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Research Article

Oscillation Criteria for Third-Order Emden–Fowler Differential Equations with Unbounded Neutral Coefficients

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New sufficient conditions for the oscillation of all solutions to a class of third-order Emden–Fowler differential equations with unbounded neutral coefficients are established. The criteria obtained essentially improve related results in the literature. In particular, as opposed to known results, new criteria can distinguish solutions of third-order differential equations with different behaviors. Examples are also provided to illustrate the results.

1. Introduction

This paper is concerned with the oscillation of solutions of the third-order Emden–Fowler neutral differential equation

$$z'''(t) + q(t)\, x^{\lambda}(g(t)) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where $z(t) := x(t) + p(t)x(\eta(t))$, $\lambda > 0$ is the ratio of odd positive integers. Throughout, the following conditions are assumed to hold:

(C1) $p, q : [t_0, \infty) \longrightarrow \mathbb{R}$ are continuous functions, $p(t) \geq 1$, $p(t) \not\equiv 1$ for large t, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;

(C2) $\eta, g : [t_0, \infty) \longrightarrow \mathbb{R}$ are continuous functions, $\eta(t) \leq t$, η is strictly increasing, and $\lim_{t\longrightarrow\infty}\eta(t) = \lim_{t\longrightarrow\infty} g(t) = \infty$.

The study of (1) is important due to the further development of the oscillation theory and its practical reasons. Emden–Fowler differential equations have numerous applications in physics (mathematical, theoretical, and chemical physics) and engineering; see, e.g., the papers by Agarwal et al. [1], Li and Rogovchenko [2–5], and Wong [6].

By a solution of (1) we mean a continuous function $x : [t_x, \infty) \longrightarrow \mathbb{R}$, $t_x \geq t_0$, such that $z \in C^3([t_x, \infty), \mathbb{R})$ and $x(t)$ satisfies (1) on $[t_x, \infty)$. We consider only proper solutions $x(t)$ of (1) that satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_x$. Furthermore, we tacitly suppose that (1) possesses such solutions. Such a solution $x(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros on $[t_x, \infty)$; i.e., for any $t_1 \in [t_x, \infty)$, there exists a $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise, it is called nonoscillatory, i.e., if it is either eventually positive or eventually negative. Equation (1) is said to be oscillatory if all its proper solutions oscillate.

In recent years, there has been much research activity concerning the oscillation and asymptotic behavior of solutions to various classes of third-order neutral differential equations. We refer the reader to the papers [2, 7–17] and the references contained therein as examples of recent results on this topic. However, the sufficient conditions established in these papers except [10, 13] ensure that every solution $x(t)$ of equations either oscillates or converges to zero as $t \longrightarrow \infty$. This means that these results cannot distinguish solutions with different behaviors. On the other hand, the papers [2, 7–17] were concerned with the case where p is bounded, i.e., the cases where $-1 < p_0 \le p(t) \le 0$, $0 \le p(t) \le p_0 < 1$, and $0 \le p(t) \le p_0 < \infty$ were considered. In view of the observations above, we wish to develop new sufficient conditions which not only ensure oscillation of (1) but also can be applied to the case where p is unbounded. We would like to point out that only a few results are known regarding oscillatory and asymptotic behavior of third-order neutral differential equations for unbounded p; see, e.g., the papers [18–20], where the Riccati transformation technique and comparison method were used to obtain the results. A similar observation as above is valid for these papers as well, i.e., the sufficient conditions established in these papers cannot distinguish solutions with different behaviors too.

Consequently, our work is of significance because of the above-mentioned reasons. Moreover, the results obtained in this paper can easily be extended to more general third-order differential equations with unbounded neutral coefficients to derive more general oscillation results. It is our belief that the present paper will contribute significantly to the study of oscillatory behavior of solutions of third-order neutral differential equations. In the sequel, all functional inequalities are supposed to hold eventually.

2. Main Results

We begin with the following lemmas that will play an important role in establishing our main results. For notational purposes, we let

$$\varphi(t) := \frac{1}{p(\eta^{-1}(t))}\left(1-\frac{1}{p(\eta^{-1}(\eta^{-1}(t)))}\right) \ge 0,$$
$$\psi(t) := \frac{1}{p(\eta^{-1}(t))}\left(1 - \frac{1}{p(\eta^{-1}(\eta^{-1}(t)))}\frac{\left(\eta^{-1}(\eta^{-1}(t))\right)^{2/l_1}}{\left(\eta^{-1}(t)\right)^{2/l_1}}\right) \ge 0, \tag{2}$$

where $l_1 \in (0,1)$ is a constant and η^{-1} is the inverse function of η.

