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Research Article

A Continuous Approximation Approach Based on Regular Hexagon Partition for the Facility Location Problem under Disruptions Risk

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Today's business environment is complex, dynamic, and uncertain which makes a supply chain facility increasingly vulnerable to disruption from various risk accidents as one of the main threats to the whole supply chain's operation. However, in general, most of the studies on facility location problems assume that the facilities, once built, will always run availably and reliably. In fact, although the probability is very low, supply chain disruptions often incur disastrous consequences. Therefore, it is critical to account for disruptions during designing supply chain networks. To accomplish planned outcomes and greater supply chain resiliency, this article proposes a continuous approximation approach based on regular hexagon partition to address the reliable facility location problems with consideration of facility disruptions risk. The optimization goal is to determine the best facility location that minimizes the expected total system cost on the premise that the supply chain network is not disrupted as a whole when one or some facilities are subject to probabilistic failure. Our numerical experiment discusses the performance of the proposed solution approaches which demonstrates that the benefits of considering disruptions in the supply chain design can be significant. In addition, considering the impact of disruption probability estimation error on the optimal decision, the misestimating of the disruption probability is also investigated in this paper.

1. Introduction

Today's business context is totally different from before. The fierce competition and the rapid changing of customers' preferences, together with the fast development of information technology and globalization, have forced firms to operate jointly instead of individually [1, 2]. However, the highly dynamic and uncertain environment often poses serious challenges to the supply chain decision-makers. In general, the optimal decisions of supply chain network design include the quantity, location, and capacity of entities in each echelon of the chain. Therefore, supply chain network design increasingly becomes critical and difficult in nature which has remarkable impacts on all other decisions concerning a supply chain's operation and performance. Particularly, it is very expensive to change and these processes become more and more complex [3]. In consequence, this complexity makes decision-making more difficult and complicated. Although great effort has been expended on trying to make supply chains leaner and reducing its operational costs, recent studies indicate that they have increased supply chain risks simultaneously [4]. Among the various types of risks related to supply chain management, facility disruption risk is highly focused due to its severe, sometimes disastrous, consequences than others [5, 6]. Therefore, in such a business context, the topic on supply chain facility location problems has gained widespread attention in the last two decades from a risk management perspective [7].

Facility location problems are among the classical OR/MS (Operations Research/ Management Science) problems which have been broadly investigated and applied in many areas [8]. In the supply chain context, the term "Facility" usually means a business entity which systematically integrates related resources, e.g., capital, material, manpower,

equipment, and technology, together to manufacture specific products and services or to achieve the planned functions. Therefore, facility location plays a critical role in the strategic design of supply chain networks to accomplish an efficient and effective supply chain [9]. In essence, a general facility location optimization involves facility location, capacity, and route designing, in a specific area with specific goals, such as minimized cost or maximized market coverage. The supply chain should be carefully designed to best satisfy their spatially distributed demand. Furthermore, the expensive facility construction cost and difficulty in modifying highlight the importance of facility location decisions [10].

The extant literature on supply chain facility location problems generally assumes the facilities are always available and perfectly reliable to serve their customers throughout the planning horizon but without considering the facility disruption risk [11]. However, the risk accidents that could disrupt the supply chain maybe exit anywhere and anytime, and failing to cope with them adequately may led to huge economic losses [12]. In today's complex, dynamic, and uncertain business environment, supply chain facilities are increasingly vulnerable to disruption partially or completely which has become one of the main threats to the whole supply chain's operation [13]. Various risk events, e.g., natural catastrophes, key-supplier failure, strikes, or purposeful terrorist attacks and trade barriers, indicated that the business environment is becoming increasingly more uncertain and fragile than ever before [14, 15]. Further, the globalized procurement, decentralized manufacturing and outsourcing, supplier reduction and minimizing inventory, etc., as well as tightly optimized, lean supply chain practices championed by practitioners and researchers, increase the vulnerability of supply chain and highlight the significance and urgency of proactive risk management [5, 16]. So, the topic on supply chain disruption risk has attracted growing attention in the last few years [12].

Designing a robust and reliable supply chain network will contribute to maintaining and enhancing its ability to better against supply chain disruptions [17]. There are some reliable facility location models that have only appeared very recently which generally addressed the ability of supply chain networks as a whole to maintain basic operations when some facilities are disrupted unexpectedly. To our knowledge, the seminal work on reliable facility location problem was carried out by Snyder & Daskin [18] which investigates facility disruption risk. Although the topic on reliable location problems has received much attention, it is still understudied, especially in the subfield of continuous facility location problem [9, 19].

In this study, focusing on facility disruption risk, a stylized reliable facility location model will be formulated in a continuous plane which divides the demand area as hexagonal grid. The optimization goal is to minimize the expected total cost on the premise that the supply chain network is not disrupted as a whole with consideration of facility disruptions risk. More specific, the expected total cost includes initial facility construction costs and expected transportation cost in normal and disruption scenarios. In addition, considering the disruption probability estimation error, the impact of misestimating the disruption probability on optimal decisions is also examined in this paper.

Our findings contribute to the literature in several ways: (1) this study adds to the growing body of literature on reliable facility location problems. As mentioned above, the supply chain failure would incur severe, even disastrous, consequences to the supply chain's operational continuity or firm's survival. However, this topic is still understudied. Some research investigated discrete reliable location problem, while very few literature addressed continuous location problem which is proposed to overcome the deficiency of discrete models in solving large-scale problems; (2) for the continuous location literature, to our knowledge, this is the first study that adopts a new method that divides the demand area as hexagonal grid in a continuous facility location model. The hexagonal partition is an optimal scheme to divide a huge plane into a number of regular subregions with equal area and it is a substantial improvement in continuous location problem; (3) the analysis on the misestimating disruption probability is very limited in the extant supply chain literature. Actually, the estimation error never vanishes. This study will shed light on it by discussing the sensitivity and robustness of the model to explore the effect of estimation error on the optimal solution.

The remainder of the paper is structured as follows. Section 2 presents a related literature review, which systematically summarizes the current research of reliable facility location problem. Section 3 proposes the research problem by formulating it into a continuous location model with consideration of facility disruptions risk and the empirical results are analyzed. In Section 4, this paper conducts a numerical experiment to analyze the algorithm's solving performance, as well as its sensitivity and robustness which are also investigated. In Section 5, we end the paper by concluding remarks and drawing managerial insights, as well as briefly discussing further research directions.

