$ and $(\delta - \delta r_1^0 + \mu r_2^0)/(\mu + \delta) < \delta/(\mu + \delta)$. Player 1, who starts proposing, has an advantage and player 2, who does not start proposing, has a disadvantage compared to Driesen et al.'s [2] case. In other words, player 1, who starts proposing, benefits from his own initial reference points and player 2 (the player who does not start proposing) suffers loss because of existing initial reference points.

Finally, it is important to note that proposals can never be below the reference points on the equilibrium path. For example, if a proposal of player 1 would be below his own reference points, then player 1 has made a higher proposal last phases and so he would improve his payoff by accepting the higher proposal.

Now, we show that the subgame perfect equilibrium is unique. For the strategy profile (f, g), it satisfies the following three conditions:

(I) The strategies f and g are stationary Markov strategies. At each time $t \in T_{odd}$, the proposal prescribed by f does not depend on time but on the reference points at time t and the initial reference point, and at each time $t \in T_{even}$, the Y/N decision prescribed by f depends on the proposal of player 2, the reference points at time t, and the initial reference point. Similarly, the strategy g for player 2 can be described.

(II) Immediate acceptance: According to f, player 1 makes any proposal that is accepted by his opponent according to g, and conversely.

(III) Indifference between acceptance and rejection: For a proposal made by player 1, his opponent is indifferent between accepting this proposal or rejecting it according to the strategy profile (f, g), and conversely.

An interesting observation is that above three conditions are satisfied by the SPE in the Rubinstein bargaining.

Theorem 3. *The pair of strategy* $(\hat{f}, \hat{g})$ *is the unique SPE, which satisfies the conditions: (I), (II), and (III).*

Proof. See Appendix B. □

The condition (I) implies that the equilibrium strategies are history-dependent despite the impact this play has on reference points of the two players. Nevertheless, it does not mean that bargainers are limited to stationary Markov strategies. In fact, the condition (II) must be satisfied by any SPE in some subgames; i.e., the reference points in these subgames are higher than the (equilibrium) payoff. Condition (III) requires for a proposal made by a player that his opponent is indifferent between accepting or rejecting this proposal.

4. Analysis of the Equilibrium

Here, we discuss the impact of loss aversion coefficients on the SPE $(\hat{f}, \hat{g})$ and investigate the SPE $(\hat{f}, \hat{g})$ with respect to the discount factor (or the probability of continuation of game) δ. Then, we analyze what happens when δ tends to 1 and discuss what happens for different continuation probabilities. Finally, we investigate what happens when the time lapse between proposals goes to zero.

Since the set $X_{3,\text{III}}$ is the relevant set at the beginning of bargaining game, we focus on this set. In fact, the comparative statics results are similar in subgames. For the strategy profile

$$x^{3,\mathrm{III}} = \left(\frac{\mu_1(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\delta r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\mu_1 r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2}, \frac{\delta(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\delta r_1^0(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} + \frac{\mu_1 r_2^0(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} \right), \quad (27)$$

we restrict ourselves to the analysis of player 1, since what player 1 gains is what his opponent losses. By differentiating with respect to λ_1 and λ_2, we have

$$\frac{dx_1^{3,\mathrm{III}}}{d\lambda_1} = -\frac{\delta^2(1-\delta)^2(1+\lambda_2)\left(1-r_1^0-r_2^0\right)}{\left(\mu_1\mu_2-\delta^2\right)^2} < 0,$$
$$\frac{dx_1^{3,\mathrm{III}}}{d\lambda_2} = \frac{\delta\mu_1(1-\delta)^2(1+\lambda_1)\left(1-r_1^0-r_2^0\right)}{\left(\mu_1\mu_2-\delta^2\right)^2} > 0. \quad (28)$$

Thus, a player is hurt by loss aversion of himself and benefits from his opponent's at given initial reference points.

Example 4. Consider that players 1 and 2 with loss aversion bargain over a pie, whose size is one. The outcomes of the bargaining are shown in Theorem 1. Let $\delta = 0.6$, $r_1^0 = 0.1$, and $r_2^0 = 0.2$. Figures 2 and 3 show player 1's equilibrium payoff with respect to loss aversion coefficients λ_1 and λ_2, respectively.

From Figure 2, it follows that player 1's equilibrium share is decreasing as λ_1. That is, player 1 is hurt by his own loss aversion. In Figure 3, player 1's equilibrium share is increasing as λ_2; i.e., player 1 benefits from player 2's loss aversion.

By differentiating with respect to r_1^0 and r_2^0, we have

$$\frac{dx_1^{3,\mathrm{III}}}{dr_1^0} = \frac{\delta(\mu_1-\delta)}{\mu_1\mu_2-\delta^2},$$
$$\frac{dx_1^{3,\mathrm{III}}}{dr_2^0} = -\frac{\mu_1(\mu_2-\delta)}{\mu_1\mu_2-\delta^2} \quad (29)$$

Thus, a player benefits from his initial reference point and is hurt by the reference point of the opponent.

4.1. Convergence of the Subgame Perfect Equilibrium for Continuation Probability. We investigate convergence of the subgame perfect equilibrium in the following two different aspects:

(1) Convergence of the subgame perfect equilibrium for a common δ

In this subsection, we analyze what happens to the SPE when δ goes to 1.

$$\lim_{\delta\longrightarrow 1} x_1^{3,\mathrm{III}} = \frac{(1+\lambda_2)\left(1-r_2^0\right)+(1+\lambda_1)r_1^0}{2+\lambda_1+\lambda_2},$$
$$\lim_{\delta\longrightarrow 1} x_2^{3,\mathrm{III}} = \frac{(1+\lambda_1)\left(1-r_1^0\right)+(1+\lambda_2)r_2^0}{2+\lambda_1+\lambda_2}. \quad (30)$$

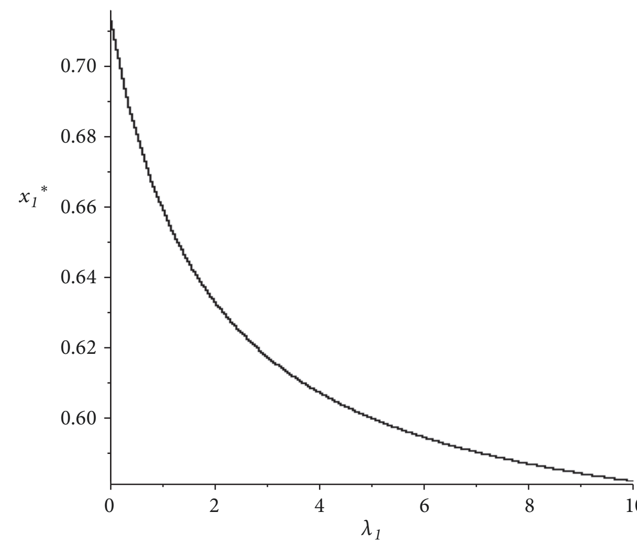

FIGURE 2: The changes of equilibrium share of player 1 as λ_1.

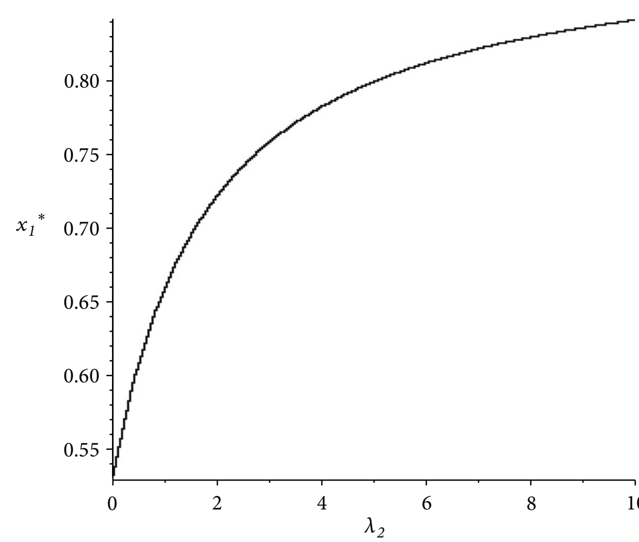

FIGURE 3: The changes of equilibrium share of player 1 as λ_2.

An interesting observation is that the limit equilibrium proposals for δ tending to 1 are equal to the limit equilibrium proposals in the result obtained by Driesen et al. [2] if the initial reference points are equal to zero.

We can repeat this for all subgames. In the limit for δ tending to 1, the nine sets of Figure 1 and the limit equilibrium proposals are shown in Figures 4, 5, 6, and 7 for the case where $\lambda_2 > \lambda_1$.

In Figure 4, the nine sets of Figure 1 and the limit equilibrium proposals are shown in the limit for δ tending to 1, for the case where $\lambda_2 > \lambda_1$ and $r := r_1^0 = r_2^0$. If $r = 0$, all of regions in Figure 4 are consistent with that of Figure 3 obtained in Driesen et al. [2]. Moreover, the limit outcome is (0.5, 0.5) in $X_{1,\mathrm{I}}$ when $\delta \longrightarrow 1$. If $r \neq 0$, the limit outcomes of player 2 in some sets are higher than that of player 2 in Driesen et al.'s [2] outcomes. In $X_{1,\mathrm{I}}$, the limit

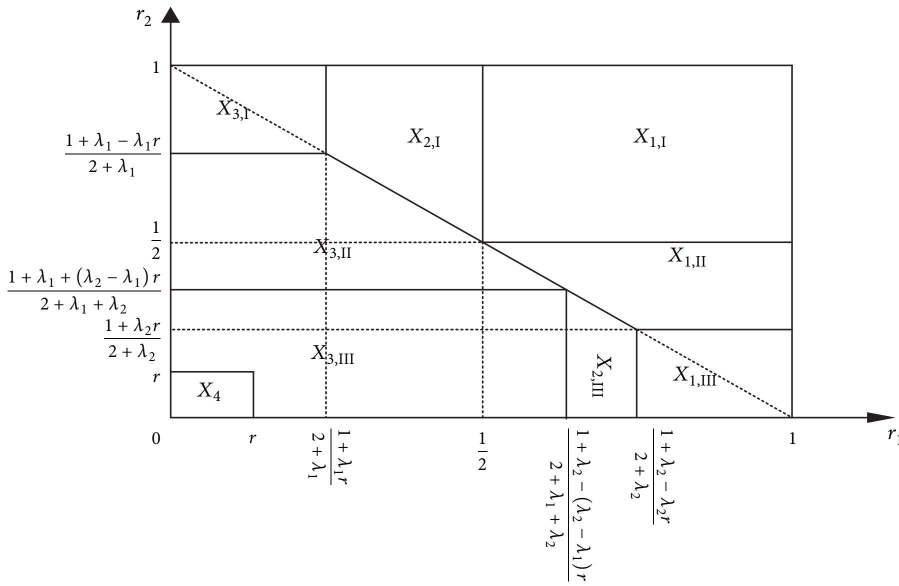

FIGURE 4: The equilibrium partitions for $\delta \longrightarrow 1$, with $\lambda_2 > \lambda_1$ and $r := r_1^0 = r_2^0$.

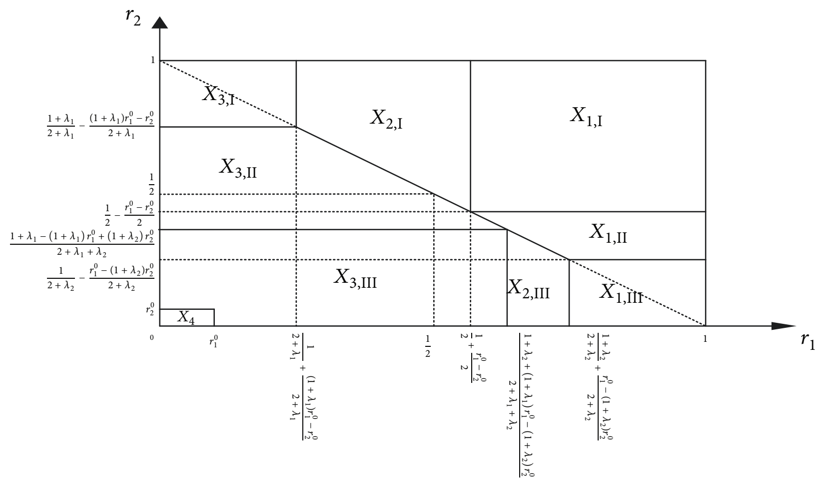

FIGURE 5: The equilibrium partitions for $\delta \longrightarrow 1$, with $\lambda_2 > \lambda_1$ and $r_1^0 > r_2^0$.

outcome is an equal share (0.5, 0.5) when $\delta \longrightarrow 1$. The limit equilibrium outcome in $X_{1,\mathrm{II}}$ and $X_{3,\mathrm{II}}$ is $(1-r_2, r_2)$, while it is $(r_1, 1-r_1)$ in $X_{2,\mathrm{I}}$ and $X_{2,\mathrm{III}}$, which are also the limit equilibrium outcome in Driesen et al.'s [2] outcomes. The limit equilibrium outcome is $((1+\lambda_1 r)/(2+\lambda_1), (1+\lambda_1-\lambda_1 r)/(2+\lambda_1))$ in $X_{3,\mathrm{I}}$ and the limit equilibrium outcome is $((1+\lambda_2-\lambda_2 r)/(2+\lambda_2), (1+\lambda_2 r)/(2+\lambda_2))$ in $X_{1,\mathrm{III}}$. The set $X_{2,\mathrm{II}}$ becomes the line $r_1+r_2=1$ where $r_1 \in ((1+\lambda_1 r)/(2+\lambda_1), (1+\lambda_2-\lambda_2 r)/(2+\lambda_2))$.

