Research Article

Dynamics and Optimal Harvesting Control for a Stochastic One-Predator-Two-Prey Time Delay System with Jumps

Tingting Ma,1 Xinzhu Meng,1,2 and Zhengbo Chang1

1College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China
2State Key Laboratory of Mining Disaster Prevention and Control Cofounded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Xinzhu Meng; mxz721106@sdust.edu.cn

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We consider a stochastic one-predator-two-prey harvesting model with time delays and Lévy jumps in this paper. Using the comparison theorem of stochastic differential equations and asymptotic approaches, sufficient conditions for persistence in mean and extinction of three species are derived. By analyzing the asymptotic invariant distribution, we study the variation of the persistent level of a population. Then we obtain the conditions of global attractivity and stability in distribution. Furthermore, making use of Hessian matrix method and optimal harvesting theory of differential equations, the explicit forms of optimal harvesting effort and maximum expectation of sustainable yield are obtained. Some numerical simulations are given to illustrate the theoretical results.

1. Introduction

Many researchers are widely focused on the complex dynamics of biological systems such as delay population systems [1–3], stochastic population systems [4–12], and impulsive population systems [13–15]. Recently, many scholars have investigated two-species models and studied the extinction and persistence [16–18]. However, in the real world, it is a common phenomenon that one predator catches two or more kinds of preys [19]. Consequently, models with three or more species which can explain the dynamical behaviors of the population accurately are investigated [20, 21]. On the other hand, it is necessary and important to consider time delay caused by the competition and predation of species. Delayed differential equations can exhibit much more complex dynamics than differential equations without delay, and stable equilibrium can become unstable with the effects of a time delay. Therefore, many researchers have studied the Lotka-Volterra time delay models with two competitive preys and one predator [22, 23]. Notice that the composite population systems with stochastic effects and time delays present some complex dynamics; thus this causes widespread researchers concern [24–30].

In order to rationalize the model, some coefficients should be modified into the existing models. The one-predator-two-prey model with time delays is described by

\[
\begin{align*}
\frac{dX_1}{dt} &= X_1(t) \left[a_1 - a_{11}X_1(t) - a_{12}X_2(t - \tau_{12})
- a_{13}X_3(t - \tau_{13})\right], \\
\frac{dX_2}{dt} &= X_2(t) \left[a_2 - a_{21}X_1(t - \tau_{21}) - a_{22}X_2(t)
- a_{23}X_3(t - \tau_{23})\right], \\
\frac{dX_3}{dt} &= X_3(t) \left[-a_3 + a_{31}e^{-a_{31}\tau_{31}}X_1(t - \tau_{31})
+ a_{32}e^{-a_{32}\tau_{32}}X_2(t - \tau_{32}) - a_{33}X_3(t)\right],
\end{align*}
\]

with initial value

\[
X_i(\theta) = \varphi_i(\theta), \quad \varphi_i(\theta) > 0, \quad \varphi_i(\theta) \in C[-\tau_0, 0],
\]

where \(X_i(t)\) denotes the size of the \(i\)-th species of the prey at time \(t\) \((i = 1, 2)\), \(X_3(t)\) denotes the size of the predator at

\begin{align*}
\end{align*}
time \( t, \tau_{ij} \geq 0 \) is the time delay, \( i, j = 1, 2, 3, \theta \in [-\tau_0, 0] \), \( \tau_0 = \max\{\tau_{ij}\} \), \( a_i > 0 \) \( i=1, 2, 3 \), and \( a_i \) \( i=1, 2 \) stands for the growth rates of two preys, and \( a_3 \) stands for the death rate of the predator, respectively. \( a_i > 0 \) is the intransitive competition rate of species \( i, i = 1, 2, 3 \). \( a_{31}e^{-\alpha_{t_1}} > 0 \) and \( a_{32}e^{-\alpha_{t_2}} > 0 \) are the efficiency of food conversion. \( a_{12} \) and \( a_{21} \) are the interspecific competition rates between 1 and 2, and \( a_{13} \) and \( a_{23} \) are the capture rates.

Many systems may suffer environment perturbation, the growth rate \( a_1 \) is affected by the white noise [31, 32], \( B = (B(t), t \geq 0) \) is a real-valued Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, P)\) with \( B(0) = 0 \).

\[
\hat{a}_i = a_i + \sigma_i dB_i(t), \quad i = 1, 2, 3. \tag{3}
\]

\( \sigma_i > 0 \) \( i = 1, 2, 3 \) denotes the coefficients of the environmental stochastic effects on the preys and the predator populations, respectively.

Generally, the dynamical behavior of the species may suffer from the sudden environmental change significantly. Nevertheless, white noise cannot explain huge, occasionally catastrophic disturbances. Therefore, applying the discontinuous stochastic process as \( \text{Lévy jump} \) to model the abrupt nature phenomenon in ecosystem is necessary [33–35]. In many cases, populations suffer sudden distribution, and this causes changes in the productivity of marine and freshwater species. How to use the discontinuous stochastic process to study these abrupt nature phenomena has been an interesting topic.

Recently, some scholars have applied \( \text{Lévy jump} \) into their models and showed that \( \text{Lévy jump} \) could describe sudden random environmental perturbations. According to the \( \text{Lévy} \) decomposition theorem [36], we get that

\[
\tilde{N}(t, dv) = N(t, dv) - \lambda(dv)t \tag{4}
\]

\( \tilde{N}(t, dv) \) is a compensated Poisson process and \( N \) is a Poisson counting measure with characteristic measure \( \lambda \) on a measurable subset \( \mathbb{Y} \) of \( (0, +\infty) \) with \( \lambda(\mathbb{Y}) < \infty \). The distribution of \( \text{Lévy jump} \) \( L_i(t) \) can be completely parameterized by \( (a_i, \sigma_i, \lambda) \) and satisfies the property of infinite divisibility. It is characterized by its characteristic function \( \Phi_{L_i(t)} \); we can get a detailed explanation from the following \( \text{Lévy-Khintchine formula} \) [37]. There are many other papers about stochastic models with \( \text{Lévy jump} \); the readers could refer to [38, 39] and references therein. Considering the inevitable situations in the real world, we assume that the intrinsic growth rates \( a_1 \) and \( a_2 \) and the death rate \( a_3 \) of the model are perturbed by the \( \text{Lévy jump} \) to signify the sudden climate change, so we introduce the \( \text{Lévy jump} \) into the underlying stochastic model (1). Taking the economic factors into account, reasonable natural resources management can increase sustainable production and profits. Therefore, harvesting models have been already used to exploited the optimal harvesting policies of renewable resources [40–43]. We only consider to harvest preys \( x_1 \) and \( x_2 \) and \( h_1 > 0 \) and \( h_2 > 0 \) are the harvesting effort rates of \( x_1 \) and \( x_2 \), respectively [42, 43]. From system (1), we can obtain the following stochastic harvesting model with \( \text{Lévy jump} \):

\[
dx_1(t) = x_1(t) \left[ a_1 - h_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_{12}) - a_{13}x_3(t - \tau_{13}) \right] dt + \sigma_1 x_1(t) dB_1(t) + x_1(t) \cdot \int_\mathbb{Y} \gamma_1(v) N(dt, dv),
\]

\[
dx_2(t) = x_2(t) \left[ a_2 - h_2 - a_{21}x_1(t - \tau_{21}) - a_{22}x_2(t) - a_{23}x_3(t - \tau_{23}) \right] dt + \sigma_2 x_2(t) dB_2(t) + x_2(t) \cdot \int_\mathbb{Y} \gamma_2(v) N(dt, dv),
\]

\[
dx_3(t) = x_3(t) \left[ -a_3 + a_{31}e^{-\alpha_{t_1}}x_1(t - \tau_{31}) + a_{32}e^{-\alpha_{t_2}}x_2(t - \tau_{32}) - a_{33}x_3(t) \right] dt + \sigma_3 x_3(t) dB_3(t) + x_3(t) \cdot \int_\mathbb{Y} \gamma_3(v) N(dt, dv). \tag{5}
\]

This paper is organized as follows. In Section 1, we formulate the model. We show the solution of model (5) is global and positive. We give the main theorems for persistence in mean and extinction under the model (5) and its proof in Section 3. In Section 4, we prove the global attractivity and stability in distribution. Our main aim of this paper is to investigate the optimal strategy of the proposed model; we give the conclusions in Section 5. Finally, we carry out numerical simulations and some figures to support the main conclusions in Section 6.

2. Global Positive Solution

In order to explore the dynamical behaviors of ecological population, we first study the positivity of the solutions of system (5).