2. Theoretical Background and Literature Review

The topic of facility location is not new which could be traced back to the classical Weber problem in 1909 [20, 21]. Basically, facility location problems explore where to configure facilities so as to minimize the cost of satisfying a certain amount of demands in an efficient way [22]. In addition, the facility location decisions can have a long-lasting impact on future business of a supply chain as well as its flexibility to satisfy these demands as they maybe evolve over time. Thereafter, facility location problem received increasing attention and has been broadly investigated and applied in many areas.

Based on a systematic review of the extant literature, we could identify a variety of classifications on facility location problems. In order to present a clearer logic, we will adopt the classification of American Mathematical Society [23] in this paper to outline and examine research related to the facility location problems, particularly reliable facility location problems. According to this classification, the facility location problems are usually sorted into discrete location

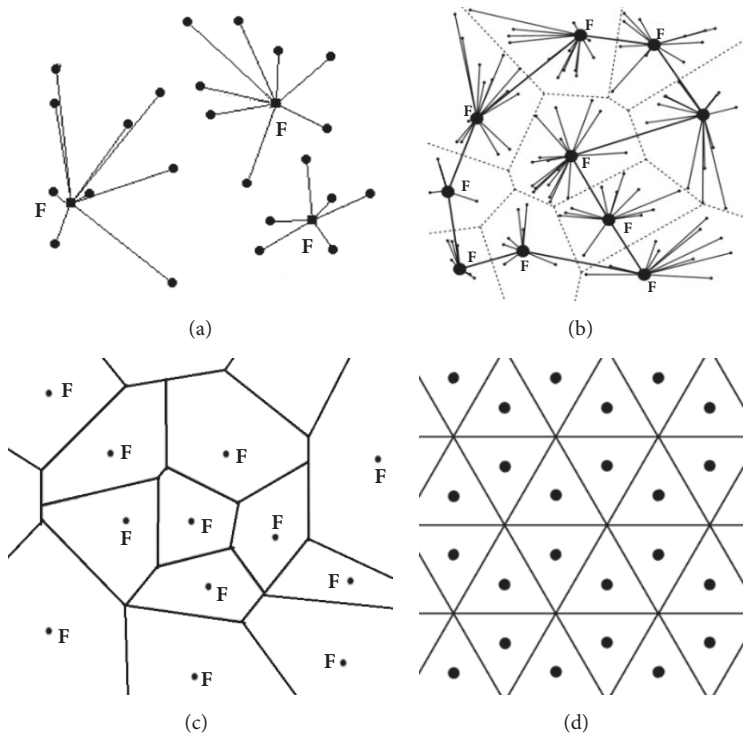

Figure 1: (a) Regular discrete location problem; (b) complicated discrete location problem; (c) regular continuous location problem; (d) stylized continuous location problem.

problem and continuous location problem which are interrelated logically. The original facility location problem was raised as a discrete problem which is usually defined as an optimal decision of the sites to locate new facilities among some available candidate locations. It seeks to determine the optimal facility locations and demand assignments to balance the tradeoff between initial fixed setup costs and routine transportation costs (as shown in Figure 1(a)). Most facility location problems focused on discrete, especially for some specific facility location decisions (e.g., [18, 24, 25]). Although the discrete models are considered more realistic in most cases, the researchers still hold negative attitudes on the practicality of the models in large-scale facility location problems [26]. The main argument is that its excessive computational burden in large-scale problems often makes it unable to obtain the optimal solution and even becomes NP-hard (e.g., Figure 1(b)). In order to overcome this deficiency, continuous approximation (CA) models are developed to obtain good approximate solutions to large-scale problems which is a good alternative to find near-optimal solutions. In most CA models, a large set of spatially distributed customers (demand points) can be approximated properly by a continuous function [27]. Accordingly, the whole target demand region could be divided continuously into a number of subregions and each one is serviced by an assigned facility. In this way, the continuous subdemand regions and facilities could be stated as Figure 1(c). This solution may be feasible when there are a limited number of subregions. However, it is almost impossible to deal with the increasing number of subregions and also becomes NP-hard as Figure 1(b). Therefore, aims to make a rational tradeoff between solution's accuracy and computational burden,improved CA models in an infinite homogeneous plane based on stylized grid emerged (e.g., Figure 1(d)). The principal idea of CA models is to approximate a large amount of discrete spatially distributed customers (demand points) as a continuously distributed service area to overcome the computational difficulties associated with large-scale problems [28]. While the CA approach does not decide the exact location of the candidate facilities, it defines a subregion for each facility in terms of continuous service areas [29].

However, in general, most studies on facility location problems assume that the facilities, once built, will always run availably and reliably. Actually, today's dynamic and uncertain environment makes facilities increasingly vulnerable to fail from time to time. Although the probability is very low, there are numerous legendary examples of significant supply chain disruptions that have resulted in heavy losses to the whole supply chain network. Therefore, the research on the reliable facility location problem which considers disruption risk is an emerging research area in recent years [5].

To increase the robustness and reliability of a supply chain in mitigating unpredicted facility disruptions, a number of reliable models have been designed. To the best of our knowledge, the seminal work on reliable facility location problem was carried out by Snyder & Daskin [18] in this field. Assuming that the facility failure probabilities are mutually independent and equal, this paper investigates an uncapacitated fixed-charge location problem (UFLP) based on p-median problem. Thereafter, many researches mainly focused