Lemma 1 (see [21]). *Let the function h satisfy $h^{(i)}(t) > 0$, $i = 0, 1, \ldots, m$ and $h^{(m+1)}(t) \le 0$ eventually. Then, for every $l \in (0,1)$, $h(t)/h'(t) \ge lt/m$ eventually.*

Lemma 2. *Let conditions* (C1) *and* (C2) *be satisfied and assume that x is an eventually positive solution of (1). Then for sufficiently large t, either*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, *and* $z'''(t) \le 0$; *or*

(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, *and* $z'''(t) \le 0$.

Proof. The proof is not difficult and so is omitted. □

Theorem 3. *In addition to conditions* (C1) *and* (C2), *assume that there exists a function $\sigma \in C([t_0,\infty),\mathbb{R})$ such that $g(t) \le \sigma(t) < \eta(t)$ for $t \ge t_0$. If for some constants $l_1, l_2 \in (0,1)$, the two first-order delay differential equations*

$$Y'(t) + \frac{(l_1 l_2)^{\lambda}}{2^{\lambda}} q(t)\psi^{\lambda}(g(t))\left(\eta^{-1}(g(t))\right)^{2\lambda} \cdot Y^{\lambda}\left(\eta^{-1}(g(t))\right) = 0 \tag{3}$$

and

$$W'(t) + \frac{1}{2^{\lambda}} q(t)\varphi^{\lambda}(g(t))\left(\eta^{-1}(\sigma(t)) - \eta^{-1}(g(t))\right)^{2\lambda} \cdot W^{\lambda}\left(\eta^{-1}(\sigma(t))\right) = 0 \tag{4}$$

oscillate, then (1) oscillates.

Proof. Let x be a nonoscillatory solution of (1). Since $-x$ is also a solution of (1), without loss of generality, we may suppose that there exists a $t_1 \in [t_0,\infty)$ such that, for $t \ge t_1$, $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$. It follows from Lemma 2 that z satisfies either case (I) or case (II).

Assume first that case (I) holds. By virtue of the definition of z, we conclude that

$$\begin{aligned} x(t) &= \frac{1}{p(\eta^{-1}(t))}\left(z\left(\eta^{-1}(t)\right) - x\left(\eta^{-1}(t)\right)\right) \\ &= \frac{z\left(\eta^{-1}(t)\right)}{p(\eta^{-1}(t))} - \frac{z\left(\eta^{-1}\left(\eta^{-1}(t)\right)\right) - x\left(\eta^{-1}\left(\eta^{-1}(t)\right)\right)}{p(\eta^{-1}(t))\,p(\eta^{-1}(\eta^{-1}(t)))} \\ &\ge \frac{z\left(\eta^{-1}(t)\right)}{p(\eta^{-1}(t))} - \frac{z\left(\eta^{-1}\left(\eta^{-1}(t)\right)\right)}{p(\eta^{-1}(t))\,p(\eta^{-1}(\eta^{-1}(t)))}. \end{aligned} \tag{5}$$

Taking into account (I) and Lemma 1 with $m = 2$, we deduce that, for every $l_1 \in (0,1)$,

$$\frac{z(t)}{z'(t)} \ge l_1 \frac{t}{2}, \tag{6}$$

which yields

$$\left(\frac{z(t)}{t^{2/l_1}}\right)' = \frac{tz'(t) - (2/l_1)z(t)}{t^{(2/l_1)+1}} \le 0. \tag{7}$$

Hence, $z(t)/t^{2/l_1}$ is nonincreasing for sufficiently large t. It follows from $\eta(t) \le t$ and the monotonicities of $\eta(t)$ and $z(t)/t^{2/l_1}$ that

$$z\left(\eta^{-1}\left(\eta^{-1}(t)\right)\right) \le \frac{\left(\eta^{-1}\left(\eta^{-1}(t)\right)\right)^{2/l_1}}{\left(\eta^{-1}(t)\right)^{2/l_1}} z\left(\eta^{-1}(t)\right). \tag{8}$$