Lemma 1. There exists an positive constant \( M_{\text{max}} \); we have

\[
\ln(1 + \gamma_i(v)) \leq M_{\text{max}}, \quad i = 1, 2, 3. \tag{6}
\]

Lemma 2 (see [44]). We assume that, for each \( m > 0 \), there exists an \( L_p \) satisfying

\[
|D_1(x, y) - D_1(y, v)|^2 K_0 dy \leq L_m|x - y|^2, \text{ and } D_1(x, v) = y_i(x) \quad i = 1, 2, 3 \text{ with } |x| \wedge |y| \leq m.
\]

Lemma 3. For any given initial value \( p(\theta) = (p_1(\theta), p_2(\theta), p_3(\theta)) \in C([-\tau_0, 0], \mathbb{R}_+^3) \), model (5) has a unique global positive solution \( x(t) \) on \( t \geq 0 \) a.s. Moreover, there is a positive constant \( K \) such that

\[
\lim_{t \to \infty} E(x_1(t)) \leq K, \quad i = 1, 2, 3. \tag{7}
\]

Proof. For any given initial value \( (x_1(0), x_2(0), x_3(0)) \), there is a unique positive \( (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}_+^3 \) for \( t \in [0, \tau_e] \), where \( \tau_e \) is the explosion time. We verify that the positive solution is global, that is \( \tau_e = \infty \) as. Let \( m_0 \geq 0 \) be sufficiently large so \( x_1(t), x_2(t), \) and \( x_3(t) \) all lie within the interval
Evidently, $\tau_\epsilon$ is strictly increasing when $m \to \infty$. Let $\tau_{\infty} = \lim_{m \to \infty} \tau_m$; thus $\tau_{\infty} \leq \tau_m$ as. If this statement is not true, then there exist pairs of constants $T > 0$, $m_1 \geq m_0$, and $0 < \epsilon < 1$ such that

$$P(\tau_\infty \leq T) \leq \epsilon,$$

where $K$ is a positive constant,

$$M_{\max} = \max \left\{ \int_\gamma [a\gamma_1 (\nu) - a\ln (1 + \gamma_1 (\nu))], \int_\gamma [b\gamma_2 (\nu) - b\ln (1 + \gamma_2 (\nu))], \int_\gamma [c\gamma_3 (\nu) - c\ln (1 + \gamma_3 (\nu))] \right\}.$$

Thus,

$$dV \leq K dt + \sigma_1 (x_1 - a) dB_1 (t) + \sigma_2 (x_2 - b) dB_2 (t)$$

$$+ \sigma_3 (x_3 - c) dB_3 (t) + \int_\gamma [\gamma_1 (\nu) x_1 (t-) - a\ln (1 + \gamma_1 (\nu)) x_1 (t)] + \int_\gamma [\gamma_2 (\nu) x_2 (t-) - b\ln (1 + \gamma_2 (\nu)) x_2 (t)] + \int_\gamma [\gamma_3 (\nu) x_3 (t-) - c\ln (1 + \gamma_3 (\nu)) x_3 (t)] + K dt.$$

Taking expectation, yields

$$EV (x_1 (\tau_m \wedge T), x_2 (\tau_m \wedge T), x_3 (\tau_m \wedge T)) \leq V (x_1 (0), x_2 (0), x_3 (0)) + E \int_0^{\tau_m \wedge T} K dt.$$

Set $\Omega_m = \tau_m \wedge T$ for $m \geq m_0$ and we can obtain $P(\Omega_m) \geq \epsilon$. For each $\omega \in \Omega_m$, there are $x_1 (\tau_m, \omega), x_2 (\tau_m, \omega), x_3 (\tau_m, \omega)$ equaling either $m$ or $1/m$ and yields

$$V (x_1 (0), x_2 (0), x_3 (0)) + KT$$

$$\geq E \left[ 1_{\Omega_m (\omega)} V (x_1 (\tau_m, \omega), x_2 (\tau_m, \omega), x_3 (\tau_m, \omega)) \right]$$

$$\geq \epsilon \left[ \left( \frac{1}{m} - 1 - \ln \frac{1}{m} \right) \wedge (m - 1 - \ln m) \right],$$

where $1_{\Omega_m (\omega)}$ stands for the indicator function of $\Omega_m (\omega)$.

Let $m \to \infty$, which implies

$$\infty > V (x_1 (0), x_2 (0), x_3 (0)) + KT = \infty$$

is a contradiction. So, we have that $\tau_\infty = \infty$. This completes the proof of Lemma 1.

3. Persistence in Mean and Extinction of the Model

For the sake of convenience, we introduce some following notations. Let $R^+_i = \{ a = (a_1, a_2, a_3) \in R^3 \mid a_i > 0, i = 1, 2, 3 \}$ and $C([-\tau_0, 0], R^3)$ stand for all continuous
functions from \([-\tau_0, 0]\) to \(R^3\). Additionally, we give some notations
\[
\eta_i = \int_{-\tau_0}^{0} \{ y_i (v) - \ln (1 + y_i (v)) \} \lambda (dv),
\]
\[
b_i = a_i - h_i - \frac{1}{2} \sigma_i^2 - \eta_i, \quad i = 1, 2,
\]
\[
b_3 = a_i + \frac{1}{2} \sigma_i^2 - \eta_3,
\]
\[
M_i(t) = \int_{-\tau_0}^{t} \ln (1 + y_i (v)) \bar{N} (ds, dv),
\]
\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
-a_{31} e^{-\sigma_1 \tau_1} - a_{32} e^{-\sigma_2 \tau_2} & a_{33}
\end{bmatrix},
\]
\[
A_1 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
-a_{31} e^{-\sigma_1 \tau_1} & a_{33}
\end{bmatrix},
\]
\[
A_2 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
-a_{31} e^{-\sigma_1 \tau_1} & -a_3 & a_{33}
\end{bmatrix},
\]
\[
A_3 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
-a_{31} e^{-\sigma_1 \tau_1} & -a_{32} e^{-\sigma_2 \tau_2} & -a_3
\end{bmatrix},
\]
\[
\bar{A}_1 = \begin{bmatrix}
\frac{1}{2} \sigma_1^2 & a_{12} & a_{13} \\
\frac{1}{2} \sigma_2^2 & a_{22} & a_{23} \\
\frac{1}{2} \sigma_3^2 & -a_{32} e^{-\sigma_2 \tau_2} & -a_{33}
\end{bmatrix},
\]
\[
\bar{A}_2 = \begin{bmatrix}
\frac{1}{2} \sigma_1^2 & a_{12} & a_{13} \\
\frac{1}{2} \sigma_2^2 & a_{22} & a_{23} \\
\frac{1}{2} \sigma_3^2 & a_{33}
\end{bmatrix},
\]
\[
\bar{A}_3 = \begin{bmatrix}
\frac{1}{2} \sigma_1^2 & a_{12} & a_{13} \\
\frac{1}{2} \sigma_2^2 & a_{22} & a_{23} \\
\frac{1}{2} \sigma_3^2 & a_{33}
\end{bmatrix},
\]
\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
-a_{31} e^{-\sigma_1 \tau_1} & -a_{32} e^{-\sigma_2 \tau_2} & -a_3
\end{bmatrix},
\]
\[
\Gamma_2 = a_{11} (a_2 - h_2) - a_{12} (a_1 - h_1),
\]
\[
\Gamma_3 = a_{11} e^{-\sigma_1 \tau_1} (a_1 - h_1) - a_{12} e^{-\sigma_2 \tau_1} (a_1 - h_1),
\]
\[
\Gamma_1 = a_{11} (a_2 - h_2) - a_{12} (a_1 - h_1),
\]
\[
\bar{\Gamma}_1 = \frac{1}{2} a_{21} \sigma_1^2 - \frac{1}{2} a_{11} \sigma_1^2,
\]
\[
\bar{\Gamma}_2 = \frac{1}{2} a_{12} \sigma_2^2 - \frac{1}{2} a_{22} \sigma_2^2,
\]
\[
\bar{\Gamma}_3 = \frac{1}{2} a_{13} \sigma_3^2 + \frac{1}{2} a_{33} \sigma_3^2.
\]

\[
dN_1(t) = \left[ a_1 - h_1 - \frac{1}{2} \sigma_1^2 - a_{11} e^{N_1(t)} - a_{12} e^{N_1(t-\tau_1)} - a_{13} e^{N_1(t-\tau_2)} - \int_{-\tau_0}^{t} \{ y_1 - \ln (1 + y_1) \} \lambda (dv) \right] dt
\]
\[
+ \sigma_1 dB_1(t) + \int_{-\tau_0}^{t} \ln (1 + y_1) \bar{N} (dr, dv),
\]
\[
dN_2(t) = \left[ a_2 - h_2 - \frac{1}{2} \sigma_2^2 - a_{21} e^{N_2(t)} - a_{22} e^{N_2(t-\tau_2)} - a_{23} e^{N_2(t-\tau_3)} - \int_{-\tau_0}^{t} \{ y_2 - \ln (1 + y_2) \} \lambda (dv) \right] dt
\]
\[
+ \sigma_2 dB_2(t) + \int_{-\tau_0}^{t} \ln (1 + y_2) \bar{N} (dr, dv),
\]
\[
dN_3(t) = \left[ a_3 - \frac{1}{2} \sigma_3^2 - a_{31} e^{N_3(t)} - a_{32} e^{N_3(t-\tau_2)} - a_{33} e^{N_3(t-\tau_3)} + \int_{-\tau_0}^{t} \{ y_3 - \ln (1 + y_3) \} \lambda (dv) \right] dt + \sigma_3 dB_3(t)
\]
\[
+ \int_{-\tau_0}^{t} \ln (1 + y_3) \bar{N} (dr, dv),
\]

with initial value \(N_i(0) = \ln \varphi_{i0}\). It is not difficult to know that the coefficients of model (5) satisfy the local Lipschitz condition; therefore, model (5) has a unique local solution \(x(t)\) on \([0, \tau_e]\), where \(\tau_e\) is the explosion time. Applying Itô’s formula, we can get the following solution:

\[
x(t) = \left( x_1(t) = e^{N_1(t)}, x_2(t) = e^{N_2(t)}, x_3(t) = e^{N_3(t)} \right)^T.
\]

It is the unique positive local solution to model (5). Now let us prove \(\tau_e = +\infty\). Thus, we introduce the following auxiliary model:

\[
dY_1(t) = Y_1(t) \left[ a_1 - h_1 - a_{11} Y_1(t) \right] dt
\]
\[
+ \sigma_1 Y_1(t) dB_1(t) + Y_1(t) \int_{-\tau_0}^{t} \{ y_1 - \ln (1 + y_1) \} \lambda (dv).
\]
\[
dY_2(t) = Y_2(t) \left[ a_2 - h_2 - a_{22} Y_2(t) \right] dt + \sigma_{Y_2}(t) dB_2(t) + Y_2(t)
\]
\[
\cdot \int_{t}^{\infty} Y_3(v) \tilde{N}(dt, dv),
\]
\[
dY_3(t) = Y_3(t) \left[ -a_3 + a_{31} e^{-a_{31} T_1} Y_1(t - T_3) + a_{32} e^{-a_{32} T_3} Y_1(t - T_3) \right] dt + \sigma_{Y_3}(t) dB_3(t) + Y_3(t)
\]
\[
\cdot \int_{t}^{\infty} Y_3(v) \tilde{N}(dt, dv),
\]

(20)

with initial value
\[
Y_i(\theta) = \phi_i(\theta), \quad i = 1, 2, 3, \quad \theta \in [-\tau_0, 0].
\]

(21)

Obviously
\[
x_i(\theta) \leq \phi_i(\theta), \quad i = 1, 2, 3.
\]

(22)

Before starting proving, we state several hypotheses. We assume that
\[
\frac{1}{2} \sigma_{a_i}^2 + \eta_i > \frac{1}{2} \sigma_{a_i}^2 + \eta_2,
\]

(23)

which implies that the persistent ability of species 1 is better than that of species 2.