In Figure 5, the nine sets of Figure 1 and the limit equilibrium proposals are shown in the limit for δ tending to 1, for the case where $\lambda_2 > \lambda_1$ and $r_1^0 > r_2^0$. In the set of $X_{1,\mathrm{I}}$, the limit equilibrium partition is $((1+r_1^0-r_2^0)/2, (1-r_1^0+r_2^0)/2)$. Since $r_1^0 > r_2^0$, player 1 benefits from the reference points compared to player 1's share in Driesen et al.'s [2] outcomes. The limit equilibrium proposal is $(1-r_2, r_2)$ in $X_{1,\mathrm{II}}$ and $X_{3,\mathrm{II}}$, while the limit equilibrium outcome is $(r_1, 1-r_1)$ in $X_{2,\mathrm{I}}$ and $X_{2,\mathrm{III}}$, which are also the limit equilibrium partition in Driesen et al.'s [2] outcomes. The limit equilibrium proposal in $X_{3,\mathrm{I}}$ is $((1+\lambda_1 r_1^0-r_2^0)/(2+\lambda_1), (1+\lambda_1-(1+\lambda_1)r_1^0+r_2^0)/(2+\lambda_1))$, where player 1 benefits from his own initial reference point compared to player 1's share in Driesen et al.'s [2] outcomes. And in $X_{1,\mathrm{III}}$, it is $((1+\lambda_2+r_1^0-(1+\lambda_2)r_2^0)/(2+\lambda_2), (1-r_1^0+(1+\lambda_2)r_2^0)/(2+\lambda_2))$, where player 1 benefits from his own initial reference point compared to player 1's share in Driesen et al.'s [2] outcome if $r_1^0 > (1+\lambda_2)r_2^0$, and player 2 benefits from his own initial reference point compared to player 2's share in Driesen et al.'s [2] outcome if $r_1^0 < (1+\lambda_2)r_2^0$. The set $X_{2,\mathrm{II}}$ becomes the line $r_1+r_2=1$, where $r_1 \in ((1+\lambda_1 r_1^0-r_2^0)/(2+\lambda_1), (1+\lambda_2+r_1^0-(1+\lambda_2)r_2^0)/(2+\lambda_2))$. In $X_{3,\mathrm{III}}$,

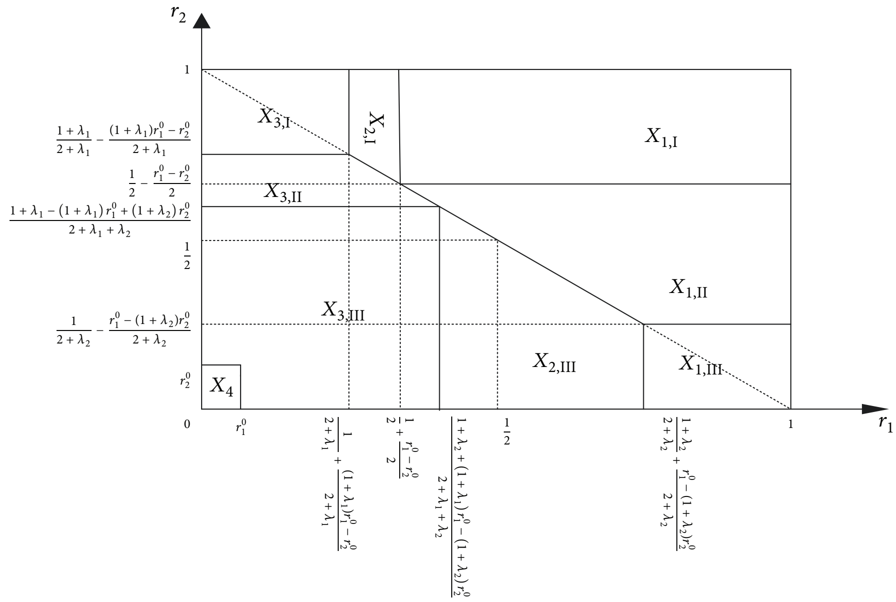

FIGURE 6: The equilibrium partitions for $\delta \longrightarrow 1$, with $r_1^0 < r_2^0$ and $\lambda_1 < \lambda_2 < \lambda_1(1-2r_1^0) + 2(r_2^0 - r_1^0)/(1-2r_2^0)$.

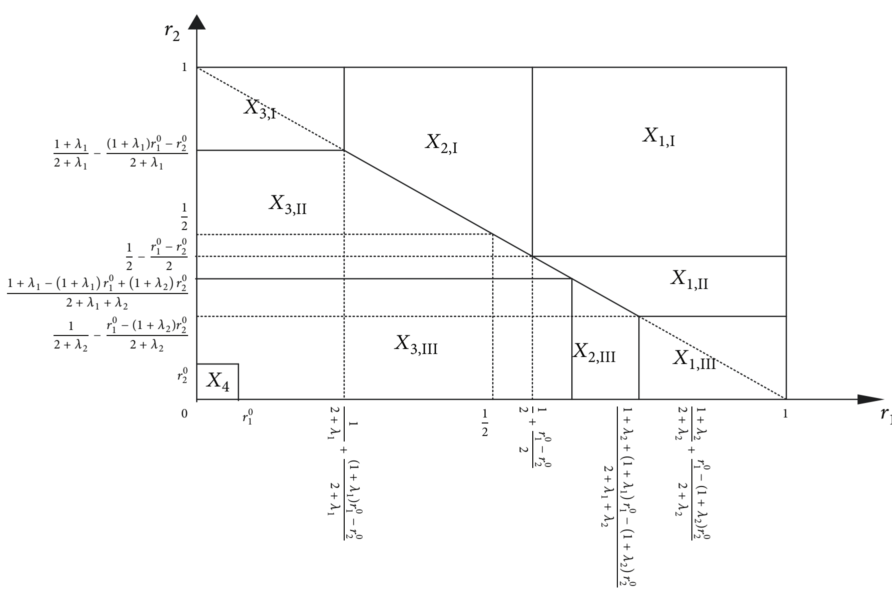

FIGURE 7: The equilibrium partitions for $\delta \longrightarrow 1$, with $r_1^0 < r_2^0$ and $\lambda_1(1-2r_1^0) + 2(r_2^0 - r_1^0)/(1-2r_2^0) < \lambda_2$.

the limit equilibrium partition is $((1+\lambda_2+(1+\lambda_1)r_1^0-(1+\lambda_2)r_2^0)/(2+\lambda_1+\lambda_2), (1+\lambda_1-(1+\lambda_1)r_1^0+(1+\lambda_2)r_2^0)/(2+\lambda_1+\lambda_2))$, where player 1 benefits from the initial reference point r_1^0 compared to player 1's share in Driesen et al.'s [2] outcome.

In Figure 6, the nine sets of Figure 1 and the limit equilibrium proposals are shown in the limit for δ tending to 1, for the case where $r_1^0 < r_2^0$ and $\lambda_1 < \lambda_2 < \lambda_1(1-2r_1^0) + 2(r_2^0 - r_1^0)/(1-2r_2^0)$. In $X_{1,\text{I}}$, the limit equilibrium partition is $((1+r_1^0-r_2^0)/2, (1-r_1^0+r_2^0)/2)$. Since $r_1^0 < r_2^0$, player 2 benefits from the reference points compared to player 2's share in Driesen et al.'s [2] outcome. The limit equilibrium outcome in $X_{1,\text{II}}$ and $X_{3,\text{II}}$ is $(1-r_2, r_2)$, while it is $(r_1, 1-r_1)$ in $X_{2,\text{I}}$ and $X_{2,\text{III}}$, which are also the limit equilibrium outcome in Driesen et al.'s outcomes [2]. The limit equilibrium partition in $X_{3,\text{I}}$ is $((1+\lambda_1 r_1^0 - r_2^0)/(2+\lambda_1), (1+\lambda_1-(1+\lambda_1)r_1^0 + r_2^0)/(2+\lambda_1))$, where player 1 benefits from his own initial reference point compared to player 1's share in Driesen et al.'s [2] outcomes if $r_1^0 < r_2^0 < \lambda_1 r_1^0$. And in $X_{1,\text{III}}$, it is $((1+\lambda_2+r_1^0-(1+\lambda_2)r_2^0)/(2+\lambda_2), (1-r_1^0+(1+\lambda_2)r_2^0)/(2+\lambda_2))$, where the share of player 1 is lower than that of player 1 in Driesen et al.'s [2] outcomes, because $r_1^0 < r_2^0$. The set $X_{2,\text{II}}$ becomes the line $r_1 + r_2 = 1$ where $r_1 \in ((1+\lambda_1 r_1^0 - r_2^0)/(2+\lambda_1), (1+\lambda_2+r_1^0-(1+\lambda_2)r_2^0)/(2+\lambda_2))$. In $X_{3,\text{III}}$, the limit equilibrium partition is $((1+\lambda_2+(1+\lambda_1)r_1^0-(1+\lambda_2)r_2^0)/(2+\lambda_1+\lambda_2)$,

$(1+\lambda_1-(1+\lambda_1)r_1^0+(1+\lambda_2)r_2^0)/(2+\lambda_1+\lambda_2))$. Since $\lambda_1<\lambda_2<\lambda_1(1-2r_1^0)+2(r_2^0-r_1^0)/(1-2r_2^0)$, player 2's share assigned by the limit equilibrium partition is higher than that of player 1. That is, although player 2 has a higher loss aversion coefficient than that of player 1, player 2 can obtain a high share of the pie since this player has a high initial reference point.

In Figure 7, the nine sets of Figure 1 and the limit equilibrium proposals are shown in the limit for δ tending to 1, for the case where $r_1^0<r_2^0$ and $\lambda_1(1-2r_1^0)+2(r_2^0-r_1^0)/(1-2r_2^0)<\lambda_2$. In $X_{3,\mathrm{III}}$, the limit equilibrium outcome is

$$\left(\frac{\left(1+\lambda_2+\left(1+\lambda_1\right)r_1^0-\left(1+\lambda_2\right)r_2^0\right)}{\left(2+\lambda_1+\lambda_2\right)},\right.\\ \left.\frac{\left(1+\lambda_1-\left(1+\lambda_1\right)r_1^0+\left(1+\lambda_2\right)r_2^0\right)}{\left(2+\lambda_1+\lambda_2\right)}\right). \tag{31}$$

Since $\lambda_1(1-2r_1^0)+2(r_2^0-r_1^0)/(1-2r_2^0)<\lambda_2$, player 2's share assigned by the limit equilibrium partition is lower than that of player 1. That is, although player 2 has a higher initial reference point than that of player 1, player 2 cannot obtain a high share of the pie because of higher loss aversion level of himself. The analysis of the limit equilibrium partition in other sets is similar to that of the limit equilibrium partition in Figure 6.

4.2. Convergence of the Subgame Perfect Equilibrium for $\delta_1\neq\delta_2$. Consider the situation where each player i has his own continuation probability δ_i $(i=1,2)$. δ_i is interpreted as the probability of the bargaining occurs at time $t+1$ if player i rejected his opponent's proposal at time t. It follows from inequalities (6) and (7) that

$$\begin{aligned}&\left(1+\lambda_2\right)\left(x_2-r_2^0\right)-\lambda_2\max\left\{r_2-r_2^0,x_2-r_2^0\right\}\\&\quad\geq\delta_2\left[\left(1+\lambda_2\right)\left(y_2-r_2^0\right)\right.\\&\quad-\lambda_2\max\left\{y_2-r_2^0,\max\left\{r_2-r_2^0,x_2-r_2^0\right\}\right\}\Big]-(1\\&\quad-\delta_2)\lambda_2\max\left\{r_2-r_2^0,x_2-r_2^0\right\}\end{aligned} \tag{32}$$

and

$$\begin{aligned}&\left(1+\lambda_1\right)\left(y_1-r_1^0\right)-\lambda_1\max\left\{r_1-r_1^0,y_1-r_1^0\right\}\\&\quad\geq\delta_1\left[\left(1+\lambda_1\right)\left(x_1-r_1^0\right)\right.\\&\quad-\lambda_1\max\left\{x_1-r_1^0,\max\left\{r_1-r_1^0,y_1-r_1^0\right\}\right\}\Big]-(1\\&\quad-\delta_1)\lambda_1\max\left\{r_1-r_1^0,y_1-r_1^0\right\}\end{aligned} \tag{33}$$

In particular, we can obtain the unique SPE by assuming the inequalities (32) and (33) are equalities, which satisfies conditions (I), (II), and (III).

We further generalize the SPE if there exists a time lapse Δ between proposals. Moreover, after the last proposal was rejected by player $i, i=1,2$, the waiting time for breakdown of the game is a probability distribution function. We assume that the waiting time is exponentially distributed with parameter β_i, which is the survival rate. After a proposal was rejected by player i, the probability that the bargaining game continues is denoted by δ_i^Δ, where $\delta_i=\exp(-1/\beta_i)$. Since the reference points in $X_{3,\mathrm{III}}$ are the relevant at the beginning of bargaining game, we restrict ourselves to analyzing this case. The outcomes are

$$\begin{aligned}x^{3,\mathrm{III}}=&\left(\frac{\mu_1\left(\mu_2-\delta_2^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}+\frac{\delta_2^\Delta r_1^0\left(\mu_1-\delta_1^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}\right.\\&-\frac{\mu_1 r_2^0\left(\mu_2-\delta_2^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta},\frac{\delta_2^\Delta\left(\mu_1-\delta_1^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}-\frac{\delta_2^\Delta r_1^0\left(\mu_1-\delta_1^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}\\&\left.+\frac{\mu_1 r_2^0\left(\mu_2-\delta_2^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}\right)\end{aligned} \tag{34}$$

and

$$\begin{aligned}y^{3,\mathrm{III}}=&\left(\frac{\delta_1^\Delta\left(\mu_2-\delta_2^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}+\frac{\mu_2 r_1^0\left(\mu_1-\delta_1^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}\right.\\&-\frac{\delta_1^\Delta r_2^0\left(\mu_2-\delta_2^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta},\frac{\mu_2\left(\mu_1-\delta_1^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}-\frac{\mu_2 r_1^0\left(\mu_1-\delta_1^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}\\&\left.+\frac{\delta_1^\Delta r_2^0\left(\mu_2-\delta_2^\Delta\right)}{\mu_1\mu_2-\delta_1^\Delta\delta_2^\Delta}\right),\end{aligned} \tag{35}$$

where $\mu_i=1+\lambda_i(1-\delta_i^\Delta)$ for $i=1,2$. For Δ tending to 0, we can derive

$$\begin{aligned}&\lim_{\Delta\longrightarrow0}x^{3,\mathrm{III}}\\&\quad=\left(\frac{\left(1+\lambda_2\right)\left(1-r_2^0\right)\ln\delta_2+r_1^0\left(1+\lambda_1\right)\ln\delta_1}{\left(1+\lambda_1\right)\ln\delta_1+\left(1+\lambda_2\right)\ln\delta_2},\right.\\&\quad\left.\frac{\left(1+\lambda_1\right)\left(1-r_1^0\right)\ln\delta_1+r_2^0\left(1+\lambda_2\right)\ln\delta_2}{\left(1+\lambda_1\right)\ln\delta_1+\left(1+\lambda_2\right)\ln\delta_2}\right)\\&\quad=\lim_{\Delta\longrightarrow0}y^{3,\mathrm{III}}.\end{aligned} \tag{36}$$

Note that this is an asymmetric Nash bargaining solution, as shown by Harsanyi and Selten (1972) and Kalai (1977). That is, this is the solution to the following optimization problem

$$\max_{z\in Z}z_1^\alpha z_2^{1-\alpha} \tag{37}$$

where α is defined as bargaining power of player 1 and $\alpha=((1+\lambda_2)(1-r_2^0)\ln\delta_2+r_1^0(1+\lambda_1)\ln\delta_1)/((1+\lambda_1)\ln\delta_1+(1+\lambda_2)\ln\delta_2)$.