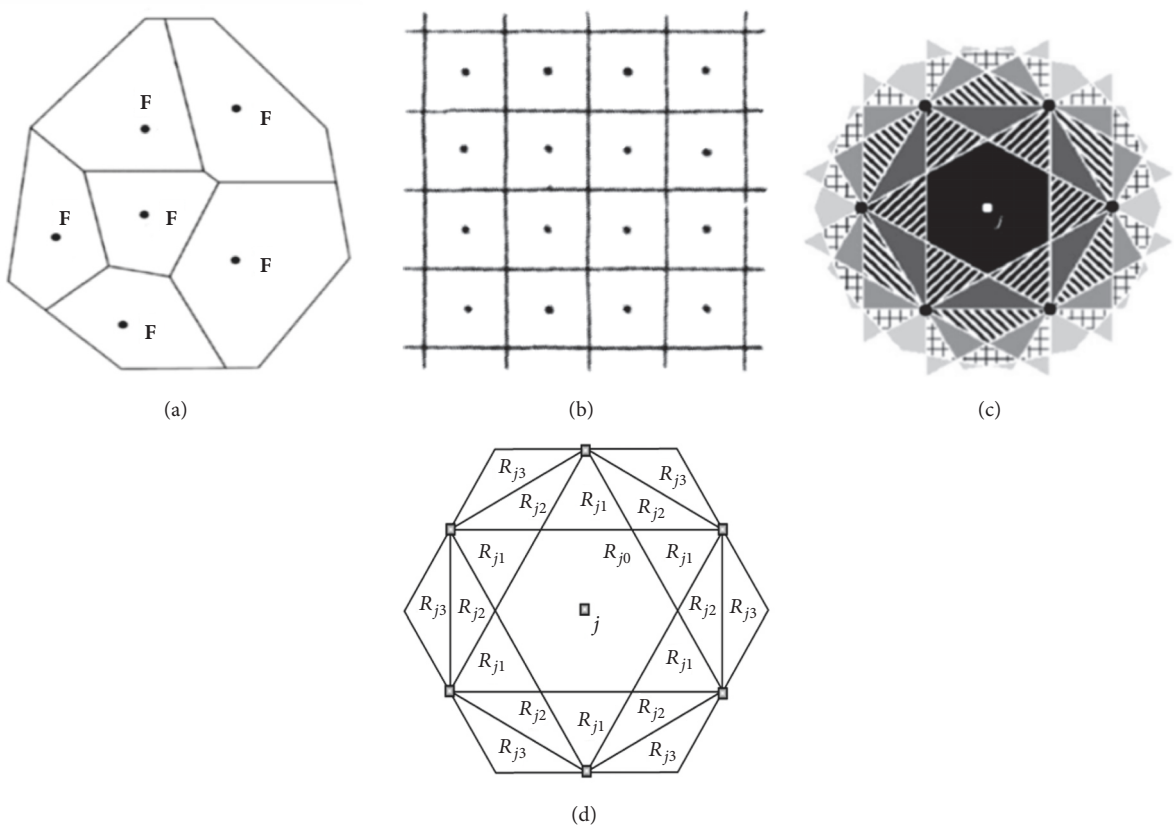

Figure 2: (a) Irregular partition of target plane; (b) chessboard grid partition of target plane; (c) quasi-regular geometric partition of target plane; (d) regular geometric partition of target plane.

on discrete location problems attracting more attention as an emerging field. However, there are only a few papers that examined facility location problems with consideration of disruption risk by introducing CA model. Lim et al. [4] addressed a continuous facility location problem under the facility disruption scenario with the option of hardening selected facilities (various protection plans). In their model, the disruption probabilities are assumed independent and the whole service area follows a continuous uniform distribution which is divided as chessboard grid. Further, Cui, Ouyang, and Shen [27] investigated the same problem by developing a CA model with a relaxed assumption of location-specific probabilities. In their model, an infinite homogeneous plane is divided as regular triangle-hexagon partition to achieve the optimal facility location decision. Li and Ouyang [30] focused his research on spatially correlated, site-dependent disruption probabilities by developing a CA model to minimize the sum of initial facility setup costs and the expected transportation costs between facilities and demands under normal and failure scenarios. This paper also discusses the impacts of spatial distribution characteristics of a continuous service area and the changes of disruption probability on the model. By formulating a disk model, Ouyang and Daganzo [31] transform the CA approach for location problems into feasible discrete designs which could be easily applied to other logistics problems. Furthermore, Wang, Lim, and Ouyang [28] extend the disk model of Ouyang and Daganzo [31] to a tube model to discretize the continuous facility density function into a set of time-varying facility location trajectories which effectively solves the dynamic location problem to a certain accuracy. In addition, Wang and Ouyang [10] propose a game-theoretical model based on a CA program to optimize supply chain network design in which the spatial competition and facility disruption risks are considered. In addition, considering three risk factors (demand, supply, and financial), Jabalameli et al. [32] studied a capacitated location problem in a continuous space as a risk model.

Briefly, there are two limitations to the existing literature about the continuous location problem. First, the partition on the target area is still understudied. In most CA models, how to divide the whole target area continuously as a key issue determines the computational performance. For example, in the extant literature, Novaes et al. [33] investigate a quasi-continuous facility location problem in a finite irregular region (Figure 2(a)). While the works of Lim et al. [4] (Figure 2(b)), Li and Ouyang [30] (Figure 2(c)), and Cui, Ouyang, and Shen [27] (Figure 2(d)) transform the specific target area to a regular geometric plane graphics approximately. As shown in Figures 2(b), 2(c), and 2(d), the regular partition on the target area aims to make the study generalized. If not, as shown in Figure 2(a), although the irregular scheme

is more realistic, it still needs to, respectively, calculate the definite integral of the 6 subregions. Accordingly, in essence, it is still a discrete thinking. Obviously, if the amount of subregions increases, the calculation process would be very difficult. Therefore, as a "compromise", the target area must be regularly divided into a number of subregions with regular geometry to achieve the feasible solution in an infinite homogeneous plane. Obviously, the partition rule of the target area is one of the key issues that impact the optimal solution. In addition, it should be pointed out that due to the low efficiency of the partition in Figure 2(b) and the extra computational complexity in Figures 2(c) and 2(d), the large-scale problems are difficult to be solved effectively and efficiently.

Second, the necessary analysis on the impact of mises-timating disruption probability lacks. In fact, the optimal solution may be changed because of market uncertainties or the parameter estimation error, etc. And even all the relevant parameters are known in advance, the supply chain network will still be subjected to disruptions due to the dynamic changing business environment [34]. Therefore, these limitations will restrict the validity of the optimal solution in application.

It is these research gaps that motivate us to conduct this study and form the main contributions of this paper. To address these gaps, this paper will investigate a continuous facility location problem by formulating a stylized reliable facility location model in a continuous plane which divides the demand area as hexagonal grid under the facility disruption scenario. The optimization goal is to minimize the expected total cost on the premise that the supply chain network is not disrupted as a whole under the facility disruption scenario. Further, considering the impact of disruption probability estimation error on the optimal decision, this paper will focus on the further discussion about the disruption probability estimation and the robustness of the model.