Assumption 4. \( A > 0, A_1 > 0, i = 1, 2, 3 \) express that all the population could coexist under the condition that the model frees from stochastic noises. \( A_{32} > 0, \Gamma_i > 0, i = 1, 2 \), which imply that the two prey populations could coexist when there is no environmental noises and the predators are absent, where \( A_{ij} \) is the complement minor of \( a_{ij} \) in the determinant \( A \), \( i, j = 1, 2, 3 \).

Assumption 5. \( \bar{A}_i > 0, A_{23} > 0, A_{31} < 0, A_{32} < 0 \).

Assumption 6. \( a_{11} > a_{12} + a_{13}, a_{22} > a_{23} + a_{33}, a_{33} > a_{31} e^{-a_{31} T_1} + a_{32} e^{-a_{32} T_3} \).

Lemma 7 (see [26, 33]). Let \( \psi \in C[\Omega \times [0, +\infty), R_+] \).

(i) If there exist three constants \( \lambda, \lambda_0 > 0 \), and \( T \geq 0 \), such that
\[
\lg \psi(t) \leq \lambda t - \lambda_0 \int_{0}^{t} \psi(s) ds + F(t), \quad \text{a.s.}
\]

(24)

for all \( t \geq 0, F \in [[0, \infty) \times \Omega, \mathbb{R}] \) and \( \lim_{t \to \infty} F(t)/t = 0 \) as.

Then, for all \( t \geq T \), where \( \alpha \) and \( \delta \) are constants, then
\[
\langle \psi(t) \rangle^\alpha \leq \frac{\lambda}{\lambda_0} \quad \text{a.s., if } \lambda \geq 0,
\]
\[
\lim_{t \to \infty} \psi(t) = 0 \quad \text{a.s., if } \lambda \leq 0.
\]

(25)

(ii) If there exist three constants \( \lambda, \lambda_0, \) and \( T \geq 0 \), such that
\[
\lg \psi(t) \geq \lambda t - \lambda_0 \int_{0}^{t} \psi(s) ds + F(t), \quad \text{a.s.}
\]

(26)

for all \( t \geq 0, F \in [[0, \infty) \times \Omega, \mathbb{R}] \) and \( \lim_{t \to \infty} F(t)/t = 0 \) as.

Then, for all \( t \geq T \), then \( \langle \psi(t) \rangle^\alpha \geq \lambda/\lambda_0, \) a.s.

Lemma 8. For arbitrarily \( \tau \geq 0 \),
\[
\lim_{t \to \infty} \frac{1}{t} \int_{t-\tau}^{t} Y_i(s) ds = 0, \quad \text{a.s., } i = 1, 2.
\]

(27)

Proof.
\[
\lim_{t \to \infty} \frac{1}{t} \int_{t-\tau}^{t} Y_i(s) ds
\]
\[
= \lim_{t \to \infty} \frac{1}{t} \left( \int_{t}^{t} Y_i(s) ds - \int_{0}^{t-\tau} Y_i(s) ds \right) = 0.
\]

(29)

If \( b_1 \geq 0 \), then
\[
\lim_{t \to \infty} \frac{1}{t} \int_{t-\tau}^{t} Y_i(s) ds
\]
\[
= \lim_{t \to \infty} \frac{1}{t} \left( \int_{t}^{t} Y_i(s) ds - \int_{0}^{t-\tau} Y_i(s) ds \right)
\]
\[
= \frac{b_1}{a_i} - \frac{b_1}{a_i} = 0.
\]

(30)

Lemma 9. For model (20), we have the following.

(a) If \( b_1, b_2 < 0 \), then
\[
\lim_{t \to \infty} Y_i(t) = 0, \quad i = 1, 2, 3, \quad \text{a.s.}
\]

(31)

(b) If \( b_1 \geq 0, b_2 < 0 \) and \( b_2 - a_{31} e^{-a_{31} T_1} b_1/a_{11} > 0 \), then
\[
\lim_{t \to \infty} \langle Y_1(t) \rangle = \frac{b_1}{a_{11}},
\]
\[
\lim_{t \to \infty} \langle Y_2(t) \rangle = 0,
\]
\[
\lim_{t \to \infty} \langle Y_3(t) \rangle = 0,
\]

(32)

a.s.

(c) If \( b_1 \geq 0, b_2 < 0 \) and \( b_2 - a_{31} e^{-a_{31} T_1} b_1/a_{11} < 0 \), then
\[
\lim_{t \to \infty} \langle Y_1(t) \rangle = \frac{b_1}{a_{11}},
\]
\[
\lim_{t \to \infty} Y_2(t) = 0,
\]
\[
\lim_{t \to \infty} Y_3(t) = 0.
\]
\[
\begin{align*}
\lim_{t \to +\infty} Y_1(t) &= \frac{a_{31}e^{-a_{12}t}b_1 - a_{11}b_3}{a_{11}a_{33}}, &\text{a.s.} \\
\lim_{t \to +\infty} Y_2(t) &= \frac{b_2}{a_{22}}, &\lim_{t \to +\infty} Y_3(t) &= 0, &\text{a.s.} \\
\end{align*}
\]

(d) If \( b_1 < 0, b_2 \geq 0 \) and \( b_3 - a_{32}e^{-a_{21}t}b_2/a_{22} > 0 \), then
\[
\begin{align*}
\lim_{t \to +\infty} \langle Y_1(t) \rangle &= 0, \\
\lim_{t \to +\infty} \langle Y_2(t) \rangle &= \frac{b_2}{a_{22}}, \\
\lim_{t \to +\infty} \langle Y_3(t) \rangle &= 0, &\text{a.s.}
\end{align*}
\]

(e) If \( b_1 \geq 0, b_2 \geq 0 \) and \( b_3 - a_{32}e^{-a_{21}t}b_2/a_{22} \geq 0 \), then
\[
\begin{align*}
\lim_{t \to +\infty} \langle Y_1(t) \rangle &= 0, \\
\lim_{t \to +\infty} Y_2(t) &= \frac{b_2}{a_{22}}, \\
\lim_{t \to +\infty} \langle Y_3(t) \rangle &= 0, &\text{a.s.}
\end{align*}
\]

(f) If \( b_1 \geq 0, b_2 \geq 0 \) and \( b_3 - a_{32}e^{-a_{21}t}b_2/a_{22} - a_{31}e^{-a_{12}t_1}b_1/a_{11} < 0 \), then
\[
\begin{align*}
\lim_{t \to +\infty} \langle Y_1(t) \rangle &= \frac{b_1}{a_{11}}, \\
\lim_{t \to +\infty} \langle Y_2(t) \rangle &= \frac{b_2}{a_{22}}, \\
\lim_{t \to +\infty} \langle Y_3(t) \rangle &= 0, &\text{a.s.}
\end{align*}
\]

(g) If \( b_1 \geq 0, b_2 \geq 0 \) and \( b_3 - a_{32}e^{-a_{21}t}b_2/a_{22} - a_{31}e^{-a_{12}t_1}b_1/a_{11} < 0 \), then
\[
\begin{align*}
\lim_{t \to +\infty} \langle Y_1(t) \rangle &= \frac{b_1}{a_{11}}, \\
\lim_{t \to +\infty} \langle Y_2(t) \rangle &= \frac{b_2}{a_{22}}, \\
\lim_{t \to +\infty} \langle Y_3(t) \rangle &= \frac{a_{22}a_{31}e^{-a_{12}t_1}b_1 + a_{11}a_{33}e^{-a_{21}t}b_2 - a_{11}a_{22}b_3}{a_{11}a_{22}a_{33}}, &\text{a.s.}
\end{align*}
\]

Proof. First, let us prove (a). Applying Itô's formula to model (20), we can get that
\[
\begin{align*}
\ln Y_1(t) - \ln Y_1(0) &= b_1t - a_{11} \int_0^t Y_1(s) \, ds + \sigma_1 B_1(t) + M_1(t), \\
\ln Y_2(t) - \ln Y_2(0) &= b_2t - a_{22} \int_0^t Y_2(s) \, ds + \sigma_2 B_2(t) + M_2(t), \\
\ln Y_3(t) - \ln Y_3(0) &= -b_3t + a_{33} \int_0^t Y_3(s) \, ds + \sigma_3 B_3(t) + M_3(t).
\end{align*}
\]