Using (8) in (5), we arrive at

$$x(t) \geq \psi(t) z\left(\eta^{-1}(t)\right), \tag{9}$$

and thus

$$x(g(t)) \geq \psi(g(t)) z\left(\eta^{-1}(g(t))\right). \tag{10}$$

Combining (1) and (10), we obtain

$$z'''(t) + q(t)\psi^{\lambda}(g(t)) z^{\lambda}\left(\eta^{-1}(g(t))\right) \leq 0. \tag{11}$$

It follows now from (6) and (11) that

$$\begin{aligned} &z'''(t) \\ &\quad + \frac{l_1^{\lambda}}{2^{\lambda}} q(t)\psi^{\lambda}(g(t))\left(\eta^{-1}(g(t))\right)^{\lambda}\left(z'\left(\eta^{-1}(g(t))\right)\right)^{\lambda} \\ &\quad \leq 0. \end{aligned} \tag{12}$$

Letting $w(t) := z'(t)$, we have

$$\begin{aligned} w(t) &> 0, \\ w'(t) &> 0, \\ w''(t) &\leq 0, \end{aligned} \tag{13}$$

and inequality (12) can be written as

$$\begin{aligned} &w''(t) \\ &\quad + \frac{l_1^{\lambda}}{2^{\lambda}} q(t)\psi^{\lambda}(g(t))\left(\eta^{-1}(g(t))\right)^{\lambda} w^{\lambda}\left(\eta^{-1}(g(t))\right) \\ &\quad \leq 0. \end{aligned} \tag{14}$$

Combining (13) and Lemma 1 with $m = 1$, we get, for every $l_2 \in (0,1)$,

$$\frac{w(t)}{w'(t)} \geq l_2 t, \tag{15}$$

and so

$$w\left(\eta^{-1}(g(t))\right) \geq l_2 \eta^{-1}(g(t)) w'\left(\eta^{-1}(g(t))\right). \tag{16}$$

Using (16) in (14), we deduce that

$$\begin{aligned} &w''(t) + \frac{(l_1 l_2)^{\lambda}}{2^{\lambda}} q(t)\psi^{\lambda}(g(t))\left(\eta^{-1}(g(t))\right)^{2\lambda} \\ &\quad \cdot \left(w'\left(\eta^{-1}(g(t))\right)\right)^{\lambda} \leq 0. \end{aligned} \tag{17}$$

Letting $Y(t) := w'(t)$, we see that Y is a positive solution of the first-order delay differential inequality

$$\begin{aligned} &Y'(t) + \frac{(l_1 l_2)^{\lambda}}{2^{\lambda}} q(t)\psi^{\lambda}(g(t))\left(\eta^{-1}(g(t))\right)^{2\lambda} \\ &\quad \cdot Y^{\lambda}\left(\eta^{-1}(g(t))\right) \leq 0. \end{aligned} \tag{18}$$

Therefore, by [22, Theorem 1], we conclude that, for every $l_1, l_2 \in (0,1)$, (3) has a positive solution, which contradicts the fact that (3) oscillates.

Next, suppose that case (II) holds. Since z is strictly decreasing and $\eta(t) \leq t$, we have

$$z\left(\eta^{-1}(t)\right) \geq z\left(\eta^{-1}\left(\eta^{-1}(t)\right)\right). \tag{19}$$

Using (19) in (5), we conclude that

$$x(t) \geq \varphi(t) z\left(\eta^{-1}(t)\right), \tag{20}$$

and thus

$$x(g(t)) \geq \varphi(g(t)) z\left(\eta^{-1}(g(t))\right). \tag{21}$$

Substitution of (21) into (1) implies that

$$z'''(t) + q(t)\varphi^{\lambda}(g(t)) z^{\lambda}\left(\eta^{-1}(g(t))\right) \leq 0. \tag{22}$$

Since $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, and $z'''(t) \leq 0$, for $v \geq u \geq t_2$, one can easily arrive at

$$z(u) \geq \frac{(v-u)^2}{2} z''(v). \tag{23}$$

By virtue of $g(t) \leq \sigma(t)$ and the fact that η is strictly increasing, we deduce that $\eta^{-1}(g(t)) \leq \eta^{-1}(\sigma(t))$. Substitute $u = \eta^{-1}(g(t))$ and $v = \eta^{-1}(\sigma(t))$ into (23) to obtain

$$\begin{aligned} &z\left(\eta^{-1}(g(t))\right) \\ &\quad \geq \frac{\left(\eta^{-1}(\sigma(t)) - \eta^{-1}(g(t))\right)^2}{2} z''\left(\eta^{-1}(\sigma(t))\right). \end{aligned} \tag{24}$$