3. Model Formulation

In this section, a CA model for the location problem in a continuous plane is formulated in which the demand area is divided as hexagonal grid under the risk of disruptions. As discussed above, the partition of target demand plane in continuous facility location problem is one of the key issues that impact the optimal solution. We draw on the research conclusions in the field of mathematics. According to the "Honeycomb conjecture" [35, 36] and Circle-packing theorem [37, 38], hexagonal partition is an optimal scheme to divide the target plane into a number of subregions of equal area (as shown in Figure 3). More specific, in supply chain facility location context, it means that the distance sum of any demand point in the grid to the center facility is minimal. Furthermore, Drezner and Zemel [39] and Trietsch [40], respectively, proved that hexagonal partition is significantly better than square partition which is better than triangular partition. Therefore, this paper adopts the hexagonal partition as an optimal scheme in addressing continuous facility location problem which is an improvement than previous research.

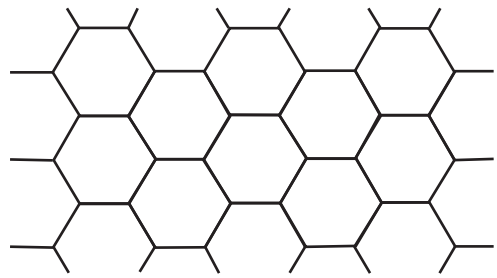

Figure 3: The division of hexagonal grid.

The following notations will be used throughout the paper:

G: the target region with area of S which subjects to a uniform distribution with a probability density of ρ

q: the facility disruption probability, $q \in (0, 1)$. Considering the focus on the study, the disruption probability is presented as a given parameter

c: the per-unit transportation costs

f_R: the setup cost of reliable facilities

f_U: the setup cost of unreliable facilities, $f_R > f_U$

r: the hardening cost factor of unreliable facility, $r = f_R/f_U d(P_1, P_2)$: the travel distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

In addition, generally, there are three ways to calculate the distance between two points, Euclidean distance, Manhattan distance and Great Circle distance (as shown in Figure 4). Considering the specific situation of the research problem, this study adopts the Manhattan distance, i.e. $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$.

The target region G is assumed uniformly distributed with a probability density of ρ. The supply chain decision-maker seeks to locate a number of facilities to meet all the demand with considering random facility disruptions. Consistent with the study of Snyder and Daskin [18], two types of facilities are designed in this paper: (1) reliable facilities, which are not subject to disruption risk by additional hardening cost input and (2) unreliable facilities, which are subject to disruption risk, where the setup cost $f_R > f_U$. To satisfy the whole target region, each subregion would be served by a reliable/ unreliable facility initially, where 1-q is the probability that the facility works availably. If an initial allocated unreliable facility in a subregion fails, there is a closest reliable facility as a backup to serve its demands. In addition, we assume the reliable facilities have no capacity constraints and the decision-makers are risk-neutral in this study. Therefore, we model the objective function as follows:

$$TC = f_R n_R + f_U n_U + d \cdot \rho S c \tag{1}$$

The objective of the model is to find solutions that minimize the expected total cost (TC) of locating n_R reliable facilities, n_U unreliable facilities, and the expected transportation cost between facilities and demands.

Consider a hexagonal subregion G_0 with area S_0 in the whole service region G that is large enough. Moreover, this

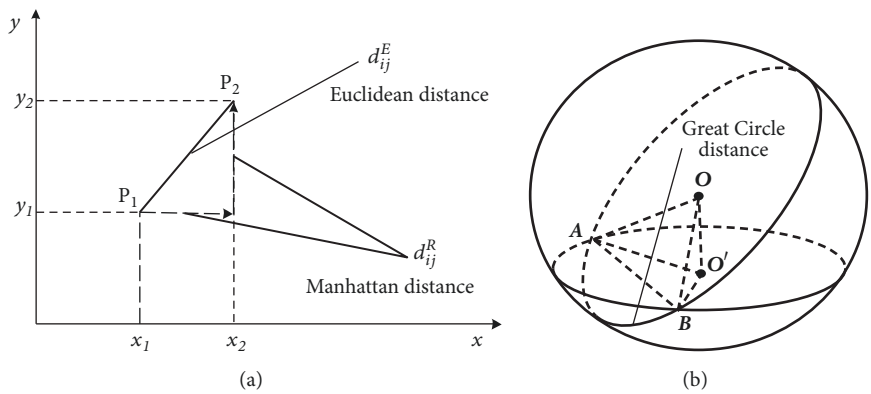

Figure 4: (a) Distance between two points; (b) great circle distance.

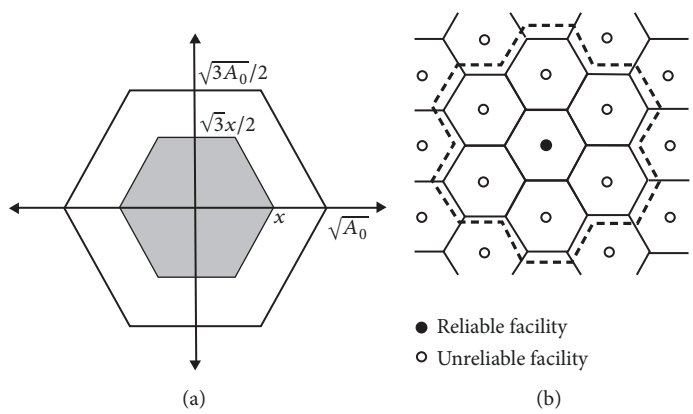

Figure 5: (a) A hexagonal subregion G_0; (b) facility location set (topo-area, $n_0 = 7$).

paper argues that the transportation cost of the central part of a subregion to its assigned facility could be ignored since its closeness. As shown in Figure 5(a), the side length of hexagon is supposed as $\sqrt{A_0}$ to make the calculation convenient. Accordingly, the whole demand region G can also be approximated as a hexagon with side length of $\sqrt{A}$. Thus, the area of the hexagon G_0 is $S_0 = (3/2)\sqrt{3}A_0$, where the shaded part of G_0 is the central area that the transportation cost is ignored. The graphical representation is shown in a Cartesian coordinate system (Figure 5(a)).