Dividing both sides of (38), (39), and (40) by \( t \), we can obtain that
\[
\begin{align*}
t^{-1} \ln \frac{Y_1(t)}{Y_1(0)} &= b_1 - a_{11} \langle Y_1(t) \rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{M_1(t)}{t}, \\
t^{-1} \ln \frac{Y_2(t)}{Y_2(0)} &= b_2 - a_{22} \langle Y_2(t) \rangle + \frac{\sigma_2 B_2(t)}{t} + \frac{M_2(t)}{t}, \\
t^{-1} \ln \frac{Y_3(t)}{Y_3(0)} &= -b_3 + a_{33} \langle Y_3(t) \rangle + a_{32} \langle Y_2(t) \rangle - a_{31} \langle Y_1(t) \rangle \\
&\quad - \frac{\sigma_3 B_3(t)}{t} + \frac{M_3(t)}{t}.
\end{align*}
\]

Note that
\[
\lim_{t \to +\infty} \frac{B_i(t)}{t} = 0, \quad \text{a.s., } i = 1, 2, 3.
\]

Firstly, we prove (a). Since \( b_1 < 0, b_2 < 0 \), using Lemma 7, yields
\[
\lim_{t \to +\infty} Y_1(t) = 0, \quad \text{a.s.}
\]

and
\[
\lim_{t \to +\infty} Y_2(t) = 0, \quad \text{a.s.}
\]
Substituting (45) and (46) into (40) gives

\[
\ln Y_3 (t) - \ln Y_3 (0) \leq -b_3 t + \epsilon t - a_{33} \int_0^t Y_3 (s) \, ds + a_1 B_3 (t) + M_3 (t),
\]

where \( \epsilon \) is small enough satisfying \( b_3 + \epsilon < 0 \). Applying (i) in Lemma 7 gives

\[
\lim_{t \to +\infty} Y_3 (t) = 0, \quad a.s.
\]  

(48)

Secondly, we prove (b) and (c). Using Lemma 7, since \( b_1 \geq 0 \) and \( b_2 < 0 \), it is easy to obtain that

\[
\lim_{t \to +\infty} \langle Y_1 (t) \rangle = \frac{b_1}{a_{11}}, \quad a.s.
\]  

(49)

and

\[
\lim_{t \to +\infty} Y_2 (t) = 0, \quad a.s.
\]  

(50)

As a consequence, we can study and discuss the following model:

\[
dY_1 (t) = Y_1 (t) [a_1 - a_{11} Y_1 (t)] \, dt + \sigma_1 Y_1 (t) \, dB_1 (t) + \int_y y (v) \, N (dt, dv),
\]

\[
dY_3 (t) = Y_3 (t) [-a_3 + a_{31} e^{-a_{31} \tau_{31}} Y_1 (t - \tau_{31}) + a_{32} e^{-a_{32} \tau_{32}} Y_2 (t - \tau_{32}) - a_{33} Y_3 (t)] \, dt
\]

\[
+ \sigma_3 Y_3 (t) \, dB_3 (t) + Y_3 (t) \cdot \int_y y (v) \, N (dt, dv).
\]

We know that

\[
\ln Y_3 (t) - \ln Y_3 (0) = -b_3 t
\]

\[
- a_{31} \int_0^t Y_1 (s - \tau_{31}) \, ds
\]

\[
+ a_{31} e^{-a_{31} \tau_{31}} \int_0^t Y_1 (s) \, ds + a_3 B_3 (t) + M_3 (t)
\]

\[
\geq 0
\]

(52)

It is not difficult to obtain the system

\[
\lim_{t \to +\infty} Y_3 (t) = 0, \quad a.s., \text{ if } a_{31} e^{-a_{31} \tau_{31}} \frac{b_1}{a_{11}} < 0.
\]

\[
\lim_{t \to +\infty} \int_0^t Y_3 (t) \, ds = \frac{a_{32} e^{-a_{32} \tau_{32}} b_1 - a_{11} b_3}{a_{11} a_{33}},
\]

\[
\text{if } a_{31} e^{-a_{31} \tau_{31}} \frac{b_1}{a_{11}} > 0.
\]

(53)

The proofs of (d) and (e) are similar to these of (b) and (c).

Then, we give the proofs of (f) and (g). Using Lemma 7, since \( b_1 \geq 0 \) and \( b_2 \geq 0 \), it is easy to obtain that

\[
\lim_{t \to +\infty} \langle Y_1 (t) \rangle = \frac{b_1}{a_{11}}, \quad a.s.
\]  

(54)

and

\[
\lim_{t \to +\infty} \langle Y_2 (t) \rangle = \frac{b_2}{a_{22}}, \quad a.s.
\]  

(55)

Using (54) and (55), then combining (44) leads to

\[
\lim_{t \to +\infty} t^{-1} \ln Y_1 (0) = 0, \quad a.s.
\]  

(56)

and

\[
\lim_{t \to +\infty} t^{-1} \ln Y_2 (0) = 0, \quad a.s.
\]  

(57)

Multiplying (41), (42), and (43) by \( a_{31} e^{-a_{31} \tau_{31}}, a_{32} e^{-a_{32} \tau_{32}}, \) and \( a_{11} a_{22}, \) respectively, and adding them, we can derive that

\[
\frac{a_{31} e^{-a_{31} \tau_{31}} a_{22}}{t} \ln \frac{Y_1 (t)}{Y_0 (t)} + \frac{a_{11} e^{-a_{32} \tau_{32}} a_{32}}{t} \ln \frac{Y_2 (t)}{Y_0 (t)}
\]

\[
= a_{22} a_{31} e^{-a_{31} \tau_{31}} b_1 + a_{11} a_{32} e^{-a_{32} \tau_{32}} b_2 - a_{11} a_{22} b_3
\]

\[
- a_{11} a_{22} a_{33} \langle Y_3 (t) \rangle
\]

\[
= \frac{-a_{11} a_{22} a_{33} \langle Y_3 (t) \rangle}{t} \int_0^t Y_1 (s) \, ds - \int_{-\tau_{31}}^0 Y_1 (s) \, ds
\]

\[
- a_{11} a_{22} \frac{a_{31} e^{-a_{31} \tau_{31}}}{t} \int_0^t Y_2 (s) \, ds - \int_{-\tau_{32}}^0 Y_2 (s) \, ds
\]

\[
+ a_{22} a_{31} e^{-a_{31} \tau_{31}} \left[ \sigma_1 B_1 (t) + M_1 (t) \right]
\]

\[
+ a_{11} a_{32} e^{-a_{32} \tau_{32}} \left[ \sigma_2 B_2 (t) + M_2 (t) \right]
\]

\[
+ a_{11} a_{22} \left[ \sigma_3 B_3 (t) + M_3 (t) \right]
\]

\[
= \frac{-a_{11} a_{22} a_{33} \langle Y_3 (t) \rangle}{t} \int_0^t Y_1 (s) \, ds - \int_{-\tau_{31}}^0 Y_1 (s) \, ds
\]

\[
- a_{11} a_{22} \frac{a_{31} e^{-a_{31} \tau_{31}}}{t} \int_0^t Y_2 (s) \, ds - \int_{-\tau_{32}}^0 Y_2 (s) \, ds
\]

\[
+ a_{22} a_{31} e^{-a_{31} \tau_{31}} b_1 + a_{11} a_{32} e^{-a_{32} \tau_{32}} b_2 - a_{11} a_{22} b_3
\]

An application of (56), (57), (58), Lemma 7, and Lemma 8 gives that

\[
\lim_{t \to +\infty} Y_3 (t) = 0,
\]

\[
a.s., \text{ if } \frac{a_{31} e^{-a_{31} \tau_{31}} b_1}{a_{11}} + \frac{a_{32} e^{-a_{32} \tau_{32}} b_2}{a_{22}} - b_3 < 0
\]

(59)

\[
\lim_{t \to +\infty} \int_0^t Y_3 (t) \, ds = \frac{a_{22} a_{31} e^{-a_{31} \tau_{31}} b_1 + a_{11} a_{32} e^{-a_{32} \tau_{32}} b_2 - a_{11} a_{22} b_3}{a_{11} a_{22} a_{33}},
\]

\[
a.s., \text{ if } \frac{a_{31} e^{-a_{31} \tau_{31}} b_1}{a_{11}} + \frac{a_{32} e^{-a_{32} \tau_{32}} b_2}{a_{22}} - b_3 > 0.
\]

This completes the proof. \( \square \)
Lemma 10. The solution of model (5) obeys
\[
\limsup_{t \to +\infty} \frac{\ln x_i(t)}{t} = 0, \text{ a.s., } i = 1, 2, 3.
\] (60)

Proof. From Lemma 9, we can obtain that either \( \lim_{t \to +\infty} \ln Y_i(t) = 0 \) or \((1/t) \lim_{t \to +\infty} Y_i(t) \) is a constant, \( i = 1, 2, 3 \). Since \( Y_i(t) \geq x_i(t) \), we only need to prove the following results.

If \( \lim_{t \to +\infty} \ln Y_i(t) = 0, \) then
\[
\limsup_{t \to +\infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t \to +\infty} \frac{\ln Y_i(t)}{t} \leq 0,
\] a.s., \( i = 1, 2, 3 \).