Using (24) in (22), we get

$$\begin{aligned} &z'''(t) + \frac{1}{2^{\lambda}} q(t)\varphi^{\lambda}(g(t))\left(\eta^{-1}(\sigma(t)) - \eta^{-1}(g(t))\right)^{2\lambda} \\ &\quad \cdot \left(z''\left(\eta^{-1}(\sigma(t))\right)\right)^{\lambda} \leq 0. \end{aligned} \tag{25}$$

Letting $W(t) := z''(t)$, we see that W is a positive solution of the first-order delay differential inequality

$$\begin{aligned} &W'(t) + \frac{1}{2^{\lambda}} q(t)\varphi^{\lambda}(g(t))\left(\eta^{-1}(\sigma(t)) - \eta^{-1}(g(t))\right)^{2\lambda} \\ &\quad \cdot W^{\lambda}\left(\eta^{-1}(\sigma(t))\right) \leq 0. \end{aligned} \tag{26}$$

The rest of the proof is similar to that of case (I) and hence is omitted. This completes the proof. □

From [23], it is well known that if

$$\liminf_{t \longrightarrow \infty} \int_{\tau(t)}^{t} R(s)\,\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{27}$$

then the first-order delay differential equation

$$x'(t) + R(t) x(\tau(t)) = 0 \tag{28}$$

oscillates, where $R, \tau \in C([t_0, \infty), \mathbb{R})$, $R(t) \geq 0$, $\tau(t) < t$, and $\lim_{t \longrightarrow \infty} \tau(t) = \infty$. Therefore, by virtue of Theorem 3, we have the following result.

Corollary 4. *Let conditions* (C1) *and* (C2) *be satisfied and* $\lambda = 1$. *Assume that there exists a function* $\sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) \leq \sigma(t) < \eta(t)$ *for* $t \geq t_0$. *If for some constant* $l_1 \in (0,1)$,

$$\liminf_{t\longrightarrow\infty}\int_{\eta^{-1}(g(t))}^{t} q(s)\,\psi(g(s))\left(\eta^{-1}(g(s))\right)^{2}\mathrm{d}s > \frac{2}{l_1\mathrm{e}} \quad (29)$$

and

$$\liminf_{t\longrightarrow\infty}\int_{\eta^{-1}(\sigma(t))}^{t} q(s)\,\varphi(g(s)) \cdot \left(\eta^{-1}(\sigma(s)) - \eta^{-1}(g(s))\right)^{2}\mathrm{d}s > \frac{2}{\mathrm{e}}, \quad (30)$$

then (1) oscillates.

Corollary 5. *Let conditions* (C1) *and* (C2) *be satisfied and* $\lambda < 1$. *Assume that there exists a function* $\sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $g(t) \leq \sigma(t) < \eta(t)$ *for* $t \geq t_0$. *If for some constant* $l_1 \in (0,1)$,

$$\int_{t_0}^{\infty} q(t)\,\psi^{\lambda}(g(t))\left(\eta^{-1}(g(t))\right)^{2\lambda}\mathrm{d}t = \infty \quad (31)$$

and

$$\int_{t_0}^{\infty} q(t)\,\varphi^{\lambda}(g(t))\left(\eta^{-1}(\sigma(t)) - \eta^{-1}(g(t))\right)^{2\lambda}\mathrm{d}t = \infty, \quad (32)$$

then (1) oscillates.

Proof. Applications of (31), (32), and [24, Theorem 2] imply that (3) and (4) oscillate. Hence, by Theorem 3, (1) oscillates. □

Next, we present the following interesting result for which we need to assume that the function g in condition $(C2)$ is nondecreasing.