It is easy to check that α is negatively related to λ_1 and positively related to λ_2. Obviously, α depends on the initial reference points of players, where α is increasing as r_1^0 and decreasing as r_2^0.

5. Conclusions

A player may transfer his expectations derived from previous opponents when he enters into a new bargaining situation with another player. That is, the initial reference points in bargaining problems are not zero. In this paper, we investigate the impact of loss aversion and the initial reference points in the classical Rubinstein bargaining problem, by constructing a SPE in Rubinstein bargaining model with loss aversion and reference dependence and making a sensitivity analysis about the SPE with respect to loss aversion coefficients of bargainers. It is found that the equilibrium share of a player is negatively related to his own loss aversion and the initial reference point of the other and positively related to the opponent's loss aversion and his own initial reference point. It is further found that the outcome converges to asymmetric Nash bargaining if the probability of breakdown tends to zero, where higher loss aversion of a player results in a higher bargaining power of the opponent and a player who has a higher initial reference point has a higher bargaining power.

We introduce the unique SPE based on the following three features: stationary Markov strategies, immediate acceptance, and indifference between acceptance and rejection. It is still an open question whether the three features are necessary conditions for uniqueness of the subgame perfect equilibrium.

Appendix

A. Proof of Theorem 2

The one-deviation property is used to prove Theorem 2. According to this property, the sufficient condition that a strategy profile $(f, g) \in F \times G$ is a SPE is no one can improve his own payoffs by deviating unilaterally only once.

The one-deviation property, as Hendon et al. [15] pointed out, holds in infinite-horizon extensive-form games. These games are continuous at infinity. In order to define the continuity at infinity, we have the following: For any $\varepsilon > 0$, there exists a number $t \in T$ such that if, for $(f, g), (f', g') \in F \times G$, we have $(f^s, g^s) = (f'^s, g'^s)$ for all $s \leq t$, then $|U_i(f, g) - U_i(f', g')| < \varepsilon$.

Lemma A.1. *The bargaining game, where bargainers are loss averse and their initial reference points are not zero, is continuous at infinity.*

Proof. Let $\varepsilon > 0$, and let the strategy profiles $(f, g), (f'^s, g'^s) \in F \times G$ satisfy $(f^s, g^s) = (f'^s, g'^s)$ for all $s \leq t$, where $t > \max_{i=1,2} \log_\delta \varepsilon/(1+\lambda_i)$. For two such strategy profiles, if player i obtains the whole pie from a strategy profile at time $t+1$, while the other one results in perpetual disagreement, we have the following.

For the former, player i would obtain $\overline{U}_i = \delta^t + (1-\delta)\sum_{s=1}^{t} \delta^{s-1} u_i(h^s, d)$. For the latter he would obtain $\underline{U}_i = (1-\delta)\sum_{s=1}^{\infty} \delta^{s-1} u_i(h^s, d)$. It follows from $u_i(h^t, d) \geq -\lambda_i$ that for all $t \in T$

$$\begin{aligned} \underline{U}_i &= (1-\delta)\sum_{s=1}^{t}\delta^{s-1}u_i(h^s,d) \\ &\quad + (1-\delta)\sum_{s=t+1}^{\infty}\delta^{s-1}u_i(h^s,d) \\ &\geq (1-\delta)\left(\sum_{s=1}^{t}\delta^{s-1}u_i(h^s,d) + \frac{\delta^t(-\lambda_i)}{(1-\delta)}\right) \\ &= -\delta^t\lambda_i + (1-\delta)\sum_{s=1}^{t}\delta^{s-1}u_i(h^s,d). \end{aligned} \tag{A.1}$$

It follows from this equality and $\overline{U}_i - \underline{U}_i \geq 0$ that

$$\begin{aligned} &\left|U_i(f,g) - U_i(f',g')\right| \leq \overline{U}_i - \underline{U}_i \\ &\quad \leq \delta^t + (1-\delta)\sum_{s=1}^{t}\delta^{s-1}u_i(h^s,d) + \delta^t\lambda_i \\ &\quad - (1-\delta)\sum_{s=1}^{t}\delta^{s-1}u_i(h^s,d) = \delta^t(1+\lambda_i) < \varepsilon. \end{aligned} \tag{A.2}$$

Thus, the game is continuous at infinity. □

By Lemma A.1, the one-deviation property can be used. We define $\Omega = \{1, I., \ldots\ldots, 3, III.\}$.

Proof of Theorem 2. The sufficient condition that the strategy profile $(\hat{f}, \hat{g})$ is a SPE is that no one can improve his share by deviating unilaterally at one point in time. The utility of share that player 1 obtains according to the following strategy $\hat{f}$ is denoted by μ_1^*.

Let $h^{t\text{-}1} \in C^{t\text{-}1}$; that is, $h^{t\text{-}1}$ is a history continuing to time t. We assume that $h^{t\text{-}1}$ satisfies $(r_1(h^{t\text{-}1}), r_2(h^{t\text{-}1})) \in X_\omega$ with $\omega \in \Omega$, and $h^t = (h^{t-1}, z)$ with $z \in Z$. If $t \in T_{\text{odd}}$ (T_{even}), then the proposal z is made by player 1 (6). If z is rejected, then the bargaining continues with probability δ or ends in disagreement with probability $1-\delta$. If the bargaining continues to moment $t+1$, it ends in accepting the proposal at $t+1$, since the strategy profile $(\hat{f}, \hat{g})$ is prevalent.

To present that the strategy $\hat{f}$ is the best response to the strategy $\hat{g}$, we distinguish the following two cases: $t \in T_{odd}$ and $t \in T_{even}$. For each case, the following three subcases are considered: (r_1, r_2) is in the set X_ω where we have the following.

Case 1. $\omega \in \{1, \text{I}, \ 1, \text{II}, \ 1, \text{III}\}$, then $r_1 > x_1^\omega > y_1^\omega$ and $y_1^\omega - r_1^0 = \delta(x_1^\omega - r_1^0)$.

Case 2. $\omega \in \{2, \text{I}, \ 2, \text{II}, \ 2, \text{III}\}$, then $x_1^\omega \geq r_1 > y_1^\omega$ and $(1+\lambda_1)(y_1^\omega - r_1^0) = \delta(x_1^\omega - r_1^0) + \delta\lambda_1(r_1 - r_1^0)$.

Case 3. $\omega \in \{3, \text{I}, \ 3, \text{II}, \ 3, \text{III}\}$, then $x_1^\omega > y_1^\omega \geq r_1$ and $\mu_1(y_1^\omega - r_1^0) = \delta(x_1^\omega - r_1^0)$.

If $t \in T_{odd}$ for Case 1, we distinguish the following cases:

(i) $z_1 = x_1^\omega$: In the case player 1 plays $\hat{f}$ and player 2 accepts it, we have

$$\begin{aligned} u_1^* &= u_1(h^t, a) \\ &= (1+\lambda_1)(x_1^\omega - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, x_1^\omega - r_1^0\} \\ &= (1+\lambda_1)(x_1^\omega - r_1^0) - \lambda_1 (r_1 - r_1^0). \end{aligned} \quad (A.3)$$

(ii) $z_1 < x_1^\omega$: Then $z_2 > x_2^\omega$. Thus, player 2 accepts it. The payoff of player 1 is given as follows:

$$\begin{aligned} u_1(h^t, a) = {} & (1+\lambda_1)(z_1 - r_1^0) \\ & - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\}. \end{aligned} \quad (A.4)$$

If $r_1 > x_1^\omega > z_1$, then we have $u_1(h^t, a) = (1+\lambda_1)(z_1 - r_1^0) - \lambda_1(r_1 - r_1^0)$. Since $z_1 < x_1^\omega$, we have $u_1(h^t, a) < u_1^*$. Thus, z is not optimal.

(iii) $z_1 > x_1^\omega$: Then $z_2 < x_2^\omega$. Thus, player 2 rejects the proposal and proposes y^ω when the bargaining game continues. The payoff of player 1 is given as follows:

$$\begin{aligned} & \delta u_1(h^{t+1}, a) + (1-\delta) u_1(h^t, d) \\ &= \delta\big((1+\lambda_1)(y_1^\omega - r_1^0) \\ &- \lambda_1 \max\{y_1^\omega - r_1^0, r_1 - r_1^0\}\big) - (1-\delta)\lambda_1 (r_1 \\ &- r_1^0) = \delta(1+\lambda_1)(y_1^\omega - r_1^0) - \delta\lambda_1(r_1 - r_1^0) - (1 \\ &- \delta)\lambda_1(r_1 - r_1^0) = \delta(1+\lambda_1)(y_1^\omega - r_1^0) - \lambda_1 (r_1 \\ &- r_1^0) \end{aligned} \quad (A.5)$$

Since $x_1^\omega > \delta^2 x_1^\omega = \delta y_1^\omega$, we have $\delta u_1(h^{t+1}, a) + (1-\delta) u_1(h^t, d) < u_1^*$. Thus, z is not optimal.

If $t \in T_{odd}$ for Case 2, the following three cases are distinguished:

(i) $z_1 = x_1^\omega$: In the case player 1 plays $\hat{f}$ and player 2 accepts it, we have

$$\begin{aligned} u_1^* &= u_1(h^t, a) \\ &= (1+\lambda_1)(x_1^\omega - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, x_1^\omega - r_1^0\} \\ &= x_1^\omega - r_1^0 \end{aligned} \quad (A.6)$$

(ii) $z_1 < x_1^\omega$: Then $z_2 > x_2^\omega$. Thus, player 2 accepts it. The payoff of player 1 is given as follows:

$$\begin{aligned} u_1(h^t, a) = {} & (1+\lambda_1)(z_1 - r_1^0) \\ & - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\}. \end{aligned} \quad (A.7)$$

It follows from $(1+\lambda_1)(z_1 - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\} \le z_1 - r_1^0 < x_1^\omega - r_1^0 = u_1^*$ that z is not optimal.

(iii) $z_1 > x_1^\omega$: Then $z_2 < x_2^\omega$. Thus, player 2 rejects the proposal and proposes y^ω when the bargaining game continues. The payoff of player 1 is given as follows:

$$\begin{aligned} & \delta u_1(h^{t+1}, a) + (1-\delta) u_1(h^t, d) \\ &= \delta(1+\lambda_1)(y_1^\omega - r_1^0) \\ &\quad - \delta\lambda_1 \max\{y_1^\omega - r_1^0, r_1 - r_1^0\} \\ &\quad - (1-\delta)\lambda_1(r_1 - r_1^0) \\ &= \delta(1+\lambda_1)(y_1^\omega - r_1^0) - \lambda_1(r_1 - r_1^0) \end{aligned} \quad (A.8)$$

It follows from $(1+\lambda_1)(y_1^\omega - r_1^0) = \delta(x_1^\omega - r_1^0) + \delta\lambda_1(r_1 - r_1^0)$ that $\delta(x_1^\omega - r_1^0) = (1+\lambda_1)(y_1^\omega - r_1^0) - \delta\lambda_1(r_1 - r_1^0)$. Therefore, $u_1^* = x_1^\omega - r_1^0 = (1+\lambda_1)(1/\delta)(y_1^\omega - r_1^0) - \lambda_1(r_1 - r_1^0) > (1+\lambda_1)\delta(y_1^\omega - r_1^0) - \lambda_1(r_1 - r_1^0)$, which means that z is not optimal.

If $t \in T_{odd}$ for Case 3, the following three cases are distinguished:

(i) $z_1 = x_1^\omega$: In the case player 1 plays $\hat{f}$ and player 2 accepts it, we have

$$\begin{aligned} u_1^* &= u_1(h^t, a) \\ &= (1+\lambda_1)(x_1^\omega - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, x_1^\omega - r_1^0\} \\ &= x_1^\omega - r_1^0 \end{aligned} \quad (A.9)$$

(ii) $z_1 < x_1^\omega$: Then $z_2 > x_2^\omega$. Thus, player 2 accepts it. The payoff of player 1 is given as follows:

$$\begin{aligned} u_1(h^t, a) = {} & (1+\lambda_1)(z_1 - r_1^0) \\ & - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\}. \end{aligned} \quad (A.10)$$

It follows from $(1+\lambda_1)(z_1 - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\} \le z_1 - r_1^0 < x_1^\omega - r_1^0 = u_1^*$ that z is not optimal.

(iii) $z_1 > x_1^\omega$: Then $z_2 < x_2^\omega$. Thus, player 2 rejects the proposal and proposes y^ω when the bargaining game continues. The payoff of player 1 is given as follows:

$$\begin{aligned} & \delta u_1(h^{t+1}, a) + (1-\delta) u_1(h^t, d) \\ &= \delta(1+\lambda_1)(y_1^\omega - r_1^0) \\ &\quad - \delta\lambda_1 \max\{y_1^\omega - r_1^0, r_1 - r_1^0\} \\ &\quad - (1-\delta)\lambda_1(r_1 - r_1^0) \\ &= \delta(y_1^\omega - r_1^0) - (1-\delta)\lambda_1(r_1 - r_1^0) \end{aligned} \quad (A.11)$$

Since $u_1^* = x_1^\omega - r_1^0 > y_1^\omega - r_1^0 > \delta(y_1^\omega - r_1^0) > \delta(y_1^\omega - r_1^0) - \lambda_1(1-\delta)(r_1 - r_1^0)$, z is not optimal.