If \( (1/t) \lim_{t \to +\infty} \int_0^t Y_i(t) = \) a constant, then
\[
\lim_{t \to +\infty} \frac{\ln Y_i(t)}{t} = 0, \text{ a.s., } i = 1, 2, 3,
\] (62)
and hence
\[
\limsup_{t \to +\infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t \to +\infty} \frac{\ln Y_i(t)}{t} = 0, \text{ a.s.} \quad \square
\] (63)

Theorem 11. For system (5), define
\[
\rho_1 = \frac{(a_i + h_i)}{(1/2) \sigma_i^2},
\rho_2 = \frac{\Gamma_2}{\Gamma_2},
\rho_3 = \frac{\Lambda_3}{\Lambda_3},
\nu_2 = \frac{\Gamma_3}{\Gamma_3},
\nu_3 = \frac{A_3}{A_3},
\] (64)

From Assumptions 4, 5, and 6, we can obtain the following results.

(I) If \( \Lambda > 0 \), then \( \rho_1 > \rho_2 > \rho_3 \); moreover,

(i) If \( \rho_1 < 1 \), then species \( i \), \( i = 1, 2, 3 \), go to extinction a.s.; i.e.,
\[
\lim_{t \to +\infty} x_i(t) = 0, \text{ a.s., } i = 1, 2, 3.
\] (65)

(ii) If \( \rho_1 > 1 > \rho_2 \), then the predator goes to extinction a.s., and one prey is persistent in mean, and another goes to extinction; i.e.,
\[
\lim_{t \to +\infty} x_i(t) = 0, \text{ a.s., } i = 2, 3,
\] (66)
\[
\lim_{t \to +\infty} \langle x_1(t) \rangle = \frac{b_1}{a_11}, \text{ a.s.}
\]

(iii) If \( \rho_2 > 1 > \rho_3 \), then species 3 goes to extinction and species 1,2 are persistent in mean; i.e.,
\[
\lim_{t \to +\infty} \langle x_i(t) \rangle = \frac{\Gamma_1 - \Gamma_i}{\Lambda_{33}}, \text{ a.s., } i = 1, 2.
\] (67)

(iv) If \( \rho_3 > 1 \), then three species are persistent in mean; i.e.,
\[
\lim_{t \to +\infty} \langle x_i(t) \rangle = \frac{A_i - A_i}{A}, \text{ a.s., } i = 1, 2, 3.
\] (68)

(II) If \( \Lambda < 0 \), then \( \rho_1 > \nu_2 > \nu_3 \); moreover,

(v) If \( \rho_1 < 1 \), then species \( i \), \( i = 1, 2, 3 \) go to extinction a.s.;

(vi) If \( \rho_1 > 1 > \nu_2 \), then the predator goes to extinction and one prey is persistent in mean, and another goes to extinction, a.s.;

(vii) If \( \nu_2 > 1 > \nu_3 \), then the species 2 goes to extinction and species 1,3 are persistent in mean, a.s.;

(viii) If \( \nu_3 > 1 \), then three species are persistent in mean, a.s.

Proof. Now we will give the proof of (I), and the proof of (II) is parallel to (I). We can get
\[
\frac{(a_i - h_i)}{\sigma_i^2/2 + \eta_i} > \frac{(a_2 - h_2)}{\sigma_2^2/2 + \eta_2},
\] (69)
\[
\Gamma_2 = a_{11}(a_2 - h_2) - a_{21}(a_1 - h_1) > 0,
\]
and, as a consequence,
\[
\Gamma_2 = a_{11}\left(\frac{1}{2}\sigma_1^2 + \eta_1\right) - a_{21}\left(\frac{1}{2}\sigma_2^2 + \eta_1\right) > 0,
\] (70)
\[
\rho_1 - \rho_2 > 0;
\]
we can also compute that
\[
\frac{\Gamma_2 - \Gamma_3}{\Gamma_2} = \frac{a_{11}A}{\Gamma_2} > 0,
\] (71)
\[
\frac{\Gamma_3 - A_3}{\Gamma_3} = \frac{a_{11}\Lambda}{\Gamma_3} > 0.
\]
So \( \rho_1 > \rho_2 > \rho_3 \) is established.

Applying Itô’s formula to model (5) yields
\[
\ln x_1(t) - \ln x_1(0) = b_1 t - a_{11} \int_0^t x_1(s) \, ds - a_{12} \int_0^t x_2(s - \tau_{12}) \, ds
\]
Firstly, we prove (i). Since $a_{11}, a_{12},$ and $a_{13}$ are positive, we can get

$$-a_{13} \int_0^t x_3 (s - \tau_{31}) \, ds + \sigma_1 B_1 (t) + M_1 (t),$$

$$-a_{22} \int_0^t x_2 (s) \, ds - a_{23} \int_0^t x_3 (s - \tau_{32}) \, ds + \sigma_2 B_2 (t) + M_2 (t),$$

$$\ln x_3 (t) - \ln x_3 (0) = -b_3 t + a_{31} e^{-\alpha_{31}} \int_0^t x_1 (s - \tau_{31}) \, ds + a_{32} e^{-\alpha_{32}} \int_0^t x_2 (s - \tau_{32}) \, ds - a_{33} \int_0^t x_3 (s) \, ds + \sigma_3 B_3 (t) + M_3 (t).$$

(72)

$$\ln x_2 (t) - \ln x_2 (0) = b_2 t - a_{21} \int_0^t x_1 (s - \tau_{21}) \, ds$$

$$-a_{22} \int_0^t x_2 (s) \, ds - a_{23} \int_0^t x_3 (s - \tau_{23}) \, ds + \sigma_2 B_2 (t) + M_2 (t),$$

$$\ln x_3 (t) - \ln x_3 (0) = -b_3 t + a_{31} e^{-\alpha_{31}} \int_0^t x_1 (s - \tau_{31}) \, ds + a_{32} e^{-\alpha_{32}} \int_0^t x_2 (s - \tau_{32}) \, ds - a_{33} \int_0^t x_3 (s) \, ds + \sigma_3 B_3 (t) + M_3 (t).$$

(73)

From Lemma 3.4 of [26], we obtain that either $\lim_{t \to +\infty} Y_3 (s) = 0$ or $\lim_{t \to +\infty} (1/t) \int_0^t Y_3 (s) \, ds = c$, $c$ is a constant; it follows that

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t Y_3 (s) \, ds = c.$$

(80)

In view of (22) and Lemma 8, we can obtain that

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t x_i (s) \, ds = 0, \quad a.s., \ i = 1, 2, 3.$$

(81)

Putting (60) and (81) into (79) leads to

$$\frac{a_{11} \ln x_2 (t)}{t \ln x_2 (0)} \leq \Gamma_2 - \tilde{\Gamma}_2 + \varepsilon - A_{33} \langle x_2 (t) \rangle$$

$$+ \frac{a_{11}}{t} (\sigma_1 B_1 (t) + M_1 (t)) - a_{21} (\sigma_1 B_1 (t) + M_1 (t)),$$

where $\varepsilon$ is small enough such that $\Gamma_2 - \tilde{\Gamma}_2 + \varepsilon < 0$. By Lemma 7, we can get

$$\lim_{t \to +\infty} x_2 (t) = 0, \quad a.s.$$

(83)

Taking advantage of the above identity, we will get the following two-species models:

$$dx_1 (t)$$

$$= x_1 (t) \left[ a_1 - a_{11} x_1 (t) - a_{13} x_3 (t - \tau_{13}) \right] \, dt.$$
\[ + \sigma x_1 (t) dB_1 (t) + x_1 (t) \int_{\gamma_1 (v)} \nu (df, dv), \]
\[ dx_3 (t) \]
\[ = x_3 (t) \left[ -a_3 + a_3 e^{-a_3 \tau_3} x_1 (t - \tau_3) - a_3 x_3 (t) \right] dt \]
\[ + \sigma x_3 (t) dB_3 (t) + x_3 (t) \int_{\gamma_3 (v)} \nu (df, dv). \] (84)

It is easy to get the result with the similar proof of that in [40–42]

\[
\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} x_1 (s) ds = \frac{b_1}{a_{11}},
\lim_{t \to \infty} x_3 (t) = 0.
\] (85)

Thirdly, we prove (iii). Denote \( m, n \) as the solution of the following equations:

\[
a_{22} m + a_{32} e^{-a_3 \tau_3} n = a_{12}, \]
\[
a_{23} m + a_{33} n = a_{13}. \] (86)

Consequently,

\[
m = \frac{A_{31}}{A_{11}} > 0, \]
\[
n = \frac{-A_{31}}{A_{11}} > 0. \] (87)

According to Lemma 7, for arbitrarily given \( \varepsilon > 0 \), there exists \( a T_1 > 0 \), for all \( t > T_1 \),

\[
t^{-1} \left( m \ln \frac{x_3 (t)}{x_2 (0)} + n \ln \frac{x_1 (t)}{x_3 (0)} \right) \leq \varepsilon. \] (88)

Multiplying (75), (76), and (77) by (−1), \( m, n \), respectively, and adding them, one can observe that for sufficiently large \( t \) such that \( t > T_1 \),

\[
t^{-1} \ln \frac{x_1 (t)}{x_1 (0)} - t^{-1} \left( m \ln \frac{x_2 (t)}{x_2 (0)} + n \ln \frac{x_1 (t)}{x_3 (0)} \right)
\[
= \frac{A_1 - \overline{A}_1}{A_{11}} - \frac{A}{A_{11}} \left( x_1 (t) \right)
\[
+ t^{-1} a_{12} \left( \int_{t-\tau_1}^{t} x_2 (s) ds - \int_{t-\tau_2}^{0} x_2 (s) ds \right)
\[
t^{-1} a_{13} \left( \int_{t-\tau_1}^{t} x_3 (s) ds - \int_{t-\tau_2}^{0} x_3 (s) ds \right)
\]
\[
= t^{-1} \left( \int_{t-\tau_1}^{t} x_1 (s) ds - \int_{t-\tau_2}^{0} x_3 (s) ds \right)
\]
\[
- t^{-1} a_{13} \left( \int_{t-\tau_1}^{t} x_3 (s) ds - \int_{t-\tau_2}^{0} x_3 (s) ds \right)
\]
\[
+ t^{-1} \left( m \sigma_1 B_1 (t) - m \sigma_2 B_2 (t) - \nu \sigma_3 B_3 (t) \right)
\]
\[
+ t^{-1} \left( -M_1 (t) + m M_2 (t) + n M_3 (t) \right). \] (89)