Accordingly, the expected distance between the central facility in a subregion and demands can be expressed according to Figure 4(a).

$$\begin{aligned} \mathrm{E}[D] &= \int_{G_0} \mathrm{P}(D > x)\, dx = \int_0^{\sqrt{A_0}} [1 - F_D(x)]\, dx \\ &= \int_0^{\sqrt{A_0}} \left(1 - \frac{x^2}{A_0}\right) dx = \left(x - \frac{x^3}{3A_0}\right)\Bigg|_0^{\sqrt{A_0}} = \frac{2}{3}\sqrt{A_0} \end{aligned} \tag{2}$$

$F(x)$ is a cumulative distribution function of D and $f(x)$ is the probability density function. More specifically, $F_D(x)$ denotes the ratio of the shaded part to the whole hexagon area; i.e., $F_D(x) = x^2/A_0$.

Furthermore, suppose that G_0 is serviced by n_0 facilities and each facility covered a demand area of S_0/n_0 which means the subregion is divided further as honeycomb grid as shown in Figure 5(b). Each subregion is a separated service area and is allocated with one reliable/unreliable facility, and the n_0 facilities make a set of facilities. We assume only one reliable facility in the facility set. Accordingly, considering the reliable facility does not fail as designed above, the optimal location decision is to locate the reliable facility in the center of G_0 which is surrounded by $n_0 - 1$ unreliable ones. Thus, each subregion will be served by a facility as its primary assignment and by the reliable facility from the center as an emergency backup when one or some of the $n_0 - 1$ unreliable facilities disrupted.

Based on formula (2), in each grid, if its own initial facility is available, the expected distance between facility and demands is $(2/3)\sqrt{A_0/n_0}$, while when an unreliable facility fails, the random distance between its demand and the adjacent reliable facility is D_R. Therefore, the expected distance between the central reliable facility and the demands of adjacent $n_0 - 1$ grid which is severed by unreliable facilities initially expressed as $\mathrm{E}[D_R]$. So, $(2/3)\sqrt{A_0} = (2/3)\sqrt{A_0/n_0}(1/n_0) + \mathrm{E}[D_R]((n_0 - 1)/n_0)$. This implies that

$$\mathrm{E}[D_R] = \left(\frac{n_0}{n_0 - 1}\right)\left[\frac{2}{3}\sqrt{A_0} - \frac{2}{3n_0}\sqrt{\frac{A_0}{n_0}}\right] \tag{3}$$

Furthermore, as discussed above, the whole service area G comprises numerous continuously distributed regular subregions G_0. Obviously, it is an optimal partition to an infinite area; but to a finite area, the partition would only be an approximate optimal solution due to its specific shape. Therefore, the whole area is divided into $n_T = (n_R + n_U)$ hexagonal subregions, and there are n_R reliable facilities and n_U unreliable facilities in total.

In each set of subregions as shown in Figure 5(b), the expected distance between facility and demands is $(2/3)\sqrt{A/n_T}$. So the expected transportation costs of satisfying the whole service region when all the unreliable facilities disrupted are the expected transportation costs for serving the whole service region by reliable facilities, $(2/3)\sqrt{A/n_T}\rho Sc(n_R/n_T)$. Accordingly, the expected transportation costs for serving the whole region G are

$$\frac{2}{3}(1-q)\sqrt{\frac{A}{n_T}}\rho Sc\left(\frac{n_U}{n_T}\right) + q\mathrm{E}[D_R]\rho Sc\left(\frac{n_U}{n_T}\right) \quad (4)$$

Furthermore, according to our network design, there is one reliable facility corresponding to n_U/n_R unreliable facilities. So, it means that each grid with a reliable facility is surrounded by n_U/n_R grids with unreliable facilities.

Set $A_0 = A/n_R$, $n_0 = n_U/n_R + 1 = (n_U + n_R)/n_R = n_T/n_R$. Therefore, $\mathrm{E}[D_R]$ could be derived from formula (3):

$$\begin{aligned}\mathrm{E}[D_R] &= \left(\frac{n_0}{n_0-1}\right)\left[\frac{2}{3}\sqrt{A_0} - \frac{2}{3n_0}\sqrt{\frac{A_0}{n_0}}\right] \\ &= \frac{2}{3}\left(\frac{n_T/n_R}{n_U/n_R}\right)\left[\sqrt{\frac{A}{n_R}} - \frac{n_R}{n_T}\sqrt{\frac{A/n_R}{n_T/n_R}}\right] \\ &= \frac{2}{3}\sqrt{\frac{A}{n_R}}\left(\frac{n_T}{n_U}\right) - \frac{2}{3}\sqrt{\frac{A}{n_T}}\left(\frac{n_R}{n_U}\right)\end{aligned} \quad (5)$$

Further, we derive the expected total cost by combining of the facility setup cost and the expected transportation cost:

$$\begin{aligned}\mathrm{E}[TC] &= f_R n_R + f_U n_U + \frac{2}{3}\sqrt{\frac{A}{n_T}}\rho Sc\left(\frac{n_R}{n_T}\right) \\ &\quad + \frac{2}{3}(1-q)\sqrt{\frac{A}{n_T}}\rho Sc\left(\frac{n_U}{n_T}\right) \\ &\quad + q\mathrm{E}[D_R]\rho Sc\left(\frac{n_U}{n_T}\right)\end{aligned} \quad (6)$$

Thus, the following equation can be obtained by necessary algebra processing:

$$\mathrm{E}[TC] = f_R n_R + f_U n_U + q\psi\sqrt{\frac{1}{n_R}} + (1-q)\psi\sqrt{\frac{1}{n_T}} \quad (7)$$

where $\psi = (2/3)\rho S^{3/2}c$ is a constant.

The objective function is to minimize the expected total cost. As a consequence, the optimal quantity of reliable and unreliable facilities when the expected total cost is minimized and can be obtained by derivative of formula (7):

$$\begin{aligned}n_R^* &= \left(\frac{\psi q}{2(f_R - f_U)}\right)^{2/3}, \\ n_U^* &= \left(\frac{\psi(1-q)}{2f_U}\right)^{2/3} - \left(\frac{\psi q}{2(f_R - f_U)}\right)^{2/3}\end{aligned} \quad (8)$$

Then, $n_T^* = n_R^* + n_U^* = (\psi(1-q)/2f_U)^{2/3}$.

Since $n_R^* \le n_T^*$, the optimal solution is expressed as follows.

When $q \le 1 - f_U/f_R$, the optimal result is (8); and when $q > 1 - f_U/f_R$, the optimal solution is deploying reliable facilities only or hardening all unreliable facilities to reliable ones; i.e.,

$$\begin{aligned}n_R^* &= \left(\frac{\psi}{2f_R}\right)^{2/3}, \\ n_U^* &= 0\end{aligned} \quad (9)$$

Therefore, the optimal solution of the problem is described as follows.