If $t \in T_{even}$ for Case 1, we assume that player 2 makes some proposal $z \in Z$. If the proposal is accepted by player 1, player 1's payoff is given by $u_1(h^t, a) = (1+\lambda_1)(z_1 - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\}$. If the proposal is rejected by player 1 and the bargaining game continues, the reference point $r_1(h^t)$ is $\max\{r_1 - r_1^0, z_1 - r_1^0\}$. Player 1 proposes x^ω and player 2 accepts the proposal. Since $r_1 > x^\omega$ and $y_1^\omega - r_1^0 = \delta(x_1^\omega - r_1^0)$, we have

$$\begin{aligned}&\delta u_1\left(h^{t+1}, a\right) + (1-\delta) u_1\left(h^t, d\right)\\ &\quad = \delta\left(1+\lambda_1\right)\left(x_1^\omega - r_1^0\right)\\ &\qquad - \delta\lambda_1 \max\left\{x_1^\omega - r_1^0, \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\right\}\\ &\qquad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &\quad = \left(1+\lambda_1\right)\left(y_1^\omega - r_1^0\right) - \lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\end{aligned} \tag{A.12}$$

Therefore, it is optimal to accept z if $z_1 \geq y_1^\omega$, or it is optimal to reject it. In other words, it is optimal to play strategy $\widehat{f}$.

If $t \in T_{even}$ for Case 2, we assume that player 2 makes some proposal $z \in Z$ with $z_1 < x_1^{\omega'}$, where $\omega' \in \{1, \mathrm{I},\ 1, \mathrm{II},\ 1, \mathrm{III}\}$. If player 1 accepts the proposal, player 1's payoff can obtain

$$\begin{aligned}u_1\left(h^t, a\right) &= \left(1+\lambda_1\right)\left(z_1 - r_1^0\right)\\ &\quad - \lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\end{aligned} \tag{A.13}$$

If player 1 rejects and the bargaining game continues, the reference point $r_1(h^t)$ is $\max\{r_1 - r_1^0, z_1 - r_1^0\}$. At time $t+1$, we play a new game where $(\max\{r_1 - r_1^0, z_1 - r_1^0\}, r_2 - r_2^0)$ is the reference point pair at $t+1$. Since $(\max\{r_1 - r_1^0, z_1 - r_1^0\}, r_2 - r_2^0) \in \omega$, where $\omega \in \{2, \mathrm{I},\ 2, \mathrm{II},\ 2, \mathrm{III}\}$, we have $x_1^\omega - r_1^0 \geq \max\{r_1 - r_1^0, z_1 - r_1^0\} \geq y_1^\omega - r_1^0$, where $(y_1^\omega - r_1^0) = \delta((x_1^\omega - r_1^0) + \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\})/(1+\lambda_1)$.

At time $t+1$, player 1 makes a proposal x_1^ω and player 2 accepts it; z is rejected, which yields

$$\begin{aligned}&\delta u_1\left(h^{t+1}, a\right) + (1-\delta) u_1\left(h^t, d\right)\\ &\quad = \delta\left(1+\lambda_1\right)\left(x_1^\omega - r_1^0\right)\\ &\qquad - \delta\lambda_1 \max\left\{x_1^\omega - r_1^0, \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\right\}\\ &\qquad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &\quad = \left(1+\lambda_1\right)\left(y_1^\omega - r_1^0\right) - \lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\end{aligned} \tag{A.14}$$

If player 2 proposes $z \in Z$ with $z_1 > x_1^{\omega'}$, then player 1 accepts it, which yields $\mu_1(h^t, a) = z_1 - r_1^0$ and player 1 rejects the proposal, which yields

$$\begin{aligned}&\delta u_1\left(h^{t+1}, a\right) + (1-\delta) u_1\left(h^t, d\right)\\ &\quad = \delta\left(1+\lambda_1\right)\left(x_1^{\omega'} - r_1^0\right)\\ &\qquad - \delta\lambda_1 \max\left\{x_1^{\omega'} - r_1^0, \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\right\}\\ &\qquad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &\quad = \left(1+\lambda_1\right)\left(y_1^{\omega'} - r_1^0\right) - \lambda_1\left(z_1 - r_1^0\right)\end{aligned} \tag{A.15}$$

Since $z_1 > x_1^{\omega'} > y_1^{\omega'}$, we have $u_1(h^t, a) > \delta u_1(h^{t+1}, a) + (1-\delta)u_1(h^t, d)$.

In general, accepting z is optimal if $z_1 \geq y_1^\omega$, or it is optimal to reject it. In other words, it is optimal to follow $\widehat{f}$.

If $t \in T_{even}$ for Case 3, we assume that player 2 makes some proposal $z \in Z$ with $z_1 < y_1^\omega$. Note that $r_1 < x_1^\omega$. Hence, if the proposal is rejected, the payoff of player 1 is given by

$$\begin{aligned}&\delta u_1\left(h^{t+1}, a\right) + (1-\delta) u_1\left(h^t, d\right)\\ &\quad = \delta\left(1+\lambda_1\right)\left(x_1^\omega - r_1^0\right)\\ &\qquad - \delta\lambda_1 \max\left\{x_1^\omega - r_1^0, \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\right\}\\ &\qquad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &\quad = \delta\left(x_1^\omega - r_1^0\right)\\ &\qquad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\end{aligned} \tag{A.16}$$

If the proposal is accepted, player 1's payoff is $u_1(h^t, a) = (1+\lambda_1)(z_1 - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\}$. Since $\mu_1(y_1^\omega - r_1^0) = \delta(x_1^\omega - r_1^0)$, we have

$$\begin{aligned}u_1\left(h^t, a\right) &= \left(1+\lambda_1\right)\left(z_1 - r_1^0\right)\\ &\quad - \lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &= \left(1+\lambda_1\right)\left(z_1 - r_1^0\right)\\ &\quad - \delta\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &\quad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &< \left(1+\lambda_1\right)\left(y_1^\omega - r_1^0\right)\\ &\quad - \delta\lambda_1 \max\left\{r_1 - r_1^0, y_1^\omega - r_1^0\right\}\\ &\quad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &= \left(1+\lambda_1\right)\left(y_1^\omega - r_1^0\right) - \delta\lambda_1\left(y_1^\omega - r_1^0\right)\\ &\quad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &= u_1\left(y_1^\omega - r_1^0\right)\\ &\quad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\\ &= \delta\left(x_1^\omega - r_1^0\right)\\ &\quad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\end{aligned} \tag{A.17}$$

Hence, it is optimal to reject z. On the other hand, if player 2 makes a proposal $z \in Z$ with $z_1 \geq y_1^\omega$, then accepting yields $u_1(h^t, a) = (1+\lambda_1)(z_1 - r_1^0) - \lambda_1 \max\{r_1 - r_1^0, z_1 - r_1^0\} = z_1$.

Let $z_1 \leq y_1^{\omega'}$, where $\omega' \in \{1, \mathrm{I},\ 1, \mathrm{II},\ 1, \mathrm{III}\}$. If player 1 rejects it and the bargaining game continues, then $r_1(h^t) = z_1$, and $(z_1, r_1) \in \omega''$, where $\omega'' \in \{2,\text{I}, 2,\text{II}, 2,\text{III}\}$. Note that then $y_1^{\omega''} \le z_1 \le x_1^{\omega''}$. Thus, we have

$$\begin{aligned}\delta u_1\left(h^{t+1}, a\right) + (1-\delta) u_1\left(h^t, d\right) &= \delta\left(1+\lambda_1\right)\left(x_1^{\omega''} - r_1^0\right) \\ &\quad - \delta\lambda_1 \max\left\{x_1^{\omega''} - r_1^0, z_1 - r_1^0\right\} \\ &\quad - (1-\delta)\lambda_1\left(z_1 - r_1^0\right) \\ &= \left(1+\lambda_1\right)\left(y_1^{\omega''} - r_1^0\right) - \lambda_1\left(z_1 - r_1^0\right).\end{aligned} \tag{A.18}$$

Since $y_1^{\omega''} \le z_1$, also $z_1 \ge (1+\lambda_1) y_1^{\omega''} - \lambda_1 z_1$. If player 2 makes a proposal $z \in Z$ with $x_1^{\omega'} < z_1$ that is rejected, then we have

$$\begin{aligned}\delta u_1\left(h^{t+1}, a\right) + (1-\delta) u_1\left(h^t, d\right) &= \delta\left(1+\lambda_1\right)\left(x_1^{\omega'} - r_1^0\right) \\ &\quad - \delta\lambda_1 \max\left\{x_1^{\omega'} - r_1^0, \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\}\right\} \\ &\quad - (1-\delta)\lambda_1 \max\left\{r_1 - r_1^0, z_1 - r_1^0\right\} \\ &= \delta\left(1+\lambda_1\right)\left(x_1^{\omega'} - r_1^0\right) - \lambda_1\left(z_1 - r_1^0\right)\end{aligned} \tag{A.19}$$

Since $x_1^{\omega'} < z_1$, we have $\delta x_1^{\omega'} < z_1$. Thus, we have that $z_1 > (1+\lambda_1)\delta x_1^{\omega'} - \lambda_1 z_1$; i.e., it is optimal to accept z.

Thus, when player 2 plays strategy $\widehat{g}$, player 1 cannot improve his payoff by unilaterally deviating from $\widehat{f}$ at any single time t. Similarly, if player 1 plays $\widehat{f}$, we can prove that player 2 cannot improve his share by deviating unilaterally from the strategy $\widehat{g}$ at any single time t. Lemma A.1 implies that $(\widehat{f}, \widehat{g})$ is an SPE. □

B. Proof of Theorem 3

We assume that conditions (I)–(III) in Section 3.2 are satisfied, which is used throughout this section.

B.1. Preliminary Lemmas. A proposal is made by player 1 and a counterproposal is made by player 2, which is defined as a bargaining round. Bargaining rounds are indexed with $i \in \{0, 1, 2, \ldots\ldots\}$. We define (r_1^i, r_2^i) as the reference point pair at the beginning of round i. By (I), we have the following definition:

$$\begin{aligned} r_1^{i+1} &:= \max\left\{r_1^i, y_1\left(r_1^i, r_2^{i+1}\right)\right\}, \\ r_2^{i+1} &:= \max\left\{r_2^i, x_2\left(r_1^i, r_2^i\right)\right\}\end{aligned} \tag{B.1}$$

Lemma B.1. *When $x(r_1^i, r_2^i)$ is player 1's SPE proposal and $y(r_1^i, r_2^{i+1})$ player 2's counterproposal, we have*

$$x_2\left(r_1^i, r_2^i\right) = \begin{cases} \delta - \delta y_1\left(r_1^i, r_2^{i+1}\right) + (1-\delta) r_2^0 & \text{if } r_2^i > y_2\left(r_1^i, r_2^{i+1}\right) > x_2\left(r_1^i, r_2^i\right) \\ \dfrac{\left(\delta\left(1+\lambda_2 r_2^{i+1}\right) - \delta y_1\left(r_1^i, r_2^{i+1}\right)\right)}{\left(1+\lambda_2\right)} + (1-\delta) r_2^0 & \text{if } y_2\left(r_1^i, r_2^{i+1}\right) \ge r_2^i > x_2\left(r_1^i, r_2^i\right) \\ \dfrac{\delta}{\mu_2} - \dfrac{\delta y_1\left(r_1^i, r_2^{i+1}\right)}{\mu_2} + \dfrac{\left(\mu_2 - \delta\right)\left(1+\lambda_2\right) r_2^0}{\mu_2} & \text{if } y_2\left(r_1^i, r_2^{i+1}\right) > x_2\left(r_1^i, r_2^i\right) \ge r_2^i \end{cases} \tag{$*$}$$

and

$$y_1\left(r_1^i, r_2^{i+1}\right) = \begin{cases} \delta - \delta x_2\left(r_1^{i+1}, r_2^{i+1}\right) + (1-\delta) r_1^0 & \text{if } r_1^i > x_1\left(r_1^{i+1}, r_2^{i+1}\right) > y_1\left(r_1^i, r_2^{i+1}\right) \\ \dfrac{\left(\delta\left(1+\lambda_1 r_1^{i+1}\right) - \delta x_2\left(r_1^{i+1}, r_2^{i+1}\right)\right)}{\left(1+\lambda_1\right)} + (1-\delta) r_1^0 & \text{if } x_1\left(r_1^{i+1}, r_2^{i+1}\right) \ge r_1^i > y_1\left(r_1^i, r_2^{i+1}\right) \\ \dfrac{\delta}{\mu_1} - \dfrac{\delta x_2\left(r_1^{i+1}, r_2^{i+1}\right)}{\mu_1} + \dfrac{\left(\mu_1 - \delta\right) r_1^0}{\mu_1} & \text{if } x_1\left(r_1^{i+1}, r_2^{i+1}\right) > y_1\left(r_1^i, r_2^{i+1}\right) \ge r_1^i \end{cases} \tag{$**$}$$

Proof. According to the two conditions (II) and (III), and the definition of the bargainers' utility functions, we can prove $(*)$ and $(**)$. □