Substituting (60) and (81) into (89) yields

\[
\frac{1}{t} \ln \frac{x_1 (t)}{x_1 (0)} \leq \frac{A_1 - \overline{A}_1}{A_{11}} + 2 \varepsilon - \frac{A}{A_{11}} \left( x_1 (t) \right)
\[
+ t^{-1} \left( \sigma_1 B_1 (t) - m \sigma_2 B_2 (t) - \nu \sigma_3 B_3 (t) \right)
\[
+ t^{-1} \left( m M_1 (t) + n M_2 (t) - M_3 (t) \right). \] (90)

For sufficiently large \( t \), by \( A_1/\overline{A}_1 > \rho_1 > \rho_2 > 1 \), and choosing \( \varepsilon > 0 \) to be sufficiently small, then we have

\[
\langle x_1 (t) \rangle \leq \frac{A_1 - \overline{A}_1}{A_{11}}, \text{ a.s.} \] (91)

Similarly, denote \( p, q \) as the solution of the following equations:

\[
a_{11} p - a_{31} e^{-a_3 \tau_3} q = a_{21}, \]
\[
a_{13} p + a_{33} q = a_{23}. \] (92)

Then we have

\[
p = \frac{A_{12}}{A_{22}} > 0, \]
\[
q = \frac{-A_{32}}{A_{22}} > 0. \] (93)

In the same way, we can choose a \( T_2 > 0 \), for arbitrarily given \( \varepsilon > 0 \), such that

\[
t^{-1} \left( p \ln \frac{x_1 (t)}{x_1 (0)} + q \ln \frac{x_3 (t)}{x_3 (0)} \right) \leq \varepsilon. \] (94)
It follows that, for any sufficiently small $\epsilon$, there exists $T_3$ and $T_4$ such that multiplying (75), (76), and (77) by $p_\epsilon (-1)$ and $q_\epsilon$, respectively, and adding them, we can obtain that

$$\lim_{t \to +\infty} x_3(t) = 0, \ a.s.$$ (99)

Consequently, model (5) reduces to the following model:

$$dx_1(t) = x_1(t)\left[ a_1 - a_{11} x_1(t) - a_{13} x_3(t - \tau_{13}) \right] dt + \sigma_1 x_1(t) dB_1(t)$$

$$+ x_1(t) \int_v y_1(v) \tilde{N}(dt, dv),$$ (100)

$$dx_2(t) = x_2(t)\left[ a_2 - a_{21} x_1(t - \tau_{21}) - a_{23} x_3(t) \right] dt + \sigma_2 x_2(t) dB_2(t)$$

$$+ x_2(t) \int_v y_2(v) \tilde{N}(dt, dv),$$

which has already been investigated in [5]. Then similarly to the proof of Theorem 5.1 in [5], the following identities can be derived:

$$\lim_{t \to +\infty} \langle x_1(t) \rangle = \frac{\bar{\Gamma}_1 - \bar{\Gamma}_1}{A_{33}},$$ (101)

$$\lim_{t \to +\infty} \langle x_2(t) \rangle = \frac{\bar{\Gamma}_2 - \bar{\Gamma}_2}{A_{33}}, \ a.s.$$

Fourthly, we prove (iv). Since $A_3 - \bar{A}_3 > 0$, for arbitrariness of $\epsilon$, the application of Lemma 7 to (98) yields

$$\langle x_3(t) \rangle^* \leq \frac{A_3 - \bar{A}_3}{A}, \ a.s.$$ (102)

$$a_{12} \langle x_2(t) \rangle \leq a_{12} \langle x_2(t) \rangle^* + \epsilon$$

$$\leq \frac{a_{12} (A_2 - \bar{A}_2)}{A} + \epsilon, \ t > T_3,$$ (103)

$$a_{13} \langle x_3(t) \rangle \leq a_{13} \langle x_3(t) \rangle^* + \epsilon$$

$$\leq \frac{a_{13} (A_3 - \bar{A}_3)}{A} + \epsilon, \ t > T_4.$$ (103)

Substituting (103) into (75), we can obtain that, for sufficiently large $t$,

$$t^{-1} \ln \frac{x_1(t)}{x_1(0)} \geq -1 - a_{11} \langle x_1(t) \rangle - \frac{a_{12} (A_2 - \bar{A}_2)}{A}$$

$$- \frac{a_{13} (A_3 - \bar{A}_3)}{A} - 2\epsilon + \frac{\sigma_1 B_1(t)}{t}$$

$$+ \frac{M_1(t)}{t}.$$
According to the arbitrariness of \( \varepsilon \) and Lemma 7, we have

\begin{equation}
\langle x_1(t) \rangle \geq \frac{A_1 - \bar{A}_1}{A}, \ a.s. (105)
\end{equation}

Analogously, we can prove that \( \langle x_2(t) \rangle \geq (A_2 - \bar{A}_2)/A \) is established.

Substituting (103) and (105) into (77) yields that when \( t \) is large enough,

\[
t^{-1} \ln \frac{x_3(t)}{x_3(0)} \geq -b_3 + a_{33} \langle x_3(t) \rangle
\]

\[
- \frac{a_{33} e^{-a_3 t}}{A} (A_1 - \bar{A}_1)
- \frac{a_{33} e^{-a_3 t}}{A} (A_2 - \bar{A}_2) - 3\varepsilon
+ \frac{\sigma_3 B_3(t) + M_3(t)}{t}

\]

\[
= \frac{a_{33} (A_3 - \bar{A}_3)}{A} - a_{33} \langle x_3(t) \rangle - 3\varepsilon
+ \frac{\sigma_3 B_3(t) + M_3(t)}{t}.
\]

According to the arbitrariness of \( \varepsilon \) and Lemma 7, we have

\begin{equation}
\langle x_3(t) \rangle \geq \frac{A_3 - \bar{A}_3}{A}, \ a.s. (107)
\end{equation}

Subsequently, we have

\begin{equation}
\lim_{t \to +\infty} \langle x_i(t) \rangle = \frac{A_i - \bar{A}_i}{A}, \ a.s., \ i = 1, 2, 3. (108)
\end{equation}

This completes the proof. \( \square \)

### 4. Stability in Distribution

For the convenience, we define the following notations:

\[
c_{11} = a_{11},
\]

\[
c_{12} = a_{12},
\]

\[
c_{13} = a_{13},
\]

\[
c_{21} = a_{21},
\]

\[
c_{22} = a_{22},
\]

\[
c_{23} = a_{23},
\]

\[
c_{31} = a_{31} e^{-a_3 t},
\]

\[
c_{32} = a_{32} e^{-a_3 t},
\]

\[
c_{33} = a_{33}.
\]

**Definition 12.** Model (5) is globally attractive if \( \lim_{t \to +\infty} E|\bar{x}_i(t) - x_i| = 0, \ a.s., \ i = 1, 2, 3, \) and \( x(t) = (x_1(t), x_2(t), x_3(t))^T \) and \( \bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^T \) are two arbitrary solutions with initial values \( x(\theta) \in \mathcal{U} \) and \( \bar{x}_0 \in \mathcal{U} \), respectively.

**Lemma 13.** For any \( p > 1 \), there exists a constant \( K = K(p) \) which makes the solution \( x(t) \) of model (5) with any given initial value satisfy the property that

\[
\lim_{t \to +\infty} E|x_i^p(t)| \leq K, \ i = 1, 2, 3. (110)
\]

**Proof.** The proof is rather standard and hence is omitted. \( \square \)

From Lemma 13, there is a \( T > 0 \) such that, for \( t \geq T, \) \( E|x_i^p(t)| \leq 2K. \) Note that \( E|x_i(t)| \) is continuous; thus there is a constant \( K_1 > 0 \) such that \( E|x_i^p(t)| < K_1, \) when \( -\tau_0 \leq t < T. \) Denote \( L = \max\{2K, K_1\}; \) then we have

\[
E|x_i^p(t)| \leq L = L(p), \ t \geq \tau_0, \ p > 0, \ i = 1, 2, 3. (111)
\]

**Lemma 14.** If \( c_{11} > c_{12} + c_{31}, c_{22} > -c_{23} + c_{32}, c_{33} > -c_{31} + c_{32}, \) then model (5) will be asymptotically stable in distribution; i.e., when \( t \to +\infty, \) there exists a unique probability measure \( \mu(\cdot) \) such that the transition probability density \( p(t, \xi, \cdot) \) of \( x(t) \) converges weakly to \( \mu(\cdot) \) with any given initial value \( \xi(t) \in C([t, 0]; R_1^3) \) [45].