The disruption probability threshold $q_{th} = 1 - f_U/f_R$ divides supply chain network design into two situations: when $q \le 1 - f_U/f_R$, both types of facilities should be deployed as formula (8) stated; when $q > 1 - f_U/f_R$, the optimal solution is deploying reliable facilities only or hardening all unreliable facilities to reliable ones as formula (9) stated.

According to the optimal solution, the threshold q_{th} will approach zero when the value of f_R approaches f_U. It indicates that if the supply chain facilities disruption probability is large or the incremental hardening cost of unreliable facilities is relatively small, the best choice is to deploy reliable facilities only whereas, if the disruption probability is relatively small or the hardening cost of unreliable facilities is large, both types of facilities should be deployed.

It is important to note that all the demands should be fully satisfied as a prerequisite in this paper which means the supply chain network as a whole is not disrupted in virtue of the reliable facilities. Therefore, the situation of all facilities is unreliable and is beyond the scope of this study.

By substituting formulas (8) and (9) into (7), the optimal expected total cost is obtained:

$$\mathrm{E}[TC(n_R^*, n_U^*)] = \begin{cases} 3\left(\frac{\psi}{2}\right)^{2/3}\left[q^{2/3}(f_R - f_U)^{1/3} + (1-q)^{2/3} f_U^{1/3}\right], & q \le q_{th} \\ 3\left(\frac{\psi}{2}\right)^{2/3} f_R^{1/3}, & q > q_{th}\end{cases} \quad (10)$$

4. Numerical Studies

4.1. The Optimal Numerical Solution. The goal of this section is to show the performance of the CA model for a reliable facility location problem in a continuous service plane by a numerical experiment.

Table 1: Optimal numerical solution.

(f_R, f_U)	r		Reliable facilities	Unreliable facilities	Total
$(1.5\times10^6, 1\times10^6)$	1.5	Facility quantity	10	34	44
		Facility cost	1.5×10^7	3.4×10^7	4.9×10^7
			Initial allocation	*Backup allocation*	
		Transportation cost	2.069×10^7	7.730×10^7	9.799×10^7
		Subtotal cost	3.569×10^7	11.130×10^7	
		Total cost		14.699×10^7	

Table 2: Sensitivity analysis of n_R^*, n_T^* to q.

	q=0.01	q=0.05	q=0.1	q=0.15	q=0.2
$\frac{dn_R^*}{dq}$	482.75	131.04	104.00	90.86	82.55
$\frac{dn_T^*}{dq}$	-30.42	-30.94	-31.50	-32.10	-32.76

Table 3: The sensitivity of expected total cost to q (0.01, 0.5) and r (1.25, 2).

E[TC]	q=0.01	q=0.1	q=0.2	q=0.3	q=0.4	q=0.5
r =1.25	$8.18*10^8$	$8.55*10^8$	$8.62*10^8$	$8.62*10^8$	$8.62*10^8$	$8.62*10^8$
r =1.5	$8.24*10^8$	$8.83*10^8$	$9.07*10^8$	$9.16*10^8$	$9.16*10^8$	$9.16*10^8$
r =1.75	$8.29*10^8$	$9.03*10^8$	$9.38*10^8$	$9.57*10^8$	$9.64*10^8$	$9.64*10^8$
r =2	$8.32*10^8$	$9.18*10^8$	$9.63*10^8$	$9.90*10^8$	$10.04*10^8$	$10.08*10^8$

Table 4: The sensitivity of optimal reliable facility quantity to q (0.01, 0.5) and r (1.25, 2).

n_R^*	q=0.01	q=0.1	q=0.2	q=0.3	q=0.4	q=0.5
r =1.25	31	145	230	301	365	423
r =1.5	20	91	145	190	230	267
r =1.75	15	70	111	145	175	204
r =2	12	57	91	120	145	168

Consider a demand region G is 150,000 km^2 which uniformly distributed with a density of ρ= 300; the setup costs of the reliable and unreliable facilities are f_R = 1,500,000 and f_U = 1,000,000 respectively; the per-unit transportation costs c = 0.75; and the facilities disruption probability q = 0.05.

Therefore, the optimal numerical solution is shown in Table 1.

It should be highlighted that the disruption probability is usually assumed accurately estimated in the literature on reliable facility location but lacked the discussion on misestimating the disruption probability. However, as discussed above, the business environmental uncertainty or misestimating the disruption probability would significantly impact the optimal solution. Therefore, this paper argues that it is essential to analyze the impact of misestimating the disruption probability to improve the robustness of the model.

4.2. Sensitivity Analysis on the Model. Firstly, sensitivity analysis on the model, i.e., the sensitivity of the optimal solution n_R^*, n_T^* to the disruption probability q is addressed.

Based on formula (8), $n_R^* = (\psi q/2(f_R - f_U))^{2/3}$, $n_T^* = (\psi(1-q)/2f_U)^{2/3}$ and their derivations of q are as follows: $dn_R^*/dq = (2/3)(\psi/2(f_R - f_U))^{2/3}q^{-1/3}$, $dn_T^*/dq = -(2/3)(\psi/2f_U)^{2/3}(1-q)^{-1/3}$.

Obviously, $dn_R^*/dq > 0$, $dn_T^*/dq < 0$. The sensitivity of n_R^*, n_T^* to q depends on the value of f_R, f_U, and q. Then the sensitivity analysis of the numerical example is shown in Table 2.

The result indicates that when q is small (e.g., $q \in (0, 0.15]$), n_R^* is relatively sensitive to q, while n_T^* is not sensitive to q. In consequence, when q is small, the total quantity of facilities is relatively stable and the quantity of reliable facilities will vary with q significantly.

Then there is the further investigation on the key parameters. This paper will discuss how the disruption probability q and the facility hardening cost factor r impact the expected total cost and the quantities of the two kinds of facilities, i.e., keeping f_U and c fixed as usual, changing q from 0.01 to 0.5, and changing r from 1.25 to 2 (1.25, 1.5, 1.75, 2). The results are shown in Tables 3–5.