Theorem 6. *In addition to conditions* (C1) *and* (C2), *assume that the function* g *with* $g(t) < \eta(t)$ *is nondecreasing on* $[t_0,\infty)$. *If for some constant* $l_1 \in (0,1)$,

$$\limsup_{t\longrightarrow\infty}\left(\eta^{-1}(g(t))\right)^{2\lambda} \cdot \int_{t}^{\infty} q(s)\,\psi^{\lambda}(g(s))\,\mathrm{d}s \begin{cases} > \dfrac{2}{l_1}, & \text{if } \lambda = 1, \\ = \infty, & \text{if } \lambda < 1, \end{cases} \quad (33)$$

and

$$\limsup_{t\longrightarrow\infty}\int_{\eta^{-1}(g(t))}^{t} q(s)\,\varphi^{\lambda}(g(s))\left(\eta^{-1}(g(t)) - \eta^{-1}(g(s))\right)^{2\lambda}\mathrm{d}s \begin{cases} > 2, & \text{if } \lambda = 1, \\ = \infty, & \text{if } \lambda < 1, \end{cases} \quad (34)$$

then (1) oscillates.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we may suppose that there exists a $t_1 \in [t_0,\infty)$ such that, for $t \geq t_1$, $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$. It follows from Lemma 2 that z satisfies either case (I) or case (II).

Assume that case (I) holds. Proceeding as in the proof of Theorem 3, we deduce that (13), (14), and (16) hold for every $l_1, l_2 \in (0,1)$. Integrating (14) from t to u, $u \geq t$ and letting $u \longrightarrow \infty$, we obtain

$$\begin{aligned} w'(t) &\geq \int_{t}^{\infty}\frac{l_1^{\lambda}}{2^{\lambda}}q(s)\,\psi^{\lambda}(g(s))\left(\eta^{-1}(g(s))\right)^{\lambda} \cdot w^{\lambda}\left(\eta^{-1}(g(s))\right)\mathrm{d}s \\ &\geq \left(\int_{t}^{\infty}\frac{l_1^{\lambda}}{2^{\lambda}}q(s)\,\psi^{\lambda}(g(s))\left(\eta^{-1}(g(s))\right)^{\lambda}\mathrm{d}s\right) \cdot w^{\lambda}\left(\eta^{-1}(g(t))\right). \end{aligned} \quad (35)$$

Using (16) in (35), we conclude that

$$w'(t) \geq \left(\int_{t}^{\infty}\frac{(l_1 l_2)^{\lambda}}{2^{\lambda}}q(s)\,\psi^{\lambda}(g(s))\left(\eta^{-1}(g(s))\right)^{\lambda}\mathrm{d}s\right) \cdot \left(\eta^{-1}(g(t))\right)^{\lambda}\left(w'\left(\eta^{-1}(g(t))\right)\right)^{\lambda}, \quad (36)$$

which yields

$$w'(t) \geq \left(\int_{t}^{\infty}\frac{(l_1 l_2)^{\lambda}}{2^{\lambda}}q(s)\,\psi^{\lambda}(g(s))\,\mathrm{d}s\right) \cdot \left(\eta^{-1}(g(t))\right)^{2\lambda}\left(w'\left(\eta^{-1}(g(t))\right)\right)^{\lambda}. \quad (37)$$

Using (13) and the fact that $\eta^{-1}(g(t)) \leq t$, we have

$$w'\left(\eta^{-1}(g(t))\right) \geq w'(t), \quad (38)$$

and so inequality (37) implies that

$$w'(t) \geq \left(\int_{t}^{\infty}\frac{(l_1 l_2)^{\lambda}}{2^{\lambda}}q(s)\,\psi^{\lambda}(g(s))\,\mathrm{d}s\right) \cdot \left(\eta^{-1}(g(t))\right)^{2\lambda}\left(w'(t)\right)^{\lambda}, \quad (39)$$

i.e.,

$$\left(w'(t)\right)^{1-\lambda} \geq \left(\int_{t}^{\infty}\frac{(l_1 l_2)^{\lambda}}{2^{\lambda}}q(s)\,\psi^{\lambda}(g(s))\,\mathrm{d}s\right)\left(\eta^{-1}(g(t))\right)^{2\lambda}. \quad (40)$$

Taking lim sup as $t \longrightarrow \infty$ in (40), we obtain a contradiction to (33).