**Proof.** Denote

\[
L_A = \left(\begin{array}{ccc}
c_{11} + c_{13} & -c_{12} & -c_{13} \\
-c_{31} & c_{12} + c_{32} & -c_{33} \\
-c_{31} & -c_{32} & c_{31} - c_{32}
\end{array}\right). (112)
\]

Define \( q_i \) as the factor of \( i \)-th diagonal element of \( L_A. \) Then applying Kirchhoff’s Matrix Tree Theorem [46, 47], we can see that \( q_i > 0, \) \( i = 1, 2, 3. \)

Define

\[
V_i(t) = q_i |\ln x_i(t) - \ln \bar{x}_i(t)| + \sum_{j \neq i, j = 1} \int_{t - \tau_{ij}} t |x_j(r) - \bar{x}_j(r)| dr,
\]

\[
V(t) = \sum_{i=1}^{3} V_i(t).
\]

\[
c_{21} = a_{21},
\]

\[
c_{22} = a_{22},
\]

\[
c_{23} = a_{23},
\]

\[
c_{31} = a_{31} e^{-a_3 t},
\]

\[
c_{32} = a_{32} e^{-a_3 t},
\]

\[
c_{33} = a_{33}.
\]
Calculating the right differential $d^\ast V(t)$ yields
\[
d^\ast V(t) = \frac{3}{3} \sum_{i=1}^{3} q_i \text{sgn} \left( (x_i(t) - \bar{x}_i(t)) \ d(\ln x_i(t) - \ln \bar{x}_i(t)) \right) \\
+ \sum_{j=1}^{3} \left( \sum_{j \neq j, j=1}^{3} q_{ij} |x_j(t) - \bar{x}_j(t)| \right) dt \\
- \frac{3}{3} \sum_{j=1}^{3} q_j |c_j| \left( |x_j(t) - \bar{x}_j(t)| \right) dt \\
\leq \frac{3}{3} \sum_{j=1}^{3} \left( -q_i c_i |x_i(t) - \bar{x}_i(t)| \right) dt \\
+ \sum_{j=1}^{3} q_j |c_j| |x_j(t) - \bar{x}_j(t)| dt \\
= -\sum_{i=1}^{3} \sum_{j=1}^{3} \left( q_i A |x_i(t) - \bar{x}_i(t)| \right) dt.
\]

From Theorem 2.3 in [48],
\[
\frac{3}{3} \sum_{i=1}^{3} \sum_{j=1}^{3} \left( q_i c_i |x_i(t) - \bar{x}_i(t)| \right) \\
= \sum_{i=1}^{3} \sum_{j=1}^{3} \left( q_i c_i |x_i(t) - y_i(t)| \right).
\]

Then we can obtain
\[
d^\ast V(t) \leq -\sum_{i=1}^{3} \sum_{j=1}^{3} \left( q_i A |x_i(t) - \bar{x}_i(t)| \right) dt \\
= -\sum_{i=1}^{3} q_i c_i \left( |x_i(t) - \bar{x}_i(t)| \right) dt.
\]

Namely,
\[
E(V(t)) \leq V(0) \\
- \int_0^t \sum_{i=1}^{3} c_i \left( |x_i(r) - y_i(r)| \right) \ dt.
\]

Subsequently,
\[
E(V(t)) \\
+ \int_0^t \sum_{i=1}^{3} q_i \left( c_i \left( |x_i(r) - \bar{x}_i(r)| \right) \right) \ dt \\
\leq V(0) < \infty.
\]

Note that $V(t) \geq 0$; then $E[x_i(s) - \bar{x}_i(s)]$ is integrable on $[0, +\infty)$.

In other words, $E|x_i(t) - \bar{x}_i(t)| \in L^{-1}[0, +\infty) \ i = 1, 2, 3$.

Moreover, from model (5), we have
\[
E(x_1(t)) = x_1(0) + \int_0^t E[q_1 x_1(s) - c_1 x_1^2(r)] \\
- c_{12} x_1(r) x_2(r - \tau_{12}) \\
- c_{13} x_1(r) x_3(r - \tau_{13}) dr, \\
E(x_2(t)) = x_2(0) + \int_0^t E[q_2 x_2(r) \\
- c_{21} x_2(r) x_1(r - \tau_{21}) - c_{22} x_2^2(r) \\
- c_{23} x_2(r) x_3(r - \tau_{23}) dr, \\
E(x_3(t)) = x_3(0) + \int_0^t E[-a_3 x_3(r) \\
+ c_{31} x_3(r) x_1(r - \tau_{31}) + c_{32} x_3(r) x_2(r - \tau_{32}) \\
- c_{33} x_3^2(r)] dr.
\]

Thus,
\[
\frac{dE(x_i(t))}{dt} \leq a_i K_i, \ i = 1, 2.
\]

\[
\frac{dE(x_i(t))}{dt} \leq -a_i K_i, \ i = 1, 2, 3.
\]

where $K_i > 0$. Therefore, $E(x_i(t), i = 1, 2, 3)$ are uniformly continuous. In other words, $E(x_i(t), i = 1, 2, 3)$ are continuously differentiable functions with respect to $t$. By Lemma 13 and the conclusion of [45], one can observe that
\[
\lim_{t \to +\infty} E(x_i(t) - y_i(t)) = 0, \ i = 1, 2, 3.
\]

Suppose that $p(t, \xi, dy)$ is the transition probability density of the process $x(t)$ and $P(t, \xi)$, $M$ denotes the probability of event $x(t, \xi) \in M$ with the initial value $\xi(\theta) \in C([-\tau_0, 0]; R^k)$. By Lemma 7 and Chebyshev’s inequality [49], we can see that the family of $p(t, \xi, dy)$ is tight. So we can get a compact subset $\mathcal{X} \in R^k$ such that $p(t, \xi, \mathcal{X}) \geq 1 - \epsilon^*$ for given $\epsilon^*$.

Let $\mathcal{P}(C([-\tau, 0]; R^k))$ be the probability measures on $C([-\tau_0, 0]; R^k)$. For arbitrary two measures $P_1, P_2 \in \mathcal{P}$, we define the following Kantorovich metric:
\[
d_L(P_1, P_2) \\
= \sup_{\|x\| \in \mathcal{L}} \left\{ \int_{R^k} l(x) P_1(dx) - \int_{R^k} l(x) P_2(dx) \right\},
\]

where
\[
\mathcal{L} = \left\{ l : C([-\tau, 0]; R^k) \to R : \|l(x) - l(y)\| \leq \|x - y\|, \|l(x)\| \leq 1 \right\}.
\]
For any \( l \in \mathcal{L} \) and \( t, u > 0 \), we get
\[
|E_l(x(t+u,\xi)) - E_l(x(t,\xi))| = |E[L_l(x(t+u,\xi) | \mathcal{F}_u)] - E_l(x(t,\xi))| \leq \int_{R^1} E_l(x(t,\xi)) p(u,\xi,d\phi) - \int_{R^1} E_l(x(t;\xi)) p(u,\xi,d\phi).
\]
From (121), there exists a \( T > 0 \) such that, for \( t \geq T \), we have
\[
\sup_{l \in \mathcal{L}} |E_l(x(t;\phi)) - E_l(x(t,\xi))| \leq \varepsilon^*.
\]
Obviously
\[
|E_l(x(t+u,\xi)) - E_l(x(t;\xi))| \leq \varepsilon^*.
\]
By the arbitrariness of \( l \), we have
\[
\sup_{g \in E} |E_l(x(t+u,\xi)) - E_l(x(t;\xi))| \leq \varepsilon^*.
\]
Thus,
\[
d_l(p(t+u,\xi,\cdot), p(t,\xi,\cdot)) \leq \varepsilon^*, \quad \forall t \geq T, \quad u > 0.
\]
Therefore, \( \{p(t,0,\cdot) : t \geq t \geq 0 \} \) is Cauchy in \( \mathcal{P} \) with metric \( d_l \).

There exists a unique \( \mu(\cdot) \in \mathcal{P}(C([-\tau_0,0];R^1_+)) \) such that
\[
\lim_{t \to +\infty}d_l(p(t,0,\cdot), \mu(\cdot)) = 0.
\]
From (132), we can obtain
\[
\lim_{t \to +\infty} d_l(p(t,\xi,\cdot), p(t,0,\cdot)) = 0.
\]
Consequently,
\[
\lim_{t \to +\infty} d_l(p(t,\xi,\cdot), \mu(\cdot)) \leq \lim_{t \to +\infty} d_l(p(t,\xi,\cdot), p(t,0,\cdot)) + \lim_{t \to +\infty} d_l(p(t,0,\cdot), \mu(\cdot)) = 0.
\]
This completes the proof of Lemma 14.

\[\square\]

**5. Optimal Harvesting**

We give the following extra notions to get the optimal harvesting policy:
\[
D = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
\]
\[
\Lambda_1 = (\lambda_1, \lambda_2)^T = \left[D(D^{-1})^T + I\right]^{-1} Q,
\]
where \( Q = (Q_1, Q_2)^T \), \( Q_i = a_i - (1/2)a_j^2 - \eta_i \), \( i = 1, 2 \), and \( I \) is the unit matrix.

**Theorem 15.** Suppose that \( a_{11} > a_{21}, a_{22} > a_{12}, A > 0, \) and \( D^{-1} + (D^{-1})^T \) is positive definite.

(i) If \( \lambda_1 \geq 0 \) and when \( h_i = \lambda_i, \ i = 1, 2, \) we have \( \Theta_1 > 0, \Theta_2 > 0; \) then the optimal harvesting effort is \( H^* = \Lambda_1 = [D(D^{-1})^T + I]^{-1} Q \) and the maximum of ESY is
\[
Y^* = \Lambda_1^T D^{-1} (Q - \Lambda_1),
\]
where \( \Theta_j = A_j - \overline{A}_j, \ i = 1, 2. \)

(ii) When \( h_i = \lambda_i, \ i = 1, 2, \) there is \( \Theta_1 \leq 0 \) or \( \lambda_i < 0, \ i = 1, 2; \) then the optimal harvesting policy does not exist.