For each $\omega \in \Omega$, the following sets P_ω and Q_ω are introduced:

$$P_{1,\text{I}} := \left\{\left(r_1^i, r_2^i\right) \mid r_2^i > y_2\left(r_1^i, r_2^{i+1}\right), r_1^i\right.$$

$$
\begin{aligned}
&> x_1\left(r_1^{i+1}, r_2^{i+1}\right)\big\} \\
P_{2,\mathrm{I}} &:= \left\{\left(r_1^i, r_2^i\right) \mid r_2^i > y_2\left(r_1^i, r_2^{i+1}\right), x_1\left(r_1^{i+1}, r_2^{i+1}\right) \right. \\
&\geq r_1^i > y_1\left(r_1^i, r_2^{i+1}\right)\big\} \\
P_{3,\mathrm{I}} &:= \left\{\left(r_1^i, r_2^i\right) \mid r_2^i > y_2\left(r_1^i, r_2^{i+1}\right), y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i\right\} \\
P_{1,\mathrm{II}} &:= \left\{\left(r_1^i, r_2^i\right) \mid y_2\left(r_1^i, r_2^{i+1}\right) \geq r_2^i > x_2\left(r_1^i, r_2^i\right), r_1^i \right. \\
&> x_1\left(r_1^{i+1}, r_2^{i+1}\right)\big\} \\
P_{2,\mathrm{II}} &:= \left\{\left(r_1^i, r_2^i\right) \mid y_2\left(r_1^i, r_2^{i+1}\right) \geq r_2^i \right. \\
&> x_2\left(r_1^i, r_2^i\right), x_1\left(r_1^{i+1}, r_2^{i+1}\right) \geq r_1^i > y_1\left(r_1^i, r_2^{i+1}\right)\big\} \\
P_{3,\mathrm{II}} &:= \left\{\left(r_1^i, r_2^i\right) \mid y_2\left(r_1^i, r_2^{i+1}\right) \geq r_2^i \right. \\
&> x_2\left(r_1^i, r_2^i\right), y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i\big\} \\
P_{1,\mathrm{III}} &:= \left\{\left(r_1^i, r_2^i\right) \mid x_2\left(r_1^i, r_2^i\right) \geq r_2^i, r_1^i \right. \\
&> x_1\left(r_1^{i+1}, r_2^{i+1}\right)\big\} \\
P_{2,\mathrm{III}} &:= \left\{\left(r_1^i, r_2^i\right) \mid x_2\left(r_1^i, r_2^i\right) \geq r_2^i, x_1\left(r_1^{i+1}, r_2^{i+1}\right) \geq r_1^i \right. \\
&> y_1\left(r_1^i, r_2^{i+1}\right)\big\} \\
P_{3,\mathrm{III}} &:= \left\{\left(r_1^i, r_2^i\right) \mid x_2\left(r_1^i, r_2^i\right) \geq r_2^i, y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i\right\} \\
Q_{1,\mathrm{I}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid r_1^i > x_1\left(r_1^{i+1}, r_2^{i+1}\right), r_2^{i+1} \right. \\
&> y_2\left(r_1^{i+1}, r_2^{i+2}\right)\big\} \\
Q_{1,\mathrm{II}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid r_1^i \right. \\
&> x_1\left(r_1^{i+1}, r_2^{i+1}\right), y_2\left(r_1^{i+1}, r_2^{i+2}\right) \geq r_2^{i+1} \\
&> x_2\left(r_1^{i+1}, r_2^{i+1}\right)\big\} \\
Q_{1,\mathrm{III}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid r_1^i \right. \\
&> x_1\left(r_1^{i+1}, r_2^{i+1}\right), x_2\left(r_1^{i+1}, r_2^{i+1}\right) \geq r_2^{i+1}\big\} \\
Q_{2,\mathrm{I}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid x_1\left(r_1^{i+1}, r_2^{i+1}\right) \geq r_1^i \right. \\
&> y_1\left(r_1^i, r_2^{i+1}\right), r_2^{i+1} > y_2\left(r_1^{i+1}, r_2^{i+2}\right)\big\} \\
Q_{2,\mathrm{II}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid x_1\left(r_1^{i+1}, r_2^{i+1}\right) \geq r_1^i \right. \\
&> y_1\left(r_1^i, r_2^{i+1}\right), y_2\left(r_1^{i+1}, r_2^{i+2}\right) \geq r_2^{i+1} \\
&> x_2\left(r_1^{i+1}, r_2^{i+1}\right)\big\} \\
Q_{2,\mathrm{III}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid x_1\left(r_1^{i+1}, r_2^{i+1}\right) \geq r_1^i \right. \\
&> y_1\left(r_1^i, r_2^{i+1}\right), x_2\left(r_1^{i+1}, r_2^{i+1}\right) \geq r_2^{i+1}\big\} \\
Q_{3,\mathrm{I}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i, r_2^{i+1} \right. \\
&> y_2\left(r_1^{i+1}, r_2^{i+2}\right)\big\} \\
Q_{3,\mathrm{II}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i, y_2\left(r_1^{i+1}, r_2^{i+2}\right) \right. \\
&\geq r_2^{i+1} > x_2\left(r_1^{i+1}, r_2^{i+1}\right)\big\} \\
Q_{3,\mathrm{III}} &:= \left\{\left(r_1^i, r_2^{i+1}\right) \mid y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i, x_2\left(r_1^{i+1}, r_2^{i+1}\right) \right. \\
&\geq r_2^{i+1}\big\}
\end{aligned}
\tag{B.2}
$$

In these sets, we can derive the following lemmas for reference point pairs.

Lemma B.2. *For all* $(r_1^i, r_2^i) \in P_\omega$*, we have* $x_2(r_1^i, r_2^i) \geq x_2(r_1^{i+1}, r_2^{i+1}) \Longleftrightarrow x_2(r_1^i, r_2^i) \leq x_2^\omega$*. Similarly, for all* $(r_1^i, r_2^{i+1}) \in Q_\omega$*, we have* $y_1(r_1^i, r_2^{i+1}) \geq y_1(r_1^{i+1}, r_2^{i+2}) \Longleftrightarrow y_1(r_1^i, r_2^{i+1}) \leq y_1^\omega$*.*

Proof. Let $\omega = 1, \mathrm{I}$ and $(r_1^i, r_2^i) \in P_\omega$. It follows from the definition of $P_{1,\mathrm{I}}$ that $r_2^i > y_2(r_1^i, r_2^{i+1})$ and $r_1^i > x_1(r_1^{i+1}, r_2^{i+1})$. It follows from Lemma B.1 that

$$
\begin{aligned}
x_2\left(r_1^i, r_2^i\right) &= \delta - \delta y_1\left(r_1^i, r_2^{i+1}\right) + (1-\delta) r_2^0 \\
&= \delta - \delta\left(\delta - \delta x_2\left(r_1^{i+1}, r_2^{i+1}\right) + (1-\delta) r_1^0\right) \\
&\quad + (1-\delta) r_2^0 \\
&= \delta - \delta^2 + \delta^2 x_2\left(r_1^{i+1}, r_2^{i+1}\right) - \delta(1-\delta) r_1^0 \\
&\quad + (1-\delta) r_2^0
\end{aligned}
\tag{B.3}
$$

If $x_2(r_1^{i+1}, r_2^{i+1}) \geq x_2(r_1^i, r_2^i)$, then according to (B.3), we have

$$
\begin{aligned}
x_2\left(r_1^i, r_2^i\right) &= \delta - \delta^2 + \delta^2 x_2\left(r_1^{i+1}, r_2^{i+1}\right) - \delta(1-\delta) r_1^0 \\
&\quad + (1-\delta) r_2^0 \\
&\geq \delta - \delta^2 + \delta^2 x_2\left(r_1^i, r_2^i\right) - \delta(1-\delta) r_1^0 \\
&\quad + (1-\delta) r_2^0,
\end{aligned}
\tag{B.4}
$$

which means that $x_2(r_1^i, r_2^i) \geq \delta/(1+\delta) - \delta r_1^0/(1+\delta) + r_2^0/(1+\delta) = x_2^{1,\mathrm{I}}$. If $x_2(r_1^{i+1}, r_2^{i+1}) < x_2(r_1^i, r_2^i)$, then by (B.3)

$$
\begin{aligned}
x_2\left(r_1^i, r_2^i\right) &= \delta - \delta^2 + \delta^2 x_2\left(r_1^{i+1}, r_2^{i+1}\right) - \delta(1-\delta) r_1^0 \\
&\quad + (1-\delta) r_2^0 \\
&< \delta - \delta^2 + \delta^2 x_2\left(r_1^i, r_2^i\right) - \delta(1-\delta) r_1^0 \\
&\quad + (1-\delta) r_2^0,
\end{aligned}
\tag{B.5}
$$

which means that $x_2(r_1^i, r_2^i) < \delta/(1+\delta) - \delta r_1^0/(1+\delta) + r_2^0/(1+\delta) = x_2^{1,\mathrm{I}}$. Thus, $x_2(r_1^{i+1}, r_2^{i+1}) \leq x_2(r_1^i, r_2^i) \Longleftrightarrow x_2(r_1^i, r_2^i) \leq x_2^{1,\mathrm{I}}$.

Similarly, let $\omega = 1,\mathrm{I}$ and $(r_1^i, r_2^{i+1}) \in Q_\omega$. By definition of $Q_{1,\mathrm{I}}$ we have $r_1^i > x_1(r_1^{i+1}, r_2^{i+1})$ and $r_2^{i+1} > y_2(r_1^{i+1}, r_2^{i+2})$. It follows from Lemma B.1 that

$$\begin{aligned} y_1\left(r_1^i, r_2^{i+1}\right) &= \delta - \delta x_2\left(r_1^{i+1}, r_2^{i+1}\right) + (1-\delta) r_1^0 \\ &= \delta \\ &\quad - \delta\left(\delta - \delta y_1\left(r_1^{i+1}, r_2^{i+2}\right) + (1-\delta) r_2^0\right) \\ &\quad + (1-\delta) r_1^0 \\ &= \delta - \delta^2 + \delta^2 y_1\left(r_1^{i+1}, r_2^{i+2}\right) \\ &\quad - \delta(1-\delta) r_2^0 + (1-\delta) r_1^0, \end{aligned} \tag{B.6}$$

which implies $y_1(r_1^i, r_2^{i+1}) \ge y_1(r_1^{i+1}, r_2^{i+2}) \Longleftrightarrow y_1(r_1^i, r_2^{i+1}) \le y_1^{1,\mathrm{I}}$

Similarly, we can prove the cases for $\omega \in \Omega$, $\omega \ne 1,\mathrm{I}$. □

Lemma B.3. *If $(r_1^k, r_2^k) \in P_\omega$ for all $k \ge i$, then $x_2(r_1^i, r_2^i) = x_2^\omega$. Similarly, if $(r_1^k, r_2^{k+1}) \in Q_\omega$ for all $k \ge i$, then $y_1(r_1^i, r_2^{i+1}) = y_1^\omega$.*

Lemma B.3 shows that if the reference point pair is in P_ω at the current time and all future time $t \in T_{\mathrm{odd}}$, then player 1 must make the proposal x^ω. The result for Q_ω is similar.

Proof. If $(r_1^k, r_2^k) \in P_\omega$ for all $k \ge i$, then it follows from Lemma B.1 that either $x_2(r_1^i, r_2^i)$ is independent of the reference point pair (r_1^i, r_2^i) or no proposal is ever made, which adjusts reference point pair. Thus, we can obtain $x_2(r_1^i, r_2^i)$ as the sum of a geometric series. Let $(r_1^k, r_2^k) \in P_{1,\mathrm{I}}$ for all $k \ge i$. It follows from Lemma B.1 that

$$\begin{aligned} x_2\left(r_1^i, r_2^i\right) &= \delta - \delta^2 + \delta^2 x_2\left(r_1^{i+1}, r_2^{i+1}\right) - \delta(1-\delta) r_1^0 \\ &\quad + (1-\delta) r_2^0 = \delta - \delta^2 + \delta^2\left(\delta - \delta^2\right. \\ &\quad \left. + \delta^2 x_2\left(r_1^{i+2}, r_2^{i+2}\right) - \delta(1-\delta) r_1^0 + (1-\delta) r_2^0\right) \\ &\quad - \delta(1-\delta) r_1^0 + (1-\delta) r_2^0 = \delta(1-\delta)\left(1 + \delta^2 + \delta^4\right. \\ &\quad \left. + \ldots\ldots\right) - \delta(1-\delta)\left(1 + \delta^2 + \delta^4 + \ldots\ldots\right) r_1^0 + (1 \\ &\quad - \delta)\left(1 + \delta^2 + \delta^4 + \ldots\ldots\right) r_2^0 = \delta(1-\delta)\frac{1}{1-\delta^2} \\ &\quad - \delta(1-\delta)\frac{1}{1-\delta^2} r_1^0 + (1-\delta)\frac{1}{1-\delta^2} r_2^0 = \frac{\delta}{1+\delta} \\ &\quad - \frac{\delta r_1^0}{1+\delta} + \frac{r_2^0}{1+\delta} = x_2^{1,\mathrm{I}} \end{aligned} \tag{B.7}$$

The proof for $\omega \in \Omega$, $\omega \ne 1,\mathrm{I}$ is analogous. □

Lemma B.4. *If $(r_1^i, r_2^i) \in P_{1,\mathrm{III}}$, then $(r_1^i, r_2^{i+1}) \notin Q_{1,\mathrm{I}}$. If $(r_1^i, r_2^i) \in P_{2,\mathrm{III}}$, then $(r_1^i, r_2^{i+1}) \notin Q_{2,\mathrm{I}}$.*

If $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$, then $(r_1^{i+1}, r_2^{i+1}) \notin P_{1,\mathrm{I}}$. If $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$, then $(r_1^{i+1}, r_2^{i+1}) \notin P_{1,\mathrm{II}}$.

Lemma B.4 shows some restrictions about how reference points change through the sets P and Q.

Proof. Let $(r_1^i, r_2^i) \in P_{1,\mathrm{III}}$ and assume $(r_1^i, r_2^{i+1}) \notin Q_{1,\mathrm{I}}$. Then it follows from the definitions of $P_{1,\mathrm{III}}$ and $Q_{1,\mathrm{I}}$ that $y_2(r_1^i, r_2^{i+1}) > x_2(r_1^i, r_2^i) \ge r_2^i, r_1^i > x_1(r_1^{i+1}, r_2^{i+1}) > y_1(r_1^i, r_2^{i+1})$ and $r_2^{i+1} > y_2(r_1^{i+1}, r_2^{i+2}) > x_2(r_1^{i+1}, r_2^{i+1})$. Since $r_1^i > y_1(r_1^i, r_2^{i+1})$ and $r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$, we have $r_1^{i+1} = r_1^i$ and $r_2^{i+2} = r_2^{i+1}$ by (B.1). It follows from (I) that $y_2(r_1^i, r_2^{i+1}) = y_2(r_1^{i+1}, r_2^{i+2})$. Furthermore, since $x_2(r_1^i, r_2^i) \ge r_2^i$, (B.1) implies $x_2(r_1^i, r_2^i) = r_2^{i+1}$. Thus $r_2^{i+1} > y_2(r_1^{i+1}, r_2^{i+2}) = y_2(r_1^i, r_2^{i+1}) > x_2(r_1^i, r_2^i) = r_2^{i+1}$, which is a contradiction.