Next, let case (II) hold. Then, we arrive at (22) and (23). For $t \geq s \geq t_2$, we see that $\eta^{-1}(g(t)) \geq \eta^{-1}(g(s))$. Putting $u = \eta^{-1}(g(s))$ and $v = \eta^{-1}(g(t))$ into (23), we get

$$z\left(\eta^{-1}(g(s))\right) \geq \frac{\left(\eta^{-1}(g(t)) - \eta^{-1}(g(s))\right)^{2}}{2}z''\left(\eta^{-1}(g(t))\right). \quad (41)$$

Integrating (22) from $\eta^{-1}(g(t))$ to t and using (41), we obtain

$$z''\left(\eta^{-1}\left(g\left(t\right)\right)\right) \geq \left(\int_{\eta^{-1}(g(t))}^{t} \frac{1}{2^{\lambda}} q\left(s\right) \varphi^{\lambda}\left(g\left(s\right)\right) \cdot \left(\eta^{-1}\left(g\left(t\right)\right) - \eta^{-1}\left(g\left(s\right)\right)\right)^{2\lambda} \mathrm{d}s\right) \cdot \left(z''\left(\eta^{-1}\left(g\left(t\right)\right)\right)\right)^{\lambda}, \tag{42}$$

which can be written as

$$\left(z''\left(\eta^{-1}\left(g\left(t\right)\right)\right)\right)^{1-\lambda} \geq \int_{\eta^{-1}(g(t))}^{t} \frac{1}{2^{\lambda}} q\left(s\right) \varphi^{\lambda}\left(g\left(s\right)\right) \cdot \left(\eta^{-1}\left(g\left(t\right)\right) - \eta^{-1}\left(g\left(s\right)\right)\right)^{2\lambda} \mathrm{d}s. \tag{43}$$

Taking lim sup as $t \longrightarrow \infty$ in (43), we obtain a contradiction to (34). The proof is complete. □

We conclude this paper with the following examples and remarks to illustrate the main results. The first example is concerned with the case where $p(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, whereas the second example is concerned with the case where p is a bounded function.

Example 7. Consider the sublinear Emden–Fowler neutral differential equation

$$\left(x\left(t\right) + tx\left(\frac{t}{2}\right)\right)''' + \frac{k}{t^{\alpha}} x^{1/3}\left(\frac{t}{4}\right) = 0, \quad t \geq 17, \tag{44}$$

where $k > 0$ and $0 < \alpha < 1$ are constants. Here $\lambda = 1/3$, $p(t) = t$, $\eta(t) = t/2$, $g(t) = t/4$, $q(t) = k/t^{\alpha}$, $\eta^{-1}(g(t)) = t/2$, and $\varphi(g(t)) = 2(1 - 1/t)/t$. Let $l_1 = 1/2$. Then $\psi(g(t)) = 2(1 - 16/t)/t$ and condition (31) becomes

$$\int_{17}^{\infty} \frac{k}{t^{\alpha}} \left(\frac{2t - 32}{t^2}\right)^{1/3} \left(\frac{t}{2}\right)^{2/3} \mathrm{d}t = \frac{1}{2^{2/3}} \int_{17}^{\infty} \frac{k}{t^{\alpha}} \left(2t - 32\right)^{1/3} \mathrm{d}t \geq \frac{1}{2^{1/3}} \int_{17}^{\infty} \frac{k}{t^{\alpha}} \mathrm{d}t = \infty. \tag{45}$$

Letting $\sigma(t) = t/3$, then $\eta^{-1}(\sigma(t)) = 2t/3$ and condition (32) reduces to

$$\int_{17}^{\infty} \frac{k}{t^{\alpha}} \left(\frac{2t - 2}{t^2}\right)^{1/3} \left(\frac{t}{6}\right)^{2/3} \mathrm{d}t = \frac{1}{6^{2/3}} \int_{17}^{\infty} \frac{k}{t^{\alpha}} \left(2t - 2\right)^{1/3} \mathrm{d}t \geq \frac{2^{5/3}}{6^{2/3}} \int_{17}^{\infty} \frac{k}{t^{\alpha}} \mathrm{d}t = \infty. \tag{46}$$

Therefore, by Corollary 5, (44) oscillates.

Example 8. Consider the linear differential equation

$$\left(x\left(t\right) + \frac{6t + 9}{t + 1} x\left(\frac{t}{2}\right)\right)''' + \frac{k}{t^3} x\left(\frac{t}{3}\right) = 0, \quad t \geq 1. \tag{47}$$

Here $\lambda = 1$, $p(t) = (6t + 9)/(t + 1)$, $\eta(t) = t/2$, $g(t) = t/3$, $q(t) = k/t^3$, and $k > 1390$ is a constant. It is easy to deduce that $6 \leq p(t) \leq 15/2$, $\eta^{-1}(g(t)) = 2t/3$, $\varphi(g(t)) \geq 1/9$, and $\psi(g(t)) \geq 1/25$ for some constant $l_1 \in (1/2, 1)$. Using $\lambda = 1$ in (33), we have

$$\left(\eta^{-1}\left(g\left(t\right)\right)\right)^{2\lambda} \int_{t}^{\infty} q\left(s\right) \psi^{\lambda}\left(g\left(s\right)\right) \mathrm{d}s \geq \frac{1}{25} \left(\frac{2t}{3}\right)^2 \int_{t}^{\infty} \frac{k}{s^3} \mathrm{d}s = \frac{2k}{225}. \tag{48}$$

That is, condition (33) with $\lambda = 1$ holds.