**Proof.** Let \( G = \{H = (h_1, h_2)^T \in R^2 \mid \Theta_{ni} > 0, \ i = 1, 2\}. \) When (iv) of Theorem 11 holds, we find that, for every \( H \in G \), and if the optimal harvesting effort \( H^* \) exists, then it must belong to \( G \).

Firstly we prove (i). It is easy to see that \( \Lambda_1 \in G \), so \( G \) is not empty. By (iv) of Theorem 11, for any \( H \in G \), we have
\[
\lim_{t \to +\infty} \int_0^t H^T x(s) \, ds = \sum_{i=1}^2 h_i \lim_{t \to +\infty} \int_0^t x_i(s) \, ds = H^T D^{-1} (Q - H).
\]
Applying to Lemma 14, model (18) has a unique invariant measure \( \mu(\cdot) \). According to Corollary 3.4.3 in [46], we can get that \( \mu(\cdot) \) is strong mixing. At the same time, it is ergodic by Theorem 3.2.6 in [46]. Hence, it then follows from (3.3.2) in [46] that
\[
\lim_{t \to +\infty} \int_0^t H^T x(s) \, ds = \int_{R^2} H^T x(t) \mu(dx).
\]
Let \( q(x) \) represent the stationary probability density of model (18); we obtain
\[
Y(H) = \lim_{t \to +\infty} \sum_{i=1}^2 \int_0^t E[h_i x_i(t)] = \lim_{t \to +\infty} E[H^T x(t)]
\]
\[
= \int_{R^2} H^T x(t) q(x) \, dx.
\]
Since the invariant measure of model (18) is unique, then, according to the one-to-one correspondence between \( q(x) \) and its corresponding invariant measure \( \mu(\cdot) \), we obtain
\[
\int_{R^2} H^T x Q(x) \, dx = \int_{R^2} H^T x q(dx).
\]
That is to say,
\[
Y(H) = H^T D^{-1} (Q - H).
\]
Assume that \( \Lambda_1 = (\lambda_1, \lambda_2)^T \) is the unique stagnation point of the following equation:
\[
\frac{dY(H)}{dH} = \frac{dH^T}{dH} D^{-1} (Q - H)
\]
\[
+ \frac{d}{dH} \left[ (Q - H)^T (D^{-1})^T \right] H
\]
\[
= D^{-1} Q - \left[ D^{-1} + (D^{-1})^T \right] H = 0.
\]
It holds that $\Lambda_1 = \left[ D(D^{-1})^T + I \right]^{-1} Q$. We can take use of the following Hessian matrix [42, 43]:

$$
\frac{d}{dH} \left[ \frac{dY(H)}{dH} \right] = \left( \frac{d}{dH} \left[ \left( \frac{dY(H)}{dH} \right)^T \right] \right)^T = \left( \frac{d}{dH} \left[ Q^T (D^{-1})^T - H^T \left[ D^{-1} + (P^{-1})^T \right] \right] \right)^T \tag{139}
$$

is negatively defined, so $\Lambda_1$ is the unique extreme point of $Y(H)$. In other words, if $\Lambda_1 \in G$, i.e., $\Lambda_i \geq 0$ and $\Theta_i > 0$, $i = 1, 2$, then the optimal harvesting effort is $H^* = \Lambda_1$ and $Y^*$ is the maximum value of ESY. We are now in the position to prove (ii). Suppose that the optimal harvesting effort $H^* = (\Lambda_1, \Lambda_2)^T$ exists. So $H^* \in G$; i.e., $\Theta_i |_{\Lambda_i = 0} \geq 0$, $\Lambda_i \geq 0$, $i = 1, 2$. That is to say, if $H^*$ is the optimal harvesting effort, then $H^*$ must be the unique solution of (138). However, $\Lambda_1 = (\lambda_1, \lambda_2)^T$ is also the solution of (138). Hence, $\lambda_i = \tilde{h}_i \geq 0$, and $\Theta_i |_{\lambda_i = 0} \geq 0$. It contradicts with the condition.

This completes the proof of Theorem 15. $\square$

6. Numerical Simulations

In this section, we carry out extensive numerical simulations using MATLAB by choosing the following parameters to check the model. And the phase diagrams are given too. Some figures are as follows: $a_1 = 4, a_2 = 2.5, a_3 = 0.7, a_{11} = 0.8, a_{12} = 0.3, a_{13} = 0.45, a_{21} = 0.1, a_{22} = 0.5, a_{23} = 0.5, a_{31} = 0.4, a_{32} = 0.5, a_{33} = 0.5, y_1 = 1, y_2 = 1, y_3 = 1, r_{12} = 0.6, r_{13} = 0.7, r_{21} = 0.5, r_{23} = 1, r_{31} = 0.4, r_{32} = 0.6$. From Assumptions 4, 5, and 6, we can obtain the following results.

(A) Persistent and Extinction. If $\Lambda > 0$, then $\rho_1 > \rho_2 > \rho_3$.

(i) If $\rho_1 < 1$, then species $i, i = 1, 2, 3$, go to extinction.

(ii) If $\rho_1 > 1 > \rho_2$, then the predator goes to extinction and one prey is persistent in mean, and another goes to extinction.

(iii) If $\rho_2 > 1 > \rho_1$, then the species 3 goes to extinction and species 1,2 are persistent in mean.

(iv) If $\rho_3 > 1$, then three species are persistent in mean.

In Figure 1, $\Lambda = 46.3305 > 0, \rho_1 = 0.3061 < 1$; it shows that all the populations are extinct.

In Figure 2, $\Lambda = 668.1985 > 0, \rho_2 = 2.7500 > 1$. We can find that only the prey is persistent in mean and another prey and predator are extinct.

In Figure 3, $\Lambda = 114.4603 > 0, \rho_1 = 2.7500 > \rho_2 = 1.4194 > 1$. We can find that only the two prey are persistent in mean and the predator is extinct.

In Figure 4, $\Lambda = 0.2990 > 0, \rho_1 = 7.0400 > \rho_2 = 4.0146 > \rho_3 = 2.5178 > 1$, and the noise is small.

(B) Optimal Harvesting. Regarding the optimal harvesting effort, when $a_{11} = 0.8 > a_{11} = 0.3, a_{22} = 0.5 > a_{12} = 0.1$, it is not difficult to estimate that $D^{-1} + (P^{-1})^T$ is positive definite. Note that $\Lambda = [D(D^{-1})^T + I]^{-1} Q$; we can observe $\Lambda = (\lambda_1, \lambda_2)^T = (0.9778, 0.9243)^T$. Then we can find $\rho_2 > 0, \rho_3 > 0$. Therefore, by Theorem 15, we can observe $h_1 = \lambda_1 = 0.9778, h_2 = \lambda_2 = 0.9243, Y^* = \Lambda^T D^{-1}(Q - \Lambda) = 3.3388$. Thus the optimal harvesting policy exists (see Figure 3).

In Figure 5, in order to check the conclusion, we choose another two types of harvesting data. It is obvious that the optimal harvesting policy leads to the maximum of expectation of sustainable yield.

7. Conclusion

This paper is about a stochastic three-species population model with time delays and Lévy jumps [49]. We also
consider the optimal harvesting of preys [50]. To begin with, we establish the modified model. In Theorem 11, we obtain the sufficient criteria for extinction and persistence in mean of each species. Lévy jump is important to study. When $\rho_1 < 1$, all species go to extinction. When $\rho_1 > 1 > \rho_2$, then the predator $x_3$ goes to extinction and $x_2$ is persistent in mean, $x_1$ goes to extinction. When $\rho_2 > 1 > \rho_3$, then $x_3$ goes to extinction and $x_1, x_2$ are persistent in mean. When $\rho_3 > 1$, then three species are persistent in mean. The main purpose of this paper is to study the optimal harvesting. After discussing the stability of distribution, we study the optimal harvesting and obtain the maximum yield of two preys.

From the numerical simulations, we list the following biological meanings.

1. We find the noise can cause the variation of species. When the noise is large, it in reality can suppress the increase of population, then it dies out.

2. Time delay and Lévy jump have important effects on the persistence in mean and the harvesting yield.

In traditional papers, scholars consider two species or three species without optimal harvesting. We consider a three-species model with Lévy jump and optimal harvesting. Time delay is ineluctable in the ecological environment and is necessary to consider delays. Recently, stochastic models with the telephone noise have been studied by many authors.

**Figure 2:** One prey persistent in mean and others are extinct as given in Theorem 11. (b) stands for the phase portrait of (a). The density of white noises is taken: $\sigma_1 = 0.5$, $\sigma_2 = 22$, $\sigma_3 = 30$.

**Figure 3:** The persistence in mean of two preys is given in Theorem 11. (b) stands for the phase portrait of (a). The density of white noises is taken: $\sigma_1 = 0.5$, $\sigma_2 = 0.9$, $\sigma_3 = 12$. 
The persistence in mean of two preys and of three species are given in Theorem 11. (b) stands for the phase portrait of (a). The density of white noises is taken: $\sigma_1 = 0.8$, $\sigma_2 = 0.65$, $\sigma_3 = 0.42$.

Figure 4

![Figure 4](image)

Figure 5: The optimal harvesting effort and the maximum of ESY [45]. Red line is with $h_1 = 0.9778$, $h_2 = 0.9243$, blue line is with $h_1 = 0.5$, $h_2 = 0.4$, and green line is with $h_1 = 0.2$, $h_2 = 0.1$.

Figure 5

![Figure 5](image)

[51, 52]. In the future research, we hope to add more realistic conditions and study more interesting topics, for example, pulse process, Markov Chain, telephone noise, and partial differential system [53–55].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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