Similarly, we can show the proofs of the other statements. □

B.2. Proof of Theorem 3. Now, some lemmas are proved which are used to prove Theorem 3.

Lemma B.5. *Let $\omega \in \{1,\mathrm{I},\ 1,\mathrm{II},\ 2,\mathrm{I},\ 2,\mathrm{II}\}$. Then for all $(r_1^i, r_2^i) \in P_\omega$, we have $x(r_1^i, r_2^i) = x^\omega$. Similarly, for all $(r_1^i, r_2^{i+1}) \in Q_\omega$, we have $y(r_1^i, r_2^{i+1}) = y^\omega$.*

Proof. Let $(r_1^i, r_2^i) \in P_\omega$ where $\omega \in \{1,\mathrm{I},\ 1,\mathrm{II},\ 2,\mathrm{I},\ 2,\mathrm{II}\}$. Then $r_1^i > y_1(r_1^i, r_2^{i+1})$ and $r_2^i > x_2(r_1^i, r_2^i)$. By (B.1) this implies $r_1^{i+1} = r_1^i$ and $r_2^{i+1} = r_2^i$. By (I), $x_2(r_1^i, r_2^i) = x_2(r_1^{i+1}, r_2^{i+1})$. By Lemma B.2 $x_2(r_1^i, r_2^i) = x_2^\omega$. The case for Q_ω is similar. □

Lemma B.6. *If $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$, we have the following two cases: (i) $y_1(r_1^i, r_2^{i+1}) \ge y_1^{3,\mathrm{I}}$; (ii) $y_1(r_1^i, r_2^{i+1}) \le y_1^{3,\mathrm{I}}$. Thus, we have $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{I}}$.*

Proof of Lemma B.6, Part i. Let $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$; it follows from the definition of $Q_{3,\mathrm{I}}$ that

$$\begin{aligned} x_1\left(r_1^{i+1}, r_2^{i+1}\right) > y_1\left(r_1^i, r_2^{i+1}\right) \ge r_1^i, \\ r_2^{i+1} > y_2\left(r_1^{i+1}, r_2^{i+2}\right) > x_2\left(r_1^{i+1}, r_2^{i+1}\right). \end{aligned} \tag{B.8}$$

We assume that $y_1(r_1^i, r_2^{i+1}) < y_1^{3,\mathrm{I}}$; it follows from Lemma B.2 that $y_1(r_1^{i+1}, r_2^{i+2}) > y_1(r_1^i, r_2^{i+1})$. Since $y_1(r_1^i, r_2^{i+1}) \ge r_1^i$, by (B.1), we have $r_1^{i+1} = y_1(r_1^i, r_2^{i+1})$. Hence, $r_1^{i+1} > y_1(r_1^{i+1}, r_2^{i+2})$. We have two possibilities:

(a) $r_1^{i+1} > x_1(r_1^{i+2}, r_2^{i+2}) > y_1(r_1^{i+1}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+1}) \in P_{1.\mathrm{I}}$.

(b) $x_1(r_1^{i+2}, r_2^{i+2}) \ge r_1^{i+1} > y_1(r_1^{i+1}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+1}) \in P_{2.\mathrm{I}}$.

It follows from Lemma B.4 that case (a) is ruled out. Hence, $(r_1^{i+1}, r_2^{i+1}) \in P_{2.\mathrm{I}}$.

By Lemma B.5, $x_2(r_1^{i+1}, r_2^{i+1}) = x_2^{2,\mathrm{I}}$. Since $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$, Lemma B.1 implies

$$y_1\left(r_1^i, r_2^{i+1}\right) = \frac{\delta}{\mu_1} - \frac{\delta x_2\left(r_1^{i+1}, r_2^{i+1}\right)}{\mu_1} + \frac{\left(\mu_1 - \delta\right) r_1^0}{\mu_1}$$
$$= \frac{\delta}{\mu_1} - \frac{\delta x_2^{2,\mathrm{I}}}{\mu_1} + \frac{\left(\mu_1 - \delta\right) r_1^0}{\mu_1} \quad \text{(B.9)}$$
$$= \frac{\delta\left(x_1^{2,\mathrm{I}} - r_1^0\right)}{\mu_1} + r_1^0.$$

B (B.1), we have $r_1^{i+1} = y_1(r_1^i, r_2^{i+1})$. Note that $x_1^{2,\mathrm{I}}$ is a function of r_1^{i+1} and therefore of $y_1(r_1^i, r_2^{i+1})$. That is,

$$y_1\left(r_1^i, r_2^{i+1}\right) = \frac{\delta}{\mu_1} \times \left(\frac{\left(\mu_1 - \delta\right) + \delta^2\lambda_1 r_1^{i+1}}{1 + \lambda_1 - \delta^2} + \frac{\delta\left(1 - \delta\right)\left(1 + \lambda_1\right) r_1^0}{1 + \lambda_1 - \delta^2} - \frac{\left(1 - \delta\right)\left(1 + \lambda_1\right) r_2^0}{1 + \lambda_1 - \delta^2} - r_1^0\right) + r_1^0 = \frac{\delta}{\mu_1} \times \frac{\left(\mu_1 - \delta\right) + \delta^2\lambda_1 y_1\left(r_1^i, r_2^{i+1}\right)}{1 + \lambda_2 - \delta^2} + \frac{\delta}{\mu_1} \times \frac{\delta\left(1 - \delta\right)\left(1 + \lambda_1\right) r_1^0}{1 + \lambda_1 - \delta^2} + \frac{\left(\mu_1 - \delta\right) r_1^0}{\mu_1} - \frac{\delta}{\mu_1} \times \frac{\left(1 - \delta\right)\left(1 + \lambda_1\right) r_2^0}{1 + \lambda_1 - \delta^2} \quad \text{(B.10)}$$

Thus, we have $y_1(r_1^i, r_2^{i+1}) = \delta/(1 + \lambda_1 + \delta) + (1 + \lambda_1)r_1^0/(1 + \lambda_1 + \delta) - \delta r_2^0/(1 + \lambda_1 + \delta) = y_1^{3,\mathrm{I}}$. Thus, $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{I}}$ is contradicting $y_1(r_1^i, r_2^{i+1}) < y_1^{3,\mathrm{I}}$. □

Before we prove Part ii, we have to show a similar argument for $Q_{3,\mathrm{II}}$.

Lemma B.7. *If $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$, then we have the following two cases: (i) $y_1(r_1^i, r_2^{i+1}) \geq y_1^{3,\mathrm{II}}$; (ii) $y_1(r_1^i, r_2^{i+1}) \leq y_1^{3,\mathrm{II}}$. Thus, we have $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{II}}$.*

Proof of Lemma B.7, Part i. Let $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$; it follows from the definition of $Q_{3,\mathrm{II}}$ that we have $x_1(r_1^{i+1}, r_2^{i+1}) > y_1(r_1^i, r_2^{i+1}) \geq r_1^i$ and $y_2(r_1^{i+1}, r_2^{i+1}) \geq r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$.

We assume that $y_1(r_1^i, r_2^{i+1}) < y_1^{3,\mathrm{II}}$. By Lemma B.2, $y_1(r_1^i, r_2^{i+1}) > y_1(r_1^{i+1}, r_2^{i+2})$. Since $y_1(r_1^i, r_2^{i+1}) \geq r_1^i$, we have from (B.1) that $r_1^{i+1} = y_1(r_1^i, r_2^{i+1})$. Hence, $r_1^{i+1} > y_1(r_1^{i+1}, r_2^{i+2})$. We have two possibilities:

(a) $r_1^{i+1} \geq x_1(r_1^{i+2}, r_2^{i+2}) > y_1(r_1^{i+1}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+1}) \in P_{1,\mathrm{II}}$.

(b) $x_1(r_1^{i+2}, r_2^{i+2}) \geq r_1^{i+1} > y_1(r_1^{i+1}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+1}) \in P_{2,\mathrm{II}}$.

It follows from Lemma B.4 that case (a) is ruled out. Hence, $(r_1^{i+1}, r_2^{i+1}) \in P_{2,\mathrm{II}}$. By Lemma B.5, $x_2(r_1^{i+1}, r_2^{i+1}) = x_2^{2,\mathrm{II}}$. Since $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$, by Lemma B.1, we have

$$y_1\left(r_1^i, r_2^{i+1}\right) = \frac{\delta}{\mu_1} - \frac{\delta x_2\left(r_1^{i+1}, r_2^{i+1}\right)}{\mu_1} + \frac{\left(\mu_1 - \delta\right) r_1^0}{\mu_1}$$
$$= \frac{\delta}{\mu_1} - \frac{\delta x_2^{2,\mathrm{II}}}{\mu_1} + \frac{\left(\mu_1 - \delta\right) r_1^0}{\mu_1} \quad \text{(B.11)}$$
$$= \frac{\delta}{\mu_1}\left(x_1^{2,\mathrm{II}} - r_1^0\right) + r_1^0$$

We have $r_1^{i+1} = y_1(r_1^i, r_2^{i+1})$ by (B.1). Note that $x_1^{2,\mathrm{II}}$ is a function of r_1^{i+1}. That is,

$$y_1\left(r_1^i, r_2^{i+1}\right) = \frac{\delta}{\mu_1} \times \left(\frac{\left(\mu_2 - \delta\right)\left(1 + \lambda_1\right) + \delta\lambda_2\left(1 - r_2^{i+1}\right) + \delta^2\lambda_1 r_1^{i+1}}{\left(1 + \lambda_1\right)\left(1 + \lambda_2\right) - \delta^2} + \frac{\delta\left(1 - \delta\right)\left(1 + \lambda_1\right) r_1^0}{\left(1 + \lambda_1\right)\left(1 + \lambda_2\right) - \delta^2} - \frac{\left(1 + \lambda_1\right)\left(\mu_2 - \delta\right) r_2^0}{\left(1 + \lambda_1\right)\left(1 + \lambda_2\right) - \delta^2} - r_1^0\right) + r_1^0 = \frac{\delta}{\mu_1} \times \left(\frac{\left(\mu_2 - \delta\right)\left(1 + \lambda_1\right) + \delta\lambda_2\left(1 - r_2^{i+1}\right) + \delta^2\lambda_1 y_1\left(r_1^i, r_2^{i+1}\right)}{\left(1 + \lambda_1\right)\left(1 + \lambda_2\right) - \delta^2} - \frac{\left(1 + \lambda_1\right)\left(\mu_2 - \delta\right) r_2^0}{\left(1 + \lambda_1\right)\left(1 + \lambda_2\right) - \delta^2}\right) + \frac{\left(1 + \lambda_1\right)\left(1 + \lambda_2\right)\left(\mu_1 - \delta\right) r_1^0}{\mu_1\left(\left(1 + \lambda_1\right)\left(1 + \lambda_2\right) - \delta^2\right)} \quad \text{(B.12)}$$

Thus, we have $y_1(r_1^i, r_2^{i+1}) = \delta(\mu_2 - \delta + \lambda_2\delta(1 - r_2^{i+1}))/(\mu_1(1 + \lambda_2) - \delta^2) + (1 + \lambda_2)(\mu_1 - \delta)r_1^0/(\mu_1(1 + \lambda_2) - \delta^2) - \delta(\mu_2 - \delta)r_2^0/(\mu_1(1 + \lambda_2) - \delta^2) = y_1^{3,\mathrm{II}}$.

This contradicts $y_1(r_1^i, r_2^{i+1}) < y_1^{3,\mathrm{II}}$. □

This result can be used to prove Part ii of Lemma B.6.

Proof of Lemma B.6, Part ii. Let $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$; it follows from the definition of $Q_{3,\mathrm{I}}$ that

$$x_1\left(r_1^{i+1}, r_2^{i+1}\right) > y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i,$$
$$r_2^{i+1} > y_2\left(r_1^{i+1}, r_2^{i+2}\right) > x_2\left(r_1^{i+1}, r_2^{i+1}\right). \quad \text{(B.13)}$$

We assume $y_1(r_1^i, r_2^{i+1}) > y_1^{3,\mathrm{I}}$. By Lemma B.2, $y_1(r_1^{i+1}, r_2^{i+2}) > y_1(r_1^i, r_2^{i+1})$. Since $y_1(r_1^i, r_2^{i+1}) \geq r_1^i$, we have from (B.1) that $r_1^{i+1} = y_1(r_1^i, r_2^{i+1})$. Hence, $y_1(r_1^{i+1}, r_2^{i+2}) > r_1^{i+1}$, which implies $(r_1^{i+1}, r_2^{i+1}) \in Q_{3,\mathrm{I}}$. Then from Lemma B.1, $y_1(r_1^i, r_2^{i+1}) \geq y_1^{3,\mathrm{II}}$ and $y_1(r_1^{i+1}, r_2^{i+2}) > y_1(r_1^i, r_2^{i+1})$, we have

$$x_2\left(r_1^{i+1}, r_2^{i+1}\right) = \delta - \delta y_1\left(r_1^{i+1}, r_2^{i+2}\right) + \left(1 - \delta\right) r_2^0$$
$$< \delta - \delta y_1\left(r_1^i, r_2^{i+1}\right) + \left(1 - \delta\right) r_2^0$$
$$< \delta - \delta y_1^{3,\mathrm{I}} + \left(1 - \delta\right) r_2^0 \quad \text{(B.14)}$$
$$= \delta\left(y_2^{3,\mathrm{I}} - r_2^0\right) + r_2^0 = x_2^{3,\mathrm{I}},$$

which means that $x_2(r_1^{i+1}, r_2^{i+1}) > x_2(r_1^{i+2}, r_2^{i+2})$ by Lemma B.2. Observe that, by (B.1), $r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$ implies $r_2^{i+1} = r_2^{i+2}$. Thus, $r_2^{i+2} = r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1}) > x_2(r_1^{i+2}, r_2^{i+2})$. This leaves two possibilities.

(a) $r_2^{i+2} > y_2(r_1^{i+2}, r_2^{i+3}) > x_2(r_1^{i+2}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{I}}$.

(b) $y_2(r_1^{i+2}, r_2^{i+3}) \geq r_2^{i+2} > x_2(r_1^{i+2}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{II}}$.