Next, using $\lambda = 1$ in (34), we obtain

$$\int_{\eta^{-1}(g(t))}^{t} q\left(s\right) \varphi^{\lambda}\left(g\left(s\right)\right) \cdot \left(\eta^{-1}\left(g\left(t\right)\right) - \eta^{-1}\left(g\left(s\right)\right)\right)^{2\lambda} \mathrm{d}s \geq \frac{1}{9} \cdot \int_{2t/3}^{t} \frac{k}{s^3} \left(\frac{2t - 2s}{3}\right)^2 \mathrm{d}s = \frac{k}{81} \cdot \int_{2t/3}^{t} \left(\frac{4t^2}{s^3} - \frac{8t}{s^2} + \frac{4}{s}\right) \mathrm{d}s = \frac{k}{81} \left(4 \ln \frac{3}{2} - \frac{3}{2}\right). \tag{49}$$

That is, condition (34) with $\lambda = 1$ holds. Therefore, by Theorem 6, (47) oscillates.

Remark 9. For a class of third-order Emden–Fowler delay differential equations with unbounded neutral coefficients (1), we established new oscillation criteria which complement and improve results in the cited papers because these criteria apply also in the case where p is unbounded and ensure that all solutions of (1) are oscillatory (that is, these results can distinguish solutions with different behaviors).

Remark 10. Using different methods, we improve results of Li and Rogovchenko [2] by removing restrictive condition $\eta \circ g = g \circ \eta$, which, in a certain sense, is a significant improvement compared to the results reported in the cited papers.

Remark 11. Combining Theorem 3 and the results obtained in [25], one can derive various oscillation criteria for (1) in the linear case. To study the oscillation of (1) in the superlinear case, it would be of interest to establish oscillation criteria for (3) and (4) assuming that $\lambda > 1$.

Remark 12. In the conclusion of Lemma 1, the existence of the constant $l \in (0, 1)$ is necessary in some cases. For instance, for $t \geq 2$, if $y(t) = t - 1/t$, then $y'(t) > 0$, $y''(t) < 0$, and $y(t) < ty'(t)$, and so the function y does not satisfy the conclusion of Lemma 1 provided that there is no $l \in (0, 1)$. On the basis of Lemma 1, one can easily revisit the results reported in [26–28].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All five authors contributed equally to this work and are listed in alphabetical order. They all read and approved the final version of the manuscript.

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References

[1] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, "Oscillation of second-order Emden-Fowler neutral delay differential equations," *Annali di Matematica Pura ed Applicata (1923 -)*, vol. 193, no. 6, pp. 1861–1875, 2014.

[2] T. Li and Yu. V. Rogovchenko, "Asymptotic behavior of higher-order quasilinear neutral differential equations," *Abstract and Applied Analysis*, vol. 2014, Article ID 395368, 11 pages, 2014.

[3] T. Li and Yu. V. Rogovchenko, "Oscillation of second-order neutral differential equations," *Mathematische Nachrichten*, vol. 288, no. 10, pp. 1150–1162, 2015.

[4] T. Li and Yu. V. Rogovchenko, "On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations," *Applied Mathematics Letters*, vol. 67, pp. 53–59, 2017.

[5] T. Li and Yu. V. Rogovchenko, "Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations," *Monatshefte für Mathematik*, vol. 184, no. 3, pp. 489–500, 2017.

[6] J. S. W. Wong, "On the generalized Emden-Fowler equation," *SIAM Review*, vol. 17, no. 2, pp. 339–360, 1975.

[7] B. Baculíková and J. Džurina, "Oscillation of third-order neutral differential equations," *Mathematical and Computer Modelling*, vol. 52, no. 1-2, pp. 215–226, 2010.