Table 3 illustrates the changing of the expected total cost with q and r. It is obvious that when q is small, the amount of

Table 5: The sensitivity of optimal unreliable facility quantity to q (0.01, 0.5) and r (1.25, 2).

n_U^*	q=0.01	q=0.1	q=0.2	q=0.3	q=0.4	q=0.5
r =1.25	234	104	0	0	0	0
r =1.5	245	157	85	21	0	0
r =1.75	250	179	119	65	14	0
r =2	253	191	139	91	45	0

Note: both figures in Tables 4 and 5 have been rounded.

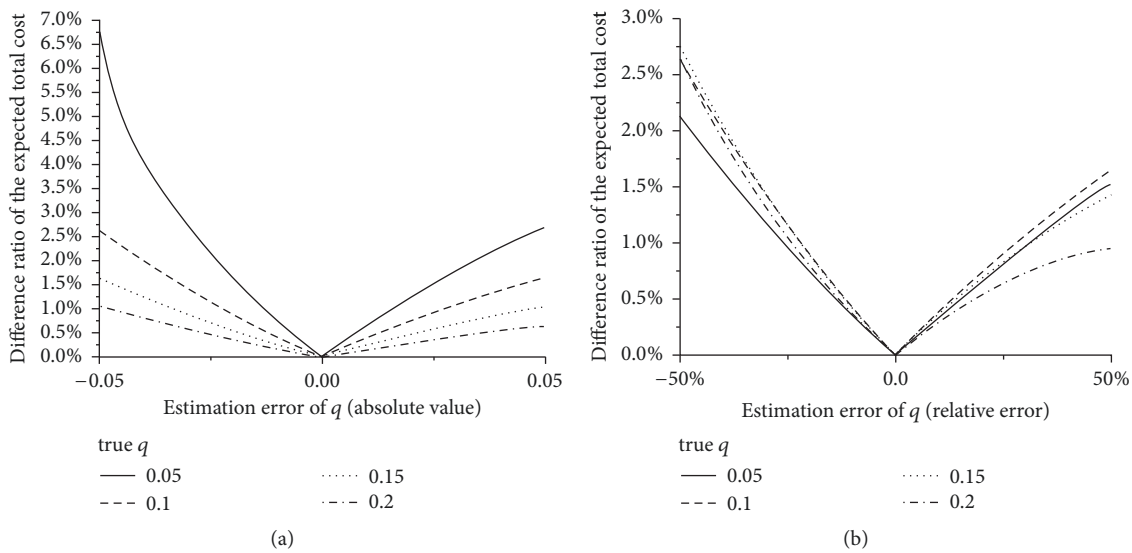

Figure 6: (a) The estimation error of q and the difference ratio of the expected total cost (absolute error); (b) the estimation error of q and the difference ratio of the expected total cost (relative error).

facilities is relatively sensitive to q; when q is large, all facilities should be hardened to reliable facilities. So, the value of $\mathrm{E}[TC]$ will keep stable when q increases to a certain extent, and the value of $\mathrm{E}[TC]$ will also increase with r.

Tables 4 and 5 illustrate the relationship of the facility quantities with q and r. As Table 4 shows, the quantity of reliable facility will increase with q, while it will decrease with r. Meanwhile, Table 5 indicates the quantity of unreliable facility will decrease with q, while it will decrease with r. More specific, all the unreliable facilities will be upgraded to reliable ones when q increases to a certain degree.

In other words, Tables 4 and 5 show that both kinds of facilities will be deployed when the disruption probability is lower than the threshold q_{th}. Nevertheless, when the disruption probability is higher than the threshold q_{th}, only the reliable facilities will be deployed. It indicates that misestimating disruption probability will considerably impact the supply chain network design, especially when the disruption probability is small. Therefore, in the following section, how the misestimation of disruption probability affects the expected total cost will be studied.

4.3. Robustness Analysis on the Model. The robustness analysis aims to investigate the impact of misestimating the disruption probability on the optimal solution. Due to the lack of historical data on disruptions and the continuous emerging of new disruption incidents, it is almost impossible to estimate the disruption probability accurately in practice. Therefore, robustness analysis on the model should be introduced to address the misestimation of disruption probability. In addition, the fault-tolerant ability of the model should also be investigated by robustness analysis to examine the practicability of the model.

Since the misestimation of disruption probability affects the expected total cost, the absolute and relative difference of the expected total cost when r=1.5 (Figure 6) will be discussed first, where q is the true disruption probability and $\hat{q}$ is the estimated disruption probability, so $\hat{q} - q$ is the estimation error and $(\hat{q}/q - 1) \times 100\%$ is the estimation error rate of q (as shown in Figure 6).

Comparing Figures 6(a) and 6(b), although the misestimation of disruption probability will affect the supply chain network design significantly, the estimation error of q would have a slight impact on the expected total cost. For instance, when we estimate the disruption probability $\hat{q}$= 0.001 while the true disruption probability is q=0.05, although it will dramatically change the optimal quantity of facilities, the impact on the expected total cost is less than 7%. And as shown in Figure 6(b), when the relative estimation error is 50%, its impact on the expected total cost is even less than

Table 6: The different ratio of expected total cost caused by estimation error of q (%).

	True q	0.05		0.1			0.15		0.2		
	Estimated q	0.01	0.1	0.05	0.15	0.2	0.1	0.2	0.1	0.15	0.3
Difference ratio of expected total cost	r=1.25	2.85	1.54	1.52	0.58	0.82	0.58	0.23	0.81	0.23	0.00
	r=1.5	4.19	2.67	2.60	1.59	2.72	1.56	1.11	2.65	1.10	0.99
	r=1.75	4.93	3.56	3.43	2.21	3.88	2.17	1.63	3.73	1.60	2.03
	r=2	5.67	4.08	3.92	2.83	4.90	2.75	2.01	4.67	1.97	2.80
	r=3	7.47	5.82	5.50	4.15	7.48	3.99	3.19	6.96	3.09	4.64

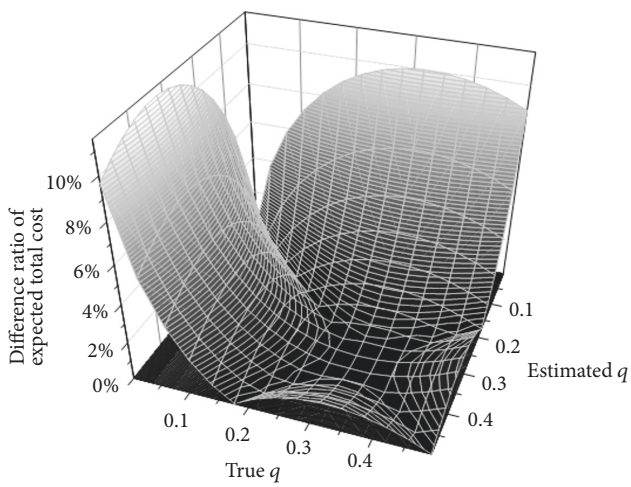

Figure 7: The misestimation of disruption probability q and the difference ratio of expected total cost.