Take case (b). Then $y_2(r_1^{i+2}, r_2^{i+3}) \geq r_2^{i+2} = r_2^{i+1} > y_2(r_1^{i+1}, r_2^{i+2})$, which implies $y_1(r_1^{i+2}, r_2^{i+3}) < y_1(r_1^{i+1}, r_2^{i+2})$. By Lemma B.2 we have $y_1(r_1^{i+1}, r_2^{i+2}) < y_1^{3,\mathrm{II}}$, which contradicts Part i of Lemma B.7. Thus, case (a) must hold; i.e., $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{I}}$. Note that $x_2(r_1^{i+2}, r_2^{i+2}) < x_2(r_1^{i+1}, r_2^{i+1}) < x_2^{3,\mathrm{I}}$. Then

$$\begin{aligned} y_1\left(r_1^{i+1}, r_2^{i+2}\right) &= \frac{\delta}{\mu_1} - \frac{\delta}{\mu_1} x_2\left(r_1^{i+2}, r_2^{i+2}\right) \\ &\quad + \frac{(\mu_1 - \delta) r_1^0}{\mu_1} \\ &> \frac{\delta}{\mu_1} - \frac{\delta}{\mu_1} x_2^{3,\mathrm{I}} + \frac{(\mu_1 - \delta) r_1^0}{\mu_1} \\ &= \frac{\delta}{\mu_1} x_1^{3,\mathrm{I}} + \frac{(\mu_1 - \delta) r_1^0}{\mu_1} = y_1^{3,\mathrm{I}}. \end{aligned} \tag{B.15}$$

If $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$ and $y_1(r_1^i, r_2^{i+1}) > y_1^{3,\mathrm{I}}$, then $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{I}}$ and $y_1(r_1^{i+1}, r_2^{i+2}) > y_1^{3,\mathrm{I}}$. Thus, for all $k \geq i$ we have $(r_1^k, r_2^{k+1}) \in Q_{3,\mathrm{I}}$. Then it follows from Lemma B.3 that $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{I}}$, which contradicts $y_1(r_1^i, r_2^{i+1}) > y_1^{3,\mathrm{I}}$. □

Proof of Lemma B.7, Part ii. Let $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$. From the definition of $Q_{3,\mathrm{II}}$ we have $x_1(r_1^{i+1}, r_2^{i+1}) > y_1(r_1^i, r_2^{i+1}) \geq r_1^i$ and $y_2(r_1^{i+1}, r_2^{i+2}) \geq r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$. We assume $y_1(r_1^i, r_2^{i+1}) > y_1^{3,\mathrm{II}}$. By Lemma B.2, $y_1(r_1^i, r_2^{i+1}) < y_1(r_1^{i+1}, r_2^{i+2})$. Since $y_1(r_1^i, r_2^{i+1}) \geq r_1^i$, by (B.1), we have $r_1^{i+1} = y_1(r_1^i, r_2^{i+1})$. Hence, $r_1^{i+1} < y_1(r_1^{i+1}, r_2^{i+2})$, which implies $(r_1^{i+1}, r_2^{i+1}) \in Q_{3,\mathrm{II}}$. From Lemma B.1, $y_1(r_1^i, r_2^{i+1}) < y_1(r_1^{i+1}, r_2^{i+2})$, $y_1(r_1^i, r_2^{i+1}) > y_1^{3,\mathrm{II}}$, and that $r_2^{i+2} = \max\{r_2^{i+2}, x_2(r_1^{i+1}, r_2^{i+1})\} = r_2^{i+1}$, we have

$$\begin{aligned} x_2\left(r_1^{i+1}, r_2^{i+1}\right) &= \frac{\delta\left(1 + \lambda_2 r_2^{i+2}\right) - \delta y_1\left(r_1^{i+1}, r_2^{i+2}\right)}{1 + \lambda_2} \\ &\quad + (1 - \delta) r_2^0 \\ &< \frac{\delta\left(1 + \lambda_2 r_2^{i+2}\right) - \delta y_1\left(r_1^i, r_2^{i+1}\right)}{1 + \lambda_2} \\ &\quad + (1 - \delta) r_2^0 \\ &< \frac{\delta\left(1 + \lambda_2 r_2^{i+2}\right) - \delta y_1^{3,\mathrm{II}}}{1 + \lambda_2} + (1 - \delta) r_2^0 \\ &= \frac{\delta\left(1 + \lambda_2 r_2^{i+1}\right) - \delta y_1^{3,\mathrm{II}}}{1 + \lambda_2} + (1 - \delta) r_2^0 \\ &= x_2^{3,\mathrm{II}}, \end{aligned} \tag{B.16}$$

which means that $x_2(r_1^{i+1}, r_2^{i+1}) > x_2(r_1^{i+2}, r_2^{i+2})$ by Lemma B.2. Since $r_2^{i+2} = r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$, thus, we have $r_2^{i+2} > x_2(r_1^{i+2}, r_2^{i+2})$. Then there are two possibilities:

(a) $r_2^{i+2} > y_2(r_1^{i+2}, r_2^{i+3}) > x_2(r_1^{i+2}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{I}}$.

(b) $y_2(r_1^{i+2}, r_2^{i+3}) \geq r_2^{i+2} > x_2(r_1^{i+2}, r_2^{i+2})$, i.e., $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{II}}$.

Take case (a); i.e., $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{I}}$. Observe that $y_2(r_1^{i+1}, r_2^{i+2}) \geq r_2^{i+1} = r_2^{i+2} > y_2(r_1^{i+2}, r_2^{i+3})$, and thus $y_1(r_1^{i+1}, r_2^{i+2}) < y_1(r_1^{i+2}, r_2^{i+3})$. Lemma B.2 now implies $y_1(r_1^{i+1}, r_2^{i+2}) > y_1^{3,\mathrm{I}}$, which contradicts Part i of Lemma B.6. Thus, case (b) must hold, which means that $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{II}}$. Since $x_2(r_1^{i+1}, r_2^{i+1}) > x_2^{3,\mathrm{II}}$ and $x_2(r_1^{i+2}, r_2^{i+2}) > x_2(r_1^{i+1}, r_2^{i+1})$, we have from Lemma B.1

$$\begin{aligned} y_1\left(r_1^{i+1}, r_2^{i+2}\right) &= \frac{\delta}{\mu_1} - \frac{\delta}{\mu_1} x_2\left(r_1^{i+2}, r_2^{i+2}\right) \\ &\quad + \frac{(\mu_1 - \delta) r_1^0}{\mu_1} \\ &> \frac{\delta}{\mu_1} - \frac{\delta}{\mu_1} x_2^{3,\mathrm{II}} + \frac{(\mu_1 - \delta) r_1^0}{\mu_1} = y_1^{3,\mathrm{II}}. \end{aligned} \tag{B.17}$$

Thus, $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$ and $y_1(r_1^i, r_2^{i+1}) > y_1^{3,\mathrm{II}}$ implies $(r_1^{i+1}, r_2^{i+2}) \in Q_{3,\mathrm{II}}$ and $y_1(r_1^{i+1}, r_2^{i+2}) > y_1^{3,\mathrm{II}}$. Then, for all $k \geq i$ we have $(r_1^{k+1}, r_2^{k+2}) \in Q_{3,\mathrm{II}}$ that implies $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{II}}$ by Lemma B.3, which contradicts $y_1(r_1^i, r_2^{i+1}) > y_1^{3,\mathrm{II}}$ □

Similar results can be obtained in $P_{2,\mathrm{III}}$ and $P_{1,\mathrm{III}}$.

Lemma B.8. *If $(r_1^i, r_2^i) \in P_{1,\mathrm{III}}$, then $x_2(r_1^i, r_2^i) = x_2^{1,\mathrm{III}}$.*

Lemma B.9. *If $(r_1^i, r_2^i) \in P_{2,\mathrm{III}}$, then $x_2(r_1^i, r_2^i) = x_2^{2,\mathrm{III}}$.*

The proofs of the two lemmas are similar to that of Lemmas B.6 and B.7, respectively.

Lemma B.10. *If $(r_1^i, r_2^i) \in P_{3,\mathrm{III}}$, then we have the following two cases: (i) $x_2(r_1^i, r_2^i) \geq x_2^{3,\mathrm{III}}$; (ii) $x_2(r_1^i, r_2^i) \leq x_2^{3,\mathrm{III}}$. Thus, we $x_2(r_1^i, r_2^i) = x_2^{3,\mathrm{III}}$.*

Proof of Lemma B.10, Part i. Let $(r_1^i, r_2^i) \in P_{3,\mathrm{III}}$. It follows from the definition of $P_{3,\mathrm{III}}$ that $y_2(r_1^i, r_2^{i+1}) > x_2(r_1^i, r_2^i) \geq r_2^i$ and $x_1(r_1^{i+1}, r_2^{i+1}) > y_1(r_1^{i+1}, r_2^{i+1}) \geq r_1^i$.

We assume $x_2(r_1^i, r_2^i) < x_2^{3,\mathrm{III}}$. Lemma B.2 implies $x_2(r_1^i, r_2^i) > x_2(r_1^{i+1}, r_2^{i+1})$. Since $x_2(r_1^i, r_2^i) \geq r_2^i$, we have from (B.1) that $x_2(r_1^i, r_2^i) = r_2^{i+1}$. Hence, $r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$. Then there are two possibilities:

(a) $r_2^{i+1} > y_2(r_1^{i+1}, r_2^{i+2}) > x_2(r_1^{i+1}, r_2^{i+1})$, i.e., $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$.

(b) $y_2(r_1^{i+1}, r_2^{i+2}) \geq r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$, i.e., $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$.

Take case (a), i.e., $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$. From Lemma B.6 we have $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{I}}$. It follows from Lemma B.1 that

$$\begin{aligned} x_2\left(r_1^i, r_2^i\right) &= \frac{\delta}{\mu_2} - \frac{\delta}{\mu_2} y_1\left(r_1^i, r_2^{i+1}\right) + \frac{(1-\delta)(1+\lambda_2)}{\mu_2} r_2^0 \\ &= \frac{\delta}{\mu_2} - \frac{\delta}{\mu_2} y_1^{3,\mathrm{I}} + \frac{(1-\delta)(1+\lambda_2) r_2^0}{\mu_2} \\ &= \frac{\delta}{\mu_2} y_2^{3,\mathrm{I}} + \frac{(1-\delta)(1+\lambda_2) r_2^0}{\mu_2}. \end{aligned} \tag{B.18}$$

$$\begin{aligned} y_2^{3,\mathrm{I}} &= \frac{1+\lambda_1}{1+\lambda_1+\delta} - \frac{(1+\lambda_1) r_1^0}{1+\lambda_1+\delta} + \frac{\delta r_2^0}{1+\lambda_1+\delta} \\ &= \frac{(1+\lambda_1)(1-\delta)}{(1+\lambda_1+\delta)(1-\delta)} - \frac{(1+\lambda_1)(1-\delta) r_1^0}{(1+\lambda_1+\delta)(1-\delta)} + \frac{\delta(1-\delta) r_2^0}{(1+\lambda_1+\delta)(1-\delta)} \\ &= \frac{1-\delta+\lambda_1(1-\delta)}{1+\lambda_1(1-\delta)-\delta^2} - \frac{(1-\delta+\lambda_1(1-\delta)) r_1^0}{1+\lambda_1(1-\delta)-\delta^2} + \frac{\delta(1-\delta) r_2^0}{1+\lambda_1(1-\delta)-\delta^2} \\ &= \frac{\mu_2}{\mu_2} \times \frac{(\mu_1-\delta)}{\mu_1-\delta^2} - \frac{\mu_2}{\mu_2} \times \frac{(\mu_1-\delta) r_1^0}{\mu_1-\delta^2} + \frac{\mu_2}{\mu_2} \times \frac{\delta(1-\delta) r_2^0}{\mu_1-\delta^2} \\ &> \frac{\mu_2(\mu_1-\delta)}{\mu_1\mu_2-\delta^2} - \frac{\mu_2(\mu_1-\delta) r_1^0}{\mu_1\mu_2-\delta^2} + \frac{\delta(\mu_1-\delta) r_2^0}{\mu_1\mu_2-\delta^2} = y_2^{3,\mathrm{III}} \end{aligned} \tag{B.19}$$

Thus $x_2(r_1^i, r_2^i) > (\delta/\mu_2) y_2^{3,\mathrm{III}} + (1-\delta)(1+\lambda_2) r_2^0/\mu_2 = x_2^{3,\mathrm{III}}$, which contradicts the initial $x_2(r_1^i, r_2^i) < x_2^{3,\mathrm{III}}$.

Take case (b), that is, $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$. By Lemma B.1, $x_2(r_1^i, r_2^i) = (\delta/\mu_2) y_2^{3,\mathrm{II}} + ((1-\delta)(1+\lambda_2)/\mu_2) r_2^0$. By (B.1) $x_2(r_1^i, r_2^i) = r_2^{i+1}$, and $y_2^{3,\mathrm{II}}$ is a function of r_2^{i+1}.

$$\begin{aligned} x_2\left(r_1^i, r_2^i\right) &= \frac{\delta}{\mu_2} y_2^{3,\mathrm{II}} + \frac{(1-\delta)(1+\lambda_2)}{\mu_2} r_2^0 \\ &= \frac{\delta}{\mu_2} \left(\frac{(1+\lambda_2)(\mu_1-\delta) + \lambda_2 \delta^2 r_2^{i+1}}{\mu_1(1+\lambda_2)-\delta^2} - \frac{(1+\lambda_2)(\mu_1-\delta) r_1^0}{\mu_1(1+\lambda_2)-\delta^2} + \frac{\delta(\mu_2-\delta) r_2^0}{\mu_1(1+\lambda_2)-\delta^2} \right) + \frac{(1-\delta)(1+\lambda_2)}{\mu_2} r_2^0 \\ &= \frac{\delta}{\mu_2} \left(\frac{(1+\lambda_2)(\mu_1-\delta) + \lambda_2 \delta^2 x_2\left(r_1^i, r_2^i\right)}{\mu_1(1+\lambda_2)-\delta^2} - \frac{(1+\lambda_2)(\mu_1-\delta) r_1^0}{\mu_1(1+\lambda_2)-\delta^2} + \frac{\delta(\mu_2-\delta) r_2^0}{\mu_1(1+\lambda_2)-\delta^2} \right) + \frac{(1-\delta)(1+\lambda_2)}{\mu_2} r_2^0 \end{aligned} \tag{B.20}$$

Solving for $x_2(r_1^i, r_2^i)$ yields $x_2(r_1^i, r_2^i) = \delta(\mu_1-\delta)/(\mu_1\mu_2-\delta^2) - \delta r_1^0(\mu_1-\delta)/(\mu_1\mu_2-\delta^2) + \mu_1 r_2^0(\mu_2-\delta)/(\mu_1\mu_2-\delta^2) = x_2^{3,\mathrm{III}}$, which contradicts $x_2(r_1^i, r_2^i) < x_2^{3,\mathrm{III}}$. □

In order to complete above proof, we need a similar result for $Q_{3,\mathrm{III}}$.