[8] B. Baculíková, B. Rani, S. Selvarangam, and E. Thandapani, "Properties of Kneser's solution for half-linear third order neutral differential equations," *Acta Mathematica Hungarica*, vol. 152, no. 2, pp. 525–533, 2017.

[9] Z. Došlá and P. Liška, "Oscillation of third-order nonlinear neutral differential equations," *Applied Mathematics Letters*, vol. 56, pp. 42–48, 2016.

[10] J. Džurina, S. R. Grace, and I. Jadlovská, "On nonexistence of Kneser solutions of third-order neutral delay differential equations," *Applied Mathematics Letters*, vol. 88, pp. 193–200, 2019.

[11] S. R. Grace, J. R. Graef, and M. A. El-Beltagy, "On the oscillation of third order neutral delay dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 63, no. 4, pp. 775–782, 2012.

[12] S. R. Grace, J. R. Graef, and E. Tunç, "Oscillatory behavior of a third-order neutral dynamic equation with distributed delays," in *Proceedings of the 10th Colloquium on the Qualitative Theory of Differential Equations*, pp. 1–14, Szeged, Hungary, 2016.

[13] S. R. Grace and I. Jadlovská, "Oscillatory behavior of odd-order nonlinear differential equations with a nonpositive neutral term," *Dynamic Systems and Applications*, vol. 27, no. 1, pp. 125–136, 2018.

[14] T. S. Hassan and S. R. Grace, "Oscillation criteria for third order neutral nonlinear dynamic equations with distributed deviating arguments on time scales," *Tatra Mountains Mathematical Publications*, vol. 61, no. 1, pp. 141–161, 2014.

[15] Y. Jiang, C. Jiang, and T. Li, "Oscillatory behavior of third-order nonlinear neutral delay differential equations," *Advances in Difference Equations*, vol. 2016, Article ID 171, 12 pages, 2016.

[16] T. Li and E. Thandapani, "Oscillation of solutions to odd-order nonlinear neutral functional differential equations," *Electronic Journal of Differential Equations*, vol. 2011, Article ID 23, 12 pages, 2011.

[17] B. Mihalíková and E. Kostiková, "Boundedness and oscillation of third order neutral differential equations," *Tatra Mountains Mathematical Publications*, vol. 43, no. 1, pp. 137–144, 2009.

[18] J. R. Graef, E. Tunç, and S. R. Grace, "Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation," *Opuscula Mathematica*, vol. 37, no. 6, pp. 839–852, 2017.

[19] T. Li and C. Zhang, "Properties of third-order half-linear dynamic equations with an unbounded neutral coefficient," *Advances in Difference Equations*, vol. 2013, Article ID 333, 8 pages, 2013.

[20] E. Tunç, "Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments," *Electronic Journal of Differential Equations*, vol. 2017, Article ID 16, 12 pages, 2017.

[21] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Academic, Dordrecht, The Netherlands, 1993, Translated from the 1985 Russian original.

[22] Ch. G. Philos, "On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays," *Archiv der Mathematik*, vol. 36, no. 1, pp. 168–178, 1981, https://link.springer.com/article/10.1007/BF01223686.

[23] R. G. Koplatadze and T. A. Chanturiya, "Oscillating and monotone solutions of first-order differential equations with deviating argument," *Differentsial'nye Uravneniya*, vol. 18, no. 8, pp. 1463–1465, 1472, 1982 (Russian).

[24] Y. Kitamura and T. Kusano, "Oscillation of first-order nonlinear differential equations with deviating arguments," *Proceedings of the American Mathematical Society*, vol. 78, no. 1, pp. 64–68, 1980.

[25] G. E. Chatzarakis and T. Li, "Oscillation criteria for delay and advanced differential equations with nonmonotone arguments," *Complexity*, vol. 2018, Article ID 8237634, 18 pages, 2018.

[26] R. P. Agarwal, M. Bohner, T. Li, and C. Zhang, "A new approach in the study of oscillatory behavior of even-order neutral delay differential equations," *Applied Mathematics and Computation*, vol. 225, pp. 787–794, 2013.

[27] T. Li and Yu. V. Rogovchenko, "Oscillation criteria for even-order neutral differential equations," *Applied Mathematics Letters*, vol. 61, pp. 35–41, 2016.

[28] E. Tunç and A. Kaymaz, "On oscillation of second-order linear neutral differential equations with damping term," *Dynamic Systems and Applications*, vol. 28, no. 2, pp. 289–301, 2019.