3%. And the different combinations of q and $\hat{q}$ are showed in Table 6 in which the expected total cost will significantly change with r.

The above analysis indicates that although the estimated disruption probability has a substantial impact on the facilities configuration, the impact on the total amount of facilities and the expected total cost is very minor; i.e., it shows strong robustness of this model. Furthermore, according to Figures 6(a) and 6(b), the impact of misestimation of q is relatively large while the true q is small. On the contrary, the impact will be reduced with the increase of q. When $q > q_{th}$, e.g., q_{th} =0.2, the changes of q will not impact the total cost anymore since all the facilities have been hardened to be reliable ones.

In addition, although the above discussion indicates the robustness of the model to the misestimation of q, the misestimation interval will increase with the disruption probability. Therefore, it is particularly necessary to study the solving performance of the model in the worst-case setting. As shown in Figure 7, the difference ratio of expected total cost will be less than 12% when r=1.25 and the misestimation of disruption probability is $\hat{q} \in [0, 0.5]$. The bottom diagonal means there is no misestimation error.

Furthermore, based on the comparison of Figures 6 and 7, this paper finds that the expected total cost difference caused by the underestimation of disruption probability is considerably higher than that of the equal overestimation. Therefore, this finding supports the viewpoint of this study: the supply chain designers should moderately overestimate the facility disruption risk based on the "pessimistic principle". Although this policy will make the number of reliable facilities increase to a certain extent, the total number of facilities is relatively stable.

5. Discussion and Conclusion

This paper investigates a continuous reliable facility location problem under facility disruption risk and presents a stylized reliable facility location model in a continuous plane. Drawing from Honeycomb conjecture, this paper proposes a new research approach that divides the target service area as the hexagonal grid to deal with spatial continuous facility location problem. This paper argues that it is an optimal scheme in addressing continuous facility location problem and an improvement compared to previous research. In addition, considering the disruption probability estimation error, the impact of misestimating the disruption probability on optimal decisions is also examined by sensitivity analysis and robustness analysis in this paper. Based on the findings, some managerial insights are summarized for supply chain designers.

(1) When the disruption probability q is large enough, the supply chain designers should upgrade all the facilities to the reliable ones with additional hardening investment. Contrarily, when the disruption probability q is small, both types of facilities should be deployed. So, the disruption probability threshold q_{th} is derived. It indicates that q_{th} is only related to the cost value of the two types of facilities, but not the disruption probability q.

(2) When the disruption probability q is relatively small, the expected total cost is not sensitive to the changes of disruption probability. Therefore, the misestimation to a certain extent is acceptable. More specifically, only if the unreliable facility hardening cost factor r and the estimation error of disruption probability are not too large, the misestimation of disruption probability just influences the expected total cost slightly. Thus, the validity of the model is further proved.

(3) The expected total cost difference caused by the underestimation of disruption probability is considerably higher than that of the equal overestimation. It means the supply chain designers should moderately overestimate the disruption risk based on the "pessimistic principle". Therefore, taking all the demands are fully satisfied as a prerequisite, i.e., the supply chain undisrupted as a whole, although this policy will make the number of reliable facilities

increases to a certain extent, the total number of facilities is relatively stable. This principle is more practicable and meaningful in actual decision-making.

More specific, this study is an extension of our previous work [41] which left some questions unaddressed. To bridge the gaps, this study carried on a thorough and comprehensive investigation, and the primary differences between two works are summarized as follows: (1) In the previous study, there is a lacking systematic theoretical discussion and literature review about the facility location problem. Without this knowledge, it is difficult to understand the theoretical connections and evolutionary processes about the location problem from the classical Weber problem (1909) [20, 21] to the topic of this paper, i.e., continuous location problem. To our knowledge, at this point, it is understudied in academia. To address this call, this paper presents a detailed theoretical discussion about the classification on location problems (e.g., [18, 24–29]) and outlines a clear logic about it in Figure 1. This part of work provides strong theoretical supports for our study. (2) To make the study more rigorous, in part 3, we add the discussion about the Euclidean distance, Manhattan distance, and Great circle distance which are shown in Figure 4. This adding and improvement lays a strong research basis to this study. (3) This research proposes a continuous approximation approach based on regular hexagon partition to address the reliable facility location problems under disruptions risk. As discussed in this paper, the partition of target demand plane in continuous facility location problem is one of the key factors that affect the optimal solution. Based on the conclusions in the field of mathematics, this paper argues that the hexagonal partition is optimal scheme in addressing continuous facility location problem which is an improvement. To make the approach more reliable and more scientific, this paper discussed its optimality and literature basis. According to the "Honeycomb conjecture" [35, 36] and Circle-packing theorem [37, 38], hexagonal partition is an optimal scheme to divide the target plane into regions of equal area (as shown in Figure 3). Furthermore, Drezner and Zemel [39] and Trietsch [40], respectively, proved that hexagonal partition is significantly better than square partition which is better than triangular partition. Nevertheless, there is lacking appropriate discussion on this issue in our last paper as an unaddressed problem. (4) In this paper, we further studied the sensitivity of expected total cost, reliable facility quantity, and unreliable facility quantity by adding Tables 3, 4, and 5 to describe their changings with different q and r. Furthermore, in order to make the analysis more comprehensive, we consider more situations with different parameter. Based on these outcomes we comprehensively describe the different ratio of expected total cost caused by estimation error of q and present the corresponding result interpretation. Comparing with the last paper, this is a remarkable improvement and makes the analysis of this paper more sufficient.

Our findings bring up new questions and motivate promising future research. First, partial failure of a facility that makes a facility partially reliable is not considered in this paper. Second, this paper supposes the disruption probability of unreliable facilities is random and independent. Last, capacity limits are not considered in this paper based on the classical UFLP. These questions are beyond the scope of our paper and are left for future research.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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