Lemma B.11. *If $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{III}}$, then we have the following two cases: (i) $y_1(r_1^i, r_2^{i+1}) \geq y_1^{3,\mathrm{III}}$; (ii) $y_1(r_1^i, r_2^{i+1}) \leq y_1^{3,\mathrm{III}}$. Thus, we have $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{III}}$.*

Proof of Lemma B.11, Part i. It is similar to the proof of Lemma B.10 Part i. □

We can continue the proof of Part ii of Lemma B.10.

Proof of Lemma B.10, Part ii. Let $(r_1^i, r_2^{i+1}) \in P_{3,\mathrm{III}}$. By the definition of $P_{3,\mathrm{III}}$ we have $y_2(r_1^i, r_2^{i+1}) > x_2(r_1^i, r_2^i) \geq r_2^i$ and $x_1(r_1^{i+1}, r_2^{i+1}) > y_1(r_1^i, r_2^{i+1}) \geq r_1^i$.

We assume $x_2(r_1^i, r_2^i) > x_2^{3,\mathrm{III}}$. Lemma B.2 implies $x_2(r_1^i, r_2^i) < x_2(r_1^{i+1}, r_2^{i+1})$. Since $x_2(r_1^i, r_2^i) \geq r_2^i$, we have from (B.1) that $x_2(r_1^i, r_2^i) = r_2^{i+1}$. Hence, $r_2^{i+1} < x_2(r_1^{i+1}, r_2^{i+1})$ which implies $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{III}}$. By the first part of Lemma B.11 this implies that $y_1(r_1^i, r_2^{i+1}) \geq y_1^{3,\mathrm{III}}$. Then by Lemma B.1 and the construction of $x^{3,\mathrm{III}}$, we have

$$\begin{aligned} x_2\left(r_1^i, r_2^i\right) &= \frac{\delta}{\mu_2} - \frac{\delta}{\mu_2} y_1\left(r_1^i, r_2^{i+1}\right) + \frac{(1-\delta)(1+\lambda_2)}{\mu_2} r_2^0 \\ &\leq \frac{\delta}{\mu_2} - \frac{\delta}{\mu_2} y_1^{3,\mathrm{III}} + \frac{(1-\delta)(1+\lambda_2)}{\mu_2} r_2^0 \\ &= x_2^{3,\mathrm{III}}, \end{aligned} \tag{B.21}$$

which contradicts $x_2(r_1^i, r_2^i) > x_2^{3,\mathrm{III}}$. □

Proof of Lemma B.11, Part ii. It is similar to the proof of Lemma B.10, Part ii. □

We can obtain the proposals made in the sets $P_{3,\mathrm{I}}$ and $P_{3,\mathrm{II}}$.

Lemma B.12. *If $(r_1^i, r_2^i) \in P_{3,\mathrm{I}}$, then $x_2(r_1^i, r_2^i) = x_2^{3,\mathrm{I}}$.*

Proof. Let $(r_1^i, r_2^i) \in P_{3,\mathrm{I}}$. It follows from the definition of $P_{3,\mathrm{I}}$ that

$$\begin{gathered} r_2^i > y_2\left(r_1^i, r_2^{i+1}\right) > x_2\left(r_1^i, r_2^i\right), \\ x_1\left(r_1^{i+1}, r_2^{i+1}\right) > y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i. \end{gathered} \tag{B.22}$$

Assuming $x_2(r_1^i, r_2^i) \neq x_2^{3,\mathrm{I}}$, we have the following three exhaustive cases, which are mutually exclusive.

(i) $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$: Since $(r_1^i, r_2^i) \in P_{3,\mathrm{I}}$, Lemma B.1 implies

$$\begin{aligned} x_2\left(r_1^i, r_2^i\right) &= \delta - \delta y_1\left(r_1^i, r_2^{i+1}\right) + (1-\delta) r_2^0 \\ &= \delta y_2\left(r_1^i, r_2^{i+1}\right) + (1-\delta) r_2^0. \end{aligned} \tag{B.23}$$

Then it follows from Lemma B.6 and the construction of $x^{3,\mathrm{I}}$ that $x_2(r_1^i, r_2^i) = \delta y_2^{3,\mathrm{I}} + (1-\delta) r_2^0 = x_2^{3,\mathrm{I}}$, which contradicts $x_2(r_1^i, r_2^i) \neq x_2^{3,\mathrm{I}}$.

(ii) $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$: Since $(r_1^i, r_2^i) \in P_{3,\mathrm{I}}$ and $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$, we have

$$\begin{aligned} y_2\left(r_1^{i+1}, r_2^{i+2}\right) \geq r_2^{i+1} &= \max\left\{r_2^i, x_2\left(r_1^i, r_2^i\right)\right\} = r_2^i \\ &> y_2\left(r_1^i, r_2^{i+1}\right). \end{aligned} \tag{B.24}$$

However, by Lemma B.7 $y_2(r_1^i, r_2^{i+1}) = y_2^{3,\mathrm{II}}$ and this implies $y_2(r_1^i, r_2^{i+1}) = y_2(r_1^{i+1}, r_2^{i+2})$ by Lemma B.2.

This contradicts the above.

(iii) $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{III}}$: By Lemmas B.1 and B.11, inequality (B.19), and the construction of $x^{3,\mathrm{I}}$, we have $x_2(r_1^i, r_2^i) = \delta - \delta y_1(r_1^i, r_2^{i+1}) + (1-\delta) r_2^0 = \delta y_2^{3,\mathrm{III}} + (1-\delta) r_2^0 < \delta y_2^{3,\mathrm{I}} + (1-\delta) r_2^0 = x_2^{3,\mathrm{I}}$. By Lemma B.2 this implies $x_2(r_1^i, r_2^i) > x_2(r_1^{i+1}, r_2^{i+1})$. Since $r_2^i > x_2(r_1^i, r_2^i)$ we have by (B.1) that $r_2^i = r_2^{i+1}$, implying $r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$. This implies $(r_1^i, r_2^{i+1}) \notin Q_{3,\mathrm{III}}$, which contradicts $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{III}}$. □

It follows that $x_2(r_1^i, r_2^i) = x_2^{3,\mathrm{I}}$.

Lemma B.13. *If $(r_1^i, r_2^i) \in P_{3,\mathrm{II}}$, then $x_2(r_1^i, r_2^i) = x_2^{3,\mathrm{II}}$.*

Proof. Let $(r_1^i, r_2^i) \in P_{3,\mathrm{II}}$. Then it follows from the definition of $P_{3,\mathrm{II}}$ that

$$\begin{gathered} y_2\left(r_1^i, r_2^{i+1}\right) \geq r_2^i > x_2\left(r_1^i, r_2^i\right), \\ x_1\left(r_1^{i+1}, r_2^{i+1}\right) > y_1\left(r_1^i, r_2^{i+1}\right) \geq r_1^i. \end{gathered} \tag{B.25}$$

Assuming $x_2(r_1^i, r_2^i) \neq x_2^{3,\mathrm{II}}$, we have the following three exhaustive cases, which are mutually exclusive.

(i) $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{I}}$: Since $r_1^i > x_2(r_1^i, r_2^i)$, we have from (B.1) that $r_2^i = r_2^{i+1}$. Hence $y_2(r_1^i, r_2^{i+1}) \geq r_2^i = r_2^{i+1} > y_2(r_1^{i+1}, r_2^{i+1})$. However, by Lemma B.6 we have $y_2(r_1^i, r_2^{i+1}) = y_2^{3,\mathrm{I}}$, which by Lemma B.2 implies $y_2(r_1^i, r_2^{i+1}) = y_2(r_1^{i+1}, r_2^{i+2})$. This contradicts the above.

(ii) $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{II}}$: By Lemma B.7 we have $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{II}}$. By Lemma B.1 and the construction of $x^{3,\mathrm{II}}$, we have $x_2(r_1^i, r_2^i) = (\delta(1+\lambda_2 r_2^{i+1}) - \delta y_1(r_1^i, r_2^{i+1}))/(1+\lambda_2) + (1-\delta) r_2^0 = (\delta(1+\lambda_2 r_2^{i+1}) - \delta y_1^{3,\mathrm{II}})/(1+\lambda_2) + (1-\delta) r_2^0 = \delta(y_2^{3,\mathrm{II}} + \lambda_2 r_2^{i+1})/(1+\lambda_2) + (1-\delta) r_2^0 = x_2^{3,\mathrm{II}}$, which contradicts $x_2(r_1^i, r_2^i) \neq x_2^{3,\mathrm{II}}$.

(iii) $(r_1^i, r_2^{i+1}) \in Q_{3,\mathrm{III}}$: By Lemma B.11 and the definition of $y_1^{3,\mathrm{III}}$ this implies $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{III}} = \delta/\mu_1 - (\delta/\mu_1) x_2^{3,\mathrm{III}} + (\mu_1 - \delta) r_1^0/\mu_1$. By Lemma B.1 we have $y_1(r_1^i, r_2^{i+1}) = y_1^{3,\mathrm{III}} = \delta/\mu_1 - (\delta/\mu_1) x_2(r_1^{i+1}, r_2^{i+1}) + (\mu_1 - \delta) r_1^0/\mu_1$.

Hence $x_2(r_1^{i+1}, r_2^{i+1}) = x_2^{3,\mathrm{III}}$. Since $r_2^i > x_2(r_1^i, r_2^i)$, we have from (B.1) that $r_2^i = r_2^{i+1}$. Thus, $x_2(r_1^i, r_2^i) < r_2^i = r_2^{i+1} \leq x_2(r_1^{i+1}, r_2^{i+1})$. By Lemma B.2 this implies $x_2(r_1^i, r_2^i) = x_2^{3,\mathrm{II}}$. Since $x_2(r_1^i, r_2^i) < r_2^i$ and $r_2^i = r_2^{i+1}$ this implies $r_2^{i+1} > x_2^{3,\mathrm{II}}$. That is, $r_2^{i+1} > \delta(\mu_1 - \delta + \lambda_2 r_2^{i+1} \mu_1)/(\mu_1(1+\lambda_2) - \delta^2) - \delta(\mu_1 - \delta) r_1^0/(\mu_1(1+\lambda_2) - \delta^2) + \mu_1(\mu_2 - \delta) r_2^0/(\mu_1(1+\lambda_2) - \delta^2)$.

This is equivalent to $r_2^{i+1} > \delta(\mu_1 - \delta)/(\mu_1\mu_2 - \delta^2) - \delta r_1^0(\mu_1 - \delta)/(\mu_1\mu_2 - \delta^2) + \mu_1 r_2^0(\mu_2 - \delta)/(\mu_1\mu_2 - \delta^2) = x_2^{3,\mathrm{III}}$. Hence, $r_2^{i+1} > x_2(r_1^{i+1}, r_2^{i+1})$, which implies $(r_1^i, r_2^{i+1}) \notin Q_{3,\mathrm{III}}$. This is a contradiction. It follows that $x_2(r_1^i, r_2^i) = x_2^{3,\mathrm{II}}$. □

We can obtain similar results for $Q_{2,\mathrm{III}}$ and $Q_{1,\mathrm{III}}$.

Lemma B.14. *If $(r_1^i, r_2^{i+1}) \in Q_{1,\mathrm{III}}$, then $y_1(r_1^i, r_2^{i+1}) = y_1^{1,\mathrm{III}}$.*

Lemma B.15. *If $(r_1^i, r_2^{i+1}) \in Q_{2,\mathrm{III}}$, then $y_1(r_1^i, r_2^{i+1}) = y_1^{2,\mathrm{III}}$.*

The proofs of the two Lemma are similar to that of Lemmas B.12 and B.13.

Proof of Theorem 3. Let $(r_1^i, r_2^i) \in P_{1,\mathrm{I}}$. According to the definition, we have $r_2^i > y_2(r_1^i, r_2^i)$ and $r_1^i > x_1(r_1^i, r_2^i)$. From Lemma B.5, we have $r_2^i > y_2^{1,\mathrm{I}}$ and $r_1^i > x_1^{1,\mathrm{I}}$. Thus, $(r_1^i, r_2^i) \in X_{1,\mathrm{I}}$, which leads to $P_{1,\mathrm{I}} \subseteq X_{1,\mathrm{I}}$.

Similarly, it follows from Lemmas B.5, B.8–B.10, B.12, and B.13 that $(r_1^i, r_2^i) \in P_\omega$ implies $(r_1^i, r_2^i) \in X_\omega$ for each $\omega \in \Omega$. Thus, we have $P_\omega \subseteq X_\omega$ for each $\omega \in \Omega$. For mutually exclusive and exhaustive sets P_ω, we have $P_\omega = X_\omega$ for each $\omega \in \Omega$. Thus, for times $t \in T_{odd}$, the unique SPE strategy of player 1 is to make the proposal x^ω if $(r_1^i, r_2^i) \in X_\omega$. In other words, the unique SPE strategy of player 1 is to follow strategy $\widehat{f}$. Similarly, we have $Q_\omega = X_\omega$ for each $\omega \in \Omega$. Thus, for times $t \in T_{even}$, the unique SPE strategy of player 2 is to make the proposal y^ω if $(r_1^i, r_2^i) \in X_\omega$; that is, the unique SPE strategy of player 2 is to follow strategy $\widehat{g}$.

By Part II of the proof of Theorem 2, the unique optimal strategy of player 1 at moments $t \in T_{even}$ is to accept offers that are at least SPE proposal of player 2 and to turn down those that are not. In other words, at time $t \in T_{even}$, the unique SPE strategy of player 1 is $\widehat{f}$. Similarly, at time $t \in T_{odd}$, the unique optimal strategy of player 2 is to accept and turn down proposals by the strategy $\widehat{g}$. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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